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Supplementary difference sets and optimal designs

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An infinite family of Hadamard matrices of order $n = 4v$ with maximum excess

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Supplementary difference sets and optimal designs

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Abstract

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D-optimal designs of order $n = 2v \equiv 2 \pmod 4$, where q is a prime power and $v = q^2 + q + 1$ are constructed using two methods, one with supplementary difference sets and the other using projective planes more directly.

An infinite family of Hadamard matrices of order $n = 4v$ with maximum excess $\sigma(n) = n\sqrt{n-3}$ where q is a prime power and $v = q^2 + q + 1$ is a prime, is also constructed.

1. Introduction

In [17–18] (Seberry) Wallis has given the following definition of *supplementary difference sets*:

If $B = \{b_1, b_2, \ldots, b_{k_1}\}$, $D = \{d_1, d_2, \ldots, d_{k_2}\}$ are two collections of k_1, k_2 residues mod v such that the congruence

$$b_i - b_j \equiv a \pmod v, \qquad d_i - d_j \equiv a \pmod v$$

has exactly λ solutions for any $a \not\equiv 0 \pmod v$ then B, D are called supplementary difference sets (abbreviated as SDS), denoted by 2-$\{v; k_1, k_2; \lambda\}$.

In [5] Elliott and Butson have given the following definition of a *relative difference set*:

A set D of k elements in a group G of order vm is a difference set of G relative to a normal subgroup F of order $m \neq vm$ if the collection of differences

$r-s$, $r, s \in D$, $r \neq s$ contains only the elements of G which are not in F, and contains every such element exactly λ times. This relative difference set (abbreviated as RDS) will be denoted by $R(v, m, k, \lambda)$.

In this paper we consider the case $m=2$, i.e. $R(v, 2, k, \lambda)$. These RDS are called also *near difference sets* (see Ryser [13]). In [5] Elliott and Butson proved that if q is an odd prime power, then we can construct cyclic relative difference sets $R(v, 2, k, \lambda)$, where

$$n = 2v = 2(q^2+q+1), \qquad k = q^2, \qquad \lambda = \tfrac{1}{2}q(q-1) \tag{1}$$

Spence [16] showed that the construction of Elliott and Butson is also valid when q is a power of 2. For the construction of these $R(v, 2, k, \lambda)$ see also [11–12].

If $n \equiv 2 \pmod 4$, $v = n/2$ and R_1, R_2 are $v \times v$ commuting matrices, with elements ± 1, such that

$$R_1R_1^T + R_2R_2^T = (2v-2)I_v + qJ_v \tag{2}$$

then the $n \times n$ matrix

$$R = \begin{bmatrix} R_1 & R_2 \\ -R_2^T & R_1^T \end{bmatrix} \tag{3}$$

has the maximum determinant (Ehlich [4]) among all $n \times n \pm 1$ matrices.

Such matrices R are called D-*optimal designs* of order n and their construction is known for the following values of n: 2, 6, 10, 14, 18, 26, 30, 38, 42, 46, 50, 54, 62, 66, 82, 86 (Ehlich [4], Yang [20–24], Chadjipantelis and Kounias [2], Chadjipantelis, Kounias and Moyssiadis [3]).

If R_1, R_2 are circulant, then pre- and post-multiplying both sides of (2) by e^T and e respectively we obtain

$$(v-2k_1)^2 + (v-2k_2)^2 = 4v-2 \tag{4}$$

where e is the $v \times 1$ matrix of 1's and k_1, k_2 is the number of -1's in every row of R_1, R_2 respectively.

If R_1, R_2 satisfy (2) so do $\pm R_1, \pm R_2$, i.e. we can always take $1 \leq k_1 \leq k_2 \leq (v-1)/2$.

In [2] Chadjipantelis and Kounias proved that the existence of 2-$\{v; k_1, k_2; \lambda\}$ SDS, where k_1, k_2 satisfy (4) and $\lambda = k_1 + k_2 - (v-1)/2$ is equivalent to the existence of D-optimal designs of order $n = 2v \equiv 2 \pmod 4$. In this paper we construct D-optimal designs for $n \equiv 2 \pmod 4$ by using SDS.

Now we give some basic definitions.

An *Hadamard matrix*, called H-*matrix*, of order n is an $n \times n$ matrix H with elements $+1, -1$ satisfying

$$H^TH = HH^T = nI_n.$$

The sum of the elements of H, denoted by $\sigma(H)$, is called *excess* of H. The *maximum excess* of H, over all H-matrices of order n, is denoted by $\sigma(n)$, i.e.

$$\sigma(n) = \max \sigma(H) \text{ for all H-matrices of order } n \tag{5}$$

An equivalent notion is the *weight* $w(H)$ which is the number of 1's in H, then $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$, see [9–10].

Kounias and Farmakis [10] proved that $\sigma(n) = n\sqrt{n}$ when $n = 4(2m+1)^2$ and a regular H-matrix exists thus satisfying the equality of Best's [1] inequality,

$$\sigma(n) \leqslant n\sqrt{n}.$$

Infinite families of H-matrices satisfying this bound have been found by Seberry [14] and Yamada [19].

Also, Kounias and Farmakis [10] proved that $\sigma(n) = n\sqrt{n-3}$ can be attained when $n = (2m+1)^2 + 3$ thus satisfying the equality of the Hammer–Levingston–Seberry [9] bound,

$$\sigma(n) \leqslant n\sqrt{n-3}$$

for this bound. This is discussed further in Section 3.

In this paper we also construct an infinite family of H-matrices of order $n = 4v$ with maximum excess $\sigma(n) = n\sqrt{n-3}$, where q is a prime power and $v = q^2 + q + 1$ is a prime.

2. On D-optimal designs of order $n \equiv 2 \pmod 4$

Spence [16] proved the following theorem.

Theorem 1 (Spence). *If there exists a cyclic projective plane of order q^2 then there exist two ± 1 matrices R_1, R_2, both circulant and of order $1 + q + q^2$, such that*

$$R_1R_1^{\mathrm{T}} + R_2R_2^{\mathrm{T}} = 2q(q+1)I + 2J \tag{6}$$

where I is the identity matrix of order $1 + q + q^2$ and J is the square matrix of order $1 + q + q^2$, all the entries of which are $+1$.

Now, by using the circulant matrices R_1, R_2 constructed by Spence in Theorem 1, and the matrix R in (3), we note the following theorem.

Theorem 2. *There exist* D-*optimal designs of order $n \equiv 2 \pmod 4$, where q is a prime power and*

$$n = 2v = 2(q^2 + q + 1).$$

Proof. Let $D = \{d_1, d_2, \ldots, d_k\}$ be a $R(v, 2, k, \lambda)$ as in (1) and $v = q^2 + q + 1$. The following two sets

$$\begin{aligned} D_1 &= \{(d+v)/2 \pmod v, \quad d \in D,\ d \text{ odd}\} \\ D_2 &= \{d/2 \pmod v, \quad d \in D,\ d \text{ even}\} \end{aligned} \tag{7}$$

constitute 2-$\{v, k_1, k_2; \lambda = k_1 + k_2 - (v-1)/2\}$ SDS, where

$$v = q^2 + q + 1, \qquad k_1 = \frac{q(q-1)}{2}, \qquad k_2 = \frac{q(q+1)}{2},$$
$$k_1 + k_2 = k = q^2, \qquad \lambda = k_1 + k_2 - \frac{v-1}{2} = k_1 \tag{8}$$

satisfying (4) (see Spence [16], Seberry Wallis and Whiteman [15]).

Since a $R(v, 2, k, \lambda)$ exists when q is a prime power, this completes the proof of Theorem 2. □

The matrices R_1, R_2 are the incidence circulant matrices of SDS described in (7) and are constructed by setting -1 in the positions indicated in D_1, D_2 respectively and $+1$ in the remaining positions. The following examples which are given in Table 1 illustrate the cases $q = 2, 3, 4, 5, 7$ of Theorem 2.

We give another proof of the above result which indicates possibilities for inequivalences and has less restrictions on the underlying structures.

First we note that a matrix, W, of order n with entries $0, +1, -1$, exactly k nonzero entries in each row and column and inner product of distinct rows zero is called a *weighing matrix* denoted $W = W(n, k)$. In fact

$$WW^{T} = kI_n,$$

and a $W(n, n)$ is an Hadamard matrix.

Theorem 3. *Let Q and P be the incidence matrices of $(q^2+q+1, q+1, 1)$ difference sets. Further suppose QP has elements $0, 1, 2$. Then $W = QP - J$ is a weighing matrix of order q^2+q+1 and weight q^2 that is $WW^T = q^2 I$ and W has entries $0, 1, -1$. Furthermore if $W = X - Y$, where X and Y have entries $0, 1$ then $R = J - X - Y$ satisfies $RR^T = qI + J$, $RJ = (q+1)J$.*

Proof. Since P and Q are incidence matrices of $(q^2+q+1, q+1, 1)$ difference sets

$$PP^T = QQ^T = qI + J, \qquad PJ = QJ = (q+1)J$$

where P, Q, I, J are of order q^2+q+1. Now

$$\begin{aligned} WW^T &= (QP-J)(P^TQ^T - J) = QPP^TQ^T - JP^TQ^T - QPJ + J^2 \\ &= Q(qI+J)Q^T - 2(q+1)^2 + J^2 = qQQ^T - (q+1)^2J + J^2 \\ &= q^2I + qJ - (q^2+2q+1-q^2-q-1)J = q^2I. \end{aligned}$$

Since PQ had entries $0, 1, 2$ $PQ - J$ must have entries $0, 1, -1$.

Now $WJ = QPJ - J^2 = (q+1)^2J - J^2 = qJ$. So $WJ = (X-Y)J = qJ$. $WW^T = q^2I$

Table 1
$R(v, 2, k, \lambda)$ where v, k, λ satisfy (1) and SDS 2-$\{v; k_1, k_2; \lambda\}$ where v, k_1, k_2, λ satisfy (8)

$n = 14,\ q = 2,\ v = 7,\ k = 4,\ k_1 = 1,\ k_2 = 3;\ \lambda = 1$

(i) $D = \{0, 1, 4, 6\}$
$D_1 = \{4\}$
$D_2 = \{0, 2, 3\}$

(ii) $D = \{0, 3, 5, 13\}$
$D_1 = \{3, 5, 6\}$
$D_2 = \{0\}$

$n = 26,\ q = 3,\ v = 13,\ k = 9,\ k_1 = 3,\ k_2 = 6;\ \lambda = 3$

(i) $D = \{0, 1, 6, 8, 10, 11, 12, 15, 18\}$
$D_1 = \{1, 7, 12\}$
$D_2 = \{0, 3, 4, 5, 6, 9\}$

(ii) $D = \{0, 1, 2, 8, 11, 18, 20, 22, 23\}$
$D_1 = \{5, 7, 12\}$
$D_2 = \{0, 1, 4, 9, 10, 11\}$

(iii) $D = \{4, 5, 7, 10, 11, 12, 15, 19, 21\}$
$D_1 = \{1, 3, 4, 9, 10, 12\}$
$D_2 = \{2, 5, 6\}$

(iv) $D = \{5, 8, 15, 17, 19, 20, 23, 24, 25\}$
$D_1 = \{1, 2, 3, 5, 6, 9\}$
$D_2 = \{4, 10, 12\}$

(v) $D = \{2, 4, 6, 7, 10, 11, 12, 18, 21\}$
$D_1 = \{4, 10, 12\}$
$D_2 = \{1, 2, 3, 5, 6, 9\}$

$n = 42,\ q = 4,\ v = 21,\ k = 16,\ k_1 = 6,\ k_2 = 10;\ \lambda = 6$

(i) $D = \{0, 1, 10, 11, 18, 20, 23, 25, 26, 29, 30, 34, 36, 37, 38, 40\}$
$D_1 = \{1, 2, 4, 8, 11, 16\}$
$D_2 = \{0, 5, 9, 10, 13, 15, 17, 18, 19, 20\}$

(ii) $D = \{0, 2, 4, 5, 6, 8, 12, 13, 16, 17, 19, 22, 24, 31, 32, 41\}$
$D_1 = \{5, 10, 13, 17, 19, 20\}$
$D_2 = \{0, 1, 2, 3, 4, 6, 8, 11, 12, 16\}$

$n = 62,\ q = 5,\ v = 31,\ k = 25,\ k_1 = 10,\ k_2 = 15;\ \lambda = 10$

(i) $D = \{0, 1, 2, 3, 5, 6, 7, 9, 10, 13, 15, 17, 23, 24, 25, 26, 30, 35, 39, 42, 45, 50, 51, 53, 58\}$
$D_1 = \{2, 4, 7, 10, 11, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28\}$
$D_2 = \{0, 1, 3, 5, 12, 13, 15, 21, 25, 29\}$

(ii) $D = \{0, 1, 2, 5, 7, 9, 10, 21, 22, 25, 29, 34, 35, 37, 39, 43, 44, 45, 46, 48, 50, 51, 54, 57, 61\}$
$D_1 = \{2, 3, 4, 6, 7, 10, 13, 15, 16, 18, 19, 20, 26, 28, 30\}$
$D_2 = \{0, 1, 5, 11, 17, 22, 23, 24, 25, 27\}$

$n = 114,\ q = 7,\ v = 57,\ k = 49,\ k_1 = 21,\ k_2 = 28;\ \lambda = 21$

(i) $D = \{0, 8, 10, 12, 15, 18, 20, 22, 23, 25, 26, 32, 34, 39, 40, 41, 43, 45, 46, 47, 50, 51, 52, 55, 56, 59, 60, 61, 62, 68, 70, 71, 73, 74, 78, 81, 84, 85, 86, 87, 88, 90, 92, 93, 94, 101, 105, 110, 111\}$
$D_1 = \{1, 2, 7, 8, 12, 14, 15, 18, 22, 24, 27, 36, 40, 41, 48, 49, 50, 51, 52, 54, 56\}$
$D_2 = \{0, 4, 5, 6, 9, 10, 11, 13, 16, 17, 20, 23, 25, 26, 28, 30, 31, 34, 35, 37, 39, 42, 43, 44, 45, 46, 47, 55\}$

(ii) $D = \{0, 2, 3, 4, 8, 10, 11, 14, 21, 22, 23, 24, 27, 28, 31, 32, 33, 34, 36, 37, 39, 40, 43, 45, 47, 48, 50, 52, 54, 55, 56, 62, 69, 70, 72, 73, 74, 75, 77, 82, 83, 86, 87, 92, 98, 101, 103, 108, 110\}$
$D_1 = \{6, 8, 9, 10, 13, 15, 22, 23, 30, 34, 39, 40, 42, 44, 45, 47, 48, 50, 51, 52, 56\}$
$D_2 = \{0, 1, 2, 4, 5, 7, 11, 12, 14, 16, 17, 18, 20, 24, 25, 26, 27, 28, 31, 35, 36, 37, 41, 43, 46, 49, 54, 55\}$

says W has q^2 entries 1 or -1 in each row, say x ones and y minus ones. Then

$$x - y = q \qquad x + y = q^2$$

and thus

$$x = \tfrac{1}{2}q(q+1), \qquad y = \tfrac{1}{2}q(q-1).$$

Now any row of W has $x = \frac{1}{2}(q^2 + q)$ ones, $y = \frac{1}{2}(q^2 - q)$ minus ones and $q + 1$ zeros.

Write any two rows of W as

$$\begin{array}{l}
1 \cdots\cdots\cdots\cdots\cdots\cdots 1 \quad - \cdots\cdots\cdots\cdots\cdots\cdots - \quad 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\
\underbrace{1\cdots1}_{a} \; \underbrace{-\cdots-}_{c} \; \underbrace{0\cdots0}_{e} \; \underbrace{1\cdots1}_{b} \; \underbrace{-\cdots-}_{d} \; \underbrace{0\cdots0}_{f} \; \underbrace{1\cdots1}_{x-a-b} \; \underbrace{-\cdots-}_{y-c-d} \; \underbrace{0\cdots0}_{q+1-e-f}
\end{array}$$

where there are, for example a columns $\binom{1}{1}$ and f columns $\binom{-1}{0}$.

Now the number of columns $\binom{0}{0}$ is $q + 1 - e - f$. Furthermore the inner product of each pair of rows is zero so $a + b - c - d = 0$. Also

$$a + c + e = x \quad \text{(number of ones in first row)}$$

$$b + d + f = y \quad \text{(number of minus ones in first row).}$$

Hence

$$\begin{aligned}
q + 1 - e - f &= q + 1 + a + c - x + b + d - y = -q^2 + q + 1 + a + c + b + d \\
&= -q^2 + q + 1 + 2c + 2d \quad (\text{using } a + b - c - d = 0) \\
&\leqslant -q^2 + q + 1 + q^2 - q \quad \text{(number of minus ones in second row)} \\
&\leqslant 1.
\end{aligned}$$

Now $1 \geqslant q + 1 - e - f \geqslant 0$. Suppose $q + 1 - e - f = 0$ then using

$$\begin{aligned}
a + b + c + d + e + f &= q^2 \\
a + b - c - d \qquad &= 0 \\
e + f &= q + 1
\end{aligned}$$

We have

$$2a + 2b = q^2 + q + 1.$$

But $q^2 + q + 1$ is always odd. So we have a contradiction and $q + 1 - e - f = 1$. In other words each row of W has $q + 1$ zeros and in each pair of rows of W exactly one zero is underneath a zero. Thus if $R = J - X - Y$ is the matrix with ones where W had zeros R is the incidence matrix of a $(q^2 + 1 + 1, q + 1, 1)$ configuration. So

$$RR^{\mathrm{T}} = qI + J \quad \text{and} \quad RJ = (q+1)J.$$

Furthermore if P and q were defined on a cyclic (abelian) group, R is defined on the same group.

Theorem 4. *There exist two matrices A and B of order* q^2+q+1 *which satisfy*

$$AA^{\mathrm{T}} + BB^{\mathrm{T}} = 2(q^2+q)I + 2J.$$

Proof. Let $A = W + R$ and $B = W - R$ be defined as above. □

Corollary 5. *There is a* D*-optimal design of order* $2(q^2+q+1)$ *whenever there is a* $(q^2+q+1,\ q+1,\ 1)$ *difference set.*

Proof. Use

$$\begin{bmatrix} A & B \\ -B^{\mathrm{T}} & A^{\mathrm{T}} \end{bmatrix}$$

as before. □

Remark 1. This construction does not require the difference set to be defined on a cyclic group. Glynn [7], Geramita and Seberry [6, p. 152] have shown the conditions of the theorem can be met, for example if $P = Q$ in theorem.

Remark 2. We note that the sets D_1 and D_2 of $2-\{v; k_1, k_2; \lambda\}$ SDS described in (7) are disjoint.

For if

$$\frac{d_i+v}{2} \equiv \frac{d_j}{2} \pmod v$$

then $d_i - d_j \equiv v \pmod{2v}$, $(d_i, d_j \in D)$ in violation of the definition of a RDS. (see Seberry Wallis and Whiteman [15]).

D-optimal designs have been constructed for $n = 14$, $n = 26$ by Ehlich [4] and Yang [22] and for $n = 42$, $n = 62$ by Yang [20, 23] and Chadjipantelis and Kounias [2]. All the other orders of D-optimal designs which are constructed by the above method are new.

3. The maximum excess of Hadamard matrices of order $n = 4v$

First we show that the Hammer–Levingston–Seberry [9, p. 246] bound for $n = (2m+1)^2 + 3$ is the same as that found by Kounias and Farmakis [10, section 4].

Hammer, Levingston and Seberry [9, p. 217] show that for H-matrices of order n, writing x for the greatest even integer $<\sqrt{n}$, $t = x$ if $|n - x^2| < |(x+2)^2 - n|$ and $t = x - 2$ otherwise, i the integer part of $n((t+4)^2 - n)/8(t+2)$, the excess of the H-matrices is bounded by

$$\sigma(n) = n(t+4) - 4i.$$

Write $n=(2m+1)^2+3=4(m^2+m+1)$. Now x, even, is the greatest even integer $<\sqrt{n}$.

Let $x=2a$, then $2a<\sqrt{n}$ and

$$4m^2 \leq 4a^2 < 4(m^2+m+1) < 4(m+1)^2$$

Hence $m \leq a < m+1$.

Thus we can write

$$x=2a=2m, \qquad t=x-2=2m-2 \quad \text{and} \quad i=m^2+m+1.$$

Hence

$$\sigma(n) \leq (2m+2)-4i = n(2m+2)-n = n(2m+1) = n\sqrt{n-3}$$

This was the result given in Kounias and Farmakis [10]. We summarize this as the following lemma.

Lemma 6. *The Hammer–Levingston–Seberry bound is equivalent to* $\sigma(n) \leq n(2m+1) = n\sqrt{n-3}$ *when* $n=(2m+1)^2+3$.

Kounias and Farmakis [10] proved that $\sigma(n)=n\sqrt{n-3}$ can be attained when $n=(2m+1)^2+3$ thus satisfying the equality of the above bound.

Spence [16] proved the following theorem.

Theorem 7 (Spence). *If there exists a cyclic projective plane of order* q^2 *and two supplementary difference sets in a cyclic group of order* $1+q+q^2$, *then there exists a Hadamard matrix of the Goethals–Seidel type of order* $4(1+q+q^2)$.

Now, from this theorem of Spence we note the following theorem.

Theorem 8. *There exist* H-*matrices of order* $n=(2q+1)^2+3$, *with maximum excess* $\sigma(n)=n\sqrt{n-3}$, *where* q *is a prime power and* $v=q^2+q+1$ *is a prime.*

Proof. It is easy to see (Spence [16], Seberry Wallis and Whiteman [15]) that if $v=q^2+q+1$ is a prime, then we can construct two sets D_3 and D_4 as

$$2-\left\{v; k_3, k_4; k_3+k_4-\frac{v+1}{2}\right\} \tag{9}$$

SDS, where D_3 is the set of quadratic residues of v, and D_4 is the set of quadratic nonresidues of v, $k_3=k_4=q(q+1)/2$, $\lambda=k_3+k_4-(v+1)/2=q(q+1)/2-1$.

By using (7) and (9) SDS, we can construct a

$$4-\left\{v; k_1, k_2, k_3, k_4; \lambda=\sum_{i=1}^{4} k_i - v\right\}$$

which may be used to construct H-matrices (H_{4v}) of the Goethals–Seidel type.

Now, it is obvious that $n = 4v = 4(q^2 + q + 1) = (2q + 1)^2 + 3$, and from Lemma 3 and the result of Kounias and Farmakis [10], we note that these H-matrices have maximum excess $\sigma(n) = n\sqrt{n-3}$. □

If we construct the R_3, R_4 incidence circulant matrices of (9) SDS, we have

$$R_3R_3^{\mathrm{T}} + R_4R_4^{\mathrm{T}} = 2(q^2 + q + 2)I_v - 2J_v. \tag{10}$$

Hence from (6) and (10) we obtain:

$$R_1R_1^{\mathrm{T}} + R_2R_2^{\mathrm{T}} + R_3R_3^{\mathrm{T}} + R_4R_4^{\mathrm{T}} = 4(q^2 + q + 1)I_v = 4vI_v. \tag{11}$$

The following matrix G, whose construction is due to Goethals and Seidel [8], is an H-matrix of order $4(q^2 + q + 1)$:

$$G = \begin{bmatrix} R_1 & R_2W & R_3W & R_4W \\ -R_2W & R_1 & -R_4^{\mathrm{T}}W & R_3^{\mathrm{T}}W \\ -R_3W & R_4^{\mathrm{T}}W & R_1 & -R_2^{\mathrm{T}}W \\ -R_4W & R_3^{\mathrm{T}}W & R_2^{\mathrm{T}}W & R_1 \end{bmatrix} \tag{12}$$

where $W = [w_{ij}]$ is the permutation matrix of order $v = q^2 + q + 1$ defined by

$$w_{ij} = \begin{cases} 1, & \text{if } i + j \equiv 1 \pmod{v}, \\ 0, & \text{otherwise.} \end{cases}$$

The circulant $(1, -1)$ matrices R_1, R_2, R_3, R_4 of order v, have row sums $2q + 1, 1, 1, 1$ respectively, then G gives the row-sum vector $(2qe_{3n/4}^{\mathrm{T}}, (2q + 4)e_{n/4}^{\mathrm{T}})$ where re_s^{T} denotes the $1 \times s$ vector $(r, r, \ldots, r)$.

Example. From Theorem 8 we obtain the following orders of H-matrices with maximum excess:

$n = 28$ $(q = 2, v = 7)$,
$n = 52$ $(q = 3, v = 13)$,
$n = 124$ $(q = 5, v = 31)$,
$n = 292$ $(q = 8, v = 73)$,
$n = 1228$ $(q = 17, v = 307)$,
$n = 3028$ $(q = 27, v = 757)$,
$n = 6892$ $(q = 41, v = 1723)$,
$n = 14164$ $(q = 59, v = 3541)$, etc.

H-matrices with maximum excess have been constructed for $n = 28$, $n = 52$, $n = 124$ from the results of Hammer, Levingston and Seberry [9] using Williamson-type matrices alone, or from the results of Kounias and Farmakis [10]. All the other orders of H-matrices with maximum excess are new.

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