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A Construction for Orthogonal Designs with Three Variables

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TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY

ABSTRACT

We show how orthogonal designs $OD(48p^2t; 16p^2t, 16p^2t, 16p^2t)$ can be constructed from an Hadamard matrix of order $4p$ and an $OD(4t; t, t, t)$. This allows us to assert that $OD(48p^2t; 16p^2t, 16p^2t, 16p^2t)$ exist for all $t, p \leq 102$ except possibly for $t \in \{67, 71, 73, 77, 79, 83, 86, 89, 91, 97\}$. These designs are new.

1. Introduction

Let $H = (h_{ij})$ be a matrix of order h with $h_{ij} \in \{1, -1\}$. H is called an *Hadamard matrix* of order n , if $HH^T = hI_h$ where I_h denotes the identity matrix order of h .

An *orthogonal design* A , of order n , type $(p_1, p_2, \dots, p_u)$, denoted $OD(n; p_1, p_2, \dots, p_u)$, on the commuting variables $(\pm x_1, \pm x_2, \dots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occur p_k times in each row and column such that the rows are pairwise orthogonal.

In other words

$$AA^T = (p_1x_1^2 + \dots + p_ux_u^2)I_n.$$

It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d$, $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

$OD(4t; t, t, t)$, otherwise called Baumert-Hall arrays, and $OD(2^p; a, b, 2^p - a - b)$ have been extensively used to construct Hadamard matrices and weighing matrices. For details see Geramita and Seberry (1979).

Geramita, Geramita and Wallis (=Seberry) observed (see Geramita and Seberry [1979, §4.3]) that if A, B, C, D are four circulant or type 1 matrices of order n satisfying

$$AA^T + BB^T + CC^T + DD^T = \left(\sum_{i=1}^u p_i x_i^2 \right) I_n$$

then A, B, C, D can be used in the Goethals-Seidel array or (J. Seberry Wallis-Whiteman array)

$$\begin{bmatrix} A & BR & CR & -DR \\ -BR & A & D^T R & -C^T R \\ CR & -D^T R & A & B^T R \\ DR & C^T R & -B^T R & A \end{bmatrix} \quad (1).$$

to form an orthogonal design $OD(4n; p_1, p_2, \dots, p_u)$.

2. Background

Kharaghani (1985) defined $C_k = [h_{ij}, h_{kj}]$ and applying that to Hadamard matrices of order $4p$, obtained matrices satisfying

$$C_i C_j = 0, i \neq j \quad (2).$$

$$\sum_{i=1}^{4p} C_i^2 = (4p)^2 I_{4p}.$$

He then used this to show there are Bush-type (blocks J_{4p} down the diagonal) and Szekeres-type ($h_{ij} = -1 \Rightarrow h_{ji} = 1$ and not necessarily vice versa) Hadamard matrices. By using a symmetric Latin square he could also have shown that regular symmetric Hadamard matrices with constant diagonal of order $(4p)^2$ could be constructed by his method.

Hammer, Sarvate and Seberry applied Kharaghani's method to $OD(n; s_1, \dots, s_u)$ and in particular $OD(4t; t, t, t, t)$ and $OD(4s; s, s, s, s)$ obtaining existence of $OD(48s^2t; 12s^2t, 12s^2t, 12s^2t, 12s^2t)$ and $OD(80s^2t; 20s^2t, 20s^2t, 20s^2t, 20s^2t)$ from $OD(4s; s, s, s, s)$ and $OD(4t; t, t, t, t)$. Seberry (to appear) extended this further to obtain $OD(16ks^2t; 4ks^2t, 4ks^2t, 4ks^2t, 4ks^2t)$ for $\{1, 3, 5, \dots\}$.

We modify their techniques to obtain new orthogonal designs.

3. Construction

Let $C_1, C_2, \dots, C_{4p}$ be the Kharaghani matrices of order $4p$ obtained from an orthogonal design $OD(4p; p, p, p, p)$.

Let a, b, c be commuting variables and write

$$\begin{aligned} &[aC_1 : aC_2 : \dots : aC_p : bC_{p+1} : \dots : bC_{2p} : cC_{2p+1} : \dots : cC_{3p}] \\ &[aC_{3p+1} : \dots : aC_{4p} : cC_{p+1} : \dots : cC_{2p} : bC_{2p+1} : \dots : bC_{3p}] \end{aligned}$$

for the first rows of two block circulant matrices W_1 and W_2 . It can be checked that $W_1 W_2^T = W_2 W_1^T$. Write

$$\begin{aligned} & [aC_{2p+1} : \dots : aC_{3p} : bC_1 : \dots : bC_p : cC_{3p+1} : \dots : cC_{4p}] \\ & [aC_{p+1} : \dots : aC_{2p} : bC_1 : \dots : bC_p : bC_{3p+1} : \dots : bC_{4p}] \end{aligned}$$

for the first rows of two block back-circulant matrices W_3 and W_4 . It can be checked that $W_3 W_4^T = W_4 W_3^T$.

Now by virtue of being circulant and back-circulant

$$W_i W_j^T = W_j W_i^T \quad i \in \{1, 2\}, \quad j \in \{3, 4\}.$$

Example. Let $p = 3$ so there are 12 matrices of order 12, $C_1, \dots, C_{12}$. Then

$$\begin{aligned} W_1 &= \begin{bmatrix} aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 & cC_7 & cC_8 & cC_9 \\ aC_3 & aC_1 & aC_2 & bC_6 & bC_4 & bC_5 & cC_9 & cC_7 & cC_8 \\ aC_2 & aC_3 & aC_1 & bC_5 & bC_6 & bC_4 & cC_8 & cC_9 & cC_7 \\ cC_7 & cC_8 & cC_9 & aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 \\ cC_9 & cC_7 & cC_8 & aC_3 & aC_1 & aC_2 & bC_6 & bC_4 & bC_5 \\ cC_8 & cC_9 & cC_7 & aC_2 & aC_3 & aC_1 & bC_5 & bC_6 & bC_4 \\ bC_4 & bC_5 & bC_6 & cC_7 & cC_8 & cC_9 & aC_1 & aC_2 & aC_3 \\ bC_6 & bC_4 & bC_5 & cC_9 & cC_7 & cC_8 & aC_3 & aC_1 & aC_2 \\ bC_5 & bC_6 & bC_4 & cC_8 & cC_9 & cC_7 & aC_2 & aC_3 & aC_1 \end{bmatrix} \\ W_2 &= \begin{bmatrix} aC_{10} & aC_{11} & aC_{12} & cC_4 & cC_5 & cC_6 & bC_7 & bC_8 & bC_9 \\ aC_{12} & aC_{10} & aC_{11} & cC_6 & cC_4 & cC_5 & bC_9 & bC_7 & bC_8 \\ aC_{11} & aC_{12} & aC_{10} & cC_5 & cC_6 & cC_4 & bC_8 & bC_9 & bC_7 \\ bC_7 & bC_8 & bC_9 & aC_{10} & aC_{11} & aC_{12} & cC_4 & cC_5 & cC_6 \\ bC_9 & bC_7 & bC_8 & aC_{12} & aC_{10} & aC_{11} & cC_6 & cC_4 & cC_5 \\ bC_8 & bC_9 & bC_7 & aC_{11} & aC_{12} & aC_{10} & cC_5 & cC_6 & cC_4 \\ cC_4 & cC_5 & cC_6 & bC_7 & bC_8 & bC_9 & aC_{10} & aC_{11} & aC_{12} \\ cC_6 & cC_4 & cC_5 & bC_9 & bC_7 & bC_8 & aC_{12} & aC_{10} & aC_{11} \\ cC_5 & cC_6 & cC_4 & bC_8 & bC_9 & bC_7 & aC_{11} & aC_{12} & aC_{10} \end{bmatrix} \end{aligned}$$

$$W_1 W_2^T = I_q \times bc \sum_{i=4}^q C_i c_i^T$$

but $C_i C_i^T$ is symmetric and so $W_1 W_2^T = W_2 W_1^T$.

W_3 and W_4 are formed similarly but are back-circulant in blocks (i.e., type 2) in the language of Seberry-Wallis and Whiteman).

Thus W_1, W_2, W_3, W_4 are Williamson-type matrices of order $12p^2$ they can be used to replace the variables of an $OD(4; 1, 1, 1, 1)$ to get an $OD(48p^2; 16p^2, 16p^2, 16p^2, 16p^2)$. In general they can be used to replace the variables of an $OD(4t; t, t, t, t)$ so we have

Theorem: If there is an Hadamard matrix of order $4p$ and an $OD(4t; t, t, t, t)$ then there is an $OD(48p^2t; 16p^2t, 16p^2t, 16p^2t, 16p^2t)$.

Since Hadamard matrices of order $4p$ exist for all $p \leq 102$ and $OD(4t; t, t, t, t)$ exist for all $t \leq 102$ except possibly for $t \in S$, $S = \{67, 71, 73, 77, 79, 83, 86, 89, 91, 97\}$ (see Seberry (1986a)) we have

Corollary: $OD(48p^2t; 16p^2t, 16p^2t, 16p^2t, 16p^2t)$ exist for all $t, p \leq 102$ except possibly for $t \in S$.

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References

- [1] J. Cooper and J. Seberry Wallis, "A construction of Hadamard arrays", *Bull. Austral Math. Soc.*, 7 (1972) 269-278.
- [2] A.V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices* Marcel Dekker, New York-Basel (1986).
- [3] J. Hammer, D.G. Sarvate, J. Seberry, "A note on orthogonal designs", *Ars Combinatoria*, submitted.
- [4] H. Kharagani, "New class of weighing matrices", *Ars Combinatoria*, Vol. 19, pp. 69-72.
- [5] J. Seberry, "More towards proving the Hadamard conjecture", (preprint).
- [6] J. Seberry-Wallis, "Hadamard matrices, Part IV of W.D. Wallis, A. P. Street, and J. Seberry-Wallis", *Combinatorics: Room Square, sum free sets and Hadamard matrices*, Lecture Notes in Mathematics, Vol. 292, Springer-Verlag, Berlin-Heidelberg New York.