Faculty of Informatics - Papers (Archive)

Faculty of Engineering and Information Sciences

1991

Existence of SBIBD(4k2, 2k2 + k, k2 + k) and Hadamard matrices with maximal excess

Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Existence of SBIBD(4k2, 2k2 + k, k2 + k) and Hadamard matrices with maximal excess

Abstract

It is shown that SBIBD($4k^2$, $2k^2 \pm k$, $k^2 \pm k$) and Hadamard matrices with maximal excess exist for k = qs, q $\sum$ {q : q $\equiv$ 1 (mod 4) is a prime power}, s $\sum$ {l, ... ,33, 37, ... ,41,45, ... ,59} U {2g + 1,g the length of a Golay sequence}.

This leaves the following odd k < 250 undecided 47,71,77,79,103,107;127,131,133,139, 141,151,163,167,177,179,191,199,209, ... ,217,223,227, 231,233,237,239,243,249. There is also a proper n dimensional Hadamard matrix of order $(4k^2)^n$. Regular symmetric Hadamard matrices with constant diagonal are obtained for orders $4k^2$ whenever complete regular 4-sets of regular matrices of order k2 exist.

Disciplines

Physical Sciences and Mathematics

Publication Details

Existence of SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) and Hadamard matrices with maximal excess

Jennifer Seberry
Department of Computer Science
University College
The University of New South Wales
Australian Defence Force Academy
Canberra ACT 2600
Australia

Abstract

It is shown that SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) and Hadamard matrices with maximal excess exist for $k = qs$, $q \in \{q : q \equiv 1 \pmod 4)$ is a prime power$\}$, $s \in \{1, ..., 33, 37, ..., 41, 45, ..., 59\} \cup \{2g + 1, g$ the length of a Golay sequence$\}$.

This leaves the following odd $k < 250$ undecided 47,71,77,79,103,107,127,131,133,139, 141,151,163,167,177,179,191,199,209,...,217,223,227, 231,233,237,239,243,249.

There is also a proper n dimensional Hadamard matrix of order $(4k^2)^n$.

Regular symmetric Hadamard matrices with constant diagonal are obtained for orders $4k^2$ whenever complete regular 4-sets of regular matrices of order k^2 exist.

1 Introduction

We refer the reader to J. Wallis [8] and A.V. Geramita and J. Seberry [2] for undefined terms.

The *excess* of an Hadamard matrix is the sum of its elements. The maximal excess of all Hadamard matrices of order $4k^2$ is $8k^3$ and this is equivalent to the existence of an SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) (see Seberry [6], Best [1]).

Theorem 1 *Suppose there exist* $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ *supplementary difference sets. Then*

(i) there is an Hadamard matrix of order $4q^2$ with maximal excess $8q^3$,

(ii) there is an $SBIBD(4q^2, 2q^2 \pm q, q^2 \pm q)$,

(iii) there is a proper n dimensional Hadamard matrix of order $(4q^2)^n$.

Proof: Form the group ± 1 incidence matrix (type 1) for each of the sets. Then each row has $\frac{1}{2}q(q-1)$ elements plus one and $\frac{1}{2}q(q+1)$ elements minus one. Thus the row sum is $-q$. Negate each matrix to form four matrices A, B, C, D each with row sum q and thus excess q^3.

Then, recalling that the $1, -1$ incidence matrices of $n - \{v, k, \lambda\}$ supplementary difference sets have inner product, given by Lemma 1.20 [8], $4(nk - \lambda)I + (nv - 4nk + 4\lambda)J = 4q^2 I$ in this case, form

$$H = \begin{bmatrix} -A & BR & CR & DR \\ BR & A & D^T R & -C^T R \\ CR & -D^T R & A & B^T R \\ DR & C^T R & -B^T R & A \end{bmatrix},$$

where R transforms each matrix into its type 2 form (see Geramita and Seberry [2]). H is an Hadamard matrix of order $4q^2$ with excess $8q^3$.

We note each row of H has $3q(q+1)/2 + q(q-1)/2 = 2q^2 + q$ elements $+1$ and since H has constant inner product zero the underlying $(0,1)$ matrix (mapping -1 to 0) has constant inner product $q^2 + q$. Thus we have the required SBIBD.

Let $0,1,2,3$ be the cyclic group of order 4. Then the elements (i, d_j), $i = 0,1,2,3$, $d_j \in D_j$, $j = 1,2,3,4$ the supplementary difference sets (in this case with the extra properties of the D_j), form an abelian group difference set and so satisfy Theorem 4 of [3]. This gives the proper higher dimensional Hadamard matrix. $\quad\square$

Now M. Xia and G. Liu [9] have reported the existence of these supplementary difference sets for every $q \equiv 1 \pmod 4$ a prime power. Thus we have

Theorem 2 *Let* $q \equiv 1 \pmod 4$ *be a prime power. Then there is an* $SBIBD(4q^2, 2q^2 \pm q, q^2 \pm q)$.

Combining these results with those of Koukouvinos and Seberry [5] we have, noting that Koukouvinos, Kounias and Sotirakoglou [4] have now found T-sequences of lengths 55 and 57 which correspond to $s^2 = s^2 + 0^2 + 0^2 + 0^2$,

Corollary 3 $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ *and Hadamard matrices with maximal excess exist for*

(i) $k \in \{1, ..., 45, 49, ..., 69, 73, 75, 81, ..., 101, 105, 109, ..., 125, 129, 135, 137, 143, ..., 149, 153,$ $..., 161, 165, 169, ..., 175, 181, ..., 189, 193, ..., 197, 201, ..., 207, 219, 221, 225, 229, 235, 241,$ $245, 247\} \cup \{p : p \equiv 1 \pmod 4$ *is a prime power*$\} \cup \{2s + 1 : s$ *the length of Golay sequences*$\}$,

(ii) $k \in \{5(2s + 1), 5.3^i(2s + 1), i > 0, s$ *the length of Golay sequences*$\}$,

(iii) $k \in \{qs : q \equiv 1 \pmod 4$ *is a prime power and* s (odd) $= 1, ..., 33, 37, ..., 41, 45, ..., 59$ *or* $s = 2g + 1$, g *the length of Golay sequences*$\}$.

We recall Theorem 6 of [5] which uses T-sequences of length s^2 corresponding to the decomposition

$$s^2 = s^2 + 0^2 + 0^2 + 0^2$$

and Williamson-type matrices of order $4w^2$ corresponding to the decomposition

$$4w^2 = w^2 + w^2 + w^2 + w^2$$

to form Hadamard matrices with maximal excess of order $4k^2 = 4s^2w^2$ and an SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$.

Hence we have

Corollary 4 *Let $q \equiv 1 \pmod 4$ be a prime power. Then there exists an Hadamard matrix with maximal excess of order $4k^2 = 4q^2s^2$ and an $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ for s (odd) $\in \{1, ..., 33, 37, ..., 41, 45, ..., 59\} \cup \{2g + 1 : g \text{ the length of a Golay sequence}\}$.*

Proof: The required T-sequences are found in §3 of [5] and [4]. □

2 Regular matrices

We now define a *complete regular 4-set of regular matrices* of order q^2 as four $1, -1$ matrices satisfying

$$A_i^T = A_i, \quad A_iA_j = aJ, \quad i \neq j, \ i,j = 1,2,3,4, \ a \text{ constant}$$
$$A_iJ = qJ,$$
$$\sum_{i=1}^{4} A_i^2 = 4q^2I.$$

Then we have:

Theorem 5 *If there exist complete regular 4-sets of regular matrices of orders s^2 and t^2 respectively then there exists a complete regular 4-set of regular matrices of order s^2t^2.*

Proof: Let the sets of order s^2 and t^2 be A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 respectively, so $A_iJ = sJ$, $B_iJ = tJ$, $i = 1,2,3,4$. Then

$$C_1 = \frac{1}{2}[A_1 \times (B_1 + B_2) + A_2 \times (B_1 - B_2)]$$

$$C_2 = \frac{1}{2}[-A_1 \times (B_3 - B_4) + A_2 \times (B_3 + B_4)]$$

$$C_3 = \frac{1}{2}[A_3 \times (B_1 + B_2) - A_4 \times (B_1 - B_2)]$$

$$C_4 = \frac{1}{2}[A_3 \times (B_3 - B_4) + A_4 \times (B_3 + B_4)]$$

is a complete regular 4-set of regular matrices of order s^2t^2. □

A complete regular 4-set of regular matrices may be constructed from the following $4 - \{9; 6; 15\}$

$$\{x, x+1, x+2, 2x, 2x+1, 2x+2\}$$
$$\{1, 2, x, x+2, 2x, 2x+1\}$$
$$\{1, 2, x+1, x+2, 2x+1, 2x+2\}$$
$$\{1, 2, x, x+1, 2x, 2x+2\}$$

given by A.L. Whiteman (see also [7]). So we have

Corollary 6 *Complete regular 4-sets of regular matrices exist for orders 9^i, $i = 1, 2, ..., $.*

If we could establish the existence of complete regular 4-sets of regular matrices of orders $q_1^2, q_2^2, ...$ with row sums $q_1, q_2, ...$ respectively we would have

Theorem 7 *Let $q_1^2, q_2^2, ...$ be the orders of complete regular 4-sets of regular matrices with row sums $q_1, q_2, ...$ respectively. Write $w^2 = q_1^2 . q_2^2 . ....$ Then there is*

(i) a complete regular 4-set of matrices of order w^2

(ii) Williamson-type matrices of order w^2

(iii) SBIBD$(4w^2, 2w^2 \pm w, w^2 \pm w)$

(iv) a regular symmetric Hadamard matrix with constant diagonal of order $4w^2$ with maximal excess $8w^3$

(v) a proper higher dimensional Hadamard matrix of order $(4w^2)^n$.

Proof: The previous theorem gives (i). Any complete regular 4-set of regular matrices of order w^2 (and row sum $q_1.q_2. \ldots$) may be used as Williamson matrices giving (ii). Let the matrices be A, B, C, D then

$$H = \begin{bmatrix} A & B & C & -D \\ B & A & -D & C \\ C & -D & A & B \\ -D & C & B & A \end{bmatrix}$$

using the $A_i A_j = aJ$, $i \neq j$ property is a regular symmetric Hadamard matrix with constant diagonal of order $4w^2$ and row sum $2w$. Hence H has excess $8w^3$ which is maximal for the order giving (iii) and (iv). The higher dimensional Hadamard matrix is constructed as in the proof of Theorem 1. □

3 Decomposition into squares

Complete regular 4-sets of regular matrices of order w^2 give Williamson-type matrices of order $4w^2$ corresponding to the decomposition

$$4w^2 = w^2 + w^2 + w^2 + w^2.$$

So recalling Theorem 6 of [5] which uses T-sequences of length s^2 corresponding to the decomposition

$$s^2 = s^2 + 0^2 + 0^2 + 0^2$$

we have

Lemma 8 *Suppose there exist regular 4-sets of regular matrices of order w^2. Then there exists an Hadamard matrix with maximal excess of order $4k^2 = 4w^2s^2$ and an SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$ for s(odd) $\in \{1, ..., 33, 37, ..., 41, 45, ..., 59\} \cup \{2g + 1 : g$ the length of a Golay sequence$\}$.*

Remark: Hadamard matrices of order $4k^2$ are quite large but the limited knowledge of the family SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$ makes these results worth pursuing.

References

[1] M.R. Best. The excess of Hadamard matrix. *Indag. Math.*, 39:357-361, 1977.

[2] A.V. Geramita and J. Seberry. *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*. Marcel Dekker, New York-Basel, 1979.

[3] J. Hammer and J. Seberry. Higher dimensional orthogonal designs and Hadamard matrices II. *Congressus Numerantium*, XXVII:23-29, 1979.

[4] C. Koukouvinos, S. Kounias and K. Sotirakoglou. On base and Turyn sequences. *Math. Comp.* (to appear).

[5] C. Koukouvinos and J. Seberry. Construction of new Hadamard matrices with maximal excess and infinitely many new SBIBD$(4k^2, 2k^2 + k, k^2 + k)$. *Discrete Mathematics*. A special edition edited by Rolf Rees in honour of Norman J. Pullman, Annals of Discrete Mathematics (to appear).

[6] J. Seberry. SBIBD$(4k^2, 2k^2 + k, k^2 + k)$ and Hadamard matrices of order $4k^2$ with maximal excess are equivalent. *Graphs and Combinatorics*, 5:373-383, 1989.

[7] J. Seberry and A.L. Whiteman. New Hadamard matrices and conference matrices obtained via Mathon's construction. *Graphs and Combinatorics*, 4:355-377, 1988.

[8] J.S. Wallis. Hadamard matrices. In *Combinatorics: Room squares, sum-free sets and Hadamard matrices*, Volume 292 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1972. Part IV of W.D. Wallis, Anne Penfold Street, and Jennifer Seberry Wallis.

[9] M. Xia and G. Liu. An infinite class of supplementary difference sets and Williamson matrices. *Journal of Combinatorial Theory (Series A)* (to appear).