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A note on orthogonal designs

Abstract
We extend a method of Kharaghani and obtain some new constructions for weighing matrices and orthogonal designs. In particular we show that if there exists an OD(s₁,...,sr), where w = \( \sum s_i \), of order n, then there exists an OD(s₁w,s₂w,...,srw) of order n(n+k) for k \( \geq 0 \) an integer. If there is an OD(t,t,t) in order n, then there exists an OD(12t,12t,12t,12t) in order 12n. If there exists an OD(s,s,s,s) in order 4s and an OD(t,t,t,t) in order 4t there exists an OD(12s²t,12s²t,12s²t,12s²t) in order 48s²t and an OD(20s²t,20s²t,20s²t,20s²t) in order 80s²t.

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A Note on Orthogonal Designs

J. Hammer, D.G. Sarvate and Jennifer Seberry*

ABSTRACT

We extend a method of Kharaghani and obtain some new constructions for weighing matrices and orthogonal designs. In particular, we show that if there exists an $OD(s_1,\ldots,s_t)$, where $w = \sum s_i$, of order $n$, then there exists an $OD(s_1w, s_2w, \ldots, s_t w)$ of order $n(n+k)$ for $k \geq 0$ an integer. If there is an $OD(t,t,t,t)$ in order $n$, then there exists an $OD(12t,12t,12t,12t)$ in order $12n$. If there exists an $OD(t,s,s,s)$ in order $4s$ and an $OD(t,t,t,t)$ in order $4t$ there exists an $OD(12s^2t,12s^2t,12s^2t,12s^2t)$ in order $48s^2t$ and an $OD(20s^2t,20s^2t,20s^2t,20s^2t)$ in order $80s^2t$.

1. Introduction.

Let $W = [w_{ij}]$ be a matrix of order $n$ with $w_{ij} \in \{0,1,-1\}$. $W$ is called a weighing matrix of weight $p$ and order $n$, if $WW^T = W^TW = pI_n$, where $I_n$ denotes the identity matrix of order $n$. Such a matrix is denoted by $W(n,p)$. If squaring all its entries gives an incidence matrix of a SBIBD then $W$ is called a balanced weighing matrix.

An orthogonal design (OD), $A$, say, of order $n$ and type $(s_1,s_2,\ldots,s_t)$ on the commuting variables $(\pm x_1, \ldots, \pm x_t)$ and $0$, is a square matrix of order $n$ with entries from $(\pm x_1, \ldots, \pm x_t)$ and $0$. Each row and column of $A$ contains $s_k$ entries equal to $s_k$ in absolute value, the remaining entries in each row and column being equal to $0$. Any two distinct rows of $A$ are orthogonal.

In other words

$$AA^T = (x_1^2 + \cdots + x_t^2)I_n.$$ 

An Hadamard matrix $W = [w_{ij}]$ is a $W(n,n)$ i.e. it is a square matrix

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of order \( n \) with entries \( w_{ij} \in \{1,-1\} \) which satisfies
\[
WW^T = W^TW = nI_n
\]

OD’s have been used to construct new Hadamard matrices. For details see Geramita and Seberry (1979).

Kharaghani (1985) defined \( C_k = [w_{k+1}, w_{k-1}] \) and with that obtained skew symmetric and symmetric \( W(n^2+2n, p^2) \) from \( W(n, p) \), where \( s \) is any positive integer such that \( n + s \) is even. Each \( C_k \) is a symmetric \( \{0,1,-1\} \) matrix of order \( n \). We define \( C_k \) by the Kronecker product and by extending Kharaghani’s method we obtain some new constructions of weighing matrices and orthogonal designs.

2. Some properties of \( C_k \)’s.

The \( C_k \)’s can be defined as a Kronecker product of the \( k \)th row of \( W \) with its transpose, in other words, if \( R_k \) denotes the \( k \)th row of \( W \), then \( C_k = R_k \times R_k^T \). Similarly, we define \( C_k \)’s corresponding to the OD, \( A \), as follows:

Let \( U \) be a weighing matrix obtained from \( A \) by replacing all the variables of \( A \) by 1. Let \( A_k \) and \( U_k \) denote the \( k \)th rows of \( A \) and \( U \) respectively. Then \( C_k = A_k \times U_k^T \).

Lemma 2.1. Let \( V_i \) be the \( i \)th row of an \( SBIBD(v, p, \lambda) \). Consider
\[
X = [V_1 \times V_1^T, \ldots, V_n \times V_n^T]
\]
then \( XX^T = p((p-\lambda)I + \lambda J) \).

Proof.
\[
XX^T = V_1 V_1^T \times V_1 V_1^T \ldots V_n V_n^T \times V_n V_n^T
= p \sum_i v_i v_i^T
= p((p-\lambda)I + \lambda J).
\]

Corollary 2.2. Given a balanced \( W(n, p) \), based on an \( SBIBD(n, p, \lambda) \), consider
\[
X = [C_1; C_2; \ldots; C_n]
\]
where \( C_i \) is obtained from \( C_1 \) by squaring all its entries. Then the inner product of any two distinct rows of \( X \) is \( \lambda p \).
3. A new construction of orthogonal designs.

Many constructions in orthogonal design theory have been expressed in terms of Kronecker products of matrices, for example see Gastineau-Hills (1983) and Gastineau-Hills and Hammer (1983). The Kronecker product of two or more designs is not in general a design since products of variables appear, for example:

\[
\begin{bmatrix}
  x_1 & x_2 \\
  -x_2 & x_1
\end{bmatrix} \times \begin{bmatrix}
  y_1 & y_2 \\
  y_2 & -y_1
\end{bmatrix} = \begin{bmatrix}
  x_1 y_1 & x_1 y_2 & x_2 y_1 & x_2 y_2 \\
  -x_2 y_1 & x_1 y_1 & -x_2 y_2 & x_1 y_2 \\
  x_1 y_2 & x_2 y_2 & -x_3 y_1 & -x_3 y_1 \\
  -x_2 y_2 & x_1 y_2 & x_2 y_1 & -x_1 y_1
\end{bmatrix}
\]

(\text{where } z_1 = x_1 y_1, z_2 = x_2 y_1, z_3 = x_1 y_2, z_4 = x_2 y_2) is not orthogonal if we take \( z_1, z_2, z_3 \) and \( z_4 \) as independent. However it is a different matter if we take a Kronecker product of an OD with a weighing matrix.

A construction of Kharaghani can be extended to give the following result:

\textbf{Theorem 3.1.} If there exists an OD, \( A \), of type \((s_1, s_2, \ldots, s_r)\), where

\[ w = \sum_{k=1}^{r} s_k, \]

and order \( n \) on the variables \((\pm x_1, \ldots, \pm x_r, 0)\) then there exist \( n \) matrices \( C_1, \ldots, C_n \) of order \( n \) satisfying

\[ \sum_{k=1}^{n} C_k C_k^T = \sum_{k=1}^{r} s_k \]

\[ C_k C_j^T = 0, \quad k \neq j. \]

\textbf{Proof.} Let \( A = (a_{ij}) \) be the OD. Replace all the variables of \( A \) by 1 making it a \((0,1,-1)\) weighing matrix \( U = (u_{ij}) \) of order \( n \) and weight \( w \). Write \( A_k \) and \( U_k \) for the \( k \)th rows of \( A \) and \( U \) respectively. Form

\[ C_k = A_k \times U_k^T. \]

Then

\[ C_k C_j^T = (A_k \times U_k^T)(A_j \times U_j^T)^T = (A_k A_j^T) \times U_k^T U_j \]

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= 0 if \( k \neq j \) because \( A \) is an orthogonal design.

Now
\[
\sum_{k=1}^{n} G_k G_k^T = \sum_{k=1}^{n} (A_k \times U_k^T) (A_k^T \times U_k)
\]
\[
= \sum A_k A_k^T \times U_k U_k^T
\]
\[
= \sum \sigma_k x_k^T (\sigma_k U_k U_k^T)
\]
\[
= \sum \sigma_k x_k^T (\omega_k)
\]
by the properties of \( U \). \( \Box \)

Example 3.2. Let
\[
A = \begin{bmatrix}
-a & b & c & -d \\
b & a & d & c \\
c & -d & a & -b \\
-d & -c & b & a \\
\end{bmatrix}
\]
\[
U = \begin{bmatrix}
-1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 \\
\end{bmatrix}
\]

Then
\[
C_1 = \begin{bmatrix}
a & -a & -a & a \\
b & b & b & -b \\
c & e & c & -c \\
d & -d & -d & d \\
\end{bmatrix}
\]
\[
C_2 = \begin{bmatrix}
b & b & b & b \\
a & a & a & a \\
d & d & d & d \\
 c & c & c & c \\
\end{bmatrix}
\]
\[
C_3 = \begin{bmatrix}
e & -c & c & -c \\
d & -d & d & d \\
a & -a & a & -a \\
b & b & b & b \\
\end{bmatrix}
\]
\[
C_4 = \begin{bmatrix}
d & d & -d & -d \\
e & c & -c & -c \\
b & -b & b & b \\
a & -a & a & a \\
\end{bmatrix}
\]

Thus we have:

Theorem 3.3. Suppose there exists an \( OD(s_1, s_2, \ldots, s_r) \), where \( w = \sum \sigma_i \) of order \( n \). Then there exists an \( OD(s_1 w, s_2 w, \ldots, s_r w) \) of order \( n(n+k) \) for \( k \geq 0 \) an integer.

Proof. Form \( C_1, \ldots, C_n \) as in the previous theorem. Form a latin square of order \( n + k \) and replace \( n \) of its elements by \( C_1, \ldots, C_n \) and the other elements by the \( n \times n \) zero matrix. \( \Box \)

For instance, using Theorem 3.3 we can construct an \( OD(4,4,4,4) \) of order \( 4n \), for \( n \geq 4 \). Using inequivalent Latin squares in Theorem 3.3 will
Corollary 3.4. If there is an OD(t,t,t,t) in order 4t, then there is an OD(4t²,4t²,4t²,4t²) in every order 4t(4t+k), k ≥ 0 an integer.

But this construction can be used in other ways.

Example 3.5. Write 1,2,3,4 for C₁,...,C₄. Define

\[
A₁ = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad A₂ = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 4 \end{bmatrix}, \quad A₃ = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 4 & 3 \\ 4 & 3 & 1 \end{bmatrix}, \quad A₄ = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}.
\]

Then \(A₄A₄ᵀ = A₄A₄ᵀ\). Thus \(A₁, A₂, A₃, A₄\) can be used to replace the variables of any \(OD(t,t,t,t)\). □

Hence we have

Theorem 3.6. Suppose there is an OD(t,t,t,t) in order \(n\). Then there exists an OD(12t,12t,12t,12t) in order 12n.

Proof. Use the OD(1,1,1,1) in order 4 to form \(C₁,...,C₄\) of order 4. Substitute these in \(A₁,...,A₄\) of Example 3.5 to obtain Williamson-type matrices of order 12, on 4 variables each repeated 12 times. Use these to replace the variables of the \(OD(t,t,t,t)\) to get the result. □

Now if we had started to construct \(C₁,...,C₄ₘ\) of order 4ₘ from an \(OD(s,s,s,s)\) in order 4ₘ we would have each of 4 variables occurring 4ₘ² times in each row of \([C₁; C₂; ...; C₄ₘ]\). But we can use these to form Williamson type matrices in a number of ways:

Let \(Aₖ\) be a circulant matrix with first row \((i+1,i+2,...,i+ₘ), i = 0, s, 2s, \text{ and } 3s\). These four matrices can be substituted in an \(OD(t,t,t,t)\). Hence we have:

Theorem 3.7. If there exists an OD(s,s,s,s) in order 4ₘ and an OD(t,t,t,t) in order 4t, then there exists an OD(4ₘ²t,4ₘ²t,4ₘ²t,4ₘ²t) in order 16ₘ²t.

Now if we write \(i\) for \(Bₖ\) we can proceed exactly as in Example 3.5 so we have:

Theorem 3.8. If there exists an OD(s,s,s,s) in order 4ₘ and an OD(t,t,t,t) in order 4t, then there exists an OD(12s²t,12s²t,12s²t,12s²t) in order 4ₘ₈²t. □

Consider the OD(5,5,5,5) in order 20. The construction gives \(C₁,C₂,...,C₂₀\) of order 20 and hence an OD(300,300,300,300) in order 1200.
Example 3.10. We suppose as before that $1, 2, 3, 4$ are matrices of order $n$ such that $ij^T = 0$ and $\sum ii^T = \sum n x_i^2 I_n$.

Define

$$
A_1 = \begin{bmatrix}
3 & 1 & 2 & -2 & 1 \\
1 & 3 & 1 & 2 & -2 \\
-2 & 1 & 3 & 1 & 2 \\
2 & -2 & 1 & 3 & 1 \\
1 & 2 & -2 & 1 & 3
\end{bmatrix}, \quad
A_2 = \begin{bmatrix}
1 & 3 & 4 & -4 & 3 \\
3 & 1 & 3 & 4 & -4 \\
4 & -4 & 3 & 1 & 3 \\
3 & 4 & -4 & 3 & 1
\end{bmatrix}
$$

$$
A_3 = \begin{bmatrix}
4 & 1 & 2 & -1 & 4 \\
1 & 2 & 2 & -1 & 4 \\
2 & 2 & -1 & 4 & 1 \\
2 & -1 & 4 & 1 & 2 \\
-1 & 4 & 1 & 2 & 2
\end{bmatrix}, \quad
A_4 = \begin{bmatrix}
2 & 3 & 4 & 4 & -3 \\
3 & 4 & 4 & -3 & 2 \\
4 & 4 & -3 & 2 & 3 \\
4 & -3 & 2 & 3 & 4 \\
-3 & 2 & 3 & 4 & 4
\end{bmatrix}
$$

Then $A_i A_j^T = A_j A_i^T$ and $\sum A_i A_j^T = \sum 5 x_i^2 I_{5n}$.

Thus if $B_i$ are as described after Theorem 3.7 we have

Theorem 3.11. Suppose there is an $OD(s, s, s, s)$ in order $4s$ and an $OD(t, t, t, t)$ in order $4t$. Then there is an $OD(20s^2 t, 20s^2 t, 20s^2 t, 20s^2 t)$ in order $80s^2 t$.

4. Method used to form inequivalent Hadamard matrices.

Construction 4.1. Let $H$ be Hadamard of order $n$. Form $C_i$, $i = 1, 2, \ldots, n$, from $H$ as before. Let $L$ and $M$ be Hadamard matrices of order $t$. Then

$$(L \times C_i)(M \times C_j) = 0, \quad i \neq j.$$

So if $H_1, \ldots, H_n$ are Hadamard matrices of order $t$ (inequivalent or just different equivalence operations applied to one) then the matrices

$$H_i \times C_i, \quad H_i^2 \times C_i, \ldots, H_i^n \times C_i, \quad i \in \{1, 2, \ldots, n\}$$

can be put into a latin square of order $n$ to form Hadamard matrices of order $n^2 t$. The method will possibly give many inequivalent Hadamard matrices. The method can be generalized to give weighing matrices and orthogonal designs which are also possibly inequivalent.
5. Method used with coloured designs to form rectangular weighing matrices.

In a recent paper Rodger, Sarvate and Seberry (1987) have studied coloured BIBDs showing every BIBD can be coloured. By definition a coloured BIBD is the incidence matrix of the \( BIBD(v,b,r,k,\lambda) \) whose nonzero entries are replaced by \( r \) fixed symbols such that each row and column has no repeated symbol. Consider a coloured symmetric \( BIBD(v,k,\lambda) \) and a \( W(k,p) \). If we replace the \( i \)th symbol by \( C_i \) for \( i = 1,2,\ldots,k \) and the 0 entries by the \( k \) by \( k \) zero matrix, we get \( W(uk,p^2) \).

In general, if we consider a coloured \( BIBD(v,b,r,k,\lambda) \) and there exists a weighing matrix \( W(r,p) \) then we form the \( C_i \), \( i = 1,\ldots,r \) and replace the \( i \)th colour by \( C_i \) and zeros by the zero matrix of order \( r \). This matrix, \( B \), has size \( vr \times vr \), \( rp \) nonzero elements in each row and \( pk \) non-zero elements in each column. Hence we have:

**Theorem 5.1.** Suppose there is a \( BIBD(v,b,r,k,\lambda) \) and a \( W(r,p) \). Then there is a \((0,1,-1)\) matrix \( B \) with \( rp \) nonzero elements in each row and \( pk \) nonzero elements in each column such that

\[
BB^T = rpI.
\]

In particular, if the BIBD is symmetric then we have a \( W(uk,p^2) \). ∎

**Remark.** If we replace entries of an \( n \)-dimensional latin cube by suitable \( C_i \)'s then we will get \( n \)-dimensional orthogonal designs.
References.


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