Gradient flow of the Dirichlet energy for the curvature of plane curves

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Gradient flow of the Dirichlet energy for the curvature of plane curves

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This thesis is presented as part of the requirements for the conferral of the degree:

Doctor of Philosophy

Supervisor:
Glen Wheeler, David Hartley, James McCoy

The University of Wollongong
School of Mathematics and Applied Statistics

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Declaration

I, Yuhan Wu, declare that this thesis is submitted in partial fulfilment of the requirements for the conferral of the degree Doctor of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

The results of Chapter 4, 5, 6 appear in papers ‘A sixth order flow of plane curves with boundary conditions’, ‘Higher order curvature flows of plane curves with generalised Neumann boundary conditions’ and ‘Evolution of closed curves by length-constrained curve diffusion’ which are under supervision of Assoc. Prof. James McCoy and Dr Glen Wheeler.

__________________________________________

Yuhan Wu

May 13, 2021
Abstract

This thesis studies curvature flows of planar curves with Neumann boundary condition and flows of closed planar curves without boundary. We describe the local existence for them. For the global existence results for curvature flows of planar curves, we consider the sixth and higher order curvature flows of planar curves with suitable associated generalised Neumann boundary condition. The conclusion is that the solution of each flow problem exists for all time and converges to a unique line segment exponentially. Moreover, we study the curve diffusion flow and ideal curve flow of planar curves with constrained length, as well as the ideal curve flow of planar curves with preserved area. Closed curve satisfying one of these flows exists for all time and converges to a unique round circle exponentially.
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Chapter 1

Introduction

In differential geometry, the study of curves and surfaces includes local and global properties. Local properties depend only on the behaviour of the curves or surfaces in the neighbourhood of a point. For instance, the curvature of a curve is a local property. The global properties consider the influence of local properties on the behaviour of the entire curve or surface. Global differential geometry is concerned with the relations between local and global properties of curves and surfaces.

The properties of curves and surfaces in differential geometry are investigated by differential and integral calculus. The calculus involves differentiable functions, vector fields, differential forms, mappings and various operations of differentiation and integration. A parabolic evolution equation is a partial differential equation describes processes which are evolving in time, for example the heat flow shows the transfer of heat from a hot area to a cool area if there is a different temperature between materials which are next to each other. A parabolic geometric evolution equation can be called a geometric heat flow which is often seen as a gradient flow of a geometric object.

The second order geometric heat flows are well studied. The mean curvature flow can be seen as the most famous second order geometric flow, it was proposed to describe the formation of grain boundaries in annealing metals which is material science in 1957 by physicist Mullins in [65] and later in 1959 [66]. Mullins was the first to write down the mean curvature flow equation. The mean curvature flow was studied by Gerhard Huisken from the perspective of partial differential equations in 1984 [36]. It is the negative steepest descent gradient flow for area functional, the surface area is decreasing monotonically and stationary if and only if the surface stays minimal. There are some related flows that have the same leading order term as mean curvature flow, for example, the volume-preserving mean curvature flow studied by Huisken [37] and area-preserving mean curvature flow considered by McCoy [59]. The curve-shortening flow is the one-dimensional case of the mean curvature flow. There has been much research on the long-time behaviour of smooth curves under this flow problem. In [26] and [27], Gage shows that if a convex plane curve evolving by curve-shortening shrinks to a point, then its limiting...
shape is circular in a weak sense. Afterwards, Gage and Hamilton in [28], Grayson in [33] and [34] study the convergence in $C^\infty$ norm.

Second-order curvature flows with free boundaries have been studied for decades. In 1989, Huisken considers mean curvature flow of graphs in cylindrical domains [38]. Afterwards, in [74] and [75], Stahl considers mean curvature flow with a free boundary on an umbilic hypersurface regarding continuation criteria and singularities. They continue to receive significant research attention, for example [9], where Buckland studies boundary monotonicity formula. Freire considers mean curvature flow of graphs with constant angle at a free boundary in [25]. Second-order curvature flows with free boundary are also studied for example by Koeller [44], Marquardt [57], Mizuno and Tonegawa [64], Edelen [19], Lambert [50], V. Wheeler [84] and [83], McCoy, Mofarreh and Williams [60]. Also some results hold for curves with mixed Neumann-Dirichlet boundary conditions in [54], [12]. Results on classification of singularities and the extension beyond singularities of the mean curvature flow are obtained by Huisken and Sinestrari in [39], [40], [41] and [42].

Higher order geometric evolution problems have drawn interest in recent years, especially, fourth order geometric equations. The surface diffusion flow is a famous fourth order geometric evolution equation. In 1957 [65], Mullins first proposed it to model the formation of tiny thermal grooves in phase interfaces. The surface diffusion flow involves second derivative of the mean curvature so is a fourth order flow. It is the gradient flow for surface area in a Sobolev space and decreases the surface area while the volume is preserved. In [10], Cahn, Elliott and Novick-Cohen prove that the only limit of the Cahn-Hilliard equation with a concentration dependent mobility is surface diffusion. Later in [21], Elliott and Garcke consider motion laws for surface diffusion flow. Giga and Ito consider self-intersection and loss of convexity of surface diffusion flow in [31] and [32] respectively. Afterwards, in [43], Ito shows the loss of convexity for surface diffusion flow equation. In [24], Escher, Mayer and Simonett consider the solutions to immersed hypersurfaces under surface diffusion flow exist and are unique. Escher and Ito [23] show the loss of convexity for intermediate surface diffusion flow. they also prove that surface diffusion flow and intermediate surface diffusion flow develop singularities in finite time. There is also research on surface diffusion flow with boundary conditions. For example [30], Garcke, Ito and Kohsaka study nonlinear stability of stationary plane curves under surface diffusion flow with specified boundary conditions. Asai and Giga consider surface diffusion flow with contact angle boundary conditions in [2]. Global analysis for the surface diffusion flow is also studied, the theory of singularities for the flow is considered for instance by G. Wheeler [79] [80]. In G. Wheeler’s PhD thesis [79], he considers the constrained surface diffusion flow which has a function of time. This function is chosen to coincide with a natural geometric restriction, for instance, different choices of the function can lead to conservation of mixed volumes or a reduction in mass and increase of free
surface energy. He proves that the two-dimensional surface diffusion flow exists for all time and converges to a round sphere exponentially. By choosing the suitable function of time, he shows the enclosed volume is monotonically increasing while the surface area is decreasing.

The curve diffusion flow is the one-dimensional case for the surface diffusion flow. It is the steepest descent $H^{-1}$-gradient flow for length and also a fourth order geometric evolution equation. The proof of local existence for curve diffusion flow refers to the standard procedure in [75] by Stahl, it solves the flow by converting the curve to a graph over the initial data. Furthermore, in [22], Elliott and Maier-Paape give that the graphs evolved as curves by curve diffusion flow become non-graphical in finite time. Eventually, it is shown in [8] by Blatt that a large class of higher order hypersurface flows lose convexity and embeddedness. In [85], G. Wheeler and V. Wheeler consider the curve diffusion flow of open plane curve with free boundary between on parallel lines. For closed curves moving by curve diffusion flow, G. Wheeler proves the area of closed curves is constant in [82]. In [81], G. Wheeler studies the solution of the flow problem by using the isoperimetric ratio and small initial oscillation of curvature. He proves the closed curve exists for all time and converges to a simple circle exponentially. We are interested in the closed curve diffusion flow with constrained length which is studied in Chapter 6.

Another fourth order flow is the Willmore flow, which is the steepest descent $L^2$-gradient flow for Willmore functional. Normally, Willmore functional is the integral of mean curvature squared. It was first studied by Sophie Germain in the 19th century. Afterwards it received significant attention from Blaschke who first presented its Euler-Lagrange operator in 1929, see Blaschke and Thomsen [7], Blaschke and Reidemeister [6]. For more references, see Simonett [72], Bauer and Kuwert [4], Bernard and Rivièere [5], Chen [11] and Kusner [46], McCoy and G. Wheeler [62]. Furthermore, the studies of Willmore conjecture with minimal surfaces can be found in, for example, Li and Yau [51], Marques and Neves [58], roughly speaking, Willmore conjecture is that there is a lower estimate of the Willmore energy for immersed surface.

Elastic flow is the one-dimensional Willmore flows and the steepest descent $L^2$-gradient flow for the elastic energy which is the integral of curvature squared. Local existence for the flow is shown in [75] by Stahl. In [85], G. Wheeler and V. Wheeler consider the elastic flow have free boundary, supported on parallel lines in the plane. Let us now define $E[\gamma]$ for a smooth closed or open planar curve $\gamma$ by $E[\gamma] = \frac{1}{2} \int k_s^2 ds$, where $k_s$ is the first arc-length derivative of the curvature. Our interest is the $L^2$-gradient flow for this functional $E[\gamma]$ which is the sixth order curvature flow studied in Chapter 4.

Fourth-order curvature flows with different boundary conditions have been studied in recent years. For example, in [29] and [30], stability results are proved by Garcke, Ito and Kohsaka for curves under curve diffusion flow that are graphical with nearby equilibria evolving in bounded domains with free boundary. Other studies include curves moving by
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gradient for elastic energy with constrained length and fixed boundary points with zero curvature by Dall’Acqua, Lin and Pozzi [13], curves moving by Helfrich energy with natural boundary conditions by Dall’Acqua and Pozzi [14], Helfrich energy for planar curves is the sum of the Willmore functional and the length functional times a positive weight factor. The elastic energy with clamped boundary conditions is studied by Lin [52] and Dall’Acqua, Pozzi and Spener [15]. In [67], Novaga and Okabe consider the steepest descent flow of open planar curves with clamped boundary conditions and symmetric Navier boundary conditions. The most relevant research for us is the study of curve diffusion and elastic flow of curves between parallel lines in [85] by G. Wheeler and V. Wheeler.

Higher-order curvature flows of closed curves without boundary have received some attention. Particularly, some works are done by Dziuk, Kuwert and Schätzle [17], Edwards, Gerhardt-Bourke, McCoy, G. Wheeler and V. Wheeler [20], Giga and Ito [32], Parkins and G. Wheeler [68]. Moreover, McCoy, G. Wheeler and Williams obtain the maximal time estimate for the closed immersed hypersurfaces which evolve by constrained surface diffusion flows in [63]. Sixth order flows of closed curves are studied in [1] by Andrews, McCoy, G. Wheeler and V. Wheeler, authors prove the convergence of the solution to an ideal curve flow. An ideal curve is a smooth curve with zero normal speed. The application of the energy in this paper appear in computer aided design [35] by Harary and Tal. In Chapter 7, we consider an ideal curve flow that preserves the length while the signed enclosed area does not decrease. This flow is obtained by adding a suitable global function of time into the flow speed.

When dealing with higher order flows, many of the tools and techniques applied to study second order curvature flow cannot be used, such as the maximum principles. The maximum principle is the important tool to study the behaviour of solutions to the second order geometric heat flows and has been used to prove many theorems related to second order flows, for example [36], [18], [38] by Huisken, Ecker, Gage and Hamilton. However, some known techniques can be applied to various higher order curvature flow under different conditions. Some inequalities like interpolation inequalities, Sobolev inequality, Young’s inequality, Cauchy-Schwarz inequality and Hölder’s inequality are often used during the studies. In [17], Dziuk, Kuwert and Schätzle give an interpolation inequality for closed curves while Dall’Acqua and Pozzi prove an inequality which can handle non-closed curves in [14].

One technique for studying fourth-order flows is to use curvature integral estimates. It was first used for the Willmore flow by Kuwert and Schätzle. They set up a general framework which can be used to study different varieties of fourth-order curvature flow in [47], [48] and [49]. The framework includes evolution equations for curvature quantities, integral estimates, short time existence and the concentration compactness alternative. These turn out to be useful for other fourth order flows, even higher order flows whether
they have or do not have a gradient structure. Therefore, this framework can be applied to related flows to obtain stability results for parabolic problems. Applications and modifications of this framework have been used for the surface diffusion flow by G. Wheeler in [80] and [81], the geometric triharmonic heat flow [61] by McCoy, Parkins and G. Wheeler, and polyharmonic flows [68] by Parkins and G. Wheeler.

About flows of higher even order than four, little research has been done so far. But motivation for them comes from various areas. Applications of sixth order curvature flows involving a thrice-iterated Laplacian are increasing, for example, Ugail and Wilson consider the modelling of ulcers in [78], computer graphics studied by Malcolm, Wilson and Hagenin [56] and Ugail in [77], interactive design is considered by Kubiesa, Ugail and Wilson in [45], and computer designs introduced by Liu and Xu in [55]. Moreover, in [86], Xu, Pan and Bajaj study several geometric flows, including sixth order flows that have been used for surface blending, N-side hole filling and free-form surface design. Fluid flows and propeller blade design are considered by Tagliabue in [76] and Dekanski in [16]. In [61], the geometric triharmonic flow for closed surfaces is considered by McCoy, Parkins and G. Wheeler, this flow involves fourth derivatives of the mean curvature so is a sixth order flow. Parkins and G. Wheeler concern even order flow of closed planar curves in [68]. These interests and applications of sixth order flows provide strong motivation for the study in Chapter 4.

1.1 Main results of thesis

We summarise the main results of this thesis as the following. In Chapter 2, we give some basic definitions and notations related to our problems and some inequalities we will use frequently. Chapter 3 describes the local existence for flows of planar curves with Neumann boundary condition and flow of closed planar curves without boundary. Some references about short time existence are mentioned in this chapter. In Chapter 4, we consider the sixth order curvature flow of open planar curves with Neumann boundary condition. We show that the solution of the flow problem exists for all time and converges to a unique line segment. We also obtain the bounded region of solution. In Chapter 5, we consider the higher order curvature flow of planar curves with suitable associated generalised Neumann boundary condition. We generalise the sixth order case where we considered the $L^2$-gradient flow for the energy $\frac{1}{2} \int k^2 ds$, and consider the $L^2$-gradient flow for the energy $\frac{1}{2} \int m k^2 ds$ with suitable associated generalised Neumann boundary conditions, where $k_m$ is the $m$th derivative of the curvature. In this chapter, we follow the similar process in Chapter 4, our conclusion is the curve moving under $(2m + 4)$th-order curvature flows with generalized Neumann boundary condition converges to a unique straight line segment when time goes to infinity. In Chapter 6, we study the curve diffusion flow of planar curves with constrained length. We assume that initial data close to a round
circle, within the sense of normalised $L^2$ oscillation of curvature, closed curve under this flow exists for all time and converges exponentially fast to a round circle. We also give the self-similar solutions for this flow problem. In Chapter 7, we study an ideal curve flow of closed curve that preserves the length while the signed enclosed area does not decrease and the ideal curve flow of closed curve with preserved area. We prove the long-time existence of the curves and exponential convergence under the smallness condition.

1.2 Suggestions for further research

In Chapter 4 and Chapter 5, we study the long-time existence for sixth order curvature flow and $(2m + 4)$th-order curvature flows of planar curves with Neumann boundary condition, and the boundaries are two parallel lines with a distance between them. For further research, the framework we use in these two chapters can be used to study other even order curvature flows with the same boundary condition. The results and framework also can be adapted to the same or different even order curvature flows of planar curves with Neumann boundary condition, but with the different boundaries, for example, a cone or a simple circle. Moreover, it can be applied to even order curvature flows of planar curves with other boundary conditions, such as Dirichlet condition. There are some open questions related to closed planar curves, for example, the global existence for sixth order curvature flow and higher order curvature flows of closed planar curves without boundary, long-time existence of length-constrained or area-preserved curves satisfying these flows.
Chapter 2

Prerequisites

We introduce some basic definitions and notations related to our problems.

Consider the open or closed curve $\gamma$ to move by the normal velocity $F$:

$$\frac{\partial}{\partial t} \gamma = F[k] \nu,$$

where $F[k]$ denotes the normal speed of the curve, $\nu$ is unit normal vector field of the curve and $k = \langle \gamma_{ss}, \nu \rangle$ is the scalar curvature, where $\langle \gamma_{ss}, \nu \rangle$ denotes the inner product of $\gamma_{ss}$ and $\nu$. $\tau = \frac{\gamma_{u}}{|\gamma_{u}|} = \gamma_{s}$ is the unit tangent vector field along $\gamma$.

Firstly, we give the open curves with free boundary. Let $\eta_{1}, \eta_{2} : \mathbb{R} \rightarrow \mathbb{R}^{2}$ denote two parallel vertical lines in $\mathbb{R}^{2}$, with a distance $|e| \neq 0$ between them. Consider a one-parameter family of smooth immersed curves $\gamma : [-1, 1] \times [0, T) \rightarrow \mathbb{R}^{2}$, the two end points of the curve lie on $\eta_{1}, \eta_{2}$ respectively and the curves meet the two parallel lines $\eta_{1}, \eta_{2}$ orthogonally. Here $e$ to be any vector such that $e$ is perpendicular to the parallel lines $\eta_{1,2}$. We wish the curve $\gamma$ to move by the normal velocity $F$. This is called open curves moving under curvature flow with Neumann boundary condition.

Here we introduce the ideal curves. We consider the energy functional

$$E[\gamma] = \frac{1}{2} \int_{\gamma} k_{s}^{2} ds,$$

where $k_{s}$ is the derivative of curvature with respect to arc-length and $ds$ the arclength element. We are interested in the $L^{2}$ gradient flow for curves of small initial energy with Neumann boundary condition. As our energy involves the first derivative of curvature, the gradient flow will be sixth order and so six boundary conditions will be needed. The corresponding gradient flow has normal speed given by $F$. Let $\gamma$ be a smooth curve satisfying $F = 0$, that is a stationary solution to the $L^{2}$-gradient flow of $E$. We call such curves ideal and they are critical points for $E$.

Secondly, for closed curves, we let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^{2}$ be a (suitably) smooth embedded regular curve. We say $\gamma$ is periodic with period $P$ if there exists a vector $V \in \mathbb{R}^{2}$ and a positive
number $P$ such that, for all $m \in \mathbb{N}$,

$$
\gamma(u+P) = \gamma(u) + V
$$

and

$$
\partial^m_u \gamma(u+P) = \partial^m_u \gamma(u).
$$

here $\partial^m_u$ denotes the $m$th derivative of $\gamma$.

If $V = 0$ then $\gamma$ is closed and we write $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. The length of $\gamma$ is

$$
L[\gamma] = \int_0^P |\gamma_u(u)| du
$$

and the signed enclosed area is

$$
A[\gamma] = -\frac{1}{2} \int_0^P \langle \gamma, v \rangle |\gamma_u| du
$$

where $v$ is a unit normal vector field on $\gamma$, $\langle \gamma, v \rangle$ denotes the inner product of $\gamma$ and $v$.

Throughout this thesis, we will keep our evolving curves $\gamma$ parametrised by arc-length $s$ and $k_{s^n}$ denotes $(k_{s^n})^2$, where $k_{s^n}$ is the $n$-th iterated derivative of $k$ with respect to arc-length and $ds = |\gamma_u| du$.

**Definition 1.** There are several definitions for both open and closed curves.

(i) The length of $\gamma$ is

$$
L[\gamma] = \int_\gamma ds.
$$

Note that for open curves, $w$ is not always an integer.

(ii) The average curvature is

$$
\bar{k}[\gamma] = \frac{1}{L} \int_\gamma k ds.
$$

(iii) The oscillation of curvature is defined as

$$
K_{osc}[\gamma] = L \int_\gamma (k - \bar{k})^2 ds.
$$

The following two additional definitions are for closed curves:

(iv) The signed enclosed area is

$$
A[\gamma] = -\frac{1}{2} \int_\gamma \langle \gamma, v \rangle ds.
$$
(v) The isoperimetric ratio of \( \gamma \) is

\[
I[\gamma] = \frac{L^2}{4\pi A}.
\]

In this thesis, we denote \( L(t) = L[\gamma(\cdot, t)] \), \( A(t) = A[\gamma(\cdot, t)] \), \( K_{osc}(t) = K_{osc}[\gamma(\cdot, t)] \) and \( I(t) = I[\gamma(\cdot, t)] \). For the convenience, we use \( L \) to denote \( L(t) \) and \( \gamma_0 \) to denote initial curve.

For the winding number \( \omega \), we have

\[
\omega = \frac{1}{2\pi} \int_{\gamma} kds.
\]

For closed curves, \( w \in \mathbb{Z} \), for example, in figure 2.1,

\[
\omega[\gamma_1] = \frac{1}{2\pi} \int_{\gamma_1} kds = 0;
\]

\[
\omega[\gamma_2] = \frac{1}{2\pi} \int_{\gamma_2} kds = 1;
\]

\[
\omega[\gamma_3] = \frac{1}{2\pi} \int_{\gamma_3} kds = 2.
\]

\[\text{Figure 2.1}\]

However, for open curves with Neumann boundary condition, the winding number must be a multiple of \( \frac{1}{2} \). For example, in Figure 2.2,

\[
\omega[\gamma_4] = \frac{1}{2\pi} \int_{\gamma_4} kds = 1,
\]

\[
\omega[\gamma_5] = \frac{1}{2\pi} \int_{\gamma_5} kds = \frac{1}{2}.
\]
We give the commutator relation between Euclidean arc-length and time derivatives of the evolving curves. Proofs appear for example in [85] by G. Wheeler and V. Wheeler, we provide them again here for completeness.

**Lemma 1.** The commutator of arc-length and time derivative is given by

\[ \partial_t \partial_s = \partial_s \partial_t + kF \partial_s \]

and the measure \( ds \) evolves by

\[ \partial_t ds = -kF ds \]

**Proof.** We do the computation

\[ \partial_t |\gamma_u|^2 = 2 |\gamma_u||\gamma_u|_t = 2 \langle \gamma_u, \gamma_u \rangle = 2 \langle \gamma_u, (F \nu)_u \rangle = 2F \langle \gamma_u, \nu_u \rangle = -2(kF) |\gamma_u|^2 \]

then

\[ \partial_t |\gamma_u| = -kF |\gamma_u|, \]

so we have

\[ \partial_t \partial_s - \partial_s \partial_t = \partial_t \left( \frac{1}{|\gamma_u|} \right) \partial_u = - \frac{\partial_t |\gamma_u|}{|\gamma_u|^2} \partial_u = \frac{-kF |\gamma_u|}{|\gamma_u|^2} \partial_u = kF \frac{\partial_u}{|\gamma_u|} \partial_s = kF \partial_s. \]

The result follows. \( \square \)

This commutator relation can make the calculation of derivative of curvature vectors easier.
Lemma 2. Some basic evolution equations:
\[
\tau_t = -F_s \nu, \quad \nu_t = F_s \tau.
\]
where \(\tau\) is a unit tangential vector field on \(\gamma\).

Proof. We apply the above Lemma 1 to the following calculation,
\[
\tau_t = \gamma_{st} = \gamma_{ss} + kF \gamma_s = (F\nu)_s + kF \tau
= F_s \nu + F \nu_s + kF \tau
= F_s \nu - kF \tau + kF \tau
= F_s \nu.
\]
By the orthonormality of \(\{\tau, \nu\}\), \(\langle \tau, \nu \rangle = 0\) and \(\langle \nu_t, \nu \rangle = 0\), so we find
\[
\nu_t = -\langle \nu, F_s \nu \rangle \tau = -F_s \tau.
\]

Lemma 3. We have the following corresponding evolution equations for various geometric quantities:

(i) \(\frac{d}{dt} L = - \int_{\gamma} kFd\tau\),
(ii) \(\frac{d}{dt} k = F_{ss} + k^2 F;\)
(iii) \(\frac{d}{dt} k_s = F_s^3 + k^2 F_s + 3kk_s F;\)
(iv) \(\frac{d}{dt} k_{st} = F_{s^t} + k^2 F_{ss} + 5kk_s F_s + 4kk_{ss} F + 3k^2 F;\)
For each \(l = 0, 1, 2, \ldots,\)
(v) \(\frac{d}{dt} k_{s^l} = F_{s^l+2} + \sum_{j=0}^{l} \partial_{s^j} (k_{s^{l-j}} F).\)

Proof. For (i), under Lemma 1 and Definition 1 (i), we have
\[
\frac{dt}{dt} L = \frac{d}{dt} \int_{\gamma} ds = - \int_{\gamma} kFd\tau.
\]
For (ii),
\[
\gamma_{sts} = \gamma_{ss} + kF \gamma_s = \tau_s + kF \gamma_s
= (F_s \nu)_s + kF \cdot k
= (F_{ss} + k^2 F) \nu - kF_s \tau.
\]
The scalar curvature $k = \langle \gamma_{ss}, \nu \rangle$, then by using Lemma 2
\[
\frac{\partial}{\partial t} k = k_t = \langle \gamma_{ss}, \nu_t \rangle + \langle \gamma_{sst}, \nu \rangle
\]
\[
= \langle \gamma_{ss}, -F_s \tau \rangle + \langle (F_{ss} + k^2 F) \nu - (kF_s) \tau, \nu \rangle
\]
\[
= F_{ss} + Fk^2.
\]

Working with the scalar quantities, we obtain the evolution equations (iii), (iv),
\[
\frac{\partial}{\partial t} k_s = \partial_s \partial_t k + kF_{ks}
\]
\[
= (F_{ss} + k^2 F)_s + kk_s F
\]
\[
= F_{ss} + k^2 F_s + 3kk_s F;
\]
\[
\frac{\partial}{\partial t} k_{ss} = \partial_s \partial_t k_s + kF_{kss}
\]
\[
= (F_{s3} + k^2 F_s + 3kk_s F)_s + kk_{ss} F
\]
\[
= F_{s4} + k^2 F_{ss} + 5kk_s F_s + 4kk_{ss} F + 3k_s^2 F.
\]

For the proof of (v), we use the induction method to get the expression of $k_{s^l}$, for all $n \in \mathbb{N} \cup \{0\}$,
for $n = 0$, we have $k_t = F_{ss} + k^2 F$;
for $n = 1$, we have $k_{st} = F_{s3} + \sum_{j=0}^1 \partial_{s^j} (kk_{s^{j+1}} F)$;
for $n = 2$, we have $k_{sst} = F_{s4} + \sum_{j=0}^2 \partial_{s^j} (kk_{s^{j+1}} F)$;

By assuming when $n = l$, for all $l \in \mathbb{N}, l \geq 3$

\[
k_{s^l} = k_{s^l t} = F_{s^{l+2}} + \sum_{j=0}^l \partial_{s^j} (kk_{s^{j+1}} F),
\]

we have for $n = l + 1$,
\[
k_{s^{l+1}} = k_{s^{l+1} t} = k_{s^{l+1} s} + kF_{k_{s^{l+1}}}
\]
\[
= \left[ F_{s^{l+2}} + \sum_{j=0}^l \partial_{s^j} (kk_{s^{j+1}} F) \right]_s + kk_{s^{l+1}} F
\]
\[
= F_{s^{l+3}} + \sum_{j=0}^l \partial_{s^{j+1}} (kk_{s^{j+1}} F) + kk_{s^{l+1}} F
\]
\[
= F_{s^{l+3}} + \sum_{j=0}^{l+1} \partial_{s^j} (kk_{s^{j+1}} F) + kk_{s^{l+1}} F
\]
\[
= F_{s^{l+3}} + \sum_{j=0}^{l+1} \partial_{s^j} (kk_{s^{j+1}} F).
\]
Here we finish the proof.

The next lemma shows that the winding numbers of curves we study do not change over time.

**Lemma 4.** For the open curves with Neumann boundary condition and closed curves, we obtain $\omega(t) = \omega(0)$ for all time.

**Proof.** First for open curves, as mentioned before, the Neumann condition is equivalent to

$$\langle \nu(\pm 1, t), e \rangle = 0,$$

differentiating it in time implies

$$F_s(\pm 1, t) \langle \tau(\pm 1, t), e \rangle = \pm |e| F_s(\pm 1, t) = 0,$$

as $|e| \neq 0$, we have that $F_s(\pm 1, t) = 0$.

We compute

$$\frac{d}{dt} \int_\gamma k ds = - \int_\gamma F_{ss} + Fk^2 - Fk^2 ds = - \int_\gamma F_{ss} ds = - F_s|_{\{0, L\}} = 0,$$

which implies

$$\omega(t) = \frac{1}{2\pi} \int_\gamma k ds = \frac{1}{2\pi} \int_\gamma k ds \bigg|_{t=0} = \omega(0).$$

Then for closed curves, we have

$$\frac{d}{dt} \int_\gamma k ds = \int_\gamma F_{ss} + Fk^2 - Fk^2 ds = \int_\gamma F_{ss} ds = 0,$$

so

$$\omega(t) = \frac{1}{2\pi} \int_\gamma k ds = \frac{1}{2\pi} \int_\gamma k ds \bigg|_{t=0} = \omega(0).$$

\[ \square \]

### 2.1 Preliminaries

Here we introduce some inequalities we will use frequently in the following Chapters. First, we state the Poincaré-Sobolev-Wirtinger [PSW] inequalities for open curves.

**Proposition 1.** Suppose function $f : [0, L] \to \mathbb{R}, L > 0$, is absolutely continuous and $\int_0^L f dx = 0$. Then

$$\int_0^L f^2 dx \leq \frac{L^2}{\pi^2} \int_0^L f_s^2 dx. \quad (2.1)$$
Suppose function $f : [0, L] \rightarrow \mathbb{R}, L > 0$, is absolutely continuous and $f(0) = f(L) = 0$. Then

$$\int_0^L f^2 dx \leq \frac{L^2}{\pi} \int_0^L f_x^2 dx.$$  

**Proposition 2.** Suppose $f : [0, L] \rightarrow \mathbb{R}, L > 0$, is absolutely continuous and $f(0) = f(L) = 0$. Then

$$\|f\|_2^2 \leq \frac{L}{\pi} \int_0^L f_x^2 dx.$$  

Suppose $f : [0, L] \rightarrow \mathbb{R}, L > 0$, is absolutely continuous and $\int_0^L f dx = 0$. Then

$$\|f\|_\infty^2 \leq \frac{2L}{\pi} \int_0^L f_x^2 dx. \quad (2.2)$$

For the closed curves studied in Chapter 6, we will frequently use the following PSW inequalities. For proofs of these see for example [68, Appendix A] by Parkins and G. Wheeler, we also put the proof in Appendix A, 10.

**Proposition 3.** Suppose function $f : \mathbb{R} \rightarrow \mathbb{R}$ is absolutely continuous and periodic with period $P$. Then if $\int_0^P f dx = 0$, we have

(i) 

$$\int_0^P f^2 dx \leq \frac{P^2}{4\pi^2} \int_0^P f_x^2 dx, \quad (2.3)$$

with equality if and only if $f(x) = A \cos \left( \frac{2\pi}{P} x \right) + B \sin \left( \frac{2\pi}{P} x \right)$ for arbitrary constants $A$ and $B$;

(ii) 

$$\|f\|_\infty^2 \leq \frac{P}{2\pi} \int_0^P f_x^2 dx. \quad (2.4)$$

To state the next interpolation inequality we will use, we first need to set up some notations. For normal tensor $S$ and $T$, we use $S \star T$ to denote any linear combination of $S$ and $T$. In our setting, $S$ and $T$ will be simply curvature $k$ or its arc-length derivatives. We use $P^m_n(k)$ to denote any linear combination of terms of type $\partial_i^1 k \times \partial_i^2 k \times \ldots \times \partial_i^m k$, where $m = i_1 + i_2 + \ldots + i_n$ is the total number of derivatives.

Here we show the following interpolation inequality for the open curves with free boundary and closed curves. For open curves, we refer to [14, Theorem 4.3] by Dall’Acqua and Pozzi; for closed curves, it appears in Dziuk, Kuwert and Schätzle’s paper [17].

**Proposition 4.** Let $\gamma : I \rightarrow \mathbb{R}^2$ be a smooth open or closed curve. Then for any term $P^m_n(k)$
with \( n \geq 2 \) that contains derivatives of \( k \) of order at most \( l - 1 \),
\[
\int_I |P_n^m(k)| ds \leq cL^{1-m-n}\|k\|_{0,2}^{n-p}\|k\|_{l,2}^p
\]
where \( p = \frac{1}{l} (m + \frac{1}{2}n - 1) \) and \( c = c(l,m,n) \). Moreover, if \( m + \frac{1}{2} < 2l + 1 \) then \( p < 2 \) and for any \( \varepsilon > 0 \),
\[
\int_I |P_n^m(k)| ds \leq \int_I |\partial_s k|^2 ds + c\varepsilon^{\frac{n}{2-p}} \left( \int_I |k|^2 ds \right)^{\frac{n-p}{2-p}} + c \left( \int_I |k|^2 ds \right)^{m+n-1}.
\]

Note that in above, \( \| \cdot \|_{0,2} \) and \( \| \cdot \|_{m,2} \) denotes scale-invariant norms, for example
\[
\|k\|_{0,2} = L^\frac{1}{2} \left( \int k^2 ds \right)^{\frac{1}{2}}
\]
and
\[
\|k\|_{1,2} = L^\frac{1}{2} \left( \int k^2 ds \right)^{\frac{1}{2}} + L^\frac{1}{2} \left( \int k^2 ds \right)^{\frac{1}{2}};
\]
Except for the statement of this proposition, in this thesis we will use the notation \( \| \cdot \|_2 \) to denote the regular unscaled norms, pointing out explicit scaling factors where relevant.

In the estimates throughout this thesis, the constants \( c \) can be different from line to line where they depend only on absolute quantities.

Lastly, we state the following standard inequalities for reference.

**Proposition 5** (Hölder inequality). Suppose functions \( f, g : I \to \mathbb{R} \). Assume \( 1 < p, q \leq \infty \) satisfies \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( f \in L^p(I) \) and \( g \in L^q(I) \), then \( fg \in L^1(I) \) satisfies
\[
\left( \int_I |f(x)g(x)| dx \right)^{\frac{1}{p}} \leq \left( \int_I |f(x)|^p ds \right)^{\frac{1}{p}} \left( \int_I |g(x)|^q ds \right)^{\frac{1}{q}}.
\]

**Proposition 6** (Minkowski inequality). Suppose functions \( f, g : I \to \mathbb{R} \). Assume \( 1 < p < \infty \) and \( f, g \in L^p(I) \). Then
\[
\left( \int_I |f(x) + g(x)|^p ds \right)^{\frac{1}{p}} \leq \left( \int_I |f(x)|^p ds \right)^{\frac{1}{p}} + \left( \int_I |g(x)|^p ds \right)^{\frac{1}{p}}.
\]
We use above two inequalities for open curves when \( I = [-1, 1] \), and apply these two inequalities to closed curves under \( I = S^1 \).

**Proposition 7** (Grönwall’s inequality). Let \( f(t) \) be a non-negative, absolutely continuous function on \([0, T]\), which satisfies for
\[
f'(t) \leq \alpha(t)f(t) + \beta(t),
\]
where \( f'(t) \) is the time derivative of \( f(t) \), \( \phi(t) \) and \( \psi(t) \) are functions on \([0, T]\). Then

\[
f(t) \leq \left[ f(0) + \int_0^t \beta(\delta)d\delta \right] e^{\int_0^t \alpha(\delta)d\delta}.
\]

**Proof.** Let \( g(t) := f(t) \cdot e^{-\int_0^t \alpha(\delta)d\delta} \), we have

\[
\frac{d}{dt} g(t) = e^{-\int_0^t \alpha(\delta)d\delta} \cdot \left[ f'(t) - \alpha(t)f(t) \right],
\]

as \( f'(t) \leq \alpha(t)f(t) + \beta(t) \), then

\[
\frac{d}{dt} \left( f(t) \cdot e^{-\int_0^t \alpha(\delta)d\delta} \right) \leq e^{-\int_0^t \alpha(\delta)d\delta} \cdot \beta(t).
\]

Now integrate and get

\[
f(t) \cdot e^{-\int_0^t \alpha(\delta)d\delta} \leq f(0) + \int_0^t \beta(\delta)d\delta.
\]

The result follows. \( \square \)

**Proposition 8** (Young’s inequality). If \( 1 < p, q < \infty \) satisfies \( \frac{1}{p} + \frac{1}{q} = 1 \). Then we have

\[
ab \leq \varepsilon a^p + \left( \varepsilon p \right)^{-\frac{q}{p}} b^q.
\]

where \( a \) and \( b \) are strictly positive real numbers.
Chapter 3

Short time existence

This chapter describes the local existence for flows of open planar curves with Neumann boundary condition and flow of closed planar curves without boundary. Some references about short time existence are mentioned in this chapter.

3.1 A sixth order flow of plane curves with boundary conditions

The sixth order curvature flow of plane curves with Neumann boundary condition is defined as follows, see more details in Chapter 4.

**Definition 2.** Let $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$ be a family of smooth immersion. $\gamma$ is said to move under sixth order curvature flow $F$ with homogeneous Neumann boundary condition, if

$$
\begin{cases}
\frac{\partial}{\partial t} \gamma(s,t) = -F \nu, & \text{for all } (s,t) \in [-1,1] \times [0,T) \\
\gamma(\cdot,0) = \gamma_0, & \\
\langle \nu, \nu_{\eta_{1,2}} \rangle(\pm1,t) = k_{s}(\pm1,t) = k_{s^3}(\pm1,t) = 0, & \text{for all } t \in [0,T)
\end{cases}
$$

(3.1)

where $F = k_{s^4} + k_{ss^2} - \frac{1}{2}k_{s^3}^2$, $\nu$ and $\nu_{\eta_{1,2}}$ are the unit normal fields to $\gamma$ and $\eta_{1,2}$ respectively.

Here we state the way to prove the short time existence for curvature flows of open planar curves with Neumann boundary condition. The first step is to convert the weakly parabolic system (3.1) together with boundary conditions to a corresponding quasilinear scalar parabolic equation. This involves fixing a graphical parametrisation over a reference curve. The reference curve here is a straight line segment. The conversion process using generalised Gaussian coordinates in the case with boundary conditions is described for example in [75, Section 2], the case of higher codimension is discussed in [73]. The
second step is for the scalar parabolic equation with boundary conditions, we consider
the corresponding linearized equation, for which existence of a unique smooth solution
is well-known. By using the solution existence of the linearized problem together with
the general result on the nonlinear evolutionary boundary value problems (for example, in
[69, Theorem 4.4]) to see that the scalar graph equation has a unique solution at least for
a short time. We then prove scalar graph equation is equivalent to the flow system (3.1),
thus a solution to (3.1) exists for a short time. The solution to (3.1) is necessarily not
unique due to the possibility of choosing different parametrizations, however the image
curve is unique. This method also works for the higher order cases (3.6).

3.1.1 Scalar quasilinear initial boundary value problem

This coordinate system (3.1) will be seen as generalized Gaussian coordinates. Let \( l([-1,1]) \)
be a straight line segment which is perpendicular to boundaries \( \eta_1, \eta_2 \). Define the flux
lines \( \Phi = \Phi(u, \cdot) \) to the curve are perpendicular to \( l([-1,1]) \) and tangential to \( \eta_1, \eta_2 \),
see Figure 3.1. Define a neighbourhood \( \mathcal{U} \subset \mathbb{R}^2 \) of \( l([-1,1]) \), \( \mathcal{U}_\epsilon := \{\Phi(u,x) : u \in [-1,1], |x| < \epsilon\} \). In \( \mathcal{U} \), let \( \rho(p_0) \) denote tangential coordinate of \( p_0 \) on \( l([-1,1]) \), then
define a smooth normal vector field \( \xi \) with properties:

\[
\langle \xi, \rho \rangle |_{l([-1,1])} = 0, \quad \xi|_{\eta_1,2} \subset T\eta_1,2, \quad \|\xi\| = 1,
\]
where \( \eta_1,2 \cap \mathcal{U}_\epsilon = \{p \in \mathbb{R}^2 : p = \Phi(u,x), u \in \eta_1,2, x \in (-\epsilon,\epsilon)\} \). Also
\[
\Phi_\sigma(u,\sigma) = \xi(\Phi(u,\sigma)), \quad \Phi(u,0) = \gamma_0(u).
\]

Therefore, for any point \( p = \Phi(u,x) \), we let \( x(p) \) is equal to the length of the flux line
through \( p \) between \( p \) and intersection point \( p_0 = \Phi(u,0) \) on \( l([-1,1]) \). We have
\[
x(p) = \int_0^x |\Phi_\sigma(u,\sigma)| d\sigma = \int_0^x |\xi(\Phi(u,\sigma))| d\sigma.
\]

We define \( M_t = \{p \in \mathbb{R}^2 : p = \Phi(u,w(u,t)), u \in [-1,1]\} \), here \( w(u,t) : [-1,1] \times [0,\sigma] \to \mathbb{R} \) and \( \sigma \in [0,T) \).

Lemma 5. Set \( \gamma(u,t) : [-1,1] \times [0,\sigma] \to \mathbb{R}^2, \gamma(u,t) := (u,w(u,t)) \), then the expression
for the evolution of \( w \) is as follows,

\[
\dot{w}_t(u,t) = v^{-6}w_{u^6} - 18v^{-7}w_{u^5} - 22v^{-7}w_{uu}w_{u^4} + 141v^{-8}w_{u^2}^2w_{u^4} - 13v^{-7}w_{u^3}w_{u^4} + 232v^{-8}w_{uu}w_{u^5} - 561v^{-9}w_{u^3}^2 - 3v^{-7}w_{u^4}w_{uu} + 69v^{-8}w_{u^3}w_{uu} + 48v^{-8}w_{uu}w_{uu} - 699v^{-9}w_{u^2}w_{uu} + 945v^{-10}w_{u^4}w_{uu} + v^{-10}w_{u^2}w_{u^4} - 4v^{-11}w_{uu}w_{u^3} - \frac{1}{2}v^{-10}w_{uu}w_{u^2} + \frac{21}{2}v^{-12}w_{uu}w_{u^3}^3 - 3v^{-11}w_{uu}w_{u^3}^3, \quad (3.2)
\]
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\[ \eta_1 \eta_2 \]

\[ \gamma([-1, 1]) \]

\[ \xi \rho \]

\[ l([-1, 1]) \]

Figure 3.1

where \( v(u, w(u)) := \sqrt{1 + (w_u)^2} \).

Proof. As \( \gamma(u, t) = (u, w(u, t)) \), then we have \( \gamma_u(u, t) = (1, w_u)(u, t) \).

Let \( v(u, t) := |\gamma_u|(u, t) = \sqrt{1 + w_u^2} \), then the tangential vector

\[ \tau(u, t) = \frac{\gamma_u}{|\gamma_u|} = v^{-1}(1, w_u). \]

From above, we can get expression of normal vector \( \nu(u, t) \) directly as follows,

\[ \nu(u, t) = v^{-1}(-w_u, 1). \]

Differentiating \( \gamma \) with respect to time, then \( \partial_t \gamma(u, t) = (0, w_t(u, t)) \) and also

\[ \partial_t \gamma(u, t) = \left[ k_{,s}(u, t) + k^2 k_{ss}(u, t) - \frac{1}{2} k k_{ss}^2(u, t) \right] \cdot v(u, t), \]

here \( s = \int_{u_0}^{u} |\gamma_u|d\bar{u} = \int_{u_0}^{u} \sqrt{1 + w_u^2}d\bar{u} \),

\[ \partial_s = \frac{\partial_u}{|\gamma_u|} = \frac{\partial_u}{v}. \]

From \( \partial_t \gamma \cdot \nu = F \), we obtain

\[ (0, w_t(u, t)) \cdot \nu = k_{,s}(u, t) + k^2 k_{ss}(u, t) - \frac{1}{2} k k_{ss}^2(u, t). \]

As \( v(u, t) = v^{-1}(-w_u, 1)(u, t) \), then

\[ v^{-1} \cdot w_t(u, t) = k_{,s}(u, t) + k^2 k_{ss}(u, t) - \frac{1}{2} k k_{ss}^2(u, t). \]

Also, we have
\[\gamma_{ss} = \frac{1}{v} \partial_u \gamma_s = \frac{1}{v} \partial_u \left( \frac{\gamma_u}{v} \right) = \frac{1}{v} \cdot \frac{\gamma_{uu} \cdot v - \gamma_u \cdot v_u}{v^2} = \frac{1}{v^2} \cdot \gamma_{uu} - \frac{1}{v^3} \gamma_u v_u = \frac{1}{v^2} (0, w_{uu}) - \frac{v_u}{v^3} (1, w_u),\]

here \(\gamma_u = (1, w_u), \gamma_{uu} = (0, w_{uu}).\)

We can get the expression for the curvature \(k(u, t),\)

\[k(u, t) = \langle \gamma_{ss}, \nu \rangle (u, t) = \frac{1}{v^3} \cdot w_{uu},\]

and its first, second, third and fourth derivatives are as follows,

\[k_s(u, t) = \frac{1}{v} \partial_u \left( \frac{w_{uu}}{v^3} \right) = \frac{1}{v} \cdot \frac{w_{uu} v^3 - w_{uu} \cdot 3 v^2 v_u}{v^6} = \frac{1}{v^4} w_{uu}^3 - \frac{3}{v^5} v_u w_{uu},\]

\[k_{ss}(u, t) = \frac{1}{v} \cdot \partial_u \left( \frac{w_{uu}^3}{v^4} - \frac{3}{v^5} v_u w_{uu} \right)
\quad = \frac{1}{v} \cdot \left( \frac{w_{uu} v^4 - 4 w_{uu}^3 v_u}{v^8} - 3 \cdot \frac{u_{uu} w_{uu} v^5 + v_u w_{uu}^3 v^5 - 5 w_{uu} v_u^2 v^4}{v^{10}} \right)
\quad = \frac{1}{v^5} w_{uu}^4 - \frac{4}{v^6} v_u w_{uu}^3 - \frac{3}{v^6} v_u w_{uu} - \frac{3}{v^6} v_u w_{uu}^3 + 15 \frac{1}{v^7} v_u^2 w_{uu}
\quad = \frac{1}{v^5} w_{uu}^4 - \frac{7}{v^6} v_u w_{uu}^3 - \frac{3}{v^6} v_u w_{uu} + 15 \frac{v_u^2}{v^7} w_{uu},\]

and

\[k_{ss}^3(u, t) = \frac{1}{v} \partial_u \left( \frac{1}{v^5} W_{uu}^4 - \frac{7}{v^6} v_u W_{uu}^3 - \frac{3}{v^6} v_u W_{uu} + 15 \frac{v_u^2}{v^7} W_{uu} \right)
\quad = \frac{1}{v} \left( \frac{w_{uu} v^5 - 5 w_{uu}^2 v^4 v_u}{v^{10}} - 7 \cdot \frac{v_{uu} w_{uu}^3 v^6 + v_u w_{uu}^3 v^6 - 6 v^5 v_u^2 w_{uu}}{v^{12}}
\quad - 3 \cdot \frac{v_{uu} w_{uu}^3 v^6 + v_{uu} w_{uu}^3 v^6 - 6 v^5 v_u v_{uu} w_{uu}}{v^{12}}
\quad + 15 \cdot \frac{2 v_u v_{uu} w_{uu} v^7 + v_u^2 w_{uu} v^7 - 7 v^6 v_u^3 w_{uu}}{v^{14}} \right).\]

Simplify above equation, we have
Then here we have $F(u,t)$ as the following equation:
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\[ F(u, t) = k_{e1}(u, t) + k^2 k_{ss}(u, t) - \frac{1}{2} k k^2 (u, t) \]

\[ = \frac{1}{v} w_{t}^6 - 18 \frac{v}{v^2} w_{u} w_{u}^6 - 22 \frac{v}{v^3} w_{uu} w_{u}^4 + \frac{141}{v^4} v^2 w_{u}^2 - 13 \frac{v}{v^5} v_{u} w_{u}^3 + 232 \frac{v}{v^6} v_{uu} w_{u}^3 \]

\[- \frac{561}{v^7} v^3 w_{u}^3 - 3 \frac{v}{v^8} v_{u} w_{u} w_{u} w_{u} + 69 \frac{v}{v^9} v_{u} v_{uu} w_{u}^2 + 48 \frac{v}{v^{10}} v_{uu} w_{u}^2 - \frac{699}{v^{11}} v^2 v_{uu} w_{u} \]

\[ + \frac{945}{v^{12}} v^4 w_{u} + \frac{1}{v^{13}} v_{uu} w_{u}^4 \cdot \left( \frac{1}{v^5} v_{u}^2 w_{u} - \frac{7}{v^6} v_{u} w_{u}^5 + \frac{3}{v^7} v_{uu} w_{u}^2 + \frac{15}{v^8} v_{uu}^2 \right) \]

\[- \frac{1}{2} \cdot \frac{1}{v^2} w_{u} w_{u}^2 \cdot \left( \frac{1}{v^5} w_{u}^2 + \frac{9}{v^6} v_{uu} w_{u} w_{u} - \frac{6}{v^7} v_{uu} w_{u} w_{u} \right) \]

\[ = \frac{1}{v} w_{t}^6 - 18 \frac{v}{v^2} w_{u} w_{u}^6 - 22 \frac{v}{v^3} w_{uu} w_{u}^4 + \frac{141}{v^4} v^2 w_{u}^2 - 13 \frac{v}{v^5} v_{u} w_{u}^3 + 232 \frac{v}{v^6} v_{uu} w_{u}^3 \]

\[- \frac{561}{v^7} v^3 w_{u}^3 - 3 \frac{v}{v^8} v_{u} w_{u} w_{u} w_{u} + 69 \frac{v}{v^9} v_{u} v_{uu} w_{u}^2 + 48 \frac{v}{v^{10}} v_{uu} w_{u}^2 - \frac{699}{v^{11}} v^2 v_{uu} w_{u} \]

\[ + \frac{945}{v^{12}} v^4 w_{u} + \frac{1}{v^{13}} v_{uu} w_{u}^4 \cdot \left( \frac{1}{v^5} v_{u}^2 w_{u} - \frac{7}{v^6} v_{u} w_{u}^5 + \frac{3}{v^7} v_{uu} w_{u}^2 + \frac{15}{v^8} v_{uu}^2 \right) \]

\[- \frac{1}{2} \cdot \frac{1}{v^2} w_{u} w_{u}^2 \cdot \left( \frac{1}{v^5} w_{u}^2 + \frac{9}{v^6} v_{uu} w_{u} w_{u} - \frac{6}{v^7} v_{uu} w_{u} w_{u} \right) \]

also

\[ w_{t}(u, t) = v \cdot F(u, t) \]

\[ = v^6 w_{t}^6 - 18 v^5 w_{u}^6 - 22 v^4 w_{uu} w_{u}^4 + 141 v^3 w_{u}^2 - 13 v^2 v_{u} w_{u}^3 \]

\[ + 232 v^2 v_{uu} w_{u}^3 - 561 v v^2 w_{u} w_{u} + 69 v v^2 v_{uu} w_{u}^2 - 48 v v^2 w_{uu} w_{u} + 945 v^2 v_{uu} w_{u} + v^2 w_{uu} w_{u} \]

\[ - 4 v^2 v_{uu} w_{u}^2 - 1 \frac{1}{2} v^2 w_{uu} w_{u}^3 + 21 \frac{2}{2} v^2 w_{uu} w_{u}^3 - 3 v^2 w_{uu} w_{u}^3 \]

Here we finish the proof. \( \square \)

Now we calculate the boundary condition for scalar initial-boundary-value graph problem. At the boundary, the Neumann boundary condition and \( v_{\eta 1,2} = (1, 0) \) imply

\[ 0 = \langle v, v_{\eta 1,2} \rangle(u, t) = -v^{-1} w_{t}, \]

which can give us \( w_{t}(\pm 1, t) = 0. \)
Also at the boundary we can have \( v(±1,t) = 1, v_u = 0 \), then
\[
k_5(±1,t) = \frac{1}{v^4}w_{u^3}(±1,t) - \frac{3}{v^3}v_uw_{uu}(±1,t) = 0
\]
implies \( w_{u^3}(±1,t) = 0 \).

The third boundary condition in (3.1) shows
\[
k_5(±1,t) = \frac{1}{v^6}w_{u^5} - \frac{12}{v^7}v_uw_{u^4} - \frac{10}{v^7}v_vw_{u^3} + \frac{57}{v^8}v_2w_{u^5}
- \frac{3}{v^7}v_uw_{uu} + \frac{48}{v^8}v_vv_wu_{uu} - \frac{105}{v^9}v_3w_{uu} \\
= 0
\]
which implies \( w_{u^5}(±1,t) = 0 \).

Thus, the boundary conditions in (3.1) can be written as
\[
w_u(±1,t) = w_{u^3}(±1,t) = w_{u^5}(±1,t) = 0.
\]

We transform the problem (3.1) to a scalar initial-boundary-value graph problem as follows,
\[
\begin{aligned}
\frac{\partial w}{\partial t}(u,t) = f(u,t) = (vF)(u,t), & \quad \text{for all } (u,t) \in [-1,1] \times [0,\sigma] \\
w(\cdot,0) = w_0, & \quad \text{for all } u \in [-1,1] \\
w_u = w_{u^3} = w_{u^5} = 0, & \quad \text{for all } (u,t) \in ±1 \times [0,\sigma]
\end{aligned}
\tag{3.3}
\]
here \( v(u,t) = \sqrt{1 + (w_u)^2} \) in Lemma 5, \( F(u,t) \) denotes the curvature flow of \( \gamma(u,t) \), and \( f(u,t) \) is shown in (3.2).

In the evolution of the graph function \( w(u,t) \), the coefficient of the principal part is \( v^{-6} \) only contains the first derivative of \( w \), the rest of the evolution is purely nonlinear, which consists the first to fifth derivative of \( w \). Therefore, the evolution of the graph function is quasilinear. Then we check the parabolicity, as the graph function is a sixth order equation, it is strictly parabolic if the coefficient of the principle part is definite positive. Again the coefficient of the principal part is \( v^{-6} = \left(\sqrt{1 + (w_u)^2}\right)^{-6} > 0 \). Thus, the evolution of the graph function is parabolic and quasilinear.

Next, we prove that problems (3.1) and (3.3) are equivalent. For any solution \( \gamma(u,t) \) to (3.1), there is only one solution \( w(u,t) \) to (3.3) and vice versa.

We let \( w(u,t) = \tilde{w}(\phi(u,t),t) \),
\[
\tilde{\gamma}(\phi(u,t),t) = (\phi(u,t),\tilde{w}(\phi(u,t),t)) := \gamma(u,t),
\]
also \( v(u,t) = \phi(\phi(u,t),t), \tau(u,t) = \tilde{\tau}(\phi(u,t),t) \) and \( F(u,t) = F(\phi(u,t),t) \),
**Definition 3.** Define \( \phi : [-1, 1] \times [0, \sigma] \rightarrow [-1, 1] \) by the following system of ordinary differential equations:

\[
\begin{align*}
\frac{d}{dt} \phi(u,t) &= -(D\bar{\gamma})^{-1} \cdot \left( \frac{\partial}{\partial t} \bar{\gamma} \right)^T \phi(u,t), t) \\
\phi(u,0) &= u,
\end{align*}
\]

(3.4)

where \( \alpha^T := \alpha - \langle \alpha, \bar{\nu} \rangle \cdot \bar{\nu} \) denotes the tangential component of a vector \( \alpha \) and \( \bar{\nu}(\phi(u,t), t) = v^{-1} \cdot (-w_\phi, 1) \) and \( \bar{\tau}(\phi(u,t), t) = v^{-1} \cdot (1, w_\phi) \) denote normal vector and tangential vector field respectively, \( w_\phi (\phi(u,t), t) = \frac{\partial}{\partial \phi}(\phi(u,t), t) \).

At least for a short time, \( \phi \) is a diffeomorphism on \([-1, 1]\), this is equivalent to that \( \frac{\partial \phi}{\partial t} \) is tangential to the boundaries \( \eta_{1,2} \), i.e.

\[
u \in \eta_{1,2} \implies \phi(u,t) \in \eta_{1,2}, \text{ for all } t \in [0, \sigma].
\]

At the boundary, we have \( \bar{\tau} = (1, 0) \) which yields

\[
\langle \frac{\partial}{\partial t} \bar{\gamma}, \bar{\tau} \rangle (\phi(\pm 1,t), t) = 0,
\]

we also have

\[
\langle \bar{\nu}, \bar{\tau} \rangle (\phi(\pm 1,t), t) = 0.
\]

Combining these two relations and using equation \( \alpha^T := \alpha - \langle \alpha, \bar{\nu} \rangle \cdot \bar{\nu} \), we obtain at the boundary

\[
\langle D\bar{\gamma} \cdot \frac{d\phi}{dt}, \bar{\tau} \rangle (\phi(\pm 1,t), t)
\]

\[
= \langle - \left( \frac{\partial}{\partial t} \bar{\gamma} \right)^T, \bar{\tau} \rangle (\phi(\pm 1,t), t)
\]

\[
= \langle - \frac{\partial}{\partial t} \gamma, \bar{\tau} \rangle (\phi(\pm 1,t), t) + \langle \frac{\partial}{\partial t} \gamma, \bar{\nu} \rangle \cdot \langle \bar{\nu}, \bar{\tau} \rangle (\phi(\pm 1,t), t)
\]

\[
= 0.
\]

This means that \( D\bar{\gamma} \cdot \frac{d\phi}{dt} \) is tangential to \( \eta_{1,2} \), thus \( \frac{d\phi}{dt} \) must be tangential to \( \eta_{1,2} \).

**Lemma 6.** If \( w(u,t) \) is a solution of (3.3). Then there exists a unique solution \( \gamma(u,t) : [-1, 1] \times [0, \sigma] \rightarrow \mathbb{R}^2 \) of (3.1).
Proof. We calculate
\[
\frac{d}{dt} \gamma(u,t) = \frac{d}{dt} \phi(u,t) + \frac{d}{dt} \tilde{w}(\phi(u,t),t) = \frac{\partial}{\partial \phi} \tilde{w}(\phi(u,t),t) \cdot \frac{d}{dt} \phi(u,t) + \frac{\partial}{\partial \tilde{w}} \gamma(u,t,\phi(u,t),t)
\]
\[
= -\left( \frac{\partial}{\partial t} \tilde{w} \right)^T \phi(u,t) + \frac{\partial}{\partial t} \gamma(u,t,\phi(u,t),t)
\]
\[
= \left( \frac{\partial}{\partial t} \tilde{w}, \tilde{v} \right) \cdot \tilde{v}(\phi(u,t),t)
\]
\[
= \left( (0, F_v), v^{-1} \begin{pmatrix} -w_{\phi} \\ 1 \end{pmatrix} \right) \cdot v(\phi(u,t),t)
\]
\[
= \bar{F} \tilde{v}(\phi(u,t),t)
\]
\[
= F \nu(u,t).
\]

Also at the boundary, \( \langle \nu, \tau \rangle(\pm 1,t) = \langle \tilde{\nu}, \tilde{\tau} \rangle(\phi(\pm 1,t),t) = 0 \) and \( k_3(\pm 1,t) = k_3(\pm 1,t) = 0 \).

Thus, there is a unique \( \gamma(u,t) \) satisfies (3.1) when \( w(u,t) \) is a solution of (3.3).

\[ \square \]

**Lemma 7.** If \( \gamma(u,t) \) satisfies (3.1), then for a short time, there is a unique \( w(u,t) : [-1,1] \times [0,\sigma] \rightarrow \mathbb{R} \) satisfies (3.3).

Proof. We do the time derivative of \( \gamma(u,t) \),
\[
\frac{d}{dt} \gamma(u,t) = \left( \frac{d}{dt} \phi(u,t), \frac{d}{dt} \tilde{w}(\phi(u,t),t) \right)
\]
also we have
\[
\phi(u,t) = \gamma(u,t) \cdot (1,0),
\]
\[
\tilde{w}(\phi(u,t),t) = \gamma(u,t) \cdot (0,1),
\]
then we get the following two equations,
\[
\frac{d\phi}{dt}(u,t) = \frac{d}{dt} \gamma(u,t) \cdot (1,0) = F \nu(u,t) \cdot (1,0) = \bar{F} \tilde{v}(\phi(u,t),t) \cdot (1,0)
\]
\[
= \bar{F} \cdot v^{-1} \cdot (-\tilde{w}_{\phi}, 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
\[
= -\bar{F} v^{-1} \tilde{w}_{\phi}(\phi(u,t),t),
\]
\[
\frac{d\bar{w}}{dt}(\phi(u,t),t) = \frac{d}{dt}v(0,1) = Fv(u,t) \cdot (0,1) = \bar{F}v(\phi(u,t),t) \cdot (0,1)
\]
\[
= -\bar{F} \cdot v^{-1} \cdot (\bar{\phi},1) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]
\[
= \bar{F}v^{-1}(\phi(u,t),t).
\]

As \( \phi \) is a diffeomorphism on \([-1,1] \), then
\[
\frac{d\bar{w}}{dt}(\phi(u,t),t) = \frac{\partial w}{\partial \phi} (\phi(u,t),t) \frac{d\phi}{dt}(u,t) + \frac{\partial \bar{w}}{\partial t}(\phi(u,t),t).
\]

Set \( y := \phi(u,t) \), for all \((y,t) \in [-1,1] \times [0,\sigma] \), \( \gamma(y,t) := (y,w(y,t)) \). Thus, we have
\[
\frac{\partial w}{\partial t}(y,t) \bigg|_{y=\phi(u,t)} = \frac{dw}{dt}(y,t) - \frac{\partial w}{\partial y}(y,t) \frac{dy}{dt}(u,t)
\]
\[
= \bar{F}v^{-1}(y,t) + w_y \bar{F}v^{-1} w_y (y,t)
\]
\[
= \bar{F}v^{-1}(1 + w_y^2) = \bar{F}v^{-1}v^2
\]
\[
= (\bar{F}v)(y,t) \bigg|_{y=\phi(u,t)}.
\]

Here we finish the proof. \( \square \)

Lemma 6 and Lemma 7 show that our original problem (3.1) and the scalar graph problem (3.3) are equivalent.

### 3.1.2 Short time existence for the quasilinear graph problem

Here we prove that (3.3) has a unique solution for a short time. Here we refer to [69]. The general Theorem 4.4 in [69] can be used to prove nonlinear evolutionary problem with boundary conditions is well posed. See this theorem and some notations in Appendix A, Theorem 16.

In order to use this result in [69], we need to check three conditions, we give the following lemma:

**Lemma 8.** The boundary conditions in (3.3) satisfy the compatibility condition that is, for all \((u,t) \in \eta_{1,2} \times [0,\sigma] \), we have

\[
\left. \frac{\partial^j w_u}{\partial t^j} \right|_{t=0} = \left. \frac{\partial^j w_u^3}{\partial t^j} \right|_{t=0} = \left. \frac{\partial^j w_u^5}{\partial t^j} \right|_{t=0} = 0, \quad j = 0,1,2,\ldots,n.
\]

**Proof.** We have \( v(\pm 1,0) = \sqrt{1 + (w_u)^2} = 1 \), \( v_u \big|_{t=0} = \frac{w_u w_{uu}}{\sqrt{1 + (w_u)^2}} = 0 \), \( v_w \big|_{t=0} = 0, n = 0,1,2,3,\ldots \)

\[
v_t \big|_{t=0} = \left. \left[ 1 + (w_u)^2 \right]^{-1/2} w_u \cdot w_{ut} \right|_{t=0} = \left. \left[ 1 + (w_u)^2 \right]^{-1/2} w_u \cdot w_{tu} \right|_{t=0} = 0
\]
\[ \nu_r |_{t=0} = 0, n = 0, 1, 2, 3, \ldots \]

also

\[ \partial_u \partial_t = \partial_s \partial_t = \partial \partial_s - k F \partial_s = \partial \partial_u. \]

\[ \mathcal{B}(t, w) = (\mathcal{B}_1(t, w), \mathcal{B}_2(t, w), \mathcal{B}_3(t, w)) = (w_u(\pm 1, t), w_{u^3}(\pm 1, t), w_{s^5}(\pm 1, t)) = 0 \]

First, we check compatibility condition for \( \mathcal{B}_1(t, w) = w_u(\pm 1, t) \), we do time derivatives of it.

\[ j = 0: \left. \partial^j \mathcal{B}_1(t, w) \right|_{t=0} = \mathcal{B}_1(0, w) = w_u(\pm 1, 0) = 0; \]

\[ j = 1: \]

\[ \left. \partial^j \mathcal{B}_1(t, w) \right|_{t=0} = \partial \mathcal{B}_1(0, w) = w_{u^3}(\pm 1, t)|_{t=0} = w_{u^3}(\pm 1, t)|_{t=0} = 0, \]

\[ = \left. (-v \tilde{F})_u \right|_{t=0} = -v \tilde{F} - v \tilde{F}_u|_{t=0} = 0 \]

\[ j = 2: \left. \partial^j \mathcal{B}_1(t, w) \right|_{t=0} = \partial^2 \mathcal{B}_1(0, w) = w_{u^5}(\pm 1, t)|_{t=0} = \partial^2 w_{u^5}(\pm 1, 0) = 0, \]

\[ j = 3: \left. \partial^j \mathcal{B}_1(t, w) \right|_{t=0} = \partial^3 \mathcal{B}_1(0, w) = w_{u^7}(\pm 1, t)|_{t=0} = \partial^3 w_{u^7}(\pm 1, 0) = 0, \]

\[ \ldots \]

\[ j = n: \left. \partial^j \mathcal{B}_1(t, w) \right|_{t=0} = \partial^n \mathcal{B}_1(0, w) = w_{u^{2n-1}}(\pm 1, t)|_{t=0} = \partial^n w_{u^{2n-1}}(\pm 1, 0) = 0. \]

Secondly, we check compatibility condition for \( \mathcal{B}_2(t, w) = w_{u^3}(\pm 1, t) \), we do time derivatives of it.

\[ j = 0: \mathcal{B}_2(0, w) = w_{u^3}(\pm 1, 0) = 0; \]

\[ j = 1: \left. \partial^j \mathcal{B}_2(t, w) \right|_{t=0} = \partial w_{u^3}(t, w)|_{t=0} = \partial^3 w_{u^3}(\pm 1, t)|_{t=0} = \partial^3 w_{u^3}(\pm 1, 0) = 0, \]

\[ j = 2: \left. \partial^j \mathcal{B}_2(t, w) \right|_{t=0} = \partial^2 w_{u^3}(t, w)|_{t=0} = \partial^3 w_{u^3}(\pm 1, t)|_{t=0} = \partial^3 w_{u^3}(\pm 1, 0) = 0, \]

\[ j = 3: \left. \partial^j \mathcal{B}_2(t, w) \right|_{t=0} = \partial^3 w_{u^3}(t, w)|_{t=0} = \partial^3 w_{u^3}(\pm 1, t)|_{t=0} = \partial^3 w_{u^3}(\pm 1, 0) = 0, \]

\[ \ldots \]

\[ j = n: \left. \partial^j \mathcal{B}_2(t, w) \right|_{t=0} = \partial^n w_{u^3}(t, w)|_{t=0} = \partial^n w_{u^3}(\pm 1, t)|_{t=0} = \partial^n w_{u^3}(\pm 1, 0) = 0. \]

Thirdly, we check compatibility condition for \( \mathcal{B}_3(t, w) = w_{s^5}(\pm 1, t) \), we do time derivatives of it.

\[ j = 0: \mathcal{B}_3(0, w) = w_{s^5}(\pm 1, 0) = 0; \]

\[ j = 1: \left. \partial^j \mathcal{B}_3(t, w) \right|_{t=0} = \partial w_{s^5}(t, w)|_{t=0} = \partial^5 w_{s^5}(\pm 1, t)|_{t=0} = \partial^5 w_{s^5}(\pm 1, 0) = 0, \]

\[ j = 2: \left. \partial^j \mathcal{B}_3(t, w) \right|_{t=0} = \partial^2 w_{s^5}(t, w)|_{t=0} = \partial^5 w_{s^5}(\pm 1, t)|_{t=0} = \partial^5 w_{s^5}(\pm 1, 0) = 0, \]

\[ j = 3: \left. \partial^j \mathcal{B}_3(t, w) \right|_{t=0} = \partial^3 w_{s^5}(t, w)|_{t=0} = \partial^5 w_{s^5}(\pm 1, t)|_{t=0} = \partial^5 w_{s^5}(\pm 1, 0) = 0, \]

\[ \ldots \]

\[ j = n: \left. \partial^j \mathcal{B}_3(t, w) \right|_{t=0} = \partial^n w_{s^5}(t, w)|_{t=0} = \partial^n w_{s^5}(\pm 1, t)|_{t=0} = \partial^n w_{s^5}(\pm 1, 0) = 0. \]

Thus we prove that our boundary condition

\[ \mathcal{B}(t, w) = (\mathcal{B}_1(t, w), \mathcal{B}_2(t, w), \mathcal{B}_3(t, w)) = (w_u(\pm 1, t), w_{u^3}(\pm 1, t), w_{s^5}(\pm 1, t)) \]
satisfies compatibility condition.

\[ \square \]

**Lemma 9.** The boundary conditions in (3.3) satisfy the normal boundary conditions.

**Proof.** We check our boundary condition satisfy the normal boundary condition. For \( B_1(t,w) = w_u(\pm 1,t), \) we have

\[
B_1 = \partial_u B^p_1 = \partial_u \gamma(u,t) = v\tau(u,t), \quad u \in \eta_{1,2},
\]

\[
B^p_1(u,v) = v\tau \cdot v_{\eta_{1,2}} = -1 \neq 0,
\]

here \( B^p_1 \) is the principal part of \( B_1, \) \( v_{\eta_{1,2}} = v_{\eta_{1,2}}(u) \) denotes the inward normal vector to \( \eta_{1,2} \) at \( u. \) Then we prove that \( B_1 \) is normal.

For \( B_2(t,w) = w_u^3(\pm 1,t), \) we have

\[
B_2 = \partial_u^3 B^p_2 = \partial_u^3 \gamma(u,t) = -v\tau(u,t), \quad u \in \eta_{1,2},
\]

\[
B^p_2(u,v) = -v\tau(u,t) \cdot v_{\eta_{1,2}} = 1 \neq 0,
\]

here \( B^p_2 \) is the principal part of \( B_2. \) Then we prove that \( B_2 \) is normal.

For \( B_3(t,w) = w_u^5(\pm 1,t), \) we have

\[
B_3 = \partial_u^5 B^p_3 = \partial_u^5 \gamma(u,t) = v\tau(u,t), \quad u \in \eta_{1,2},
\]

\[
B^p_3(u,v) = v\tau(u,t) \cdot v_{\eta_{1,2}} = -1 \neq 0,
\]

here \( B^p_3 \) is the principal part of \( B_3. \) Then we prove that \( B_3 \) is normal. Thus, our boundary condition satisfies the normal boundary condition. \( \square \)

Now we do the linearization at any \( a \in \{ w : [-1, 1] \times [0, \sigma] \rightarrow \mathbb{R} \} \) for the factors in the quasilinear problem is (3.3). Here \( z \in C_0^\infty([0,T],[-1,1]). \)

\[
v(w) = \sqrt{1 + (w_u)^2}, \quad v(a + \varepsilon z) = \sqrt{1 + (a + \varepsilon z)^2_u}
\]

For general constant \( n, \) we have

\[
\frac{d}{d\varepsilon} v^{-n}(a + \varepsilon z) \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left[ 1 + [(a + \varepsilon z)_u]^2 \right]^{-n/2} \bigg|_{\varepsilon=0}
\]

\[
= \frac{d}{d\varepsilon} \left[ 1 + a^2_u + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2 \right]^{-n/2} \bigg|_{\varepsilon=0}
\]

\[
= -\frac{n}{2} \left[ 1 + a^2_u + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2 \right]^{-n/2-1} \cdot (2a_u z_u + 2\varepsilon z_u^2) \bigg|_{\varepsilon=0}
\]

\[
= -n(1 + a^2_u)^{-\left(n+2\right)/2} a_u \cdot z_u.
\]
As
\[ \frac{d}{du} v(w) = \frac{w_u w_{uu}}{\sqrt{1 + w_u^2}}, \]
then we have
\[ v_u(a + \varepsilon z) = \frac{(a + \varepsilon z)_u \cdot (a + \varepsilon z)_{uu}}{\sqrt{1 + [(a + \varepsilon z)_u]^2}}, \]
here we linearize \( v_u \) at \( a \),
\[
\begin{align*}
\frac{d}{d\varepsilon} v_u(a + \varepsilon z) &\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left( (a + \varepsilon z)_u \cdot (a + \varepsilon z)_{uu} \right) \bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon} \left( a_u a_{uu} + \varepsilon a_u z_{uu} + \varepsilon a_{uu} z_u + \varepsilon^2 z_a z_{uu} \right) \bigg|_{\varepsilon=0} \\
&= \frac{(a_u z_{uu} + a_{uu} z_u + 2\varepsilon z_u z_{uu}) \cdot \sqrt{1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2}}{1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2} \bigg|_{\varepsilon=0} \\
&= \left[ \begin{array}{c}
\left( a_u a_{uu} + \varepsilon a_u z_{uu} + \varepsilon a_{uu} z_u + \varepsilon^2 z_a z_{uu} \right) \cdot \left( 1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2 \right)^{-\frac{1}{2}} \\
\frac{2a_u z_u + 2\varepsilon z_u^2}{2}
\end{array} \right] \bigg|_{\varepsilon=0} \\
&= \frac{(a_u z_{uu} + a_{uu} z_u) \cdot \sqrt{1 + a_u^2 - a_u^2 a_{uu} (1 + a_u^2)^{-\frac{1}{2}} z_u}}{1 + a_u^2} \\
&= a_u \left( 1 + a_u^2 \right)^{-\frac{1}{2}} z_{uu} + \left[ a_{uu} \left( 1 + a_u^2 \right)^{-\frac{1}{2}} - a_u^2 a_{uu} (1 + a_u^2)^{-\frac{3}{2}} \right] \cdot z_u.
\end{align*}
\]
Computing the second derivatives of \( v(u, t) \) with respect to \( u \),
\[
\begin{align*}
v_{uu}(a + \varepsilon z) &= \partial_u v_u(a + \varepsilon z) \\
&= \partial_u a_u a_{uu} + \varepsilon a_u z_{uu} + \varepsilon a_{uu} z_u + \varepsilon^2 z_a z_{uu} \\
&= \left[ \frac{\left( a_{uu}^2 + a_u a_{uu, u}^3 + 2\varepsilon a_{uu} z_{uu, u} + \varepsilon a_{uu} z_u + \varepsilon a_u w_u + \varepsilon^2 z_{uu}^2 \right)}{1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2} \right] \\
&\cdot \left[ \frac{\left( a_{uu} a_u + \varepsilon a_u z_{uu} + \varepsilon a_{uu} z_u + \varepsilon^2 z_a z_{uu} \right) \left( 1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2 \right)^{-\frac{1}{2}}}{1 + a_u^2 + 2\varepsilon a_u z_u + \varepsilon^2 z_u^2} \\
&\cdot \left( a_{uu} a_u + \varepsilon a_{uu} z_u + \varepsilon a_u z_{uu} + \varepsilon^2 z_a z_{uu} \right) \right],
\end{align*}
\]
then do the linearization at any $a$, we have

$$
\left. \frac{d}{d \varepsilon} v_{uu}(a + \varepsilon z) \right|_{\varepsilon=0} = \left( 2a_{uu}z_{uu} + a_ua_{u\varepsilon} + a_{u\varepsilon}z_u \right) \cdot \sqrt{1 + a_u^2} \cdot (1 + a_u^2) + \frac{a_u^2 + a_ua_{u\varepsilon}}{(1 + a_u^2)^{\frac{3}{2}}} \cdot \frac{1}{2} \left( 1 + a_u^2 \right)^{-\frac{1}{2}} \cdot \frac{(2a_u z_u) \cdot (1 + a_u^2)}{(1 + a_u^2)^2} - \frac{(a_u^2 + a_ua_{u\varepsilon}) \cdot \sqrt{1 + a_u^2} + 2a_u z_u}{(1 + a_u^2)^{\frac{3}{2}}} \cdot \frac{(a_{uu} + a_u z_u) \cdot (1 + a_u^2)}{(1 + a_u^2)^2} - \frac{a_{uu} \cdot (-\frac{1}{2}) \cdot (1 + a_u^2)^{-\frac{1}{2}} \cdot 2a_u z_u \cdot a_{uuu} \cdot (1 + a_u^2)}{(1 + a_u^2)^{\frac{3}{2}}} + \frac{a_{uu} \cdot (1 + a_u^2)^{-\frac{1}{2}} \cdot a_{uuu} \cdot 2a_u z_u}{(1 + a_u^2)^{\frac{3}{2}}} = a_u \left( 1 + a_u^2 \right)^{-\frac{1}{2}} \cdot z_{u3} + \left[ 2a_{uu} \left( 1 + a_u^2 \right)^{-\frac{1}{2}} - 2a_u^2 a_{uuu} \cdot (1 + a_u^2)^{-\frac{1}{2}} \right] \cdot z_{uu} + \left[ a_{u3} \left( 1 + a_u^2 \right)^{-\frac{1}{2}} - a_u^2 a_{u3} \cdot (1 + a_u^2)^{-\frac{1}{2}} - 3a_u a_{uuu} \cdot (1 + a_u^2)^{-\frac{1}{2}} \right] + 3a_{u3}^2 a_{uuu} \cdot (1 + a_u^2)^{-\frac{5}{2}} \right] \cdot z_u.
$$

Moreover,

$$
\left. \frac{d}{d \varepsilon} v_{u\varepsilon}^3(a + \varepsilon z) \right|_{\varepsilon=0} = g(a_u) z_{u4} + g(a_u, a_{uu}) z_{u5} + g(a_u, a_{uu}, a_{u\varepsilon}) z_{uu} + g(a_u, a_{uu}, a_{u3}, a_{u4}) z_u,
$$

$$
\left. \frac{d}{d \varepsilon} v_{\varepsilon t}^3(a + \varepsilon z) \right|_{\varepsilon=0} = g(a_u) z_{u5} + g(a_u, a_{uu}) z_{u6} + g(a_u, a_{uu}, a_{u\varepsilon}) z_{uu} + g(a_u, a_{uu}, a_{u3}, a_{u4}) z_u,
$$

where $g$ are different functions depending only on derivatives of $a(u,t)$.

Then the linearization for $f$ at any $a \in \{ w : [-1,1] \times [0, \sigma] \rightarrow \mathbb{R} \}$ is
\[
\frac{d}{d\varepsilon} f(a + \varepsilon z) \bigg|_{\varepsilon = 0} = f_a(a)z(u, t)
\]
\[
= - \frac{d}{d\varepsilon} v^{-6}(a + \varepsilon z) \bigg|_{\varepsilon = 0} a_{a^6} - v^{-6}(a)z_{a^6} - 18\frac{d}{d\varepsilon} v^{-7}(a + \varepsilon z) \bigg|_{\varepsilon = 0} v_u(a)a_{a^5}
\]
\[
- 18v^{-7}(a) \frac{d}{d\varepsilon} v_u(a + \varepsilon z) \bigg|_{\varepsilon = 0} a_{a^5} - 18v^{-7}(a)v_u(a)z_{a^5} + \ldots
\]
\[
= -v^{-6}(a) \cdot z_{a^6} + g_5(a, a_u, a_{uu}) \cdot z_{a^5} + g_4(a, a_u, a_{uu}, a_{u^2}) \cdot z_{a^4}
\]
\[
+ g_3(a, a_u, a_{uu}, \ldots, a_{u^3}) \cdot z_{a^3} + g_2(a, a_u, a_{uu}, \ldots, a_{u^4}) \cdot z_{aa}
\]
\[
+ g_1(a, a_u, a_{uu}, \ldots, a_{u^5}) \cdot z_{u},
\]

here \(g\) are different functions only depending on derivatives of \(a(u, t)\) and all smooth in space and time.

As the boundary condition
\[
\mathcal{B}(t, z) = (z_u(\pm 1, t), z_{a^3}(\pm 1, t), z_{a^5}(\pm 1, t)),
\]
We set
\[
\mathcal{B}(t, a(u, t)) = (a_u(\pm 1, t), a_{a^3}(\pm 1, t), a_{a^5}(\pm 1, t)),
\]
\[
\mathcal{B}_a(t, a(u, t))z(u, t) = (z_u(\pm 1, t), z_{a^3}(\pm 1, t), z_{a^5}(\pm 1, t)).
\]

Therefore, we get the linear problem

\[
\begin{aligned}
\frac{\partial}{\partial t} z(u, t) &= f_a(a)z(u, t) + g(t), & \text{for all } (u, t) \in [-1, 1] \times [0, \sigma] \\
\mathcal{B}_a(t, a(u, t))z(t) &= (z_u, z_{a^3}, z_{a^5})(\pm 1, t) = 0, & \text{for all } t \in [0, \sigma] \\
z(\cdot, 0) &= 0, & \text{for all } (u, t) \in [-1, 1] \times \{t = 0\}
\end{aligned}
\]

(3.5)

We can obtain that there is a unique solution \(z \in C^\infty([-1, 1], [0, T])\) of the linearized problem by using the classical results on linear parabolic boundary value problem ([53], Ch IV, 6.4). This result is shown in Appendix A, Theorem 17. Here we state the classical result as the following theorem for our problem:

Proposition 9. Let \(z_0 : [-1, 1] \to \mathbb{R}\) be a smooth immersion. There exists a maximal \(T \in (0, \infty)\) such that the linear sixth order problem (3.5) admits a unique solution in the space \(C^\infty([-1, 1], [0, T])\).

For the unique solution of (3.5) \(z \in C^\infty([-1, 1], [0, T])\), we can see that for suitable constants \(c_k > 0\) and integers \(b, k \geq 0\), it satisfies \(\|z\|_k \leq c_k [a, g]_{b, k}\), the definition of \([a, g]_{b, k}\) can be found in Appendix A. From Lemma 8, Lemma 9 and Proposition 9, we prove (3.3) satisfies the conditions in Theorem 16 in Appendix A. Then (3.3) has a unique
smooth solution $w$ for some $T_0 > 0$.

As we proved that (3.3) is equivalent to (3.1), thus (3.1) has a unique solution for finite time. Short time existence for sixth order curvature flow of curves with Neumann boundary condition is given here.

**Theorem 1.** There exists a smooth solution $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$, unique up to parametrisation, of the flow (3.1) with speed $F$, satisfying Neumann boundary conditions and $k_s = k_{3s} = 0$ at the boundary and with initial curve $\gamma(\cdot, 0) = \gamma_0$ compatible with the boundary conditions.

### 3.2 Higher order flows of plane curves with boundary conditions

The definition for $(2m + 4)$th order curvature flow of open curves with generalized Neumann boundary condition is as follows, for more details in Chapter 5.

**Definition 4.** Let $\gamma : [-1, 1] \times [0, T] \to \mathbb{R}^2$ be a family of smooth immersion.

$\gamma$ is said to move under $(2m + 4)$th order curvature flow $F$ with generalized Neumann boundary condition, if

\[
\begin{align*}
\frac{\partial}{\partial t} \gamma(s, t) &= -F \nu, \quad \text{for all } (s, t) \in [-1, 1] \times [0, T) \\
\gamma(\cdot, 0) &= \gamma_0, \quad \text{for all } s \in [-1, 1] \\
\langle \nu, \nu_{\eta_{1,2}} \rangle &= (\pm 1, t) = 0, \quad \text{for all } t \in [0, T) \\
k_s = \ldots = k_{2m-1} = k_{2m+1} = 0, \quad \text{for all } (s, t) \in \pm 1 \times [0, T)
\end{align*}
\]

(3.6)

where $F = (-1)^{m+1}k_{2m+2} + \sum_{j=1}^{m}(-1)^{j+1}kk_{j}k_{m-j} - \frac{1}{2}k_{2m}^2$, $m \in \mathbb{N} \cup \{0\}$, $\nu$ and $\nu_{\eta_{1,2}}$ are the unit normal fields to $\gamma$ and $\eta_{1,2}$ respectively.

We are also interested in one-parameter families of curves $\gamma(\cdot, t)$ satisfying the poly-harmonic curvature flow

\[
\frac{\partial}{\partial t} \gamma(s, t) = (-1)^{m+1}k_{2m+2}\nu,
\]

(3.7)

here general $m \in \mathbb{N} \cup \{0\}$. Above $\nu$ is the smooth choice of unit normal such that the above flow is parabolic in the generalised sense.

**Lemma 10.** While a solution to the flow (3.7) with generalised Neumann boundary conditions exists, we have

\[
\frac{d}{dt} L = -\int_{\gamma} k_{2m+1}^2 ds,
\]

where $L$ denotes the length of the curve.
In view of this lemma and the separation of the supporting parallel lines η_{1,2}, the length \( L \) of the evolving curve \( γ(\cdot, t) \) remains bounded above and below under the flow (3.7).

### 3.2.1 An equivalent quasilinear scalar graph problem

Firstly, we transform the given problem into an equivalent quasilinear scalar graph problem. See section 3.1.1 for the definitions of Gaussian coordinates and \( w \).

**Lemma 11.** Set \( γ : [-1, 1] \times [0, σ] \to \mathbb{R}^2, γ(u, t) := (u, w(u, t)) \).

(i) Tangential vector field is \( τ(u) = v^{-1}(1, w_u) \), here \( v(u, w(u)) := \sqrt{1 + (w_u)^2} \).

(ii) Normal vector field is \( ν(u) = v^{-1}(-w_u, 1) \).

(iii) The evolution for \( w(u, t) \) is

\[
w_t(u, t) = \left( -1 \right)^{m+1} v \sum_{q=2}^{2m+4} \tilde{g}_{2m+4-q} w_{u^q} - \frac{1}{2} v^{-2} w_{uu} \left( \sum_{p=2}^{m+2} \tilde{g}_{m+2-p} w_{u^p} \right)^2 \\
+ \sum_{j=1}^{m} (-1)^{j+1} v^{-2} w_{uu} \sum_{l=2}^{m+j} \tilde{g}_{m+j+2-l} w_{u^l} \sum_{n=2}^{m-j} \tilde{g}_{m-j+2-n} w_{u^n}
\]

here \( \tilde{g}_{2m+4-q} = \tilde{g}_{2m+4-q}(v, v_u, ..., v_{u^{2m+4-q}}) \) is a function only depending on \( v, v_u, ..., v_{u^{2m+4-q}} \), \( v_{u^{2m+4-q}} \), similarly, \( \tilde{g}_{m+j+2-l} = \tilde{g}_{m+j+2-l}(v, v_u, ..., v_{u^{m+j+2-l}}) \), \( \tilde{g}_{m+2-p} = \tilde{g}_{m+2-p}(v, v_u, ..., v_{u^{m+2-p}}) \).

(iv) The boundary conditions are

\[
w_u(±1, t) = w_{u^1}(±1, t) = w_{u^2}(±1, t) = ... = w_{u^{2m+3}}(±) = 0.
\]

**Proof.** We give the proof of (iii) directly,

\[
∂_t γ(u, t) = \left[ \left( -1 \right)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1+j} k_{2m-j} - \frac{1}{2} k_{2m}^2 \right] \cdot ν(u, t).
\]

So

\[
(0, w_t(u, t)) \cdot ν = \left( -1 \right)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1+j} k_{2m-j} - \frac{1}{2} k_{2m}^2,
\]

\[
\frac{1}{v} \cdot w_t(u, t) = \left( -1 \right)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1+j} k_{2m-j} - \frac{1}{2} k_{2m}^2.
\]

For the calculations for \( k, k_x, ..., k_p \), see Lemma 5. By looking for the pattern, we can
have that for $m \in \mathbb{N},$

\[
\begin{align*}
k_{gm} &= g_0(v)w_{u^{m+2}} + g_1(v, v_u)w_{u^{m+1}} + g_2(v, v_u, v_{uu})w_{u^m} + g_s(v, v_u, v_{uu}, v_u^3)w_{u^{m-1}} + \\
& \quad + g_m(v, v_u, \ldots, v_{u^m})w_{uu} \\
& = \sum_{p=2}^{m+2} g_{m+2-p}(v, v_u, \ldots, v_{u^{m+2-p}})w_{u^p},
\end{align*}
\]

here $g_0 = v^{-m-3}$, $g_{m+2-p} = g_{m+2-p}(v, v_u, \ldots, v_{u^{m+2-p}})$ is a function only depending on $v, v_u, \ldots, v_{u^{m+2-p}}$.

Furthermore, we can write $k_{2m+2}, k_{m+j}, k_{m-j}$ as follows:

\[
\begin{align*}
k_{2m+2} &= \sum_{q=2}^{2m+4} \bar{g}_{2m+4-q}(v, v_u, \ldots, v_{u^{2m+4-q}})w_{u^q}, \\
k_{m+j} &= \sum_{l=2}^{m+j} \bar{g}_{m+j+2-l}(v, v_u, \ldots, v_{u^{m+j+2-l}})w_{u^l}, \\
k_{m-j} &= \sum_{n=2}^{m-j} \bar{g}_{m-j+2-n}(v, v_u, \ldots, v_{u^{m-j+2-n}})w_{u^n}.
\end{align*}
\]

where $\bar{g}, \bar{g}, \bar{g}$ are functions only depending on derivatives of $v$.

\[
\begin{align*}
\frac{1}{v} \cdot w_t(u, t) &= (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} kk_{m+j}k_{m-j} - \frac{1}{2}kk_{2m} \\
& = (-1)^{m+1} \sum_{q=2}^{2m+4} \bar{g}_{2m+4-q}w_{u^q} - \frac{1}{2}v^{-3}w_{uu} \left( \sum_{p=2}^{m+2} g_{m+2-p}w_{u^p} \right)^2 \\
& \quad + \sum_{j=1}^{m} (-1)^{j+1} v^{-3}w_{uu} \sum_{l=2}^{m+j} \bar{g}_{m+j+2-l}w_{u^l} \sum_{n=2}^{m-j} \bar{g}_{m-j+2-n}w_{u^n} \\
& = -F(u, t).
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
\frac{1}{v} \cdot w_t(u, t) &= (-1)^{m+1} v \sum_{q=2}^{2m+4} \bar{g}_{2m+4-q}w_{u^q} - \frac{1}{2}v^{-2}w_{uu} \left( \sum_{p=2}^{m+2} g_{m+2-p}w_{u^p} \right)^2 \\
& \quad + \sum_{j=1}^{m} (-1)^{j+1} v^{-2}w_{uu} \sum_{l=2}^{m+j} \bar{g}_{m+j+2-l}w_{u^l} \sum_{n=2}^{m-j} \bar{g}_{m-j+2-n}w_{u^n}.
\end{align*}
\]

We define a scalar initial-boundary-value problem: Denote $w : [-1, 1] \times [0, \sigma] \to \mathbb{R},$
The boundary conditions in Lemma 12.

use this theorem, we need to prove the following two lemmas and one proposition.

Here we prove that scalar quasilinear initial-boundary-value problem (3.9) has a unique solution for a short time. We refer to [69]. See Theorem 16 in Appendix A. In order to use this theorem, we need to prove the following two lemmas and one proposition.

**Lemma 12.** The boundary conditions in (3.9) satisfy the compatibility condition, for all $(u, t) \in \eta_{1, 2} \times [0, \sigma]$, we have

$$\frac{\partial^j w}{\partial t^j}|_{t=0} = \frac{\partial^j w_u}{\partial t^j}|_{t=0} = \ldots = \frac{\partial^j w_{u^{2m+3}}}{\partial t^j}|_{t=0} = 0, \quad j = 0, 1, 2, \ldots, n.$$  

**Proof.** In scalar graph equation (3.9), we can see that when $t = 0,$

$$v(\pm 1, 0) = \sqrt{1 + (w_u)^2} = 1, \quad v_u|_{t=0} = \frac{w_u w_{uu}}{\sqrt{1 + (w_u)^2}} = 0, \quad v_{t^j}|_{t=0} = 0, n = 0, 1, 2, 3, \ldots$$

$$v_1|_{t=0} = [1 + (w_u)^2]^{-1/2} \cdot w_u \cdot w_{uu}|_{t=0} = [1 + (w_u)^2]^{-1/2} \cdot w_u \cdot w_{tu}|_{t=0} = 0.$$  

The boundary conditions are

$$B(t, w) = (B_1(t, w), B_2(t, w), B_3(t, w), \ldots, B_{m+2}(t, w))$$

$$= (w_u(\pm 1, t), w_u(\pm 1, t), w_u(\pm 1, t), \ldots, w_{u^{2m+3}}(\pm 1, t))$$

$$= 0.$$  

First, we check compatibility condition for $B_1(t, w) = w_u(\pm 1, t), B_2(t, w) = w_u(\pm 1, t), B_3(t, w) = w_u(\pm 1, t)$, similarly see Lemma 6.

Generally, we check compatibility condition for $B_{m+2}(t, w) = w_{u^{2m+3}}(\pm 1, t)$, we do time derivatives of it.
\(j = 0:\ \mathcal{B}_{m+2}(0, w) = w_{a^2m+3}(\pm 1) = 0;\)
\(j = 1:\ \partial^j_j \mathcal{B}_{m+2}(t, w)\big|_{t=0} = \partial^2_{a^2} w_{a^2m+3}(t, w)\big|_{t=0} = \partial^2_{a} w_{a^2+3}(\pm 1, t)\big|_{t=0} = 0;\)
\(j = 2:\ \partial^j_j \mathcal{B}_{m+2}(t, w)\big|_{t=0} = \partial^2_{a^2} w_{a^2m+3}(t, w)\big|_{t=0} = \partial^2_{a} w_{a^2+3}(\pm 1, t)\big|_{t=0} = 0;\)
\(j = 3:\ \partial^j_j \mathcal{B}_{m+2}(t, w)\big|_{t=0} = \partial^2_{a^2} w_{a^2m+3}(t, w)\big|_{t=0} = \partial^2_{a} w_{a^2+3}(\pm 1, t)\big|_{t=0} = 0;\)
......
\(j = n:\ \partial^j_j \mathcal{B}_{m+2}(t, w)\big|_{t=0} = \partial^n_{a^2} w_{a^2m+3}(t, w)\big|_{t=0} = \partial^n_{a} w_{a^2+3}(\pm 1, t)\big|_{t=0} = 0.

Thus, we have proved that our boundary condition

\[ \mathcal{B}(t, w) = (\mathcal{B}_1(t, w), \mathcal{B}_2(t, w), \mathcal{B}_3(t, w), \ldots, \mathcal{B}_{m+2}(t, w)) \]

\[ = (w_3(\pm 1, t), w_3(\pm 1, t), w_3(\pm 1, t), \ldots, w_{2m+3}(\pm 1, t)) \]

satisfies compatibility condition.

\[ \square \]

**Lemma 13.** The boundary conditions in (3.9) satisfy the normal boundary conditions.

*Proof.* For \(\mathcal{B}_1(t, w) = w_u(\pm 1, t),\) we have the operator \(B_1 = \partial_u, \quad B_1^p = \partial_u \tau = v \tau, \quad B_1^p(u, v) = v \tau \cdot \nu_{1,2} \neq 0, u \in \eta_{1,2},\) here \(B_1^p\) is the principal part of \(\mathcal{B}_1, \quad \nu_{1,2} = \nu_{1,2}(u)\) denotes the inward normal vector to \(\eta_{1,2}\) at \(u.\) Thus, we prove that \(B_1\) is normal.

For \(\mathcal{B}_2(t, w) = w_u^3(\pm 1, t),\) we have \(B_2 = \partial^3_u, \quad B_2^p = \partial^3_u \tau = -v \tau, \quad B_2^p(u, v) = -v \tau \cdot \nu_{1,2} \neq 0, u \in \eta_{1,2}.\) Thus, we get that \(B_2\) is normal.

For \(\mathcal{B}_3(t, w) = w_u^5(\pm 1, t),\) we have \(B_3 = \partial^5_u, \quad B_3^p = \partial^5_u \tau = v \tau, \quad B_3^p(u, v) = v \tau \cdot \nu_{1,2} \neq 0, u \in \eta_{1,2}.\) Thus, we prove that \(B_3\) is normal.

......

For \(\mathcal{B}_{m+2}(t, w) = w_{a^2m+3}(\pm 1, t),\) we have \(B_{m+2} = \partial^m_{a^2}, \quad B_{m+2}^p = \partial^m_{a} \tau = (-1)^{m+1} v \tau, \quad B_{m+2}^p(u, v) = (-1)^{m+1} v \tau \cdot \nu_{1,2} \neq 0, u \in \eta_{1,2},\) here \(B_{m+2}^p\) is the principal part of \(\mathcal{B}_{m+2}.\)

Thus we prove that \(B_{m+2}\) is normal.

Then our boundary condition satisfies the normal boundary condition. \[ \square \]

In problem (3.9), we let \(f(w) := w_t(u, t)\) in (3.8). When \(m = 0,\)

\[ f(w) = w_t = -v^{-4} w_u^4 + 7v^{-5} v_u w_u^3 + 3v^{-5} v_{uu} w_u^2 - 15v^{-6} v_u^2 w_u. \]

Now we do the linearization of \(f(w)\) at \(a\) when \(m = 0,\)

\[ v(w) = \sqrt{1 + (w_u)^2}, \quad v(a + \varepsilon z) = \sqrt{1 + (a + \varepsilon z)^2}. \]

When \(m = 0,\) from above calculations, we can get
\[
\frac{d}{d\varepsilon} f(a + \varepsilon z) \bigg|_{\varepsilon=0} = -\frac{d}{d\varepsilon} v^{-4}(a + \varepsilon z) \bigg|_{\varepsilon=0} \cdot a_{u^4} - v^{-4}(a) \cdot z_{u^4} + 7 \frac{d}{d\varepsilon} v^{-5}(a + \varepsilon z) \bigg|_{\varepsilon=0} \cdot v_u(a) a_{u^5} + 7v^{-5}(a) \frac{d}{d\varepsilon} v_u(a + \varepsilon z) \bigg|_{\varepsilon=0} a_{u^3} + 3 \frac{d}{d\varepsilon} v^{-5}(a + \varepsilon z) \bigg|_{\varepsilon=0} \cdot v_{uu}(a) a_{uu} + 3v^{-5}(a) \frac{d}{d\varepsilon} v_{uu}(a + \varepsilon z) \bigg|_{\varepsilon=0} a_{uu} - 15 \frac{d}{d\varepsilon} v^{-6}(a + \varepsilon z) \bigg|_{\varepsilon=0} v_u^2(a) a_{uu} - 15v^{-6}(a) \frac{d}{d\varepsilon} v_u^2(a + \varepsilon z) \bigg|_{\varepsilon=0} a_{uu} + 7v^{-5}(a)v_u(a) \cdot z_{u^3} + 3v^{-5}(a)v_{uu}(a) \cdot z_{uu} - 15v^{-6}(a)v_u^2(a) \cdot z_{uu} \\
\frac{d}{d\varepsilon} f(a + \varepsilon z) \bigg|_{\varepsilon=0} = -v^{-4}(a) \cdot z_{u^4} + 7v^{-5}v_u(a) \cdot z_{u^3} + 3v^{-5}v_{uu}(a) \cdot z_{uu} - 15v^{-6}v_u^2(a) \cdot z_{uu} + 4d_{u^4}(1 + a_{u^2})^{-3} a_u \cdot z_{u^2} - 35v_u(a) a_{u^3} (1 + a_{u^2})^{-2} a_u \cdot z_u \\
-15v_{uu}(a) a_{uu}(1 + a_{u^2})^{-2} a_u \cdot z_u + 90v_u^2(a) a_{uu} (1 + a_{u^2})^{-4} a_u \cdot z_u + 7v^{-5}(a) a_{u^3} a_u (1 + a_{u^2})^{-\frac{1}{2}} \cdot z_{uu} + 3v^{-5}(a) a_{uu} \cdot a_u (1 + a_{u^2})^{-\frac{1}{2}} \cdot z_{u^3} + 7v^{-5}(a) a_{u^3} \left[a_{uu} (1 + a_{u^2})^{-\frac{1}{2}} - a_u^2 a_{uu} (1 + a_{u^2})^{-\frac{1}{2}} \right] \cdot z_u + 3v^{-5}(a) a_{uu} \left[2a_{uu} (1 + a_{u^2})^{-\frac{1}{2}} - 2a_u^2 a_{uu} \cdot (1 + a_{u^2})^{-\frac{1}{2}} \right] \cdot z_{uu} + 3v^{-5}(a) a_{uu} \left[a_{u^3} (1 + a_{u^2})^{-\frac{1}{2}} - a_u^2 a_{u^3} \cdot (1 + a_{u^2})^{-\frac{3}{2}} - 3a_u a_{uu}^2 \cdot (1 + a_{u^2})^{-\frac{1}{2}} \right] + 3a_u^2 a_{uu}^2 \cdot (1 + a_{u^2})^{-\frac{5}{2}} \cdot z_u \\
-30v^{-6}(a)v_u(a) a_{uu} a_u (1 + a_{u^2})^{-\frac{1}{2}} \cdot z_{uu} - 30v^{-6}(a)v_u(a) a_{uu} \left[a_{uu} (1 + a_{u^2})^{-\frac{1}{2}} - a_u^2 a_{uu} (1 + a_{u^2})^{-\frac{3}{2}} \right] \cdot z_u \\
= -v^{-4}(a) \cdot z_{u^4} + g[a, a_u, a_{uu}] \cdot z_{u^3} + g[a, a_u, a_{uu}, a_{u^3}] \cdot z_{uu} + g[a, a_u, a_{uu}, a_{u^3}, a_{u^4}] \cdot z_u.
\]

Then it is natural to get the linearization for general \(m\) is

\[
\frac{d}{d\varepsilon} f(a + \varepsilon z) \bigg|_{\varepsilon=0} = -v^{-2m+4}(a) \cdot z_{u^{2m+4}} + g_{2m+3}[a, a_u, a_{uu}] \cdot z_{u^{2m+3}} + g_{2m+2}[a, a_u, a_{uu}, a_{u^3}] \cdot z_{u^{2m+2}} + \ldots + g_3[a, a_u, a_{uu}, \ldots, a_{u^{2m+2}}] \cdot z_{u^3} + g_2[a, a_u, a_{uu}, \ldots, a_{u^{2m+3}}] \cdot z_{uu} + g_1[a, a_u, a_{uu}, \ldots, a_{u^{2m+4}}] \cdot z_u = f_u(a) z(u, t).
\]
The boundary condition is

\[ B(t, w) = (w_u(\pm 1, t), w_{u^3}(\pm 1, t), \ldots, w_{u^{2m+3}}(\pm 1, t)). \]

We set

\[ B(t, a(t)) = (a_u(\pm 1, t), a_{u^3}(\pm 1, t), \ldots, a_{u^{2m+3}}(\pm 1, t)), \]

\[ B_a(t, a(t))z\bigl(t\bigr) = (z_u(\pm 1, t), z_{u^3}(\pm 1, t), \ldots, z_{u^{2m+3}}(\pm 1, t)). \]

Thus, our linearized scalar problem is

\[
\begin{aligned}
\frac{\partial z}{\partial t}(u, t) &= f_a(a)z(u, t) + g(t), \quad \text{for all } (u, t) \in [-1, 1] \times [0, \sigma] \\
(z_{u^3}, \ldots, z_{u^{2m+3}})(\pm 1, t) &= 0, \quad \text{for all } t \in [0, \sigma] \\
z(\cdot, 0) &= 0.
\end{aligned}
\] (3.10)

Referring to the classical results on linear parabolic boundary value problem ([53], Ch IV, 6.4) which is shown in Appendix A, Theorem 17, we obtain the solution of the linear problem (3.10) is unique and exists.

**Proposition 10.** There is always a unique solution for the linear problem (3.10) in the space \( C^\infty([-1, 1], [0, T]) \).

For the unique solution of (3.10) \( z \in C^\infty([-1, 1], [0, T]) \), we can see that for suitable constants \( c_k > 0 \) and integers \( b, k \geq 0 \), it satisfies \( \|z\|_k \leq c_k [a, g]_{b,k} \). From Theorem 16 in Appendix A, the problem (3.9) has a unique solution for finite time. As we proved that graph scalar problem (3.9) is equivalent to our original problem (3.6) in Lemma 6 and Lemma 7, thus (3.6) has a unique solution for finite time.

Short time existence for \((2m + 4)\)th order curvature flow with Generalized Neumann boundary condition is proved.

**Theorem 2.** There exists a smooth solution \( \gamma: [-1, 1] \times [0, T) \to \mathbb{R}^2 \), unique up to parametrisation, of the system (3.6) with speed \( F \) and smooth initial curve \( \gamma(\cdot, 0) = \gamma_0 \) compatible with the boundary conditions.

Directly, we can get the short time existence for flow (3.7) satisfying Neumann boundary condition and \( k_s = \ldots = k_s^{2m-1} = k_s^{2m+1} = 0 \) at the boundary and with smooth initial curve \( \gamma(\cdot, 0) = \gamma_0 \) compatible with the boundary conditions, the solution is also unique up to parametrisation and smooth for \( 0 \leq t < \infty \).

### 3.3 The closed length-constrained curve diffusion flow

The framework of short time existence for flow of closed planar curves without boundary is that we first write the length-constrained curve diffusion flow as a graph over the initial
curve for unknown function of time, we have the scalar quasilinear parabolic problem. Secondly, we prove there is a unique solution for the graph problem and the length-constrained curve diffusion flow is invariant under tangential diffeomorphisms. Then these is a unique solution for length-constrained flow with the unknown time function. Thirdly, we use the Schauder fixed point theorem to prove the unique solution exists for our original problem with specific $h(t)$.

We consider one-parameter families of immersed closed curves $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$. The energy functional

$$L(\gamma) = \int_\gamma |\gamma_u| \, du.$$  

The curve diffusion flow is the steepest descent gradient flow for length in $H^{-1}$. We define the constrained curve diffusion flow here, see more details about this flow in Chapter 6.

**Definition 5.** Let $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a $C^4,\alpha$-regular immersed curve. The length constrained curve diffusion flow

$$\left\{ \begin{array}{ll}
\partial_t \gamma = -(k_{ss} - h(t))\nu, & \text{for all } (s,t) \in \mathbb{S}^1 \times [0, T) \\
\gamma|_{t=0} = \gamma_0, & \text{for all } s \in \mathbb{S}^1
\end{array} \right.$$  

(3.11)

where $\nu$ denotes a unit normal vector field on $\gamma$.

To preserve length of the evolving curve $\gamma(\cdot, t)$, we take

$$h(t) = -\frac{\int_\gamma k_{ss}^2 \, ds}{2\pi \omega},$$

where $\omega$ denotes the winding number of $\gamma(\cdot, t)$.

### 3.3.1 Scalar quasilinear parabolic graph function

Firstly we write $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ as a graph for unknown function of time $\tilde{h}(t)$ over the initial curve $\gamma_0$, using $\nu(u,t)$, $\tau(u,t)$ to denote the normal and tangential vector fields of the curve $\gamma(u,t)$ respectively, then $(\nu, \tau)(u,t) = 0$, $\nu(u,t) = \text{rot}_{\pi/2}\tau(u,t)$, see in Figure 3.2. Let $f : \mathbb{R} \times [0, T) \to \mathbb{R}$, $\nu_0(u) = \nu(u,0)$ write

$$\gamma(u,t) = \gamma_0(u) + f(u,t)\nu_0(u).$$

then take the derivative on both sides with respect to $t$, we have

$$(\partial_t \gamma)(u,t) = (\partial_t f)(u,t) \cdot \nu_0(u),$$

then

$$(\partial_t f)(u,t)\nu_0(u) = (\tilde{h}(t) - k_{ss}(u,t))\nu(u,t),$$
calculating the inner product with $\nu_0$ on both sides,

$$(\partial_t f)(u, t) = (\bar{h}(t) - k_{ss}(u, t))\langle\nu_0(u), v_0(u)\rangle.$$  

Let $V := |\gamma_u|$, then $\partial_s = \frac{\partial}{|\gamma_u|} \frac{\partial}{\partial s}$, $\tau(u, t) = \gamma(u, t) = \frac{\gamma_u(u, t)}{V}$ and $V_0 = |\partial_u \gamma_0|$, $\tau_0 = \frac{\partial \gamma_0}{|\partial_u \gamma_0|} = \frac{\partial \gamma_0}{V_0}$, $(V_0)_u = \langle\tau_0, (\gamma_0)_{uu}\rangle$. We also have

$$v_u = |\gamma_u| \cdot v_s = -k \tau \cdot |\gamma_u|, \quad \partial_u v_0 = -k_0 \tau_0 |\partial_u \gamma_0|.$$  

As $\gamma(u, t) = \gamma_0(u) + f(u, t)v_0(u)$, differentiating with respect with $u$,

$$\gamma_u = \partial_u \gamma_0 + (\partial_{u)f}v_0 + (\partial_{u,v_0})f = \partial_u \gamma_0 + (\partial_{u,f})v_0 + (-k_0 \tau_0 |\partial_u \gamma_0|)f$$

$$= \partial_u \gamma_0 + f_u v_0 - k_0 \tau_0 v_0 f = \tau_0 V_0 + f_u v_0 - k_0 \tau_0 V_0 f$$

$$= \tau_0 V_0 (1 - k_0 f) + f_u v_0,$$

we get $V^2 = |\gamma_u|^2 = V_0^2 (1 - k_0 f)^2 + f_u^2$, then

$$V = |\gamma_u| = \sqrt{V_0^2 (1 - k_0 f)^2 + f_u^2}$$

and

$$\tau(u, t) = \frac{\gamma_u(u, t)}{|\gamma_u(u, t)|} = \frac{\tau_0 V_0 (1 - k_0 f) + f_u v_0}{V}.$$  

Differentiating $V^2$ with respect with $u$,

$$2V v_u = 2V_0 (V_0)_u (1 - k_0 f)^2 + 2V_0^2 (1 - k_0 f) - (k_0)_{uu} - k_0 f_u + 2f_u f_{uu},$$

$$\frac{\partial}{\partial s} \gamma(u, t) = \frac{\partial}{\partial s} \gamma_0(u) + (\partial_{u,f}v_0 + (\partial_{u,v_0})f)\frac{\partial}{\partial s} f = \frac{\partial}{\partial s} \gamma_0(u) + (\partial_{u,f})v_0 + (-k_0 \tau_0 |\partial_u \gamma_0|)f\frac{\partial}{\partial s} f$$

$$= \frac{\partial}{\partial s} \gamma_0(u) + f_u \frac{\partial}{\partial s} v_0 - k_0 \tau_0 v_0 f\frac{\partial}{\partial s} f$$

$$= \tau_0 V_0 (1 - k_0 f) + f_u v_0 f\frac{\partial}{\partial s} f.$$
\[ V_u = \frac{V_0(V_0)_a(1 - k_0 f)^2 + V_0^2(1 - k_0 f)[-(k_0)_u f - k_0 f_u] + f_u f_{uu}}{V} . \]

Here we give several calculations \( \tau_u = k v, v_s = -k \tau, (\tau_0)_u = V_0 k_0 v_0, (v_0)_u = -V_0 k_0 \tau_0, \) then calculating the derivatives of \( \gamma \) of \( u, \)

\[ \gamma_{uu} = (\tau_0)_u V_0(1 - k_0 f) + \tau_0 (V_0)_u (1 - k_0 f) + \tau_0 V_0[-(k_0)_u f - k_0 f_u] + f_{uu} v_0 + f_u (v_0)_u \]
\[ = V_0^2 k_0 V_0(1 - k_0 f) + \tau_0 (V_0)_u (1 - k_0 f) + \tau_0 V_0[-(k_0)_u f - k_0 f_u] + f_{uu} v_0 \]
\[ + f_u V_0(-k_0 \tau_0) \]
\[ = V_0^2 k_0 V_0(1 - k_0 f) + \tau_0 (V_0)_u (1 - k_0 f) + \tau_0 V_0[-(k_0)_u f - 2k_0 f_u] + f_{uu} v_0, \]

\[ \gamma_{uua} = 2V_0 (V_0)_u k_0 \cdot V_0(1 - k_0 f) + V_0^2 (k_0)_u \cdot v_0(1 - k_0 f) + V_0^2 k_0 v_0(1 - k_0 f) \]
\[ + V_0^2 k_0 v_0[-(k_0)_u f - k_0 f_u] + (\tau_0)_u (V_0)_u (1 - k_0 f) + \tau_0 (V_0)_u (1 - k_0 f) \]
\[ + \tau_0 (V_0)_u[-(k_0)_u f - k_0 f_u] + (\tau_0)_u V_0[-(k_0)_u f - 2k_0 f_u] \]
\[ + \tau_0 (V_0)_u[-(k_0)_u f - 2k_0 f_u] + \tau_0 V_0[-(k_0)_u f - 2k_0 f_u] + f_{uua} v_0 + f_{uu} (v_0)_u \]
\[ = 2V_0 (V_0)_u k_0 V_0(1 - k_0 f) + V_0^2 (k_0)_u \cdot v_0(1 - k_0 f) + V_0^2 k_0 V_0(1 - k_0 f) \]
\[ + V_0^2 k_0 V_0[-(k_0)_u f - k_0 f_u] + V_0 k_0 (V_0)_u v_0(1 - k_0 f) + \tau_0 (V_0)_u (1 - k_0 f) \]
\[ + \tau_0 (V_0)_u[-(k_0)_u f - k_0 f_u] + V_0^2 k_0 V_0[-(k_0)_u f - 2k_0 f_u] \]
\[ + \tau_0 (V_0)_u[-(k_0)_u f - 2k_0 f_u] + \tau_0 V_0[-(k_0)_u f - 3(k_0)_u f_u - 2k_0 f_u] \]
\[ + f_{uua} v_0 - f_{uu} V_0 k_0 \tau_0, \]

\[ \gamma_{uuaa} = 2(v_0)_u \cdot V_0 \cdot (V_0)_u k_0(1 - k_0 f) + 2v_0[V_0 (V_0)_u k_0(1 - k_0 f)]_u \]
\[ + (v_0)_u \cdot V_0^2 (k_0)_u (1 - k_0 f) + v_0[V_0^2 (k_0)_u (1 - k_0 f)]_u - (\tau_0)_u \cdot V_0^3 k_0^2 (1 - k_0 f) \]
\[ - \tau_0 \cdot [V_0^3 k_0^2 (1 - k_0 f)]_u - (v_0)_u \cdot V_0^2 k_0[(k_0)_u f + k_0 f_u] \]
\[ - v_0 \cdot V_0^2 k_0((k_0)_u f + k_0 f_u)]_u + (v_0)_u v_0 k_0 (V_0)_u (1 - k_0 f) \]
\[ + v_0 \cdot [V_0 k_0 (V_0)_u (1 - k_0 f)]_u + (\tau_0)_u (v_0)_{uu} (1 - k_0 f) + \tau_0 \cdot [(V_0)_{uu} \cdot (1 - k_0 f)]_u \]
\[ - (\tau_0)_u \cdot (V_0)_u [(k_0)_u f + k_0 f_u] - \tau_0 \cdot [(V_0)_u \cdot ((k_0)_u f + k_0 f_u)]_u \]
\[ - (v_0)_u \cdot V_0^2 k_0[(k_0)_u f + 2k_0 f_u] - v_0 \cdot [V_0^2 k_0 ((k_0)_u f + 2k_0 f_u)]_u \]
\[ - (\tau_0)_u \cdot (V_0)_u [(k_0)_u f + 2k_0 f_u] - \tau_0 \cdot [(V_0)_u ((k_0)_u f + 2k_0 f_u)]_u \]
\[ - (\tau_0)_u \cdot V_0((k_0)_{uu} f + 3(k_0)_u f_u + 2k_0 f_{uu}] \]
\[ - \tau_0 \cdot [V_0((k_0)_{uu} f + 3(k_0)_u f_u + 2k_0 f_{uu})]_u + (v_0)_{uu} V_0^3 + v_0 f_{uu} - (\tau_0)_{uu} V_0 k_0 \]
\[ - \tau_0 (f_{uu} V_0 k_0)_u. \]

Simplify above equation, we have
\[\gamma_{uuv} = -\tau_0 \cdot 2 \cdot V_0^2 (V_0)_u k_0^2 (1 - k_0 f) + 2 \nu_0 [V_0 (V_0)_u k_0 (1 - k_0 f)]_u \\
- \tau_0 \cdot 2 \cdot V_0^3 k_0 (k_0)_u (1 - k_0 f) + \nu_0 [V_0^2 \cdot (k_0)_u (1 - k_0 f)]_u - \nu_0 \cdot V_0^4 k_0^3 (1 - k_0 f) \\
- \tau_0 \cdot [V_0^3 \cdot k_0^2 (1 - k_0 f)]_u + \tau_0 \cdot V_0^3 \cdot k_0^3 [(k_0)_u f + k_0 f_u] \\
- \nu_0 \cdot [V_0^3 k_0 ((k_0)_u f + k_0 f_u)]_u - \tau_0 \cdot [V_0^2 k_0^3 (V_0) (1 - k_0 f)]_u \\
+ \nu_0 \cdot [V_0 k_0 (V_0) (1 - k_0 f)]_u + \nu_0 V_0 k_0 (V_0) (1 - k_0 f) + \tau_0 \cdot [(V_0)_{uu} (1 - k_0 f)]_u \\
- \nu_0 V_0 k_0 (V_0)_u ((k_0)_u f + k_0 f_u) - \tau_0 \cdot [(V_0)_u ((k_0)_u f + k_0 f_u)]_u \\
+ \tau_0 V_0^2 k_0^2 [(k_0)_u f + 2 k_0 f_u] - \nu_0 \cdot [V_0^2 k_0 ((k_0)_u f + 2 k_0 f_u)]_u \\
- \nu_0 V_0 k_0 (V_0)_u ((k_0)_u f + 2 k_0 f_u) - \tau_0 \cdot [(V_0)_u ((k_0)_u f + 2 k_0 f_u)]_u \\
- \nu_0 V_0^2 k_0^2 ((k_0)_{uu} f + 3 (k_0)_u f_u + 2 k_0 f_{uu}) \\
- \tau_0 \cdot [(V_0)_{uu} (1 - k_0 f)]_u + \nu_0 V_0 k_0 f_u^3 + \nu_0 f_u^4 - \nu_0 V_0^2 k_0^2 f_{uu} \\
- \tau_0 \cdot (f_{uu} V_0 k_0)_u.\]

Now we show the expressions of the curvature \(k(u,t)\) and its derivatives, as

\[\gamma_{ss} = \frac{1}{V} \partial_u (\gamma_u) = \frac{1}{V} \partial_u \left( \frac{\gamma_u (u,t)}{V} \right) = \frac{1}{V} \frac{\gamma_{uu} \cdot V - \gamma_u \cdot V_u}{V^2} = \frac{\gamma_{uu} V - \gamma_u V_u}{V^3},\]

first we calculate the curvature,

\[k = \langle \gamma_{ss}, V \rangle = \langle \gamma_{ss}, \text{rot}_{\pi/2} (\gamma_u) \rangle = \langle \frac{\gamma_{uu} V - \gamma_u V_u}{V^3}, \text{rot}_{\pi/2} \left( \frac{\gamma_u}{V} \right) \rangle = \langle \frac{\gamma_{uu} V}{V^2} - \frac{\gamma_u V_u}{V^3}, \frac{1}{V} \cdot \text{rot}_{\pi/2} (\gamma_u) \rangle = \frac{1}{V^3} \langle \gamma_{uu}, \text{rot}_{\pi/2} (\gamma_u) \rangle,\]

here are the first and second derivatives of the curvature,

\[k_s = \frac{1}{V} \cdot \partial_u k = \frac{1}{V} \cdot \frac{1}{V^6} \left[ \langle \gamma_{uu}, \text{rot}_{\pi/2} (\gamma_u) \rangle \cdot V^3 + \langle \gamma_{uu}, \partial_u \text{rot}_{\pi/2} (\gamma_u) \rangle \cdot V^3 \right. \]
\[\left. - \langle \gamma_{uu}, \text{rot}_{\pi/2} (\gamma_u) \rangle \cdot 3 V^2 V_u \right] \]
\[= \frac{\langle \gamma_u^3, \text{rot}_{\pi/2} (\gamma_u) \rangle}{V^4} + \frac{\langle \gamma_{uu}, \partial_u \text{rot}_{\pi/2} (\gamma_u) \rangle}{V^4} - \frac{3 \langle \gamma_{uu}, \text{rot}_{\pi/2} (\gamma_u) \rangle}{V^5} \cdot V_u,\]
Finally, we can get the time derivative of the function \( f(u,t) \),

\[
\partial_t f(u,t) = (\hat{h}(t) - k_{ss}(u,t)) \langle v(u,t), v_0(u) \rangle
\]

\[
= (\hat{h}(t) - k_{ss}(u,t)) \cdot \frac{1}{V} \cdot \langle \text{rot}_{\pi/2}(\gamma_a), v_0(u) \rangle
\]

\[
= \frac{1}{V} \left[ \hat{h}(t) \left( \begin{array}{c}
\langle \gamma_a^3, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^5 + \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^4 \\
- \langle \gamma_{auu}, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 4 \cdot V^3 \cdot V_u \\
+ \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 3 \cdot V^2 \cdot V_u \\
- \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 5 \cdot V^4 \cdot V_u \\
\end{array} \right) \langle \text{rot}_{\pi/2}(\gamma_a), v_0(u) \rangle \right]
\]

\[
= \frac{1}{V} \left[ \hat{h}(t) \left( \begin{array}{c}
\langle \gamma_a^3, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^5 + \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^4 \\
- \langle \gamma_{auu}, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 4 \cdot V^3 \cdot V_u \\
+ \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 3 \cdot V^2 \cdot V_u \\
- \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 5 \cdot V^4 \cdot V_u \\
\end{array} \right) \langle \text{rot}_{\pi/2}(\gamma_a), v_0(u) \rangle \right]
\]

\[
= \frac{1}{V} \left[ \hat{h}(t) \left( \begin{array}{c}
\langle \gamma_a^3, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^5 + \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot V^4 \\
- \langle \gamma_{auu}, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 4 \cdot V^3 \cdot V_u \\
+ \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 3 \cdot V^2 \cdot V_u \\
- \langle \gamma_{auu}, \partial_u \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot 5 \cdot V^4 \cdot V_u \\
\end{array} \right) \langle \text{rot}_{\pi/2}(\gamma_a), v_0(u) \rangle \right]
\]
In $Q(f)$, the highest order term about $f$ is in the last term
\[ -\frac{1}{V^6} \langle \gamma_a, \text{rot}_{\pi/2}(\gamma_a) \rangle \cdot \langle \text{rot}_{\pi/2}(\gamma_a), v_0(u) \rangle, \]
as $\gamma_a = \tau_0 V_0(1 - k_0 f) + f_a v_0$, then $\text{rot}_{\pi/2} \gamma_a = V_0(1 - k_0 f) - f_a \tau_0$, calculating the inner product we get
\[ \langle \text{rot}_{\pi/2}(\gamma_a), v_0 \rangle = V_0(1 - k_0 f), \]
and
\[ \langle \gamma_a, \text{rot}_{\pi/2}(\gamma_a) \rangle = \langle \gamma_a, v_0 V_0(1 - k_0 f) - f_a \tau_0 \rangle = 2[V_0(\nu_0) a_0 k_0(1 - k_0 f)]_a \cdot V_0 \cdot (1 - k_0 f) + [V_0^2(k_0)_a \cdot (1 - k_0 f)]_a \cdot V_0 \cdot (1 - k_0 f) - V_0^2 k_0^2(1 - k_0 f)^2 - [V_0^2 k_0((k_0)_a f + k_0 f_a)]_a \cdot V_0 \cdot (1 - k_0 f) + [V_0 k_0(\nu_0)_a (1 - k_0 f)]_a \cdot V_0 \cdot (1 - k_0 f) + V_0^2 k_0(\nu_0)_a [(k_0)_a f + k_0 f_a] \cdot (1 - k_0 f) + f_a v_0(1 - k_0 f) - V_0^2 k_0[(k_0)_a f + 3(k_0)_a f_a + 2k_0 f_{aa}][1 - k_0 f] - V_0^2 k_0^2[(k_0)_a f + 2k_0 f_a]_a f_a + 2V_0(\nu_0) a_0 k_0^2(1 - k_0 f)_a f_a + V_0^2 k_0(\nu_0)_a (1 - k_0 f)_a f_a + [V_0^2 k_0^2(1 - k_0 f)]_a f_a - [V_0(\nu_0)(1 - k_0 f)]_a f_a + [(\nu_0)_a [(k_0)_a f + k_0 f_a]]_a f_a + V_0^2 k_0^2[(k_0)_a f + 2k_0 f_a]_a f_a + [(\nu_0)_a [(k_0)_a f + 2k_0 f_a]]_a f_a + [V_0((k_0)_a f + 3(k_0)_a f_a + 2k_0 f_{aa})]_a f_a + V_0 k_0 f_{aa} f_a + (f_{aa} V_0 k_0)_a f_a. \]

It is easy to see that the highest order term in above is $f_a^4 V_0(1 - k_0 f)$. Then the highest order term in $Q(f)$ in (3.12) is $-\frac{1}{V^6} \cdot f_a^4 V_0(1 - k_0 f) \cdot V_0(1 - k_0 f) = -\frac{V_0^2(1 - k_0 f)^2}{V^6} \cdot f_a^4$.

Thus, we can write (3.12) as follows
\[
\partial_t f(u,t) = Q(f) = -\frac{V_0^2(1 - k_0 f)^2}{V^6} \cdot f_a^4 + g(\tilde{h}, f, f_u, f_{uu}, f_{u^2}),
\]
where $V = |\gamma_a| = \sqrt{V_0^2(1 - k_0 f)^2 + f_a^2}$ and $g$ depends only on $\tilde{h}(t), f, f_u, f_{uu}, f_{u^2}$.

Our quasilinear parabolic problem is as follows,
\[
\begin{cases}
(\partial_t f)(u,t) = Q(f) = -\frac{V_0^2(1 - k_0 f)^2}{V^6} \cdot f_a^4 + g(\tilde{h}, f, f_u, f_{uu}, f_{u^2}), (u,t) \in \mathbb{S}^1 \times [0, T) \\
f(\cdot, 0) = 0,
\end{cases}
\]
(3.13)
where $g$ depends only on $\tilde{h}(t), f, f_u, f_{uu}, f_{u^3}$. We write $g$ as

$$g = g_3 (\tilde{h}, f, f_u, f_{uu}, f_{u^3}) f_{u^3} + g_2 (\tilde{h}, f, f_u, f_{uu}) f_{uu} + g_1 (\tilde{h}, f, f_u) f_u + g_0 (\tilde{h}, f) f,$$

here $g_l$ are functions depending on $\tilde{h}(t), f, \ldots, f_{u^l}, l = 0, 1, 2, 3$. Also

$$V = |\gamma_u| = \sqrt{V_0^2 (1-k_0 f)^2 + f_u^2}.$$

To prove that there is a unique solution for quasilinear parabolic problem (3.13), we refer to [3, Main Theorem 5], see Appendix A, Theorem 18.

In the evolution of the graph function $f(u, t)$, the coefficient of the principal part (see Definition 10) is $-\frac{V_0^2 (1-k_0 f)^2}{V^6}$ only contains the first derivative of $f$, the rest of the evolution is purely nonlinear, which consists the first, second, and third derivative of $f$. Therefore, the evolution of the graph function is quasilinear. Then we check the parabolicity, as the graph function is a fourth order equation, it is strictly parabolic if the coefficient of the principle part is definite negative. Again the coefficient of the principal part is $-\frac{V_0^2 (1-k_0 f)^2}{V^6} < 0$ when $f \neq \frac{1}{k_0}$, then the fourth order equation is strictly parabolic. Thus, the evolution of the graph function is parabolic and quasilinear.

Now we linearized $Q(f)$ at $f_0 = 0$, here $f_0 = f(u, 0)$.

$$V(f_0 + \epsilon f) = \sqrt{V_0^2 (1-k_0 f)^2 + \epsilon f_u^2},$$

$$\frac{d}{d\epsilon} Q(f_0 + \epsilon f) \Big|_{\epsilon = 0} = \frac{d}{d\epsilon} Q(\epsilon f) \Big|_{\epsilon = 0}$$

$$= -\frac{d}{d\epsilon} \left( \frac{V_0^2 (1-k_0 f)^2}{V^6} \right)_{\epsilon = 0} (\epsilon f)_{u^4} \Big|_{\epsilon = 0} - \frac{V_0^2 (1-k_0 f)^2}{V^6} \epsilon \cdot \frac{d}{d\epsilon} (\epsilon f)_{u^4} \Big|_{\epsilon = 0}$$

$$+ \sum_{l=0}^{3} \frac{d}{d\epsilon} g_l(\epsilon f) \Big|_{\epsilon = 0} (\epsilon f)_{u^l} \Big|_{\epsilon = 0} + \sum_{l=0}^{3} g_l(\epsilon f) \Big|_{\epsilon = 0} \cdot \frac{d}{d\epsilon} (\epsilon f)_{u^l} \Big|_{\epsilon = 0}$$

$$= -V_0^{-4} f_{u^4} + \sum_{l=0}^{3} g_l(\epsilon f) \Big|_{\epsilon = 0} \cdot f_{u^l}.$$

Then our linearized scalar graph problem at $f_0 = 0$ is

$$\begin{cases}
(\partial_t f)(u, t) = -V_0^{-4} f_{u^4} + g_{\epsilon} f_{u^l} + g(t), \text{ for all } (u, t) \in \mathbb{S}^1 \times [0, T) \\
f(\cdot, 0) = 0.
\end{cases}$$

As $f \in C^{4,1,\alpha} [\mathbb{S}^1 \times [0, T)]$, thus the leading coefficient $-V_0^{-4}$ and $g_l(\epsilon f)|_{\epsilon = 0}$, $l = 0, 1, 2, 3$ and $g(t)$ are continuous at $u, t$ and uniformly bounded. We also can see that the leading coefficient satisfies Legendre-Hadamard condition in Definition 11. Therefore, we can refer to Theorem 18 in Appendix A and proof that there is a unique solution...
for the quasilinear scalar graph problem (3.13) when $\tilde{h}(t)$ is an unknown function of time.

We need to prove this coincides with the problem:

$$\begin{cases} 
\partial_t \gamma = -(k_{ss} - \tilde{h}(t)) \nu, & \text{for all } (s,t) \in S^1 \times [0,T) \\
\gamma|_{t=0} = \gamma_0, & \text{for all } s \in S^1 
\end{cases} \quad (3.14)$$

Here we use the method of applying a tangential diffeomorphism in Definition 3 in Chapter 3 at each time step to our flow to ensure that the domain of our graph function is independent of time.

**Lemma 14.** Length-constrained curve diffusion flow is invariant under tangential diffeomorphisms.

**Proof.** The immersion $\gamma(u,t) = \gamma_0(u) + f(u,t) v_0(u)$ can be written as

$$\gamma(u,t) = (\phi(u,t), f(\phi(u,t),t)).$$

We define $\phi : S^1 \times [0,\sigma] \rightarrow S^1$ by the system of ordinary differential equation as (3.4).

We have already shown that the graph $f$ satisfies a quasilinear fourth order evolution. However, in computing the evolution of $\gamma(u,t)$, we only used that the normal part of the speed is equal to $-k_{ss} + \tilde{h}$. That is, this formulation of constrained curve diffusion flow is

$$\left( \frac{\partial}{\partial t} \gamma \right)^\perp = (-k_{ss} + \tilde{h}) \cdot \nu,$$

where $(\cdot)^\perp$ denotes normal projection. This differs from our desired evolution by a tangential diffeomorphisms $\phi$ satisfying

$$D\gamma \cdot \left( \frac{\partial \phi}{\partial t} \right) = -\left( \frac{\partial \gamma}{\partial t} \right)^\top,$$

where $(\cdot)^\top$ denotes tangential projection, see (3.4).

Consider now an evolution $\left( \frac{\partial}{\partial t} \gamma \right)^\perp = (-k_{ss} + \tilde{h}) \cdot \nu$. As $\phi(s,t) : S^1 \times [0,\sigma] \rightarrow S^1$ satisfies

$$D\gamma(\phi(u,t),t) \cdot \left( \frac{\partial \phi}{\partial t}(u,t) \right) = -\left( \frac{\partial \gamma}{\partial t}(\phi(u,t),t) \right)^\top.$$

We set $\tilde{\gamma}(u,t) = \gamma(\phi(u,t),t)$ then

$$\frac{\partial}{\partial t} \tilde{\gamma} = \frac{d}{dt} \gamma(\phi(s,t),t) = D\gamma \cdot \frac{\partial \phi}{\partial t} + \frac{\partial \gamma}{\partial t} = -\left( \frac{\partial \gamma}{\partial t} \right)^\top + \frac{\partial \gamma}{\partial t} = \left( \frac{\partial}{\partial t} \gamma \right)^\perp = (-k_{ss} + \tilde{h}) \cdot \nu.$$

Here we finish the proof. \qed
Therefore, for a short time, there is unique solution for the constrained curve diffusion flow (3.14).

### 3.3.2 Fixed point argument

We still need to prove that there is a fixed point argument for \( h(t) \) in our problem. Before giving the fixed point argument, we calculate \( \frac{d}{dt} h(0) \leq c(\gamma_0) \) first.

In Definition 5, we have that

\[
h(t) = -\frac{1}{2\pi \omega} \int k_\gamma^2 ds.
\]

We do the first derivative of \( h(t) \) with respect to time, here \( \omega \) doesn’t change under the flow. We use \( k_{\gamma t} \) to denote \( \frac{\partial}{\partial t} k_{\gamma l}, l = 0, 1, 2, \ldots \)

\[
\frac{d}{dt} h(t) = -\frac{1}{2\pi \omega} \cdot \frac{d}{dt} \int k_\gamma^2 ds
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left( \int k_\gamma^2 \frac{\partial}{\partial t} ds + \int \frac{\partial}{\partial t} k_\gamma^2 ds \right)
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left( k_\gamma^2 kF ds + \int 2k_\gamma k_s k_t ds \right)
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left[ k_\gamma^2 k(-k_{ss} + h(t))ds + 2 \int k_\gamma(-F_3 - F_k^2 - 3 k_\gamma k) ds \right]
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left[ - \int k_{\gamma s}^2 k_{ss} ds + h(t) \int k_{\gamma s}^2 k ds + 2 \int k_\gamma k_s k_{ss} ds + 2 \int k_\gamma k_{ss}^2 k ds + 2 \int k_{ss}^2 k_{t} ds + 2 \int 2k_\gamma k_{ss}^2 ds + 6 \int k_\gamma^2 k_{ss}^2 ds - 6 \int k_\gamma^2 k_{ss} h(t) ds \right]
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left[ 5 \int k_{ss} k_\gamma^2 k ds - 5h(t) \int k_\gamma^2 k ds + 2 \int k_{\gamma s}^2 k ds + 2 \int k_\gamma k_s k_{ss} ds + 2 \int k_\gamma k_{ss} k_{ss} ds - 2 \int k_{ss}^2 k_{st} ds \right]
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left[ 5 \int k_{ss} k_\gamma^2 k ds + 2 \int k_{\gamma s}^2 k ds + 2 \int k_\gamma k_{ss} k_{ss} ds - 2 \int k_{ss}^2 k_{st} ds \right]
\]

\[
= -\frac{1}{2\pi \omega} \cdot \left[ \int k_{ss} k_\gamma^2 k ds + 2 \int k_{\gamma s}^2 k ds + 2 \int k_\gamma k_{ss} k_{ss} ds - 2 \int k_{ss}^2 k_{st} ds \right]
\]

As we need to have the second time derivative of \( h(0) \) is bounded, so we only care about the highest order of \( \frac{d^2}{dt^2} h(t) \), here we check the first term in above,

\[
\frac{d}{dt} \int k_{ss}^2 ds = \int k_{ss}^2 \frac{\partial}{\partial t} ds + 2 \int k_\gamma k_{ss} k_{ss} ds,
\]
where
\[
\begin{align*}
k_{s,t} &= \partial_s \partial_t k_{s,t} = \partial_t k_{s,t} = k F k_{s,t} - k k_{s,t} = \partial_s (\partial_t k_{s}) k_s - k F k_{s,t} \\
&= \partial_s (\partial_t k_{st} - k F k_{s,t}) - k F k_{s,t} = \partial_s^2 k_{st} - \partial_s (k F k_{s,t}) - k F k_{s,t} \\
&= \partial_s^2 (-F_{s,t} - F_k k^2 - 3 F_k k) - \partial_s (k F k_{s,t}) - k F k_{s,t} \\
&= k_{s,t} + g_1 (t, k, k_s, k_{s,s}, k_{s,t}, k_{s,t}, k_{s,t}) ,
\end{align*}
\]

where \(g_1\) only depends on \(k, k_s, ..., k_{s,t}\) and \(t\) from \(h(t)\). then
\[
\frac{d}{dt} \int_{\gamma} k_{s,t}^2 ds = \int_{\gamma} \frac{\partial}{\partial t} k_{s,t}^2 ds + 2 \int_{\gamma} k_{s,t} k_{s,t}^t ds \\
= -2 \int_{\gamma} k_{s,t}^2 ds + g_2 (t, k, k_s, k_{s,s}, k_{s,t}, k_{s,t}, k_{s,t}) ,
\]

where \(g_2\) only depends on \(k, k_s, ..., k_{s,t}\) and \(t\) from \(h(t)\). Thus, the second derivative of \(h(t)\) with respect to time is
\[
\frac{d^2}{dt^2} h = \frac{1}{2 \pi \omega} \cdot \frac{d}{dt} \left[ -2 \int_{\gamma} k_{s,t}^2 ds + \frac{1}{3} \int_{\gamma} k_{s,t}^4 ds - \frac{5}{2} h(t) \int_{\gamma} k_{s,s} k_{s,t}^2 ds + 2 \int_{\gamma} k_{s,s} k_{s,s}^2 ds \right] \\
= \frac{1}{2 \pi \omega} \cdot \left[ 4 \int_{\gamma} k_{s,t}^2 ds + g (t, k, k_s, k_{s,s}, k_{s,t}) \right],
\]

where \(g\) is a function only depending on \(k, k_s, k_{s,s}, k_{s,t}, k_{s,t}\) and \(t\) from \(h(t)\).

The highest order term in above is easily seen to be \(\int k_{s,t}^2 ds\). We can see that \(\frac{d^2}{dt^2} h(0)\) is bounded if \(\gamma_0 \in C^{7,\alpha} (S^1)\). Before we show that at least one of the functions \(\tilde{h}\) coincide with our given constrained function \(h\), we need the following fixed point theorem, the proof of Theorem 3 can refer to the proof of Theorem 2.7 in [79].

**Theorem 3. (Schauder fixed point theorem)** Let \(I\) be a compact, convex subset of a Banach space \(B\) and let \(J\) be a continuous map of \(I\) into itself. Then \(J\) has a fixed point.

Then, by referring to [79, Theorem 2.7], we get the short time existence for the length-constrained curve diffusion flow.

**Theorem 4.** Let \(\gamma_0 : S^1 \to \mathbb{R}^2\) be a \(C^{7,\alpha}\)-regular immersed curve. Then there exists a maximal \(T \in (0, \infty]\) such that the constrained curve diffusion flow \(\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2\)
\[
\begin{align*}
\partial_t \gamma &= -(k_{s,s} - h(t)) \nu, \quad \text{for all } (s, t) \in \mathbb{S} \times [0, T) \\
\gamma|_{t=0} &= \gamma_0, \quad \text{for all } s \in \mathbb{S}^1
\end{align*}
\]
is uniquely solvable with \(\gamma\) of degree \(C^{4,1,\alpha} (\mathbb{S}^1 \times [0, T))\).

For the proof of this theorem, see Appendix A.
3.4 The closed constrained ideal curve flow

The framework of short time existence for closed constrained ideal curve flow is similar to the previous section (the short time existence for the length-constrained curve diffusion flow). We consider one-parameter families of immersed closed curves $\gamma: S^1 \times [0, T) \rightarrow \mathbb{R}^2$. The energy functional is

$$E(\gamma) = \int_{\gamma} k^2 ds.$$

We define the constrained ideal curve flow here, see more details about this flow in Chapter 7.

**Definition 6.** Let $\gamma: S^1 \times [0, T) \rightarrow \mathbb{R}^2$ be a $C^{6,\alpha}$-regular immersed curve. The constrained ideal curve flow

$$\begin{align*}
\partial_t \gamma &= (k_\nu + k^2 k_{ss} - \frac{1}{2}kk_s^2 + h(t))\nu, \quad \text{for all } (s,t) \in S^1 \times [0, T) \\
\gamma|_{t=0} &= \gamma_0, \quad \text{for all } s \in S^1
\end{align*}$$

(3.15)

where $\nu$ denotes a unit normal vector field on $\gamma$.

To preserve length and area of the evolving curve $\gamma(\cdot, t)$, we take $h(t)$ as

$$\frac{1}{2\pi a_0} \left(-f_\gamma k_{ss}^2 ds + \frac{7}{2}f_\gamma k_s^2 ds\right)$$

and

$$\frac{5}{2\pi} \int_{\gamma} k k_s^2 ds$$

respectively.

3.4.1 Scalar quasilinear parabolic graph function

Firstly we write $\gamma: S^1 \times [0, T) \rightarrow \mathbb{R}^2$ as a graph for unknown function of time $\tilde{h}(t)$ over the initial curve $\gamma_0$, using $\nu(u, t), \tau(u, t)$ to denote the normal and tangential vector fields of the curve $\gamma(u, t)$ respectively, then take the derivative on both sides with respect to $t$, we have

$$\partial_t f(u, t) = (k_\nu + k^2 k_{ss} - \frac{1}{2}kk_s^2 + \tilde{h}(t))\langle \nu(u, t), v_0(u) \rangle.$$

Referring to Section 3.3.1, we have the expressions for $\gamma_u, \gamma_{uu}, \gamma_{ud}$ and $\gamma_{ud}$. Also we calculate $k, k_s$ and $k_{ss}$. It is easy to see that the highest order term in $\gamma_{ud}$ is $v_0f_{u^6}$.

As $\text{rot}_{\pi/2}\gamma_u = v_0V_0(1-k_0f) - f_u\gamma_0$ and $\langle \text{rot}_{\pi/2}(\gamma_u), v_0 \rangle = V_0(1-k_0f)$. Then, we can get the time derivative of the function $f(u, t)$,

$$\begin{align*}
\partial_t f(u, t) &= (k_\nu + k^2 k_{ss} - \frac{1}{2}kk_s^2 + \tilde{h}(t))\langle \nu(u, t), v_0(u) \rangle \\
&= \frac{1}{V} \left( \frac{1}{V^2} \langle \gamma_u, \text{rot}_{\pi/2}(\gamma_u) \rangle + g(\gamma, \gamma_u, ..., \gamma_{ud}) \right) \cdot \langle \text{rot}_{\pi/2}(\gamma_u), v_0(u) \rangle \\
&= \frac{V_0^2}{V^8} (1-k_0f)^2 f_{u^6} + g(\tilde{h}, f, f_u, ..., f_{u^6}) \\
&= Q(f).
\end{align*}$$

(3.16)
Here \( V = |\gamma_u| = \sqrt{V_0^2(1 - k_0f)^2 + f_u^2} \) and \( g \) depends only on \( \tilde{h}(t), f, f_u, \ldots, f_{u^5} \).

To prove that there is a unique solution for quasilinear parabolic problem, we refer to [3, Main Theorem 5], see Appendix A, Theorem 18. Our quasilinear parabolic problem satisfies the conditions in Theorem 18.

\[
\begin{align*}
\left\{ \begin{array}{l}
(\partial_t f)(u, t) = Q(f) = \frac{V_0^2(1 - k_0f)^2}{V^8} f_{u^6} + g(\tilde{h}, f, f_u, \ldots, f_{u^5}), (u, t) \in S^1 \times [0, T) \\
\quad f(\cdot, 0) = 0,
\end{array} \right.
\end{align*}
\]

(3.17)

where \( g \) depends only on \( \tilde{h}(t), f, f_u, \ldots, f_{u^5} \). We write \( g \) as

\[
g = g_5 \left( \tilde{h}, f, f_u, f_{uu}, f_{u^5} \right) f_{u^5} + \ldots + g_1 \left( \tilde{h}, f, f_u \right) f_u + g_0(\tilde{h}, f) f,
\]

\( g_l \) are functions depending on \( \tilde{h}(t), f, \ldots, f_{u^l} \). We refer to Theorem 18 and proof that there is a unique solution for quasilinear scalar graph problem (3.17) when \( \tilde{h}(t) \) is an unknown function of time.

In the evolution of the graph function \( f(u, t) \), the coefficient of the principal part (see Definition 10) is \( \frac{V_0^2(1 - k_0f)^2}{V^8} \) only contains the first derivative of \( f \), the rest of the evolution is purely nonlinear, which consists the first, second, and third derivative of \( f \). Therefore, the evolution of the graph function is quasilinear. Then we check the parabolicity, as the graph function is a fourth order equation, it is strictly parabolic if the coefficient of the principal part is positive. Again the coefficient of the principal part is \( \frac{V_0^2(1 - k_0f)^2}{V^8} > 0 \) when \( f \neq \frac{1}{k_0} \), then the fourth order equation is strictly parabolic. Thus, the evolution of the graph function is parabolic and quasilinear.

Now we linearized \( Q(f) \) at \( f_0 = 0 \), here \( f_0 = f(u, 0) \). Then our linearized scalar graph problem at \( f_0 = 0 \) is

\[
\left\{ \begin{array}{l}
(\partial_t f)(u, t) = V_0^{-6} \cdot f_{u^6} + g_l \cdot f_{u^l} + g(t), \text{ for all } (u, t) \in S^1 \times [0, T) \\
\quad f(\cdot, 0) = 0.
\end{array} \right.
\]

As \( f \in C^{6, 1, \alpha} [S^1 \times [0, T)] \), thus the leading coefficient \( V_0^{-6} \) and \( g_l(\varepsilon f)|_{\varepsilon = 0}, l = 0, 1, \ldots, 5 \) and \( g(t) \) are continuous at \( u, t \) and uniformly bounded. We also can see that the leading coefficient satisfies Legendre-Hadamard condition in Definition 11. Therefore, we can refer to Theorem 18 and proof that there is a unique solution for the quasilinear scalar graph problem (3.17) when \( \tilde{h}(t) \) is an unknown function of time.

We need to prove this coincides with the problem:

\[
\begin{align*}
\partial_t \gamma = (k_{s^2} + k^2 k_{ss} - \frac{1}{2} k k_{s^2} + \tilde{h}(t)) \nu, \quad \text{for all } (s, t) \in S^1 \times [0, T) \\
\gamma|_{t=0} = \gamma_0, \quad \text{for all } s \in S^1
\end{align*}
\]

(3.18)

By applying a tangential diffeomorphism in Definition 3 in Chapter 3 at each time step to our flow, we can have that the domain of our graph function is independent of time.
Lemma 15. Constrained ideal curve flow is invariant under tangential diffeomorphisms.

Therefore, for a short time, there is unique solution to the constrained ideal curve flow (3.18).

3.4.2 Fixed point argument

We still need to prove that there is a fixed point argument for \( h(t) \) in our problems. Before giving the fixed point argument, we calculate \( \frac{d^2}{dt^2} h(0) \leq c(\gamma_0) \).

The length-constrained ideal curve flow

In Definition 6, for length-constrained ideal curve flow, we have that

\[
h(t) = \frac{1}{2\pi \omega} \left( -\int_{\gamma} k_s^2 ds + \frac{7}{2} \int_{\gamma} k_s^2 k^2 ds \right),
\]

see more details about this flow in Section 7.1.

We do the first derivative of \( h(t) \) with respect to time, here \( \omega \) doesn’t change under the flow. See Lemma 4 in Chapter 2.

\[
\frac{d}{dt} h(t) = \frac{-1}{2\pi \omega} \cdot \frac{d}{dt} \left( \int_{\gamma} k_s^2 ds - \frac{7}{2} \int_{\gamma} k_s^2 k^2 ds \right)
\]

\[
= \frac{-1}{2\pi \omega} \cdot \left[ -2 \int_{\gamma} k_s^2 ds + g(t, k, k_s, ..., k_s^4) \right]
\]

where \( g \) only depends on \( t, k, k_s, ..., k_s^4 \).

As we need to have the second time derivative of \( h(0) \) is bounded, so we only care about the highest order of \( \frac{d^2}{dt^2} h(t) \). Thus, the second derivative of \( h(t) \) with respect to time is

\[
\frac{d^2}{dt^2} h = \frac{-1}{2\pi \omega} \cdot \left[ 4 \int_{\gamma} k_s^2 ds + g(t, k, k_s, ..., k_s^7) \right],
\]

where \( g \) is a function only depending on \( k, k_s, ..., k_s^7 \) and \( t \) from \( h(t) \).

The highest order term in above is easily seen to be \( \frac{-2}{\pi \omega} \int k_s^2 ds \). We can see that \( \frac{d^2}{dt^2} h(0) \) is bounded if \( \gamma_0 \in C^{10,\alpha}(S^1) \). Before we show that at least one of the functions \( \tilde{h} \) coincide with our given constrained function \( h \), we need the fixed point theorem, Theorem 3.

Then, by referring to [79, Theorem 2.7], we get the short time existence for the length-constrained ideal curve flow.
Theorem 5. Let \( \gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2 \) be a \( C^{10,\alpha} \)-regular immersed curve. Then there exists a maximal \( T \in (0,\infty] \) such that the length-constrained curve diffusion flow \( \gamma : \mathbb{S} \times [0,T) \to \mathbb{R}^2 \)

\[
\begin{align*}
\partial_t \gamma &= (k_s + k^2 k_{ss} - \frac{1}{2} kk_s^2 + h(t)) \nu, \quad \text{for all } (s,t) \in \mathbb{S} \times [0,T) \\
\gamma|_{t=0} &= \gamma_0, \quad \text{for all } s \in \mathbb{S}^1
\end{align*}
\]

where \( h(t) = \frac{1}{2\pi \rho_0} \left( -\int_{\gamma} k_{ss}^2 ds + \frac{2}{3} \int_{\gamma} k_s^2 k^2 ds \right) \), is uniquely solvable with \( \gamma \) of degree \( C^{6,1,\alpha} \left( \mathbb{S}^1 \times [0,T) \right) \).

The proof of this theorem is similar to the proof of Theorem 4.

The area-preserving ideal curve flow

In Definition 6, for area-preserving ideal curve flow, we let that

\[
\gamma(t) = \gamma_0 \cdot \frac{5}{2L} \int_{\gamma} k k_s^2 ds,
\]

see more details about this flow in Section 7.2.

We do the first derivative of \( h(t) \) with respect to time, here \( w \) doesn’t change under the flow.

\[
\frac{d}{dt} h(t) = \frac{d}{dt} \left( \frac{5}{2L} \int_{\gamma} k k_s^2 ds \right)
\]

\[
= \frac{5}{2} \left[ -L^{-2} \left( \frac{d}{dt} L \right) \int_{\gamma} k k_s^2 ds + L^{-1} \frac{d}{dt} \int_{\gamma} k k_s^2 ds \right]
\]

\[
= \frac{5}{2} \left[ L^{-2} \int_{\gamma} k F ds \int_{\gamma} k k_s^2 ds + L^{-1} \int_{\gamma} (F_{ss} + F k^2) k_s^2 ds \right]
\]

\[
+ 2L^{-1} \int_{\gamma} k k_s (k_{s3} + k^2 F_s + 3kk_s F) ds - L^{-1} \int_{\gamma} k k_s^2 \cdot k F ds \right]
\]

\[
= \frac{5}{2} \left[ -5L^{-1} \int_{\gamma} k_s k_s k_s - L^{-1} \int_{\gamma} k_s^2 k_s ds + g(t,k,k_s,k_{ss},k_{s3}) \right]
\]

where \( g \) only depends on \( t,k,k_s,k_{ss},k_{s3} \).

As we need to have the second time derivative of \( h(0) \) is bounded, so we only care about the highest order of \( \frac{d^2}{dt^2} h(t) \). Thus, the second derivative of \( h(t) \) with respect to time is

\[
\frac{d^2}{dt^2} h = 5 \cdot \left[ 3L^{-1} \int_{\gamma} k_s k_s k_s k_s ds + L^{-1} \int_{\gamma} k_s^2 k_s ds + g(t,k,k_s,\ldots,k_{s6}) \right]
\]
where \( g \) is a function only depending on \( k, k_s, \ldots, k_s^6 \) and \( t \) from \( h(t) \).

The highest order factor in above is easily seen to be \( k_s^7 \). We can see that \( \frac{d^2}{dt^2} h(0) \) is bounded if \( \gamma_0 \in C^{9, \alpha}(S^1) \). Before we show that at least one of the functions \( \tilde{h} \) coincide with our given constrained function \( h \), we need the fixed point theorem, Theorem 3.

Then, by referring to [79, Theorem 2.7], we get the short time existence for the area-preserving ideal curve flow.

**Theorem 6.** Let \( \gamma_0 : S \to \mathbb{R}^2 \) be a \( C^{9, \alpha} \)-regular immersed curve. Then there exists a maximal \( T \in (0, \infty] \) such that the area-preserving ideal curve flow \( \gamma : S \times [0, T) \to \mathbb{R}^2 \)

\[
\begin{align*}
\partial_t \gamma &= (k_s^4 + k^2 k_{ss} - \frac{1}{2} k k_s^2 + h_2(t)) \nu, \\
\gamma|_{t=0} &= \gamma_0
\end{align*}
\]

where \( h(t) = \frac{5}{32} \int_S k_s^2 ds \), is uniquely solvable with \( \gamma \) of degree \( C^{6,1, \alpha} \) \((S^1 \times [0, T))\).
Chapter 4

A sixth order flow of plane curves with boundary conditions

4.1 Introduction

Let \( \eta_1, \eta_2 : \mathbb{R} \to \mathbb{R}^2 \) denote two parallel vertical lines in \( \mathbb{R}^2 \), with distance \( |e| \neq 0 \) between them. We call \( \eta_1, \eta_2 \) supporting lines for the flow. Here \( e \) to be any vector such that \( e \) is perpendicular to the parallel lines \( \eta_{1,2} \), see Figure 4.1. We consider one-parameter families of smooth immersed curves \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) meeting two parallel lines with Neumann boundary condition together with other boundary conditions and

\[ \gamma(-1, \cdot) \in \eta_1(\mathbb{R}), \gamma(1, \cdot) \in \eta_2(\mathbb{R}). \]

The energy functional is

\[ E(\gamma) = \frac{1}{2} \int_{\gamma} k_s^2 ds, \]

where \( k_s^2 = (k_s)^2 \).

Figure 4.1
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We are interested in the $L^2$-gradient flow for curves of the energy functional $E$ with Neumann boundary conditions. As our energy involves the first derivative of curvature, the gradient flow will be sixth order and so three pairs boundary conditions will be needed.

This chapter is organised as follows. Firstly, we calculate the normal variation of the energy and obtain the $L^2$-gradient flow under Neumann and other chosen boundary conditions. We show that the length of the curve does not increase under the small energy assumption. Moreover, under the small energy condition and Neumann boundary condition, the winding number of curves remains zero. By examining the oscillation of curvature under the flow we are able to show that small energy curves with zero winding number remain embedded. We then show that curvature and all curvature derivatives in $L^2$ are bounded under the flow in section 4.2. We can see that these bounds are independent of time which means the solutions exist for all time. Furthermore, in section 4.3 we give a smaller energy condition and obtain that the $L^2$-norm of the second derivative of curvature decays exponentially under this condition. Then naturally, we show that the curvature and all curvature derivatives decay to zero exponentially. A stability argument implies that the solution converges to a straight line segment which is unique in section 4.4. By the exponential convergence of the flow speed, we show that the limiting line segment is in a bounded region of the initial curve.

Throughout this chapter, we use $c$ to denote constants which can be absolute constants or depend on some parameters under different circumstances.

The corresponding gradient flow has normal speed given by $F$, that is

$$\partial_t \gamma = F \nu,$$  \hspace{1cm} (4.1)

where $\nu$ denotes a unit normal vector field on $\gamma$ and has to be chosen so that the flow is parabolic in the generalised sense.

Here we give the evolution of $k_s$,

$$k_{st} = k_{ss} + F k_s k = (F_{ss} + F k^2)_s + F k_s k = F_s^3 + F_s k^2 + 3 F k_s k,$$

$$\frac{d}{dt} \int_\gamma k_s^2 ds = 2 \int_\gamma k_s k_{st} ds - \int_\gamma k_s^2 (F k) ds$$

$$= 2 \int_\gamma k_s F_{s3} ds + 2 \int_\gamma k_s k^2 F_s ds + 5 \int_\gamma k_s^2 F ds$$

$$= -2 \int_\gamma F \left( k_{st} + k_{ss} k^2 - \frac{1}{2} k_s^2 k \right) ds + 2 k_s F_{s} |_{\partial \gamma} - 2 k_{ss} F_s |_{\partial \gamma}$$

$$+ 2 k_s F |_{\partial \gamma} + 2 k_s k^2 F |_{\partial \gamma},$$
where \( \partial \gamma \) denotes the two end points of \( \gamma \) on \( \eta_1, \eta_2 \).

Then we state the evolution equation of the energy \( E \),

\[
\frac{d}{dt} \frac{1}{2} \int_\gamma k_s^2 \, ds = - \int_\gamma F \left( k_s^4 + k_{ss} k^2 - \frac{1}{2} k_s^2 k \right) \, ds + k_s F_{ss} |_{\partial \gamma} \\
- k_{ss} F_s |_{\partial \gamma} + k_3 F |_{\partial \gamma} + k_s k^2 F |_{\partial \gamma}.
\]  

(4.2)

Thus, the normal variation \( \tilde{\gamma} = \gamma + \varepsilon F \nu \) yields

\[
\frac{d}{d\varepsilon} E[\tilde{\gamma}] \bigg|_{\varepsilon=0} = - \int_\gamma \left( k_s^4 + k_{ss} k^2 - \frac{1}{2} k_s^2 k \right) F |_{\partial \gamma} + \left[ k_s F_{ss} - k_{ss} F_s + k_3 F + k^2 k_s F \right] |_{\partial \gamma}.
\]

In view of the integral in the above, we wish to take

\[
F = k_s^4 + k_{ss} k^2 - \frac{1}{2} k_s^2 k
\]

and we prove that the above boundary term is equal to zero in Lemma 16, i.e.

\[
\left[ k_s F_{ss} - k_{ss} F_s + k_3 F + k^2 k_s F \right] |_{\partial \gamma} = 0.
\]  

(4.3)

**Lemma 16.** Under boundary condition \( \langle \nu, \nu_{\eta_1, 2} \cdot \gamma \rangle (\pm 1, t) = k_s (\pm 1, t) = k_3 (\pm 1, t) = 0 \), (4.3) holds.

**Proof.** Differentiating the Neumann boundary condition \( \langle \nu, \nu_{\eta_1, 2} \cdot \gamma \rangle (\pm 1, t) = 0 \), i.e. \( \langle \nu (\pm 1, t), e \rangle = 0 \) in time yields \( F_s (\pm 1, t) < \tau (\pm 1, t), e \rangle = \pm |e| F_s (\pm 1, t) = 0 \). As the distance between \( \eta_1 \) and \( \eta_2 \) is \( |e| \), \( |e| \neq 0 \) so we must have that

\[
F_s (\pm 1, t) = 0.
\]

In addition to the Neumann boundary condition, we also assume the no curvature flux condition \( k_s (\pm 1, t) = 0 \). and \( k_{sss} (\pm 1, t) = 0 \) at the boundary. We obtain

\[
k_s F_{ss} |_{\partial \gamma} - k_{ss} F_s |_{\partial \gamma} + k_3 F |_{\partial \gamma} - k_s k^2 F |_{\partial \gamma} = 0.
\]

Under the condition in Lemma 16, (4.2) becomes

\[
\frac{d}{dt} \frac{1}{2} \int_\gamma k_s^2 \, ds = - \int_\gamma F \left( k_s^4 + k_{ss} k^2 - \frac{1}{2} k_s^2 k \right) \, ds = - \int_\gamma F^2 \, ds
\]  

(4.4)

where \( F = k_s^4 + k_{ss} k^2 - \frac{1}{2} k_s^2 k \).
We have
\[ \frac{d}{dt} E = -\|\partial_t \gamma\|_2^2 \]
which yields the flow \( F \) is the steepest descent gradient flow of \( E \) in \( L^2 \).

Any multiple of \( F \) is parallel to the gradient flow. For convenience we normalise the
coefficient of the highest order term, then we study the corresponding gradient flow
\[
\partial_t \gamma = F \nu = \left( k_s^4 + k_{ss}s^2 - \frac{1}{2} k_s^2 k_{ss} \right) \nu.
\]

**Definition 7.** Let \( \gamma \) be a smooth curve satisfying \( F = k_s^4 + k_{ss}s^2 - \frac{1}{2} k_s^2 k_{ss} \) that is a
stationary solution to the \( L^2 \)-gradient flow of \( E \). We call such curves ideal.

Thus, the free boundary value problem that we wish to consider for the flow is the
following:
\[
\begin{align*}
(\partial_t \gamma)(u,t) &= (F \nu)(u,t), \quad \text{for all } (u,t) \in [-1,1] \times [0,T) \\
\gamma(-1,t) &\in \eta_1(\mathbb{R}), \gamma(1,t) \in \eta_2(\mathbb{R}), \quad \text{for all } t \in [0,T) \\
\langle \nu, v_{\eta_1}(-1,t) \rangle = \langle \nu, v_{\eta_2}(1,t) \rangle = 0, \quad \text{for all } t \in [0,T) \\
k_{s}(\pm 1,t) = k_{ss}(\pm 1,t) = 0. \quad \text{for all } t \in [0,T) 
\end{align*}
\]

The following is the main theorem in this chapter:

**Theorem 7.** Let \( \gamma_0 \) be a smooth embedded regular curve, \( \gamma : [-1,1] \times [0,T) \rightarrow \mathbb{R}^n \) be a
solution to (4.1). If the initial curve \( \gamma_0 \) satisfies \( \omega = 0 \) and
\[
L^3(0)\|k_s(0)\|^2 \leq \frac{\sqrt{74} - 8}{10} \cdot \pi^3,
\]
where \( L(0) \) is the length of \( \gamma_0 \) and \( k(0) = k(\cdot, 0) \) is the curvature, then the flow exists for
all time \( T = \infty \) and \( \gamma(\cdot,t) \) converges exponentially to a horizontal line segment \( \gamma_\infty \) in the
\( C^\infty \) topology.

In order to prove Theorem 7, we need to have the short time existence proved in Theorem 1 in Chapter 3.

**Lemma 17.** The hypothesis of Theorem 7 implies that \( \omega(t) = \omega(0) = 0 \).

It follows immediately that the average curvature \( \bar{k} \) satisfies
\[
\bar{k} := \frac{1}{L} \int_\gamma kds \equiv 0.
\]

**Proof.** From Lemma 4, we obtain
\[
\omega(t) = \omega(0) = 0.
\]
Then we have
\[ \int_{\gamma} k ds \bigg|_{t=0} = 2\omega\pi = 0. \]

It follows immediately that the average curvature \( \bar{k} \) satisfies
\[ \bar{k} = \frac{1}{L} \int_{\gamma} k ds \equiv 0. \]

Here we show that for small energy the length of the evolving curve does not increase.

**Lemma 18.** Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a solution to (4.1). Under the assumptions of Theorem 7, then
\[ L(t) \leq L(0), \]
for all \( t \in [0, T) \).

**Proof.** Using Lemma 3, \( \frac{d}{dt} L = \frac{d}{dt} \int_{\gamma} ds = -\int_{\gamma} kF ds \).

Now using integration by parts and the boundary conditions, we have
\[
\frac{d}{dt} L = - \int_{\gamma} k_{ss} ds + \frac{7}{2} \int_{\gamma} k_s^2 k^2 ds \\
\leq - \int_{\gamma} k_{ss} ds + \frac{7}{2} \|k\|_\infty^2 \int_{\gamma} k_s^2 ds \\
\leq - \int_{\gamma} k_{ss} ds + \frac{7L^3}{\pi^3} \|k_{ss}\|^2_\frac{3}{2} \int_{\gamma} k_s^2 ds \\
\leq - \left( 1 - \frac{7L^3}{\pi^3} \|k_s\|^2_\frac{3}{2} \right) \cdot \int_{\gamma} k_{ss} ds,
\]
where we have also used the PSW inequalities (2.1) and (2.2) in Proposition 1 and Proposition 2. We use \( \|k_i\|_{L^2}^2 \) to denote \( \int_0^t k_i^2 ds \), \( l = 0, 1, 2, \ldots \). As \( \int_0^t k ds = \int_0^t k_s ds = 0 \), then \( \|k\|_{L^2}^2 \leq \frac{2t}{\pi} \|k_s\|_{L^2}^2 \leq \frac{2t^2}{\pi} \|k_{ss}\|_{L^2}^2 \).

The assumption \( L^2(0) \|k_s(0)\|_{L^2}^2 \leq \frac{\sqrt{74} - 8}{10} \pi^3 \leq \frac{\pi^3}{8} \) is that the energy is small at \( t=0 \). We know that for all \( t \) is that the energy is decreasing in (4.4) since the flow is the \( L^2 \) gradient flow. We obtain that \( L^3 \int_0^t k_s^2 ds \leq \frac{\pi^3}{8} \) for all time. The claim follows. \( \square \)

The following lemma shows the expression for \( \partial_t k_s \), where \( l \in \mathbb{N} \cup \{0\} \).

**Lemma 19.** Let \( \gamma : [-1, 1] \times [0, T) \rightarrow \mathbb{R}^2 \) be a solution to (4.1). The evolution of the \( \ell \)-th derivative of curvature

\[
\partial_t k_{s\ell} = k_{s\ell+6} + \sum_{q+r+u=\ell} (c_1 k_{s(q+1)} k_{s\ell} + c_2 k_{s(q+3)} k_{s\ell+1} k_{s\ell} + c_3 k_{s(q+2)} k_{s\ell+2} k_{s\ell}) + \sum_{a+b+c+d+e=\ell} c_5 k_{s\ell} k_{s\ell} k_{s\ell} k_{s\ell}
\]

for constants \( c_1, c_2, c_3, c_4, c_5 \in \mathbb{R} \) and \( a, b, c, d, e, q, r, u \geq 0 \).

**Proof.** First under Lemma 3, we calculate the evolution of the curvature

\[
\partial_t k = k_{s_0} + \left( k_{s_3} k^2 - \frac{1}{2} k_s^2 \right) + \left( k_{s_4} + k_{s_5} k^2 - \frac{1}{2} k_s^2 \right) k^2
\]

\[
= k_{s_0} + \left( k_{s_3} k^2 + 2 k_{s_5} k k_s + k_{s_4} k_s k - \frac{1}{2} k_s^2 \right) + k_{s_4} k^2 + k_{s_5} k^2 - \frac{1}{2} k_s^2 k^3
\]

\[
= k_{s_0} + k_{s_3} k^2 + 2 k_{s_5} k k_s + k_{s_4} k_s k + k_{s_4} k_s k^2 + k_{s_5} k^2 - \frac{3}{2} k_{s_5} k_s^2 + k_s^2 k^4 - \frac{1}{2} k_s^2 k^3
\]

\[
= k_{s_0} + 2 k_{s_5} k^2 + 3 k_{s_3} k_s k + k_{s_4} k_s k^2 - \frac{1}{2} k_{s_5} k_s^2 + k_s^2 k^4 - \frac{1}{2} k_s^2 k^3
\]

(4.6)

and the derivative of \( k_s \) with respect to \( t \),

\[
\partial_t k_s = \partial_t \partial_s k = \partial_s \partial_t k + kF_k
\]

\[
= \partial_s \partial_t k + k \left( k_{s_4} + k_{s_5} k^2 - \frac{1}{2} k_s^2 \right) k_s
\]

\[
= \partial_s \left( k_{s_0} + 2 k_{s_5} k^2 + 3 k_{s_3} k_s k + k_{s_4} k_s k^2 - \frac{1}{2} k_{s_5} k_s^2 + k_s^2 k^4 - \frac{1}{2} k_s^2 k^3 \right) + k_{s_4} k_s k
\]

\[
+ k_{s_5} k_s k^2 - \frac{1}{2} k_s^2 k^2
\]

\[
= k_{s_0} + 2 k_{s_5} k^2 + 8 k_{s_4} k_s k + 5 k_{s_3} k_s k + k_{s_4} k_s k^2 + k_{s_5} k_s k^4 + 4 k_{s_5} k_s k^3 - 2 k_s^2 k^2 \quad (4.7)
\]
then we calculate the derivative of \( k_{ss} \) with respect to \( t \),
\[
\partial_t k_{ss} = \partial_t \partial_s k_s = \partial_s \partial_s k_s + k \left( k_s + k_{ss} k_s^2 - \frac{1}{2} k_s^2 k_s \right) k_s
\]
\[
= \partial_s \left( k_{ss} + 2k_s^2 + 8k_s k_s k_s + 5k_s^3 k_{ss} k_s + \frac{5}{2} k_s^3 k_s^2 + k_{ss} k_s^3 + 4k_{ss} k_s k_s^3 - 2k_s^3 k_s^2 \right)
\]
\[
+ k_s + k_{ss} k_s + k_s^2 k_s^3 - \frac{1}{2} k_s^2 k_s^2
\]
\[
= k_{ss} + 2k_s^2 + 4k_s k_s k_s + 8k_s^3 k_s + 8k_s^3 k_{ss} k_s + 5k_s^3 k_s k_s + 5k_s^3 k_s + 5k_s^2 k_s
\]
\[
+ 12k_{ss} k_s^2 k_s^2 - 4k_s^3 k_s + 5k_s^3 k_s - \frac{1}{2} k_s^2 k_s^2
\]
\[
= k_{ss} + 2k_s^2 + 12k_s k_s k_s + 14k_s^3 k_s k_s + \frac{21}{2} k_s^4 k_s + 5k_s^4 k_{ss} k_s + 10k_s^2 k_s + k_s k_s^3
\]
\[
+ 8k_s^3 k_s^3 + 5k_{ss}^2 k_s^2 - 4k_s^4 k_s.
\]
\( (4.8) \)

It is clear to see that when \( l \geq 2 \) we will have
\[
\partial_t k_{jl-1} = \partial_t \partial_s k_{jl-2} = \partial_s \partial_s k_{jl-2} + k \left( k_{jl-2} + k_{ss} k_{jl-2} - \frac{1}{2} k_{jl-2} \right) k_{jl-1}
\]
\[
= k_{jl-2} \sum_{q+r+u=l-1} (c_1 k_{q+1} k_{s+1} k_{u} + c_2 k_{q+2} k_{s+1} k_{u} + c_3 k_{q+1} k_{s+2} k_{u})
\]
\[
+ c_4 k_{q+1} k_{s+1} k_{u} + \sum_{a+b+c+d+e=l+1} c_5 k_{q+e} k_{s+d} k_{u} k_{s}
\]
for constants \( c_1, c_2, c_3, c_4, c_5 \in \mathbb{R} \) and \( a, b, c, d, e, q, r, u \geq 0 \).

Thus, we have
\[
\partial_t k_s = k_s + \sum_{q+r+u=l} (c_1 k_{q+1} k_{s+1} k_{u} + c_2 k_{q+2} k_{s+1} k_{u} + c_3 k_{q+1} k_{s+2} k_{u})
\]
\[
+ c_4 k_{q+1} k_{s+1} k_{u} + \sum_{a+b+c+d+e+l+2} c_5 k_{q+e} k_{s+d} k_{u} k_{s}
\]
\( (4.9) \)
for constants \( c_1, c_2, c_3, c_4, c_5 \in \mathbb{R} \) and \( a, b, c, d, e, q, r, u \geq 0 \).

This is the conclusion of this Lemma. □

Lemma 20. All odd derivatives of the curvature are equal to zero at the boundary.

Proof. Our boundary conditions are \( \langle v(\pm 1, t), e \rangle = k_s(\pm 1, t) = k_{s3}(\pm 1, t) = 0 \).

Differentiating the Neumann boundary condition in time implies
\[
F_s(\pm 1, t) \langle \tau(\pm 1, t), e \rangle = \pm |e| F_s(\pm 1, t) = 0
\]
0 = F_{3}(±1,t) = -k_s t - k_s^2 - 2k_{ss}k_s k + k_{ss}k_s^2 + \frac{1}{2}k_s^3
= -k_s t - k_s^2 - k_{ss}k_s k + \frac{1}{2}k_s^3.

As k_s(±1,t) = 0, k_{3}(±1,t) = 0, we have k_s(±1,t) = 0. Also

F_{3}(±1,t) = -k_s t - (k_s k_s^2) - (k k_{s} k_{s}^2) + \frac{1}{2}k_s^3
= -k_s t - (k_s^2 + 2k_{ss}k_s k - \frac{3}{2}k_s^2) + \frac{1}{2}k_s^3
= -k_s^2 k_s - 2k_{ss}k_s k - k_s^2 - k_{ss}k_s k - k_s k_s k + 3k_{ss}k_s + \frac{3}{2}k_s^2
= k_{3} t(±1,t),

we get k_{3} t(±1,t) = 0.

Let us give the induction argument, we assume that for all p ∈ 0,1, ..., n,

k_{s2p-1}(±1,t) = 0.

The evolution of the l–th derivative of curvature is given in Lemma 19 by

∂_t k_s^l = k_s^{l+6} + \sum_{q+r+n=l} (c_1k_{s}^qk_{s}^r k_{s}^u + c_2 k_{s}^{q+3} k_{s}^{r+1} k_{s}^u + c_3 k_{s}^{q+2} k_{s}^{r+2} k_{s}^u + c_4 k_{s}^{q+2} k_{s}^{r+1} k_{s}^u + \sum_{a+b+c+d+e=l} c_{s} k_{s}^{a} k_{s}^{b} k_{s}^{c} k_{s}^{d} k_{s}^{e}),

for constants c_1, c_2, c_3, c_4, c_5 ∈ \mathbb{R} and a, b, c, d, e, q, r, u ≥ 0.

The inductive hypothesis implies that, for l odd and less than or equal to 2n − 5, for all n ≥ 3, the derivative k_s^l vanished at the boundary. Take l = 2n − 5. Then we have

k_{s2(2n+1)} = \partial_t k_{2n-5} - \sum_{2q_1+2r_1+2a_1=2n-5} (k_{s}(2q_1+4) * k_{s}(2r_1) * k_{s}(2a_1) + k_{s}(2q_1+3) * k_{s}(2r_1+1) * k_{s}(2a_1) + k_{s}(2q_1+2) * k_{s}(2r_1+2) * k_{s}(2a_1))
= \sum_{2q_1+2r_1+2a_1=2n-5} (k_{s}(2q_1) * k_{s}(2r_1) * k_{s}(2a_1) * k_{s}(2r_1+1) * k_{s}(2a_1+1)

here * denotes a linear combination of terms with absolute coefficient. We assume that

...
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$k_{s^{2p-1}}(\pm 1, t) = 0$, we remove all terms with an odd number of derivatives of $k$. We conclude at the boundary

$$k_{s(2e+1)} = 0,$$

as required.

\[4.2\] Controlling the geometry of the flow

Under Lemma 19, we can now show that $\int_\gamma k^2_{s}ds$ are all bounded, $l = 0, 1, 2, ...$

For $l = 1$, from the energy $E = \frac{1}{2} \int_\gamma k^2_{s}ds$ is decreasing, it is clear that

$$\left. \int_\gamma k^2_{s}ds \right|_{t=0} \leq \int_\gamma k^2_{s}ds. \tag{4.10}$$

For $l = 0$, using the PSW inequality (2.1) in Proposition 1,

$$\left. \int_\gamma k^2_{s}ds \right|_{t=0} \leq \frac{L^2}{\pi} \int_\gamma k^2_{s}ds \leq \frac{L^2}{\pi} \int_\gamma k^2_{s}ds. \tag{4.11}$$

Then, under the conditions in Theorem 7, we obtain that $\int_\gamma k^2_{s}ds$ and $\int_\gamma k^2_{s}ds$ are bounded.

For $l = 2$, we give the following lemma.

Lemma 21. Let $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$ be a solution to (4.1). Under the assumptions of Theorem 7, then there exists a universal constant $c \in (0, \infty)$ such that for all $t \in [0, T)$

$$\|k_{ss}\|^2_2 \leq \|k_{ss}\|^2_2|_{t=0} + c.$$

Proof. Applying (4.8) in Lemma 19, we have

$$\frac{d}{dt} \int_\gamma k^2_{ss}ds = -2 \int_\gamma k^2_{s}\,ds + 4 \int_\gamma k^2_{s}k^2_{s}ds - 8 \int_\gamma k_{s}k_{s}k_{s}k_{s}kds + 11 \int_\gamma k_{s}k_{ss}k_{ss}k_{s}ds$$

$$+ 5 \int_\gamma k_{s}k_{ss}k_{s}k_{s}\,ds + 10 \int_\gamma k^2_{s}k_{ss}kds + 20 \int_\gamma k_{s}k^2_{s}k_{s}ds + 2 \int_\gamma k_{s}k_{ss}k_{s}k^4ds$$

$$+ 16 \int_\gamma k^3_{s}k_{s}s^3ds + 9 \int_\gamma k^3_{s}k_{s}^3ds + \frac{23}{2} \int_\gamma k^2_{s}k^2_{s}s^2ds$$

$$- 8 \int_\gamma k_{ss}k^4_{s}kds. \tag{4.12}$$

We estimate above terms separately as the following,

$$9 \int_\gamma k^3_{s}k^3_{s}ds = -9 \int_\gamma (k^2_{ss}k^3_{s})s_kds = -18 \int_\gamma k^3_{s}k_{ss}k_{s}k^3_{s}ds - 27 \int_\gamma k^2_{ss}k^2_{s}k^2_{s}ds.$$
As
\[ \int \gamma k_\gamma^2 k_\gamma^3 ds = - \int \gamma^2 k_\gamma^3 k_\gamma^3 ds = -2 \int \gamma^2 k_\gamma^2 k_\gamma^3 ds - \int \gamma^2 k_\gamma^3 ds, \]
we can see that
\[ 20 \int \gamma k_\gamma^2 k_\gamma^3 ds = - \frac{20}{3} \int \gamma^2 ds. \]

As
\[ \int \gamma^2 k_\gamma^2 k_\gamma^3 ds = - \int \gamma (k_\gamma^2 k_\gamma^3) ds = - \int \gamma k_\gamma^2 k_\gamma^3 ds = \int \gamma^2 k_\gamma^3 ds - \int \gamma k_\gamma^2 k_\gamma^3 ds, \]
after previous equation, we have
\[ 10 \int \gamma^2 k_\gamma^3 ds = -5 \int \gamma k_\gamma^2 k_\gamma^3 ds - 5 \int \gamma k_\gamma^2 k_\gamma^3 ds, \]
\[ -8 \int \gamma^2 k_\gamma^3 ds = \frac{8}{5} \int \gamma^3 ds = 4 \int \gamma^2 k_\gamma^3 ds + 12 \int \gamma k_\gamma^2 k_\gamma^3 ds. \]

Moreover,
\[ -8 \int \gamma^2 k_\gamma^3 ds = 4 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds + 2 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds. \]
Substituting above five calculations into (4.12), we get
\[ \frac{d}{dt} \int \gamma^2 k_\gamma^3 ds = -2 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds - 2 \int \gamma^2 k_\gamma^3 ds + 2 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds + 11 \int \gamma^2 k_\gamma^3 ds + 5 \int \gamma^2 k_\gamma^3 ds - 5 \int \gamma^2 k_\gamma^3 ds - 5 \int \gamma^2 k_\gamma^3 ds - 5 \int \gamma^2 k_\gamma^3 ds - \frac{20}{3} \int \gamma^2 k_\gamma^3 ds + 2 \int \gamma^2 k_\gamma^3 ds + 16 \int \gamma^2 k_\gamma^3 ds - 18 \int \gamma^2 k_\gamma^3 ds - 18 \int \gamma^2 k_\gamma^3 ds - 18 \int \gamma^2 k_\gamma^3 ds - 27 \int \gamma^2 k_\gamma^3 ds + \frac{23}{2} \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds + 12 \int \gamma^2 k_\gamma^3 ds - 2 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 4 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds + 13 \int \gamma^2 k_\gamma^3 ds. \]
here we wish to absorb \(-2 \int_{\gamma} k_\gamma k_s k_\gamma k^3 \, ds\) into other terms, we calculate
\[
-2 \int_{\gamma} k_\gamma k_s k_\gamma k^3 \, ds = \frac{1}{2} \int_{\gamma} k_\gamma k_s k_\gamma k^4 \, ds + \frac{1}{2} \int_{\gamma} k^2 \gamma k^4 \, ds,
\]
then
\[
\frac{d}{dt} \int_{\gamma} k^2 \gamma ds = -2 \int_{\gamma} k^2 \gamma k^3 ds + 4 \int_{\gamma} k_\gamma k^2 ds + 4 \int_{\gamma} k_\gamma k^2 \gamma k ds - \frac{13}{3} \int_{\gamma} k^4 \gamma ds + 4 \int_{\gamma} k^2 \gamma k^2 ds
\[
+ 5 \int_{\gamma} k_\gamma k^2 \gamma k ds + 5 \int_{\gamma} k_\gamma k_s k^2 \, ds + \frac{1}{2} \int_{\gamma} k^2 \gamma k^4 ds - \frac{7}{2} \int_{\gamma} k^2 \gamma k_s^2 \, ds
\[
+ 4 \int_{\gamma} k_\gamma k^2 \gamma k^2 \, ds.
\]
(4.13)

We estimate the seven positive terms \(4 \int_{\gamma} k^2 \gamma k^2 ds, 4 \int_{\gamma} k_\gamma k^2 \gamma k ds, 5 \int_{\gamma} k_\gamma k_s k^2 \, ds,\)
\(\frac{5}{2} \int_{\gamma} k_\gamma k_s k^4 \, ds,\) \(\frac{1}{2} \int_{\gamma} k^2 \gamma k^4 \, ds,\) \(4 \int_{\gamma} k_\gamma k^2 \gamma k^2 \, ds\) above by using interpolation inequality in Proposition 4, we obtain
\[
4 \int_{\gamma} k^2 \gamma k^2 ds + 4 \int_{\gamma} k_\gamma k^2 \gamma k ds + 5 \int_{\gamma} k_\gamma k_s k^2 \, ds = \int_{\gamma} P^8_4 (k) ds \leq \epsilon \int_{\gamma} k^2 \gamma ds + c \|k\|_2^2,
\]
and
\[
\frac{5}{2} \int_{\gamma} k_\gamma k_s k^4 \, ds = \int_{\gamma} P^6_6 (k) ds \leq \epsilon \int_{\gamma} k^2 \gamma ds + c \|k\|_2^2.
\]

Here we use interpolation inequality,
\[
\int_{\gamma} P^6_4 (k) ds \leq \epsilon \|k_\gamma\|_2^2 + c \|k\|_2^{18},
\]
together with PSW inequality (2.2) in Proposition 2, \(\int_{\gamma} k ds = 0,\)
\[
\|k\|_\infty \leq \frac{2L}{\pi} \|k_\gamma\|_2^2,
\]
we have as the following two estimates,
\[
\frac{1}{2} \int_{\gamma} k^2 \gamma k^4 \, ds \leq \frac{1}{2} \|k\|_\infty^2 \int_{\gamma} k^2 \gamma k^3 \, ds \leq \frac{1}{2} \|k\|_\infty^2 \int_{\gamma} P^6_4 (k) ds
\]
\[
\leq \|k\|_\infty^2 \left( \epsilon \|k_\gamma\|_2^2 + c \|k\|_2^{18} \right)
\]
\[
\leq \frac{2L}{\pi} \|k_\gamma\|_2^2 \left( \epsilon \frac{L^2}{\pi^2} \|k_\gamma\|_2^2 + c \|k\|_2^{18} \right)
\]
\[
= \epsilon \frac{2L^3}{\pi^2} \|k_\gamma\|_2^2 \|k_\gamma\|_2^2 + c \frac{2L}{\pi} \|k_\gamma\|_2^2 \|k\|_2^{18},
\]
and
\[
4 \int_{\gamma} k^3 s^2 k^2 ds \leq 4\|k\|_\infty^2 \int_{\gamma} |k_s k^3 s| ds \leq 4\|k\|_\infty^2 \int_{\gamma} P^4_s (k) ds
\]
\[
\leq \|k\|_\infty^2 (\varepsilon \|k_s\|_2^2 + c \|k\|_2^{18})
\]
\[
\leq \frac{2L}{\pi} \|k_s\|_2^2 \left( \varepsilon \frac{L^2}{\pi^2} \|k_s s\|_2^2 + c \|k\|_2^{18} \right)
\]
\[
= \varepsilon \frac{2L^3}{\pi^3} \|k_s\|_2^2 \|k_s s\|_2^2 + c \frac{2L}{\pi} \|k_s\|_2^2 \|k\|_2^{18}.
\]

Applying PSW inequalities (2.1) and (2.2) in Proposition 1 and Proposition 2, as \( \int_{\gamma} k^3 s ds = \int_{\gamma} k_s ds = 0 \), we have
\[
\|k_s s\|_\infty^2 \leq \frac{2L}{\pi} \|k_s\|_2^2 \leq \frac{2L^2}{\pi^2} \|k_s s\|_2^2,
\]
we estimate the fifth term in (4.13),
\[
4 \int_{\gamma} k^2 s^2 k^2 ds \leq \|k_s s\|_\infty^2 \int_{\gamma} k^2 ds \leq \frac{8L^3}{\pi^3} \|k_s s\|_2^2 \|k_s\|_2^2,
\]
then using the Hölder inequality, we estimate the fourth term in (4.13),
\[
-\frac{13}{3} \int_{\gamma} k^4 ds \leq -\frac{13}{3L} \left( \int_{\gamma} k^2 ds \right)^2.
\]
Substituting the above calculations into (4.13), we obtain
\[
\frac{d}{dt} \int_{\gamma} k^2 ds \leq \left[ -2 + 4\varepsilon + (8 + 4\varepsilon) \frac{L^3}{\pi^3} \|k_s s\|_2^2 \right] \|k_s s\|_2^2
\]
\[
+ c \|k\|_2^2 + c \frac{L}{\pi} \|k_s s\|_2^2 \|k\|_2^{18} - \frac{13}{3L} \left( \int_{\gamma} k^2 ds \right)^2.
\]
Assume that
\[
-2 + 4\varepsilon + (8 + 4\varepsilon) \frac{L^3}{\pi^3} \|k_s s\|_2^2 \leq 0,
\]
then
\[
\frac{(8 + 4\varepsilon) L^3}{\pi^3} \|k_s\|_2^2 \leq 2 + 4\varepsilon,
\]
we get that under condition
\[ \|k_s\|_2^2 \leq \frac{(2 + 4\epsilon)\pi^3}{(8 + 4\epsilon)L^3}, \]
the following inequality holds
\[ \frac{d}{dt} \int_{\gamma} k_s^2 \, ds \leq -c\|k_{ss}\|_2^2 + c\|\|k\|_2^2, \]
i.e. \( \int_{\gamma} k_s^2 \, ds \) is bounded, which is the result of this lemma. \( \square \)

We then show that all curvature derivatives in \( L^2 \) are bounded under the flow.

**Lemma 22.** Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a solution to (4.1). Under the assumption of Theorem 7, we have \( T = \infty \) and there exist absolute constants \( c \) such that
\[ \|k_{\gamma}'\|_2^2 \leq \|k_{\gamma}'\|_2^2_{t=0} + c. \]

**Proof.** Applying (4.9) in Lemma 19, we have
\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 \, ds = \int_{\gamma} 2k_{\gamma} \partial_t k_{\gamma} \, ds + \int_{\gamma} k_{\gamma}^2 \partial_s k_{\gamma} \, ds \\
= \int_{\gamma} 2k_{\gamma} k_{\gamma,16} \, ds + 2 \int_{\gamma} \sum_{q+r+l=1} (c_{1}k_{\gamma}k_{\gamma,q}k_{\gamma,r}k_{\gamma} + c_{2}k_{\gamma}k_{\gamma,q}k_{\gamma,r+1}k_{\gamma} \nonumber \\
+ c_{3}k_{\gamma}k_{\gamma,q+1}k_{\gamma,r}k_{\gamma} + c_{4}k_{\gamma}k_{\gamma,q}k_{\gamma,r+1}k_{\gamma+1}) \, ds \\
+ 2 \int_{\gamma} \sum_{a+b+c+d+e+l=2} c_{5}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma} \, ds - \int_{\gamma} k_{\gamma}^2 k_{\gamma}k_{\gamma} \, ds \\
- \int_{\gamma} k_{\gamma}^2 k_{\gamma}k_{\gamma} \, ds + \frac{1}{2} \int_{\gamma} k_{\gamma}^2 k_{\gamma}k_{\gamma} \, ds \\
= \int_{\gamma} 2k_{\gamma} k_{\gamma,16} \, ds + 2 \int_{\gamma} \sum_{q+r+l=1} (c_{1}k_{\gamma}k_{\gamma,q}k_{\gamma,r}k_{\gamma} + c_{2}k_{\gamma}k_{\gamma,q}k_{\gamma,r+1}k_{\gamma} \nonumber \\
+ c_{3}k_{\gamma}k_{\gamma,q+1}k_{\gamma,r}k_{\gamma} + c_{4}k_{\gamma}k_{\gamma,q}k_{\gamma,r+1}k_{\gamma+1}) \, ds \\
+ 2 \int_{\gamma} \sum_{a+b+c+d+e+l=2} c_{5}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma} \, ds.
\]

We have already proved that when \( l = 0, 1, 2 \), the results hold in (4.11), (4.10) and Lemma 21. So here we let \( l \geq 3 \).

It is easy to see that when \( l \geq 3 \), in the term
\[ 2 \int_{\gamma} \sum_{a+b+c+d+e+l=2} c_{5}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma}k_{\gamma} \, ds \]
always has \( a \geq 1 \).
By integration by parts, when \( a \geq 1 \), we have

\[
\frac{d}{dt} \int_\gamma k^2_{t+3} = -2 \int_\gamma k_{t+3}ds + 2 \int_\gamma \sum_{g+r+a=1} (c_1 k_{g+2} k_{g'} k_{s'} k_{s''} + c_1 k_{g+1} k_{g'} k_{s'} k_{s''} + c_2 k_{g+1} k_{g'} k_{s'} k_{s''}) ds
\]

\[+ c_2 k_{g+1} k_{g'} k_{s'} k_{s''} + c_3 k_{g} k_{g'} k_{s'} k_{s''} + c_4 k_{g} k_{g'} k_{s'} k_{s''}) ds.
\]

By using interpolation inequality, we have

\[
\int_\gamma \sum_{g+r+a=1} (c_1 k_{g+2} k_{g'} k_{s'} k_{s''} + c_1 k_{g+1} k_{g'} k_{s'} k_{s''} + c_2 k_{g+1} k_{g'} k_{s'} k_{s''}) ds
\]

\[= \int_\gamma p^{2l+4}(k) ds \leq \epsilon \int_\gamma k^2_{t+3} ds + c \|k\|_2^{4l+14}.
\]

Again, we can get

\[
2 \int_\gamma \sum_{a+b+c+d+e=1+2} (c_5 k_{g+1} k_{g'} k_{s'} k_{s''} + c_5 k_{g} k_{g'} k_{s'} k_{s''} + c_5 k_{g} k_{g'} k_{s'} k_{s''}) ds
\]

\[= \int_\gamma p^{2l}(k) ds \leq \epsilon \int_\gamma k^2_{t+2} ds + c \|k\|_2^{2l+10}.
\]

Above estimates give us

\[
\frac{d}{dt} \int_\gamma k^2_{t+3} = -2 \int_\gamma k_{t+3}ds + p^{2l+4}(k) + p^{2l}(k) + p^{2l}(k) ds
\]

\[\leq -2 \int_\gamma k_{t+3}ds + \epsilon \int_\gamma k^2_{t+3} ds + c \|k\|_2^{4l+14} + \epsilon \int_\gamma k^2_{t+3} ds + c \|k\|_2^{2l+10}
\]

\[+ \epsilon \int_\gamma k^2_{t+2} ds + c \|k\|_2^{2l+10}
\]

\[\leq \left( -1 + \epsilon + \frac{L^2}{\pi^2} \epsilon \right) \int_\gamma k^2_{t+3} ds + c \|k\|_2^{4l+14} + c \|k\|_2^{2l+10}
\]

\[\leq -c \int_\gamma k^2_{t} ds + c \left( \|k\|_2^2 \right).
\]

Then

\[
\|k_{r+3}\|_2^2 \leq \|k_{r+2}\|_2^2 + c.
\]

as required.

Pointwise bounds on all derivatives of curvature now follow from PSW inequality (2.1) in Proposition 1. From PSW inequality (2.2) in Proposition 2 and \( \int_\gamma k_{t} ds = 0 \), we have \( \|k_{t}\|_{\infty}^2 \leq \frac{24}{\pi} \|k_{t+1}\|_2^2 \); then \( \int_\gamma k_{t}^2 ds \) bounds in \( L^\infty \). It follows that the solution of the flow remains smooth up to and including the final time \( T \) from which we apply local existence
if $T < \infty$. This shows that the flow exists for all time, that is, $T = \infty$.

### 4.3 Exponential convergence

We show that under a smaller energy assumption, the second derivative of curvature decays exponentially in $L^2$.

**Lemma 23.** Let $\gamma : [-1, 1] \times [0, \infty) \to \mathbb{R}^2$ be a solution to (4.1). Assume $\omega(0) = 0$ and if at some time $t_0$ the energy satisfies

$$\|k_s\|_2^2 < \frac{\sqrt{74} - 8}{10} \cdot \frac{\pi^3}{L^3(0)}.$$  \hfill (4.14)

Then there exists a universal constant $\delta \in (0, \infty)$ such that for all $t > t_0$

$$\|k_{ss}\|_2^2 \leq e^{-\delta (t - t_0)} \|k_{ss}\|_2^2 \bigg|_{t = t_0}.$$  

**Proof.** The condition (4.14) remains true for all $t > t_0$ by following the monotonicity of the energy. Applying (4.13) in Lemma 21,

$$\frac{d}{dt} \int_\gamma k^2_s ds = -2 \int_\gamma k^2_s ds + 4 \int_\gamma k^2_s k^2 ds + 4 \int_\gamma k_s k^2_{ss} k ds - \frac{13}{3} \int_\gamma k^4_s ds$$

$$+ 4 \int_\gamma k^2_s k^2_s ds + 5 \int_\gamma k_s k_{ss} k^2_s ds + \frac{5}{2} \int_\gamma k_s k_{ss} k^4_s ds + \frac{1}{2} \int_\gamma k^2_s k^4_s ds$$

$$- \frac{7}{2} \int_\gamma k^2_s k^2_s ds + 4 \int_\gamma k_s k^3_s k^2 ds.$$

Here we handle term $4 \int_\gamma k_s k^2_{ss} k ds$ by doing integration by parts,

$$4 \int_\gamma k_s k^2_{ss} k ds = -4 \int_\gamma k_s (k^2_{ss})_s ds = -8 \int_\gamma k^2_s k_{ss} k ds - 4 \int_\gamma k_s k^2_{ss} k_s,$$

here we notice that

$$-4 \int_\gamma k_s k^2_{ss} k_s ds = 4 \int_\gamma k_{ss} (k^2_s)_s ds = 8 \int_\gamma k_s k^2_{ss} k_s ds - 4 \int_\gamma k^4_s ds,$$

$$-4 \int_\gamma k_s k^2_{ss} k_s ds = \frac{4}{3} \int_\gamma k^4_s ds,$$

then

$$4 \int_\gamma k_s k^2_{ss} k ds = -8 \int_\gamma (k^2_s k)_s k ds - 4 \int_\gamma k_s k^2_{ss} k_s$$

$$= 16 \int_\gamma k_s k^3_s k ds + 8 \int_\gamma k^2_s k^2_s ds + \frac{4}{3} \int_\gamma k^4_s ds.$$
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So, we can get

\[
\frac{d}{dt} \int_{\gamma} k_{ss}^2 \, ds = -2 \int_{\gamma} k^2_{xx} \, ds + 4 \int_{\gamma} k^2_{x} k^2 \, ds + 4 \int_{\gamma} k_{x}^2 k^2_{x} \, ds + 4 \int_{\gamma} k^2_{x} k^2_{x} \, ds + 5 \int_{\gamma} k_x k_{ss} k^2_x \, ds - 3 \int_{\gamma} k^4_x \, ds + 2 \int_{\gamma} k_x k_{ss} k^4_x \, ds - \frac{7}{2} \int_{\gamma} k^2_{x} k^2_{x} \, ds + 4 \int_{\gamma} k_x k_{ss} k^2_x \, ds
\]

Again, by applying integration by parts to four terms in above (4.15), for the fourth term,

\[
16 \int_{\gamma} k_x k_{3x} k^3_x kds = -16 \int_{\gamma} k_x k_{3x} k^3_x kds
\]

then

\[
16 \int_{\gamma} k_x k_{3x} k^3_x kds = -8 \int_{\gamma} k^2_{x3} k^2_{xx} kds - 8 \int_{\gamma} k^2_{x3} k^2_{x} ds.
\]

Also we have

\[
5 \int_{\gamma} k_x k_{ss} k^2_x \, ds = -5 \int_{\gamma} k^2_{x} k^2_x \, ds - 10 \int_{\gamma} k_{x} k^2_{ss} k_x ds
\]

\[
= -5 \int_{\gamma} k^2_{x} k^2_x ds + \frac{10}{3} \int_{\gamma} k^4_x ds,
\]

\[
\frac{5}{2} \int_{\gamma} k_x k_{ss} k^4 ds = -\frac{5}{2} \int_{\gamma} k^2_{x3} k^4 ds - 10 \int_{\gamma} k_{x} k_{ss} k^3 ds
\]

\[
= -\frac{5}{2} \int_{\gamma} k^2_{x3} k^4 ds + 5 \int_{\gamma} k^3_{x3} k^3 ds + 15 \int_{\gamma} k^2_{ss} k^2_x k^2 ds,
\]

and

\[
4 \int_{\gamma} k_{x} k^2_{x} k^2 ds = -12 \int_{\gamma} k^2_{ss} k^2_x k^2 ds - 8 \int_{\gamma} k_{ss} k^4_x kds
\]

\[
= -12 \int_{\gamma} k^2_{ss} k^2_x k^2 ds + \frac{8}{5} \int_{\gamma} k^5_x ds.
\]
Substituting above equations into (4.15), we get
\[
\frac{d}{dt} \int_{\gamma} k^2 ds = -2 \int_{\gamma} k^2 ds + 4 \int_{\gamma} k_2^2 k^2 ds - \int_{\gamma} k_3^3 k_3 ds - 8 \int_{\gamma} k_3^3 k_3 ds - 8 \int_{\gamma} k_3^3 k_3 ds - 2 \int_{\gamma} k^2 ds
\]
\[
+ \frac{1}{3} \int_{\gamma} k_3^3 ds - \frac{1}{2} \int_{\gamma} k_3^3 k_3^2 ds + 5 \int_{\gamma} k_3^3 k_3^2 ds + \frac{8}{5} \int_{\gamma} k^3 ds. \tag{4.16}
\]
Again, we calculate several terms in above by using integration by parts, for the fourth term in (4.16),
\[
-8 \int_{\gamma} k_3^3 k_3 ds = 8 \int_{\gamma} k_3^3 k_3^2 ds - 8 \int_{\gamma} k_3^3 k_3^2 ds - 8 \int_{\gamma} k_3^3 k_3^2 ds.
\]
For the sixth term,
\[
\frac{1}{3} \int_{\gamma} k_3^3 ds = - \int_{\gamma} k_3 k_3 k_3^2 ds = \int_{\gamma} k_3^2 k_3 k_3 ds + \int_{\gamma} k_3 k_3 k_3^2 ds + \int_{\gamma} k_3^2 k_3 k_3 ds,
\]
then
\[
\frac{1}{3} \int_{\gamma} k^3 ds = - \int_{\gamma} k_3 k_3 k_3^2 ds = \frac{1}{2} \int_{\gamma} k_3 k_3 k_3^2 ds + \frac{1}{2} \int_{\gamma} k_3 k_3 k_3^2 ds.
\]
The last two terms are rewritten as follows,
\[
\int_{\gamma} k_3^3 k_3^3 ds = -3 \int_{\gamma} k_3^2 k_3^2 k_3^2 ds + \frac{1}{2} \int_{\gamma} k_3^2 k_3^2 d^4 ds + \frac{1}{2} \int_{\gamma} k_3 k_3 k_3^4 ds,
\]
\[
\frac{8}{5} \int_{\gamma} k^3 ds = 4 \int_{\gamma} k_3 k_3 k_3^2 ds + 12 \int_{\gamma} k_3^2 k_3^2 k_3^2 ds.
\]
Hence, we obtain
\[
\frac{d}{dt} \int_{\gamma} k^2 ds = -2 \int_{\gamma} k^2 ds - 4 \int_{\gamma} k_2^2 k^2 ds + \frac{15}{2} \int_{\gamma} k_3^3 k_3^2 ds + \frac{1}{2} \int_{\gamma} k_3 k_3 k_3^2 ds
\]
\[
- 8 \int_{\gamma} k_3^3 k_3^2 ds - 8 \int_{\gamma} k_3 k_3 k_3^2 ds + \frac{1}{2} \int_{\gamma} k_3^3 k_3^2 ds - \frac{7}{2} \int_{\gamma} k_3^3 k_3^2 ds
\]
\[
+ 4 \int_{\gamma} k_3 k_3 k_3^2 ds + \frac{5}{2} \int_{\gamma} k_3 k_3 k_3^4 ds.
\]
By using PSW inequalities (2.1) in Proposition 1 and (2.2) in Proposition 2, as \(\int_{\gamma} k_3 ds = 0, l = 0, 1, 2, 3, 4,\) we have \(\|k_3\|_{\infty}^2 \leq \frac{2}{7} \|k_3\|_{2}^2\) and \(\|k_3\|_{2}^2 \leq \frac{2}{7} \|k_3\|_{2}^2,\).
We can estimate the terms in above equation separately as follows,

\[
\frac{15}{2} \int_{\gamma} k_s^2 k_s^2 ds \leq \frac{15}{2} \|k_s\|_2^2 \|k_s\|_2^2 \leq \frac{15L^3}{\pi^3} \cdot \|k_s\|_2 \|k_s\|_2^2,
\]

\[
\frac{1}{2} \int_{\gamma} k_s k_s k_s^2 ds \leq \frac{1}{2} \|k_s\|_\infty \|k_{ss}\|_\infty \|k_s\|_2^2 \leq \frac{L^3}{\pi^2} \|k_s\|_2 \|k_s\|_2^2,
\]

and

\[
-8 \int_{\gamma} k_s k_s k_s^2 ds \leq 8 \|k\|_\infty \|k_s\|_2 \|k_s\|_2 \|k_{ss}\|_2 \leq \frac{16L^3}{\pi^3} \|k_s\|_2 \|k_s\|_2^2,
\]

\[
\frac{1}{2} \int_{\gamma} k_s k_s k_s^2 ds \leq \frac{1}{2} \|k_s\|_\infty \|k_{ss}\|_\infty \|k_{ss}\|_2 \leq \frac{2L^6}{\pi^5} \|k_s\|_2 \|k_s\|_2^2,
\]

also

\[
4 \int_{\gamma} k_s^3 k_s^2 ds \leq 4 \|k_s\|_\infty \|k_s\|_\infty \|k_s^2\|_\infty \|k_s\|_2 \|k_s\|_2 \|k_s\|_2 \leq \frac{8L^6}{\pi^5} \|k_s\|_2 \|k_s\|_2^2,
\]

\[
\frac{5}{2} \int_{\gamma} k_s k_s k_s^4 ds \leq \frac{5}{2} \|k_s\|_\infty \|k_{ss}\|_\infty \|k_{ss}\|_2 \|k_{ss}\|_2 \leq \frac{10L^6}{\pi^5} \|k_s\|_2 \|k_s\|_2^2.
\]

Finally, by combining above calculations we obtain

\[
\frac{d}{dt} \int_{\gamma} k_s^2 ds \leq \left( -2 + \frac{32L^3}{\pi^3} \|k_s\|_2^2 + \frac{20L^6}{\pi^5} \|k_s\|_2^2 \right) \|k_s\|_2^2.
\]

If the coefficient of \(\|k_s\|_2^2\) satisfies

\[
-2 + \frac{32L^3}{\pi^3} \|k_s\|_2^2 + \frac{20L^6}{\pi^5} \|k_s\|_2^2 < 0
\]

i.e.

\[
\|k_s\|_2^2 < \frac{\sqrt{74} - 8}{10} \cdot \frac{\pi^3}{L^3},
\]

Estimating \(L\) from above by \(L(0)\), if at time \(t_0\), we have

\[
\|k_s\|_2^2 < \frac{\sqrt{74} - 8}{10} \cdot \frac{\pi^3}{L^3(0)},
\]
\[
\frac{d}{dt} \int_{\gamma} k_{ss}^2 ds \leq -\bar{c} \|k_{ss}\|_2^2 \leq c \left( \frac{L^6}{\pi^6} \right) \|k_{ss}\|_2^2 \leq -\delta \|k_{ss}\|_2^2,
\]

Therefore, we obtain
\[
\|k_{ss}\|_2^2 \leq \|k_{ss}\|_2^2 \bigg|_{t=0} e^{-\delta (t-t_0)}.
\]

This is the end of the proof. \(\square\)

Secondly, under Lemma 23, we obtain exponential decay of all curvature derivatives to zero by a standard induction argument involving integration by parts and the curvature bounds.

**Lemma 24.** Let \(\gamma: [-1, 1] \times [0, \infty) \to \mathbb{R}^2\) be a solution to (4.1). Under the assumption of Lemma 23. So \(\|k_s\|_2^2\) and \(\|k_s\|_{\infty}^2\) exponentially decay for \(t > t_0\).

**Proof.** As we have proved that when \(t \to \infty\), \(\int_{\gamma} k_s^2 ds\) exponentially decays in Lemma 23, we can obtain
\[
\|k\|_2^2 \leq \frac{L^2}{\pi^2} \|k_s\|_2^2 \leq \frac{L^4(0)}{\pi^4} \|k_{ss}\|_2^2 \leq \frac{L^4(0)}{\pi^4} e^{-\delta t} \|k_{ss}\|_2^2 \bigg|_{t=0},
\]
then \(\int_{\gamma} k_s^2 ds\) exponentially decays. Similarly
\[
\|k_s\|_2^2 \leq \frac{L^2}{\pi^2} \|k_{ss}\|_2^2 \leq \frac{L^2(0)}{\pi^2} \|k_{ss}\|_2^2 \leq \frac{L^2(0)}{\pi^2} e^{-\delta t} \|k_{ss}\|_2^2 \bigg|_{t=0},
\]
then \(\int_{\gamma} k_s^2 ds\) exponentially decays.

\[
\int_{\gamma} k_s^2 ds = -\int_{\gamma} k_{ss} k_s^2 ds \leq \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_s^2 ds \right)^{\frac{1}{2}}
\]
as \(\left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \leq \left( e^{-\delta t} \|k_s\|_2^2 \bigg|_{t=0} \right)^{\frac{1}{2}}\), and \(\left( \int_{\gamma} k_s^2 ds \right)^{\frac{1}{2}}\) is bounded in \(L^2\), then \(\int_{\gamma} k_s^2 ds\) exponentially decays.

\[
\int_{\gamma} k_s^2 ds = -\int_{\gamma} k_{ss} k_{ss}^2 ds \leq \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}}
\]
as \(\left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}}\) is bounded in \(L^2\), then \(\int_{\gamma} k_{ss}^2 ds\) exponentially decays.

By induction argument, we assume that \(\int_{\gamma} k_s^2 ds\) exponentially decays,
\[
\int_{\gamma} k_{ss}^2 ds = -\int_{\gamma} k_{ss} k_{ss}^2 ds \leq \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}}
\]
We prove this by induction. First, \(\left(\int_T k_{i+1}^2 \, ds\right)^{\frac{1}{2}}\) is bounded in \(L^2\), then \(\int_T k_{i+1}^2 \, ds\) exponentially decays.

Thus, we have \(\int_T k_{i}^2 \, ds\) exponentially decays in \(L^2\) for all \(t > t_0\). As \(\|k_{i}\|_\infty \leq \frac{T}{R} \|k_{i+1}\|_2\), we have \(\int_T k_{i}^2 \, ds\) exponentially decays in \(L^\infty\) for all \(t > t_0\).

Note that under the assumption of Theorem 7, we have \(T = \infty\). From above lemma, we can obtain uniform bounds for all derivatives of the evolving curve \(\gamma\).

**Proposition 11.** Suppose \(\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2\) solves (4.1) and satisfies the conditions of Theorem 7. Then for all \(l \in \mathbb{N}_0\),

\[
\|\partial_l \gamma\|_\infty \leq c(l) + c \sum_{p=0}^l \|\partial_p \gamma_0\|_\infty,
\]

where \(c(l)\) is a constant only depending on \(l, E(0), L(0)\).

**Proof.** We claim that for \(l, p \in \mathbb{N}_0\),

\[
\partial_l \partial_p \gamma = v \sum_{p=0}^l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) \quad (4.17)
\]

We prove this by induction. First, \(F = P_1^{4}(k) + P_3^{2}(k)\) so the equation holds for \(l = 0\). For \(q \in \mathbb{N}_0\), we do the differentiation

\[
\partial_l \partial_{l+1} \gamma = kF \cdot \partial_{l+1} \gamma + \partial_l (\partial_l \gamma) = kF \cdot \partial_{l+1} \gamma + \partial_l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) \]

\[
= kF \sum_{p+q=l} P_{q}^{p}(k) + kF \sum_{p+q=l} P_{q}^{p}(k) + v \sum_{p=0}^l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{1+i-l-p}^{4+p} + P_{3+i-l-p}^{2+p} \right) \]

\[
= v \sum_{p+q=l} \left( P_{q+2}^{p+4} + P_{q+4}^{p+2} \right) + v \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) \]

\[
= v \sum_{p+q=l} \left( P_{q+2}^{p+4} + P_{q+4}^{p+2} \right) + v \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) \]

\[
= v \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) + \tau \sum_{p=0}^l \left( P_{2+i-l-p}^{4+p} + P_{4+i-l-p}^{2+p} \right) \]
as required. Integrating (4.17) and using Lemma 24, we find
\[ \| \partial s \gamma \|_\infty \leq \| \partial s \gamma_0 \|_\infty + c \int_0^t e^{-ct'} \, dt' \leq \| \partial s \gamma_0 \|_\infty + \bar{c}(l). \]

As \( u \) is the initial space parameter before reparametrisation by arc-length, set \( v = |\partial_u \gamma| \). Referring to the proof of Theorem 3.1 in [17], then for any function \( \Phi : \mathbb{S} \to \mathbb{R} \), we have
\[ \partial_u \Phi = v \partial s \Phi + P_l(|\partial_u \Phi|, ..., |\partial_u \Phi|, \Phi, ..., |\partial_u \Phi|) \]
where \( P_l \) is a polynomial. We can get
\[ \| \partial u \gamma \|_\infty \leq c \| \partial s \gamma \| \| \partial s \gamma \|_\infty + \| P_l \|_\infty. \]

Then we obtain
\[ \| \partial u \gamma \|_\infty \leq c \| \partial s \gamma_0 \|_\infty + c(l). \]
as required. \( \square \)

The next lemma states that along a subsequence of times, there is convergence to a straight line segment.

**Lemma 25.** Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a solution to (4.1). Under the assumption of Theorem 7, there exist a subsequence of time \( t_j \) such that
\[ \gamma(\cdot, t_j) \to \gamma_\infty \]
uniformly with \( \gamma_\infty \) a straight line segment.

**Proof.** Global existence implies
\[ \lim_{t \to \infty} \int_0^t \int_\gamma F^2 \, ds \, dt = E(0) - \lim_{t \to \infty} E(t) \]
\[ = \| k_s \|_2^2 |_{t=0} - \lim_{t \to \infty} \| k_s \|_2^2 |_{t=0} \]
So there exists a subsequence, \( t_j \to \infty \) such that
\[ \int_\gamma |F(\gamma(\cdot, t_j))|^2 \, ds \to 0, \text{ as } j \to \infty. \]

For any \( \varepsilon > 0 \), \( \int_\gamma F^2 \, ds \) is eventually smaller than \( \varepsilon \). Let \( T_0, N \) be such that for \( j > N \), we have \( t_j > T_0 \) and
\[ \int_\gamma F^2(\gamma(\cdot, t_j)) \, ds < \varepsilon. \]
Then \( \|k_s\|_2^2 \) decays and becomes eventually arbitrarily small on the full sequence. This implies there exist a limit curve \( \gamma_\infty \) that the flow converges to which satisfies \( k = 0 \), that is, \( \gamma_\infty \) is a straight line segment.

### 4.4 The unique limit

As we have proven that the derivative of curvature decays to zero in \( L^2 \) along the flow, standard theory implies that the flow converges to a solution of \( k(\gamma_\infty) = 0 \). The only thing left to prove is that the limit is unique.

**Lemma 26.** The image of limiting straight line segment is unique.

**Proof.** In order to prove that \( \gamma_\infty \) is the unique limit, we refer to [1, Theorem A.1], Suppose there exists a sequence \( \{s_j\} \subset (0, \infty) \), \( s_j \to \infty \), such that \( \gamma(\cdot, s_j) \to \bar{\gamma}_\infty \neq \gamma_\infty \), in \( C^\infty \).

Consider the functional

\[
G[\gamma] = \int_\gamma |\gamma - \gamma_\infty|^2 ds.
\]

Since \( \bar{\gamma}_\infty \) and \( \gamma_\infty \) are smooth, it follows that

\[
\lim_{s_j \to \infty} G[f(\cdot, s_j)] \neq 0. \tag{4.18}
\]

We estimate by using Lemma 1,

\[
\left| \frac{d}{dt} G \right| = \frac{d}{dt} \int_\gamma |\gamma - \gamma_\infty|^2 ds
\]

\[
= 2 \int_\gamma |\gamma - \gamma_\infty| \cdot \frac{d}{dt} |\gamma - \gamma_\infty| ds + \int_\gamma |\gamma - \gamma_\infty|^2 \cdot \frac{d}{ds} ds
\]

\[
= 2 \int_\gamma F \cdot |\gamma - \gamma_\infty| ds + \int_\gamma kF \cdot |\gamma - \gamma_\infty|^2 ds
\]

then we have

\[
\left| \frac{d}{dt} G \right| = \int_\gamma F \cdot |\gamma - \gamma_\infty| \cdot (2 + k |\gamma - \gamma_\infty|) ds
\]

\[
\leq (\int_\gamma F^2 ds)^{\frac{1}{2}} \cdot \left[ \int_\gamma \left(|\gamma - \gamma_\infty|^2 + 2 + k |\gamma - \gamma_\infty| \right) ds \right]^{\frac{1}{2}}
\]

\[
\leq (\int_\gamma F^2 ds)^{\frac{1}{2}} \cdot \left[ \int_\gamma \left(4 + k^2 |\gamma - \gamma_\infty|^2 \right) ds \right]^{\frac{1}{2}}
\]

\[
\leq c (\int_\gamma F^2 ds)^{\frac{1}{2}} \cdot \left( \int_\gamma |\gamma_\infty|^2 ds + \int_\gamma |\gamma|^2 ds + \int_\gamma |k|^2 \cdot |\gamma|^4 ds + \int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds \right)^{\frac{1}{2}}.
\]
Now we show that $\int_\gamma |\gamma|^2 ds$, $\int_\gamma |\gamma_\infty|^2 ds$, $\int_\gamma |k|^2 \cdot |\gamma|^4 ds$ and $\int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds$ are uniformly bounded.

First, we can see that under the exponential decay of curvature and its derivatives in $C^\infty$, we obtain that
\[
\|\gamma(\cdot, \bar{t})\|_{\infty}^2 = \left( \int_\bar{t}^\infty |F| dt \right)^2 \leq \frac{\bar{c}}{\delta} e^{-\delta \bar{t}} \leq c_1,
\]
then
\[
\int_\gamma |\gamma|^2 ds \leq \|\gamma\|_{\infty}^2 \int_\gamma ds \leq c_1 L(0) \leq c_2,
\]
\[
\int_\gamma |k|^2 \cdot |\gamma|^4 ds \leq \|\gamma\|_{\infty}^4 \int_\gamma |k|^2 ds \leq c_3.
\]

From above two estimates, it is clear that $\int_\gamma |\gamma_\infty|^2 ds$ and $\int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds$ are bounded as well.

Thus,
\[
\left| \frac{d}{dt} G \right| \leq c \|F\|_2,
\]

By the exponential decay of curvature and its derivatives in $L^2$, we have that
\[
G[f(\cdot, s_j)] \leq c \int_{s_j}^\infty \|F\|_2 dt \leq c \int_{s_j}^\infty e^{-\delta t} dt = ce^{-\delta s_j}
\]
it follows that
\[
\lim_{s_j \to \infty} G[f(\cdot, s_j)] = 0,
\]
which is in contradiction with (4.18).

This proves that there does not exist a sequence $\{s_j\}$, so the flow must converge in $L^2$ to a unique line segment. We can obtain the exponential convergence in $C^\infty$ to a unique line segment as the curvature and all its derivatives exponentially decay.

This finishes the proof.

Then we have the following lemma,

**Lemma 27.** The flow exists globally ($T = \infty$) and converges smoothly to a straight line segment.

**Remark 1.** While we don’t know the precise height of the limiting straight horizontal line segment for the curve. However, $\|k_{x, t}\|_\infty$ decays exponentially in Lemma 24 shows that the solution curve remains within a bounded distance of the initial curve: for any $x$,
\[ |\gamma(x,t) - \gamma(x,0)| \leq \int_0^t \left| \frac{\partial \gamma}{\partial t}(x,t) \right| \, dt \]

\[ = \int_0^t \left| k_s^4 + k_{ss}k^2 - \frac{1}{2}k_s^2k \right| \, dt \]

\[ \leq c \int_0^t e^{-\delta t} \, dt = \frac{c}{\delta} \left( 1 - e^{-\delta t} \right) \leq \frac{c}{\delta}. \]
Chapter 5

Higher order flows of plane curves with boundary conditions

Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a smooth immersed regular curve, \( \eta_1, \eta_2 \) denote two parallel vertical lines in \( \mathbb{R}^2 \), with a distance \( |e| \neq 0 \) between them. Two end points \( \gamma(\pm 1, t) \) of the curve meet two parallel lines \( \eta_{1,2} \) orthogonally,

\( \gamma(-1, \cdot) \in \eta_1(\mathbb{R}), \gamma(1, \cdot) \in \eta_2(\mathbb{R}). \)

This chapter considers one-parameter families of smooth immersed curves \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) moving by either the \( L^2 \)-gradient flow for any \( m \in \mathbb{N} \cup \{0\} \)

\[
\frac{\partial \gamma}{\partial t} = \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+j} k_{2m-j} - \frac{1}{2} k_{2m}^2 \right] \nu,
\]

(5.1)

or polyharmonic curve diffusion flow

\[
\frac{\partial \gamma}{\partial t} = (-1)^{m+1} k_{2m+2} \nu
\]

(5.2)

for the energy

\[
\frac{1}{2} \int_{\gamma} k_{2m}^2 ds,
\]

(5.3)

with \( \gamma(\cdot, 0) := \gamma_0 \) and generalised Neumann boundary conditions. Here \( k_{2m} \) denotes the \( m \)-th derivative of curvature with respect to the arc length parameter \( s \) and \( \nu \) is the unit normal vector of curve \( \gamma \).
5.1 The gradient flow for the energy

In order to get the gradient flow, first we calculate the evolution of \( f_\gamma k_{s,m}^2 ds \),

\[
\frac{d}{dt} \int_\gamma k_{s,m}^2 ds = 2 \int_\gamma k_{s,m}^2 k_{s+1,m} ds - \int_\gamma k_{s,m}^2 (kF_1) ds. \tag{5.4}
\]

By using (ii) in Lemma 3, the first term in (5.4) is

\[
2 \int_\gamma k_{s,m}^2 k_{s+1,m} ds = 2 \int_\gamma k_{s,m}^2 \left[ (F_1)_{s,m+2} + \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) \right] ds
\]

\[
= 2 \int_\gamma k_{s,m}^2 \cdot (F_1)_{s,m+2} ds + 2 \int_\gamma k_{s,m}^2 \cdot \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) ds
\]

\[
= 2 \int_\gamma F \cdot \left[ (-1)^{m+2} k_{s,m+2} + \sum_{j=0}^{m} (-1)^j kk_{s,m+j} k_{s,m-j} \right] ds. \tag{5.5}
\]

The first term in (5.5)

\[
2 \int_\gamma k_{s,m}^2 \cdot (F_1)_{s,m+2} ds
\]

\[
= -2 \int_\gamma k_{s,m+1} (F_1)_{s,m+1} ds + 2 k_{s,m} (F_1)_{s,m+1} \bigg|_{\partial \gamma}
\]

\[
= 2 \int_\gamma k_{s,m+2} (F_1)_{s,m} ds - 2 k_{s,m+1} (F_1)_{s,m} \bigg|_{\partial \gamma} + 2 k_{s,m} (F_1)_{s,m+1} \bigg|_{\partial \gamma}
\]

\[
= -2 \int_\gamma k_{s,m+3} (F_1)_{s,m-1} ds + 2 k_{s,m+2} (F_1)_{s,m-1} \bigg|_{\partial \gamma} - 2 k_{s,m+1} (F_1)_{s,m} \bigg|_{\partial \gamma} + 2 k_{s,m} (F_1)_{s,m+1} \bigg|_{\partial \gamma}
\]

\[
= \ldots \ldots
\]

\[
= 2 \int_\gamma (-1)^{m+2} k_{s,m+2} F_1 ds + 2 \sum_{j=0}^{m+1} (-1)^j k_{s,m+j} (F_1)_{s,m+1-j} \bigg|_{\partial \gamma}. \tag{5.6}
\]

The second term in (5.5)

\[
2 \int_\gamma k_{s,m} \cdot \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) ds
\]

\[
= 2 \int_\gamma \frac{d}{ds} \left[ \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) \right] ds + 2 \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) \bigg|_{\partial \gamma}
\]

\[
= 2 \int_\gamma \frac{d}{ds} \left[ \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) \right] ds + 2 \sum_{j=0}^{m} \partial_{s,j-2} (kk_{s,m-j}F_1) ds
\]

\[
= 2 \int_\gamma \sum_{j=0}^{m} \partial_{s,j} (kk_{s,m-j}F_1) ds
\]

\[
= \ldots \ldots
\]

\[
= 2 \int_\gamma \sum_{j=0}^{m} \left( -1 \right)^j kk_{s,m+j} k_{s,m-j} F_1 ds + 2 \sum_{l=1}^{m} \left( -1 \right)^{l-1} \sum_{j=l}^{m} \partial_{s,j-l} (kk_{s,m-j}F_1) \bigg|_{\partial \gamma}. \tag{5.7}
\]
Substituting (5.6) and (5.7) into (5.5) we have

\[
2 \int_{\gamma} k^m k'^m d = 2 \int_{\gamma} F_1 \cdot \left[ (-1)^{m+2} k_{2m+2} + \sum_{j=0}^{m} (-1)^j k k_j + k_{m-j} \right] ds \\
+ 2 \sum_{j=0}^{m+1} (-1)^j k_{m+1-j} \left. \frac{\partial}{\partial \gamma} \right|_{\gamma} \\
+ 2 \sum_{l=1}^{m} (-1)^{l-1} k_{m+l-1} \sum_{j=1}^{m} \partial_s \left. \left( k k_{m-j} F_1 \right) \right|_{\partial \gamma}, \tag{5.8}
\]

here in order to have the $L^2$ gradient flow, we make one choice of boundary conditions such that the following equation is satisfied,

\[
\sum_{j=0}^{m+1} (-1)^j k_{m+1-j} \left. \frac{\partial}{\partial \gamma} \right|_{\gamma} + \sum_{l=1}^{m} (-1)^{l-1} k_{m+l-1} \sum_{j=1}^{m} \partial_s \left. \left( k k_{m-j} F_1 \right) \right|_{\partial \gamma} = 0. \tag{5.9}
\]

Substituting (5.8) and (5.9) into (5.4),

\[
\frac{d}{dt} \frac{1}{2} \int_{\gamma} k^m ds = \int_{\gamma} F_1 \cdot \left[ (-1)^{m+2} k_{2m+2} + \sum_{j=0}^{m} (-1)^j k k_j + k_{m-j} - \frac{1}{2} kk^2 \right] ds \\
= - \int_{\gamma} F_1 \cdot \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=0}^{m} (-1)^{j+1} k k_j + k_{m-j} + \frac{1}{2} kk^2 \right] ds \\
= - \int_{\gamma} F_1 \cdot \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k k_j + k_{m-j} - \frac{1}{2} kk^2 \right] ds.
\]

Under the condition that (5.9) holds, for the corresponding $L^2$ gradient flows of the associated curvature-dependent energy

\[
\frac{1}{2} \int_{\gamma} k^m ds,
\]

we would take the normal flow speed

\[
F_1 = (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k k_j + k_{m-j} - \frac{1}{2} kk^2. \tag{5.10}
\]

We can see that the gradient flow is of order $2m+4$ from above equation. It should have $2m+4$ boundary conditions for a unique solution. Our choice of boundary conditions are the classical Neumann condition and all odd curvature derivatives up to order $2m+1$ equal to zero at both boundaries. Note that there are other possible boundary conditions. Here
we see each pair of boundary conditions at ±1 as one condition, then the total number of conditions is $m + 2$. Therefore, we give the following lemma.

**Lemma 28.** The boundary condition (5.9) holds under (5.28), the number of the conditions is $2m + 4$ on both boundaries which is the same as the highest order of the derivatives of the curvature.

**Proof.** Differentiating the Neumann boundary condition $< v, ν_{η1,2} > (±1, t) = 0$, i.e. $v(±1, t), e > = 0$ in time yields $(F_1)_s(±1, t) < τ(±1, t), e > = ±|e|(F_1)_s(±1, t) = 0$. Now $|e| ≠ 0$ so we must have that

$$(F_1)_s(±1, t) = 0.$$ 

The evolution of $k_s(±1, t) = 0$ in time gives

$$\partial_t k_s = k_{ts} + kF_1k_s = [(F_1)_{ss} + Fk^2]_s + kF_1k_s = (F_1)_{s3} + F_3k^2 + 3F_1k_s k,$$

as $k_s(±1, t) = (F_1)_s(±1, t) = 0$, we must have that

$$(F_1)_{s3}(±1, t) = 0.$$ 

Again, differentiating $k_{s3}(±1, t) = 0$ in time yields

$$\partial_t k_{sss} = k_{ssss} + kF_1k_{sss} = [(F_1)_{s4} + (k^2F_1)_{ss} + (kk_sF_1)_s + kk_{ss}F_1]_s + kF_1k_{sss} = (F_1)_{s5} + (k^2F_1)_{s3} + (kk_sF_1)_{ss} + (kk_{ss}F_1)_s + kk_{s3}F_1 = (F_1)_{s5} + 6k_s k_{ss} F_1 + 7k^2_s (F_1)_s + 4k_s k_{ss} F_1 + 5k k_{s3} F_1 + 9k k_{ss} (F_1)_s + 7k k_{ss} (F_1)_{ss} + k^2 (F_1)_{s3} + k^2_s F_1,$$

as $k_s(±1, t) = k_{s3}(±1, t) = (F_1)_s(±1, t) = (F_1)_{s3}(±1, t) = 0$, we obtain that

$$(F_1)_{s5}(±1, t) = 0.$$ 

Generally, for each odd derivative of k equal to zero on the boundary, we get that the next odd derivative of $F_1$ is also equal to zero on the boundary by the similar argument. We assume that all odd derivative of $k$ up to order $2m + 1$ are equal to zero on the boundary, then keep differentiating $k_{s3}(±1, t) = k_{s5}(±1, t) = ... = k_{2m+1}(±1, t) = 0$ in time, we obtain

$$(F_1)_{s3}(±1, t) = (F_1)_{s5}(±1, t) = ... = (F_1)_{s2m+3} = 0.$$
Thus
\[(F_1)_s(\pm 1, t) = (F_1)_{s^3}(\pm 1, t) = (F_1)_{s^5}(\pm 1, t) = \ldots = (F_1)_{s^{2m+3}} = 0. \quad (5.11)\]

When \(m + j\) is odd, from (5.28), we have \(k_{s^m+j}(\pm 1, t) = 0\); when \(m + j\) is even, from (5.11), we have \((F_1)_{s^{m+1-j}}(\pm 1, t) = 0\). Then for any \(m, j \in \mathbb{N} \cup \{0\}\), the first term in (5.9),
\[
\sum_{j=0}^{m+1} (-1)^j k_{s^{m+j}}(F_1)_{s^{m+1-j}} \bigg|_{\gamma} = 0.
\]

When \(m + l\) is even, from (5.28), we have \(k_{s^{m+l-1}}(\pm 1, t) = 0\); when \(m + l\) is odd, the number of derivatives in \(\partial_{s^{j-l}}(kk_{s^{m-j}}F_1)\) is \(m - j + (j - l) = m - l\) which is odd, then from (5.28) and (5.11),
\[
\sum_{j=1}^{m} \sum_{j=0}^{m} \partial_{s^{j-l}}(kk_{s^{m-j}}F_1) \bigg|_{\gamma} = 0,
\]
then for any \(m, j \in \mathbb{N} \cup \{0\}\), the second term in (5.9),
\[
\sum_{j=1}^{m} (-1)^{l-1} k_{s^{m+l-1}} k_{s^{m-j}}(F_1) \bigg|_{\gamma} = 0.
\]

Thus, we prove that under (5.28), (5.9) holds. We choose (5.28) to be the ‘generalised Neumann boundary conditions’ for our curves.

Here we check our boundary condition satisfies special cases:

When \(m = 0\), the boundary condition (5.28) is \(<v, \nu_{s^1,v}> = k_s(\pm 1, t) = 0\) in [85];

When \(m = 1\), we need \(<v, \nu_{s^2,v}> = k_s(\pm 1, t) = k_s(\pm 1, t) = 0\) in Chapter 3. \(\square\)

The definition for \(2m + 4\)th order curvature flow with natural boundary condition is as follows,

**Definition 8.** Let \(\gamma : [-1, 1] \times [0, T] \to \mathbb{R}^2\) be a family of smooth immersion, which meeting two parallel lines \(\eta_{1,2}\) with \(m+2\) generalised Neumann boundary conditions (5.28). \(\gamma\) is said to move under (\(2m + 4\))th order curvature flow (5.1), if

\[
\begin{cases}
\frac{\partial}{\partial s}\gamma(s, t) = -F_1 v, & \text{for all } (s, t) \in [-1, 1] \times [0, T) \\
\gamma(\cdot, 0) = \gamma_0, & \\
<v, \nu_{s^2,v}> = k_s = \ldots = k_{s^{2m-1}} = k_{s^{2m+1}} = 0, & \text{for all } (s, t) \in \eta_{1,2} \times [0, T)
\end{cases}
\]

where \(F_1 = (-1)^{m+1} k_{s^{2m+2}} + \sum_{j=1}^{m} (-1)^{j+1} k_{s^{m+j}} k_{s^{m-j}} - \frac{1}{2} k_{s^{m+2}}^2\) denotes the \(2m + 4\)th order
curvature vector of the immersions, \( m \in \mathbb{N} \cup \{0\} \), \( \nu \) and \( \nu_{\eta_1,2} \) are the unit normal fields to \( \gamma \) and \( \eta_{1,2} \) respectively.

The main result in this section is:

**Theorem 8.** Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a family of smooth embedded or immersed curves as in Definition 8. If the initial curve \( \gamma_0 \) satisfies \( \omega = 0 \) and has sufficiently small energy, that is

\[
L^{2m+1}(0) \int_{\gamma} k^2_m(0) ds \leq \varepsilon, \tag{5.12}
\]

where \( k(0) = k(\cdot, 0) \), for some positive \( \varepsilon \) depending only on \( m \), then there exists a smooth solution \( \gamma : [-1, 1] \times [0, \infty) \to \mathbb{R}^2 \) to the \( L^2 \)-gradient flow for the energy (5.3) with \( \gamma(\cdot, 0) = \gamma_0 \). The solution \( \gamma \) converges to a straight line segment exponentially and is unique. The distance between the limiting curve and \( \gamma_0 \) is finite.

The \( \varepsilon \) in above theorem can be calculated from to (5.33).

**Lemma 29.** The hypothesis of Theorem 8 implies that \( \omega(t) = \omega(0) = 0 \).

It follows immediately that the average curvature \( \bar{k} \) satisfies

\[
\bar{k} := \frac{1}{L} \int_{\gamma} k ds \equiv 0.
\]

For the proof, see Lemma 17.

**Lemma 30.** With the natural Neumann boundary condition (5.28), a solution to the flow (5.1) satisfies \( k_{x2l+1} = 0 \) and \( (F_1)_{x2l+1} = 0 \) on the boundary for all \( l \in \mathbb{N} \cup \{0\} \).

**Proof.** Calculate the first derivative of the flow equation (5.10) with respect to the arc-length \( s \),

\[
(F_1)_s = \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} - \frac{1}{2} k k_{2m} \right] s
\]

\[
= (-1)^{m+1} k_{2m+3} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j}
\]

\[
+ \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} = \frac{1}{2} k k_{2m} - k k_{2m} k_{2m+1}.
\]

From (5.28), we must have

\[
k_{x2m+3}(\pm 1, t) = 0,
\]
\[(F_1)_s = (-1)^{m+1}k_{2m+5} + \sum_{j=1}^{m} (-1)^{j+1} \partial_s^2 (k_s k_{m+j} k_{m-j}) + \sum_{j=1}^{m} (-1)^{j+1} \partial_s^2 (kk_{m+j} k_{m-j}) + \sum_{j=1}^{m} \partial_s^2 (kk_{m+j} k_{m-j}) - \frac{1}{2} k_{m+1}^2 k_{m}^2 \]

Again from (5.28), we obtain

\[k_{2m+5}(\pm 1, t) = 0.\]

Let us give the induction argument, assuming that for \(n = 2, 3, \ldots\)

\[k_{2n+2m-1}(\pm 1, t) = 0,\]

\[\partial_t k_{s,l} = (F_1)_{s+l} + \sum_{h=0}^{l} \partial_s^h (kk_{s+h} F_1)\]

\[= (-1)^{m+1} k_{2m+l+4} + \sum_{j=1}^{m} (-1)^{j+1} \partial_s^{j+1} (kk_{m+j} k_{m-j}) - \frac{1}{2} \partial_s^{j+2} (kk_{m}^2)\]

\[+ \sum_{h=0}^{l} \partial_s^h \left( (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} kk_{m+j} k_{m-j} - \frac{1}{2} kk_{m}^2 \right)\]

\[= (-1)^{m+1} k_{2m+l+4} + \sum_{j=0}^{m} (-1)^{j+1} \sum_{q+r+u=l+2} c_{qru} k_{q+l} k_{r+l} k_{u+l}\]

\[\quad + (-1)^{m+1} \sum_{h=0}^{l} \sum_{q+r+u=h} c_{qru} k_{q+l} k_{r+l} k_{u+l}\]

\[+ \sum_{h=0}^{m} \sum_{j=0}^{m} (-1)^{j+1} \sum_{q+r+u+w=h} c_{qruw} k_{q+l} k_{r+l} k_{u+l} k_{w+l},\]

for constants \(c_{qru}, c_{qru}, c_{qruw} \in \mathbb{R}\) with \(q, r, u, v, w \geq 0.\)

The inductive hypothesis implies that for \(l\) odd and less than or equal to \(2n + 2m - 1\), the derivatives \(k_{s,l}\) vanishes on the boundary. Here we take \(l = 2n - 3\), then under the hypothesis \(k_{2n+2m-1}(\pm 1, t) = 0\), we removed all terms with an odd number of derivatives of \(k\), we conclude

\[k_{2n+2m+1} = 0, n = 2, 3, \ldots\]

Above result together with boundary conditions (5.28) yields \(k_{2l+1}(\pm 1, t) = 0, p = 0, 1, 2, \ldots\)

We have

\[(F_1)_{s}(\pm 1, t) = (F_1)_{s}(\pm 1, t) = (F_1)_{s}(\pm 1, t) = \ldots = (F_1)_{2m+3} = 0\]
and later prove that all odd derivatives of the curvature are equal to zero at the boundary.

\[(F_1)_\gamma = (-1)^{m+1} k_{2m+2+n} + \sum_{j=1}^{m} (-1)^{j+1} (k k_{2m+1} k_{2m-j})_\gamma - \frac{1}{2} (k k_{2m})_\gamma \]

\[= (-1)^{m+1} k_{2m+2+n} + \sum_{j=1}^{m} (-1)^{j+1} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3} - \frac{1}{2} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3}, \]

(5.13)

Here \(n = 2l + 1, l \in \mathbb{N} \cup \{0\}\).

If \(m\) is even, \(j\) is even, then \(m+j, m-j\) are even. \(l_1, l_2, l_3\) have at least one is odd. So we know \(2m+2+n\) is odd, \(l_1, m+j+l_2, m-j+l_3\) has at least one is odd, also \(l_1, m+l_2, m+l_3\) has at least one is odd. Thus, all three terms in (5.13) are zero at the boundary: \(k_{2m+2+n} = 0, \sum_{j=1}^{m} (-1)^{j+1} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3} = 0\) and \(\frac{1}{2} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3} = 0\).

If \(m\) is even, \(j\) is odd, then \(m+j, m-j\) are odd. \(l_1, l_2, l_3\) have one or three are odd. So \(l_1, m+j+l_2, m-j+l_3\) has at least one is odd, also \(l_1, m+l_2, m+l_3\) has at least one is odd. Thus, all three terms in (5.13) are zero at the boundary: \(k_{2m+2+n} = 0, \sum_{j=1}^{m} (-1)^{j+1} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3} = 0\) and \(\frac{1}{2} \sum_{l_1+l_2+l_3=n} k_{l_1} k_{2m+j+l_2} k_{2m-j+l_3} = 0\).

If \(m\) and \(j\) are odd, the result is the same as when \(m, j\) are even. If \(m\) is odd, \(j\) is even, the result is the same as when \(m\) is even, \(j\) is odd.

Thus \((F_1)_\gamma = 0\) holds for \(n = 2l + 1, \) any \(l, m, j \in \mathbb{N} \cup \{0\}\) and \(j \leq m\). \(\square\)

Next lemma for the flow (5.1) shows that the length of the evolving curve does not increase under the smallness assumption of initial energy (5.12).

**Lemma 31.** Suppose \(\gamma_0\) satisfies the conditions of Theorem 8. Then, under the flow (5.1) with normal speed (5.10), the length of \(\gamma\) does not increase.

**Proof.** We use Lemma 3 (i), by using integration by parts and the boundary conditions, we get

\[\frac{d}{dt} \int_L kF_1 ds = - \int_L kF_1 ds\]

\[= - \int_L \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k k_{2m+1} k_{2m-j} - \frac{1}{2} k k_{2m}^2 \right] ds\]

\[= - \int_L \left[ (-1)^{2m+2} k_{2m+1} + \sum_{j=1}^{m} (-1)^{j+1} k^2 k_{2m+1} k_{2m-j} - \frac{1}{2} k^2 k_{2m}^2 \right] ds\]

\[= - \int_L k^2 k_{2m+1} + \sum_{j=1}^{m} (-1)^{j+1} \int_L k^2 k_{2m+1} k_{2m-j} ds + \frac{1}{2} \int_L k^2 k_{2m}^2 ds. \]

(5.14)

Estimating the second and third terms in (5.14), here \(P_d^{2m}(k)\) is defined in Chapter 2,
where \( c \) is sufficiently small for the initial curve, then

\[
\sum_{j=1}^{m} (-1)^j \int_{\gamma} k^2 k_{m+j} k_{m-j} ds + \frac{1}{2} \int_{\gamma} k^2 k_{m}^2 ds
\]

\[
= \sum_{j=1}^{m} (-1)^{2j} \int_{\gamma} k_{m} \left( k^2 k_{m-j} \right)_s ds + \frac{1}{2} \int_{\gamma} k^2 k_{m}^2 ds
\]

\[
= \int_{\gamma} P_4^{2m}(k) ds \leq \int_{\gamma} |P_4^{2m}(k)| ds,
\]

In the above, integration by parts gives us that highest order derivative of \( k \) in all terms is \( k_{m} \), we can write these terms as the form \( \int_{\gamma} P_4^{2m}(k) ds \). Thus, we estimate using interpolation inequality in Proposition 4,

\[
\int_{\gamma} |P_4^{2m}(k)| ds \leq c L^{1-2m-4} \| k \|_{2}^{2m+3} \| k \|_{m+1}^{2m+1} \| k \|_{m+1}^{2m+1} + \| k \|_{m+1}^{4}
\]

\[
\leq c \| k \|_{m+1}^{2m+3} \| k \|_{m+1}^{2m+1} + c L^{1-2m-4/2} \| k \|_{m+1}^{4}
\]

\[
= c \| k \|_{2}^{2m+1} \| k \|_{m+1}^{2m+1} + c L^{2m-1} \| k \|_{3} \| k \|_{2}
\]

\[
\leq c \| k \|_{2}^{2m+1} \left( \frac{L^2}{\pi^2} \right)^{m+1} \| k \|_{m+1}^{2m+1} + c L^{2m-1} \| k \|_{3} \| k \|_{2}
\]

\[
= c L \| k \|_{2}^{2m+1} \| k \|_{m+1}^{2m+1}
\]

\[
= c_1(m) L^{2m+1} \| k \|_{3} \| k \|_{m+1}^{2}
\]

Substituting (5.15) into (5.14), we have the length of the curve does not increase.

\[
\frac{d}{dt} L = - \int_{\gamma} k^2 k_{m+1} ds + \sum_{j=1}^{m} (-1)^j \int_{\gamma} k^2 k_{m+j} k_{m-j} ds + \frac{1}{2} \int_{\gamma} k^2 k_{m+1}^2 ds
\]

\[
\leq - \left[ 1 - c_1(m) L^{2m+1} \int_{\gamma} k^2 ds \right] \int_{\gamma} k_{m+1}^2 ds
\]

\[
\leq 0,
\]

(5.16)

where \( c_1(m) L^{2m+1} \| k \|_{3} \| k \|_{m+1}^{2} < 1 \) which is the small energy condition.

Since \( \int_{\gamma} k_{m}^2 ds \) does not increase under the flow, then if the term \( L^{2m+1} \int_{\gamma} k_{m}^2 ds \) is sufficiently small for the initial curve, then \( L \) is nonincreasing under the flow (5.1). \( \square \)
CHAPTER 5. HIGHER ORDER FLOWS OF CURVES

Here we show that $L^2$ norm of all curvature derivatives are bounded.

**Proposition 12.** Suppose $\gamma_0$ satisfies the conditions of Theorem 8. Then, under the flow (5.1), we have

$$\int_\gamma k^2 ds \leq c(l,m),$$

for all $l \in \mathbb{N} \cup \{0\}$.

**Proof.** As when $l \leq m$, $\int_\gamma k^2 ds$ is obviously bounded via PSW inequality (2.1) in Proposition 1 and Lemma 31. As $\int_\gamma k ds = 0$, we have $\|k'_d\|^2 \leq \frac{t^2}{\pi^2} \|k_{d+1}'\|^2$.

For any $l$, we have via Lemma 3,

$$\begin{align*}
\frac{d}{dt} \int_\gamma k^2 ds &= 2 \int_\gamma k\dot{k} ds - \int_\gamma k^2 \cdot k F_1 ds \\
&= -2 \int_\gamma \gamma \cdot \left[ (-1)^{l+1} k_{2l+2} + \sum_{d=1}^{l} (-1)^{d+1} k_{d+1} k_{d-1} - \frac{1}{2} k^2 \right] ds \\
&= -2 \int_\gamma \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{j+1} k_{j-1} - \frac{1}{2} k^2 \right] \cdot \\
&\quad \left[ (-1)^{l+1} k_{2l+2} + \sum_{d=1}^{l} (-1)^{d+1} k_{d+1} k_{d-1} - \frac{1}{2} k^2 \right] ds.
\end{align*}$$

Simplify above equation, we have

$$\begin{align*}
\frac{d}{dt} \int_\gamma k^2 ds &= (-1)^{m+l+2} \int_\gamma k_{2m+2} k_{2l+2} ds \\
&\quad + 2 \sum_{j=1}^{m} (-1)^{j+1} \int_\gamma k_{j+1} k_{j-1} k_{2m+2} k_{2l+2} ds \\
&\quad + 2 \sum_{j=1}^{l} (-1)^{d+j+1} \int_\gamma k_{d+1} k_{d-1} k_{2m+2} k_{2l+2} ds \\
&\quad + \sum_{d=1}^{l} (-1)^{d+1} \int_\gamma k_{d+1} k_{d-1} k_{2m+2} k_{2l+2} ds \\
&\quad + \sum_{j=1}^{m} (-1)^{j+1} k_{j+1} k_{j-1} - \frac{1}{2} \int_\gamma k_{2m+2} k_{2l+2} ds.
\end{align*}$$

(5.17)
For each $l > m$, we examine each of the nine terms on the right-hand side of (5.17) in turn.

$$(-1)^{m+l+1} 2 \int_{\gamma} k_{2m+2} k_{2l+2} ds = (-1)^{2l+1} 2 \int_{\gamma} k_{2m+l+2} ds,$$

here as $2m + 2$ and $2l + 2$ are both even, so these boundary terms all have odd derivatives of curvature, i.e. they are all zero.

$$2 \sum_{j=1}^{m} (-1)^{j+l+2} \int_{\gamma} k_{k_{m+j} k_{m-j}} k_{2l+2} ds = \sum_{j=1}^{m} (-1)^{2l+j-m+3} \int_{\gamma} (kk_{m+j} k_{m-j})_{d-m+1} k_{d+m+1} ds,$$

here $2l + 2$ is even, the order in $kk_{m+j} k_{m-j}$ is $2m$ which is even as well, so we can make sure that the boundary terms all have odd derivatives of the curvature, they are all zero.

$$2(-1)^{l+1} \int_{\gamma} k_{k_{m+l} k_{2l+2}} ds = 2(-1)^{2l-m+2} \int_{\gamma} (kk_{m+l})_{d-m+1} k_{d+m+1} ds,$$

here $2l + 2$ is even, the order in $kk_{m+l}$ is $2m$ which is even as well, so we can make sure that the boundary terms all have odd derivatives of the curvature, they are all zero.

$$2 \sum_{d=1}^{l} (-1)^{d+m+1} \int_{\gamma} k_{k_{d+l} k_{d-l} k_{2m+2} ds}$$

$$= 2 \sum_{d=1}^{m} (-1)^{d+m+1} \int_{\gamma} k_{k_{d+l} k_{d-l} k_{2m+2} ds} + 2 \sum_{d=m+1}^{l} (-1)^{d+m+1} \int_{\gamma} k_{k_{d+l} k_{d-l} k_{2m+2} ds}$$

$$= 2 \sum_{d=1}^{m} (-1)^{2m+2} \int_{\gamma} k_{k_{d+l} k_{d-l} (k k_{d-l})_{2m+2} ds}$$

$$+ 2 \sum_{d=m+1}^{l} (-1)^{2d} \int_{\gamma} k_{k_{d+l} k_{d-l} (k k_{d-l})_{2m+2} ds}$$

$$= 2 \sum_{d=1}^{m} \int_{\gamma} k_{k_{d+l} k_{d-l} (k k_{d-l})_{2m+2} ds} + 2 \sum_{d=m+1}^{l} \int_{\gamma} k_{k_{d+l} k_{d-l} (k k_{d-l})_{2m+2} ds},$$

here in the second line in above, for the first term, $2m + 2$ is even, the order in $kk_{d+l} k_{d-l}$ is $2l$ which is even as well, so the boundary terms all have odd derivatives of the curvature, they are all zero;

For the second term, when $l + d$ and $l - d$ are even, then the reason is the same, when $l + d$ and $l - d$ are odd, as the total order in $\int_{\gamma} k_{k_{d+l} k_{d-l} k_{2m+2} ds}$ is $2m + 2l + 2$, the order in boundary terms are always 1 less than $2m + 2l + 2$: $2m + 2l + 1$ which is odd, then at least one term in boundary terms has odd order, i.e. all the boundary terms are zero.
here as the total order in \( \int_{\gamma} k^2 k_{j+\sigma} k_{j+\sigma} k_{j+m+1} k_{j+m-1} ds \) is \( 2m + 2l + 2 \), the order in boundary terms are always 1 less than \( 2m + 2l + 2 \): \( 2m + 2l + 1 \) which is odd, then at least one factor in each boundary term has odd order, i.e. all the boundary terms are zero.

For the same reason as above, the boundary terms in the following four equalities are all zero as well.

\[
\sum_{d=1}^{l} (-1)^{d+1} \int_{\gamma} k^2 k_{j+\sigma} k_{j+\sigma} k_{j+m} ds
\]

\[
= \sum_{d=1}^{m} (-1)^{d+1} \int_{\gamma} k^2 k_{j+\sigma} k_{j+\sigma} k_{j+m} ds + \sum_{d=m+1}^{l} (-1)^{d+1} \int_{\gamma} k^2 k_{j+\sigma} k_{j+\sigma} k_{j+m} ds
\]

\[
= \sum_{d=1}^{m} (-1)^{m+2} \int_{\gamma} k_{j+\sigma} (k^2 k_{j+\sigma} k_{j+m})_{j+m-1} ds
\]

\[
+ \sum_{d=m+1}^{l} (-1)^{d-m} \int_{\gamma} k_{j+\sigma} (k^2 k_{j+\sigma} k_{j+m})_{j+m-1} ds,
\]

\[
(-1)^{m+1} \int_{\gamma} kk_{j+\sigma} k^2 ds = (-1)^{2m+2} \int_{\gamma} k_{j+m+1} (kk^2)_{j+m+1} ds,
\]

and

\[
\sum_{j=1}^{m} (-1)^{j+1} k^2 k_{j+m} k_{j-m} k_{j+m} = \sum_{j=1}^{m} (-1)^{j+m+2} k_{j-1} (k^2 k_{j+m} k_{j+m})_{j+m+1},
\]

\[
\frac{1}{2} \int_{\gamma} k^2 k_{j+m} k_{j} ds = -\frac{1}{2} (-1)^{m+1} \int_{\gamma} k_{j-m} (k^2 k_{j+m} k_{j+m})_{j+m+1} ds.
\]
Substituting above nine equations into (5.17), we have for each \( l > m \),

\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 ds = (-1)^{2l+2} \int_{\gamma} k_{\gamma}^2 ds + \sum_{j=1}^{m} (-1)^{2l-j+m+3} \int_{\gamma} (kk_{\gamma}^2 k_{\gamma}^{m-j})_{j-m+1} k_{\gamma}^{j+1} ds + 2 \sum_{d=1}^{m} \int_{\gamma} k_{\gamma}^{m+d+1} (kk_{\gamma}^2 k_{\gamma}^{d+2})_{d-m+1} ds
\]

\[
+ 2 \sum_{j=1}^{m} \sum_{d=1}^{m} (-1)^{j+m+2} \int_{\gamma} (k^2 k_{\gamma}^2 k_{\gamma}^{d+2})_{j-m+1} k_{\gamma}^{j+1} ds
\]

\[
+ 2 \sum_{j=1}^{m} \sum_{d=1}^{l} (-1)^{2d+j-m} \int_{\gamma} k_{\gamma}^{d+1} (k^2 k_{\gamma}^2 k_{\gamma}^{d+2})_{d-m+1} ds
\]

\[
+ \sum_{d=1}^{m} (-1)^{m+2} \int_{\gamma} k_{\gamma}^{m+1} (k^2 k_{\gamma}^2 k_{\gamma}^{m+2})_{m+1} ds + \sum_{j=1}^{m} (-1)^{j+m+2} k_{\gamma}^{j+1} (k^2 k_{\gamma}^{m-j})_{j+2}^{m+1} ds
\]

\[
- \frac{1}{2} (-1)^{m+1} \int_{\gamma} k_{\gamma}^{d+1} (k^2 k_{\gamma}^2 k_{\gamma}^{d+2})_{d+1} ds.
\]

Simplify above equation, we have

\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 ds = (-1)^{2l+2} \int_{\gamma} k_{\gamma}^2 ds + \int_{\gamma} p_{4}^{2m+2l+2} (k) ds + \int_{\gamma} p_{6}^{2m+2l} (k) ds
\]

\[
\leq (-1)^{2l+2} \int_{\gamma} k_{\gamma}^2 ds + \int_{\gamma} p_{4}^{2m+2l+2} (k) ds + \int_{\gamma} p_{6}^{2m+2l} (k) ds
\]

\[
= -2 \int_{\gamma} k_{\gamma}^2 ds + \int_{\gamma} p_{4}^{2m+2l+2} (k) ds + \int_{\gamma} p_{6}^{2m+2l} (k) ds, \quad (5.18)
\]

where no higher than \((m+l+1)\)-th order derivative appears in the \( \int_{\gamma} p_{4}^{2m+2l+2} (k) ds \) term and no higher than \((m+l)\)-th derivative appears in \( \int_{\gamma} p_{6}^{2m+2l} ds \). Note that \( l + m + 1 \geq 2m + 2 \) because of \( l > m \). Then we estimate then using interpolation inequality in Proposition 4.

For the second term \( \int_{\gamma} p_{4}^{2m+2l+2} (k) ds \) in (5.18), we have
\[
\int_{\gamma} |p_{4}^{2m+2l+2}(k)| \, ds \leq \epsilon \int_{\gamma} k_{g^{m+l+2}}^{2} \, ds + c e^{-(2m+2l+3)} \cdot \left( \int_{\gamma} k^{2} \, ds \right)^{2m+2l+5} + c \left( \int_{\gamma} k^{2} \, ds \right)^{2m+2l+5}.
\]

(5.19)

For the third term \( \int_{\gamma} |p_{6}^{2m+2l}(k)| \, ds \) in (5.18), we obtain

\[
\int_{\gamma} |p_{6}^{2m+2l}(k)| \, ds \leq \epsilon \int_{\gamma} k_{g^{m+l+2}}^{2} \, ds + c e^{-(m+l+1)} \cdot \left( \int_{\gamma} k^{2} \, ds \right)^{2m+2l+5} + c \left( \int_{\gamma} k^{2} \, ds \right)^{2m+2l+5}.
\]

(5.20)

Substituting (5.19) and (5.20) into (5.18), therefore we obtain

\[
\frac{d}{dt} \int_{\gamma} k_{y}^{2} \, ds
= -2 \int_{\gamma} k_{g^{m+l+2}}^{2} \, ds + \int_{\gamma} |p_{4}^{2m+2l+2}(k)| \, ds + \int_{\gamma} |p_{6}^{2m+2l}(k)| \, ds
\leq -(2 - 2\epsilon) \cdot \left( \frac{\pi^{2}}{L^{2}} \right)^{m+2} \int_{\gamma} k_{y}^{2} \, ds + c \left( \|k\|_{2}^{2} \right)
= -c \int_{\gamma} k_{y}^{2} \, ds + c \left( \|k\|_{2}^{2} \right),
\]

(5.21)

from which the result follows.

\[\Box\]

5.1.1 Exponential convergence

We give that all derivatives of the curvature decay exponentially for all time.

Remark 2. In view of above Proposition 12, in fact the solution to (5.1) exist for all time, \( T = \infty \).

Proof. From (5.18) in Proposition 12, we have

\[
\frac{d}{dt} \int_{\gamma} k_{y}^{2} \, ds
= -2 \int_{\gamma} k_{g^{m+l+2}}^{2} \, ds + \int_{\gamma} |p_{4}^{2m+2l+2}(k)| \, ds + \int_{\gamma} |p_{6}^{2m+2l}(k)| \, ds
\leq -(2 - 2\epsilon) \cdot \left( \frac{\pi^{2}}{L^{2}} \right)^{m+2} \int_{\gamma} k_{y}^{2} \, ds + c \left( \|k\|_{2}^{2} \right)
= -c \int_{\gamma} k_{y}^{2} \, ds + c \left( \|k\|_{2}^{2} \right).
\]
Let \( l = 0 \), then we obtain

\[
\frac{d}{dt} \int_{\gamma} k^2 \, ds \leq c \left( \int_{\gamma} k^2 \, ds \right)^{2m+5},
\]

from which it follows that

\[
\int_{\gamma} k^2 \, ds \geq c (T - t)^{-\frac{1}{2m+4}}.
\]

We show that as in [17, Theorem 3.1] that if the maximal existence time \( T \) of a solution (5.1) is finite, then the curvature must blow up in \( L^2 \). We get a contradiction with the exponential decay of \( \int_{\gamma} k^2 \, ds \) in Proposition 13. Thus, we prove that the solution to (5.1) exists for all time, that is, \( T = \infty \).

**Proposition 13.** Suppose \( \gamma_0 \) satisfies the conditions of Theorem 8. Then, under the flow (5.1), \( \int_{\gamma} k^2 \, ds \) decays exponentially for all time.

**Proof.** We use Lemma 3, the boundary condition and (5.10)

\[
\frac{d}{dt} \int_{\gamma} k^2 \, ds = 2 \int_{\gamma} k k_s \, ds - \int_{\gamma} k^2 \cdot k F_1 \, ds
\]

\[
= 2 \int_{\gamma} k \left[ (F_1)_{ss} + k^2 F_1 \right] \, ds - \int_{\gamma} k^3 F_1 \, ds
\]

\[
= 2 \int_{\gamma} k_s k \, ds + \int_{\gamma} k^3 F_1 \, ds
\]

\[
= 2 \int_{\gamma} k_s \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} - \frac{1}{2} k^2 \right] \, ds
\]

\[
+ \int_{\gamma} k^3 \left[ (-1)^{m+1} k_{2m+2} + \sum_{j=1}^{m} (-1)^{j+1} k_{2m+1} k_{2m-j} - \frac{1}{2} k^2 \right] \, ds
\]

\[
= 2(-1)^{2m+1} \int_{\gamma} k_{2m+2}^2 \, ds + 2 \sum_{j=1}^{m} (-1)^{j+1+1} k_{2m+1} (k_{2m+1} k_{2m-j})_{s,j-1}
\]

\[
+ \int_{\gamma} (k^2)_{s} k_s \, ds + (-1)^{2m+2} \int_{\gamma} (k^3)_{s} k_{s+m+1} \, ds
\]

\[
+ \sum_{j=1}^{m} (-1)^{j+1+1} k_{2m+1} \left( k^4 k_{2m-j} \right)_{j-1} + \frac{1}{2} \int_{\gamma} (k^2 k^4)_{s} k_{s+m-1} \, ds
\]

\[
= -2 \int_{\gamma} k_{2m+2}^2 \, ds + 2 \sum_{j=1}^{m} k_{2m+1} (k_{2m+1} k_{2m-j})_{s,j-1} + \int_{\gamma} (k^2 k^4)_{s} k_s \, ds
\]

\[
+ \int_{\gamma} (k^3)_{s} k_{s+m+1} \, ds + \sum_{j=1}^{m} k_{s+m+1} (k^4 k_{2m-j})_{j-1}
\]

\[
+ \frac{1}{2} \int_{\gamma} (k^2 k^4)_{s} k_{s+m-1} \, ds. \quad (5.22)
\]

For the first term in (5.22), we get it by \( m \) integrations by parts and using Lemma 30. Next, we will deal with each of the above terms separately. The second, third and fourth
terms in (5.22)
\[ 2 \sum_{j=1}^{m} k_{m+1} (kk_{m}k_{m-1}) s_{j-1} + \int_{\gamma} (k_{m}^2 k) s \cdot k_s ds + \int_{\gamma} (k^3) s_{m+1} k_{m+1} ds \]
\[ = \int_{\gamma} P_{4}^{2m+2} (k) ds \leq \int_{\gamma} |P_{4}^{2m+2} (k)| ds. \]

Estimating it by the interpolation inequality in Proposition 4,
\[ \int_{\gamma} |P_{4}^{2m+2} (k)| ds \]
\[ \leq cL^{1-2m-2-4} ||k|| \frac{2m+5}{m+2} \cdot ||k|| \frac{2m+3}{m+2},2 \]
\[ \leq cL^{1-2m-2-4} \left( ||k|| \frac{2m+5}{m+2} \cdot ||k|| \frac{2m+3}{m+2},2 + ||k|| \frac{4}{2} \right) \]
\[ \leq c||k|| \frac{2m+5}{m+2} \cdot ||k|| \frac{2m+3}{m+2},2 + cL^{1-2m-2-4/2} ||k|| \frac{4}{2} \]
\[ = c||k|| \frac{2}{m+2} \cdot ||k|| \frac{2m+3}{m+2},2 + cL^{1-2m-2-3} ||k|| \frac{2}{2} \cdot ||k|| \frac{2}{2} \]
\[ \leq c||k|| \frac{2}{2} \cdot \left( \frac{L^2}{\pi^2} \right)^m \cdot ||k|| \frac{2m+3}{m+2},2 + cL^{1-2m-2-3} ||k|| \frac{2}{2} \cdot \left( \frac{L^2}{\pi^2} \right)^m ||k|| \frac{2}{2} \]
\[ = cL ||k|| \frac{2}{2} \cdot ||k|| \frac{2m+3}{m+2},2 \leq c \left( \frac{L^2}{\pi^2} \right)^m ||k|| \frac{2}{2} \cdot ||k|| \frac{2}{2} \]
\[ = c_2 (m) L^{2m+1} ||k|| \frac{2}{2} \cdot ||k|| \frac{2}{2}. \] (5.23)

For the fifth and sixth terms in (5.22), we have
\[ \sum_{j=1}^{m} k_{m+1} (k^4 k_{m-1}) s_{j-1} + \frac{1}{2} \int_{\gamma} (k_{m}^4 k) s \cdot k_s ds = \int_{\gamma} P_{6}^{2m} (k) ds \leq \int_{\gamma} |P_{6}^{2m} (k)| ds, \]
using the interpolation inequalities, we also have
\[ \int_{\gamma} |P_{6}^{2m} (k)| ds \leq cL^{1-2m-6} ||k|| \frac{4m+10}{m+2} \cdot ||k|| \frac{2m+2}{m+2},2 \]
\[ \leq cL^{1-2m-6} \left( ||k|| \frac{4m+10}{m+2} \cdot ||k|| \frac{2m+2}{m+2},2 + ||k|| \frac{6}{2} \right) \]
\[ \leq c||k|| \frac{4m+10}{m+2} \cdot ||k|| \frac{2m+2}{m+2},2 + cL^{1-2m-6} ||k|| \frac{6}{2} \]
\[ = c||k|| \frac{4}{2} \cdot ||k|| \frac{2m+2}{m+2},2 + cL^{1-2m-6} ||k|| \frac{4}{2} \cdot ||k|| \frac{2}{2} \]
\[ \leq c||k|| \frac{4}{2} \cdot \left( \frac{L^2}{\pi^2} \right)^m \cdot ||k|| \frac{2m+2}{m+2},2 + cL^{1-2m-6} ||k|| \frac{4}{2} \cdot \left( \frac{L^2}{\pi^2} \right)^m ||k|| \frac{2}{2} \]
\[ = cL^2 ||k|| \frac{4}{2} \cdot ||k|| \frac{2m+2}{m+2},2 \]
\[ \leq cL^2 \cdot \left( \frac{L^2}{\pi^2} \right)^m ||k|| \frac{4}{2} \cdot ||k|| \frac{2m+2}{m+2},2 \]
\[ = c_3 (m) L^{4m+2} ||k|| \frac{4}{2} \cdot ||k|| \frac{2m+2}{m+2},2. \] (5.24)
Substituting (5.23) and (5.24) into (5.22), we obtain

\[
\frac{d}{dt} \int_\gamma k^2 ds = -2 \int_\gamma k_{\gamma}^2 ds + 2 \sum_{j=1}^{m} k_{i+1} (k k_{x} k_{m-j})_{x_{i+1}} + \int_\gamma (k_{m} k_{x})_{x} k_{s} ds \\
+ \int_\gamma (k^3)_{x_{i+1}} + \sum_{j=1}^{m} k_{i+1} (k^4 k_{m-j})_{x_{i+1}} + \frac{1}{2} \int_\gamma (k_{m} k^4 x_{m-1}) ds \\
\leq -2 - c_2 (m) L^{2m+1} ||k_m||_2^2 - c_3 (m) L^{4m+2} ||k_m||_4^4 \cdot \int_\gamma k_{m+2}^2 ds. \quad (5.25)
\]

From Lemma 31, we know that \( L \) does not increase when the initial curve has sufficiently small energy. Also, the energy does not increase, so the coefficient on the right hand side in (5.25) is smaller than \(-\delta\) if it holds initially, for some \( \delta > 0 \). Then we have

\[
\frac{d}{dt} \int_\gamma k^2 ds \leq -\delta \int_\gamma k_{m+2}^2 ds \\
\leq -\tilde{\delta} (L(0), m) \int_\gamma k^2 ds
\]

where

\[
c_2 (m) L^{2m+1} ||k_m||_2^2 + c_3 (m) L^{4m+2} ||k_m||_4^4 < 2,
\]

then we get

\[
\int_\gamma k^2 ds \leq ce^{-\delta t},
\]

hence the result.

We have that \( \int_\gamma k^2 ds \) decays exponentially. By induction argument, all derivatives of the curvature decay exponentially.

**Proposition 14.** Suppose \( \gamma_0 \) satisfies the conditions of Theorem 8. Then, under the flow (5.1), \( \int_\gamma k^2 ds \) and \( \|k_s\|_\infty \) exponentially decay for all time and any \( l \in \mathbb{N} \cup \{0\} \).

**Proof.** We calculate the curvature derivative decay in \( L^2 \) by using integration by parts,

\[
\int_\gamma k^2 ds = - \int_\gamma k k_{x} ds \leq \|k\|_2 \|k_s\|_2 \leq ce^{-\delta_1 t},
\]

where we have eliminated the boundary term as it contains an odd derivative of curvature \( k_s \) which is equal to zero. The \( \int_\gamma k^2 ds \) is bounded by Proposition 12, the exponential convergence follows from Proposition 16.

We next compute

\[
\int_\gamma k^2 ds = - \int_\gamma k_{s} k ds \leq \|k_s\|_2 \|k_{s}\|_2 \leq ce^{-\delta_2 t},
\]
where similarly $\int_\gamma k_s^2 ds$ is bounded by Proposition 12, so then the exponential convergence follows from the previous step.

Moreover for any $l \in \mathbb{N} \cup \{0\}$, as from Proposition 12, we know that all derivatives of the curvature are bounded. We have

$$\int_\gamma k_s^2 ds = -\int_\gamma kk_{s+1} ds \leq \|k\|_2 \|k_{s+1}\|_2 \leq ce^{-\delta t}.$$  

Above $c$ are all different constants. Now we obtain that all derivatives of the curvature have exponential convergence. Moreover, exponential convergence in $L^\infty$ of $\gamma$ to $\gamma_\infty$ follows, as $\|k_s\|^2_\infty \leq \frac{c}{\tau} \|k_{s+1}\|_2^2$ in Proposition 2, we have $\int_\gamma k_s^2 ds$ exponentially decays in $L^\infty$.  

From above proposition, we can also obtain uniform bounds for all derivatives of the evolving curve $\gamma$.

**Proposition 15.** Suppose $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$ solves (5.1) and satisfies the conditions of Theorem 8. Then for all $l \in \mathbb{N}_0$,

$$\|\partial_s \gamma\|_\infty \leq c(l) + \sum_{p=0}^l \|\partial_{s} \gamma_0\|_\infty,$$

where $c(l)$ is a constant only depending on $l$, $E(0)$, $L(0)$.

**Proof.** We claim that for $l, p \in \mathbb{N}_0$,

$$\partial_l \partial_s \gamma = \nu \sum_{p=0}^l \left( P_{1+l-p}^{2m+2+p}(k) + P_{3+l-p}^{2m+p}(k) \right) + \tau \sum_{p=0}^l \left( P_{1+l-p}^{2m+2+p}(k) + P_{3+l-p}^{2m+p}(k) \right). \quad (5.26)$$

We prove this by induction. First, $F = P_1^{2m+2}(k) + P_3^{2m}(k)$ so the equation holds for $l = 0$. For $q \in \mathbb{N}_0$, we do the differentiation

$$\partial_l \partial_{s+1} \gamma = kF \cdot \partial_{s+1} \gamma + \partial_s (\partial_s \partial_s \gamma)$$

$$= \partial_s \left[ \nu \sum_{p=0}^l \left( P_{1+l-p}^{2m+2+p}(k) + P_{3+l-p}^{2m+p}(k) \right) + \tau \sum_{p=0}^l \left( P_{1+l-p}^{2m+2+p}(k) + P_{3+l-p}^{2m+p}(k) \right) \right]$$

$$+ kF \cdot \partial_{s} \tau$$

$$= kF \nu \sum_{p+q=l} P_{q}^p(k) + kF \tau \sum_{p+q=l} P_{q}^p(k) + \nu \sum_{p=0}^l \left( P_{2+l-p}^{2m+2+p}(k) + P_{4+l-p}^{2m+p}(k) \right)$$

$$+ \tau \sum_{p=0}^l \left( P_{2+l-p}^{2m+2+p}(k) + P_{4+l-p}^{2m+p}(k) \right) + \nu \sum_{p=0}^l \left( P_{2+l-p}^{2m+3+p}(k) + P_{3+l-p}^{2m+1+p}(k) \right)$$

$$+ \tau \sum_{p=0}^l \left( P_{2+l-p}^{2m+3+p}(k) + P_{3+l-p}^{2m+1+p}(k) \right).$$
\[ \begin{align*}
= \nu \sum_{p+q=l} \left( p^{p+2m+2}_{q+2}(k) + p^{p+2m}_{q+4}(k) \right) + \nu \sum_{p=0}^{l} \left( p^{2m+2+p}_{2+l-p}(k) + p^{2m+p}_{4+l-p}(k) \right) \\
+ \nu \sum_{p=1}^{l+1} \left( p^{2m+2+p}_{2+l-p}(k) + p^{2m+p}_{4+l-p}(k) \right) + \tau \sum_{p=0}^{l} \left( p^{p+2m+2}_{q+2}(k) + p^{p+2m}_{q+4}(k) \right) \\
+ \tau \sum_{p=0}^{l} \left( p^{2m+2+p}_{2+l-p}(k) + p^{2m+p}_{4+l-p}(k) \right) + \tau \sum_{p=0}^{l+1} \left( p^{2m+2+p}_{2+l-p}(k) + p^{2m+p}_{4+l-p}(k) \right)
\end{align*} \]

as required. Integrating (5.26) and using Proposition 14, we find

\[ \| \partial_s \gamma \|_{\infty} \leq \| \partial_s \gamma_0 \|_{\infty} + c \int_0^t e^{-ct'} dt' \leq \| \partial_s \gamma_0 \|_{\infty} + \bar{c}(l). \]

As \( u \) is the initial space parameter before reparameterization by arc-length, set \( v = |\partial_u \gamma| \).

Referring to the proof of Theorem 3.1 in [17], then for any function \( \Phi : \mathbb{S} \rightarrow \mathbb{R} \), we have

\[ \partial_u^l \Phi = v \partial_s^l \Phi + P^l(v, ... , \partial_u^{l-1} \gamma, \Phi, ... \partial_s^{l-1} \Phi) \]

where \( P^l \) is a polynomial. Then we obtain

\[ \| \partial_u^l \gamma \|_{\infty} \leq \| \partial_s^l \gamma_0 \|_{\infty} + c(l). \]

as required. \( \square \)

To finish the proof of Theorem 8, we need two lemmas in Section 5.3. Lemma 36 gives that there is a subsequence \( t_i \rightarrow \infty \) such that \( \gamma(\cdot, t_i) \rightarrow \gamma_\infty \) in \( C^\infty([1, 1], \mathbb{R}^2) \). Lemma 38 proves that the exponential convergence of \( \gamma \) to a unique horizontal straight segment. This completes the proof of Theorem 8.

### 5.2 The polyharmonic curve diffusion flow

This section considers the dual spaces to the Sobolev spaces \( H^{m+1} \), which are denoted by \( H^{-(m+1)} \) and consist of the bounded linear functionals \( L : H^{m+1} \rightarrow \mathbb{R} \). We establish our result for the polyharmonic curve diffusion flow (5.2) for each fixed \( m \in \mathbb{N} \cup \{0\} \) which is the gradient flow for length in the Sobolev spaces \( H^{-(m+1)} \).

Since curvature depends on second derivatives of \( \gamma \) with respect to \( s \), the flow (5.2) has order \( 2m+4 \). For a solution to the flow (5.2), we have \( F_2 = (-1)^{m+1} k_s^{2m+2} \) is the \( H^{-(m+1)} \) gradient flow of the length \( \int_\gamma |\gamma_0| du \), then using Lemma 3 (i) and \( m+1 \) integrations by
parts, we obtain

$$\frac{d}{dt} L(\gamma) = - \int_{\gamma} k F_2 ds$$

$$= (-1)^{m+2} \int_{\gamma} k k_{2m+2} ds$$

$$= (-1)^{m+1} \int_{\gamma} k k_{2m+1} ds + (-1)^{m} k_{2m+1} |_{\partial \gamma}$$

$$= (-1)^{m} k_{m} k_{2m} ds + (-1)^{m-1} k_{k,2m} |_{\partial \gamma} + (-1)^{m} k_{2m+1} |_{\partial \gamma}$$

$$= .....$$

$$= (-1)^{2} \int_{\gamma} k_{m} k_{m+2} ds + (-1)^{1} k_{m-1} k_{m+2} |_{\partial \gamma} + ... + (-1)^{m-1} k_{2m+1} |_{\partial \gamma}$$

$$+ (-1)^{m} k_{2m+1} |_{\partial \gamma}$$

$$= - \int_{\gamma} k_{m+1} ds + (-1)^{0} k_{m+1} |_{\partial \gamma} + (-1)^{1} k_{m-1} k_{m+2} |_{\partial \gamma} + ...$$

$$+ (-1)^{m-1} k_{2m+1} |_{\partial \gamma} + (-1)^{m} k_{2m+1} |_{\partial \gamma}$$

$$= - \int_{\gamma} k_{m+1} ds + \sum_{j=0}^{m} (-1)^{m-j} k_{j} k_{2m+1-j} |_{\partial \gamma}.$$  \hspace{1cm} (5.27)

We can see from above, each of the boundary terms contains an odd derivative of $k$ with the order up to $2m + 1$.

**Lemma 32.** The boundary terms $\sum_{j=0}^{m} k_{j} k_{2m+1-j} |_{\partial \gamma} = 0$ in above holds under the following:

$$< \mathbf{v}, \mathbf{v}_{\eta_{1,2}} > (\pm 1, t) = k_{s}(\pm 1, t) = ... = k_{2m-1}(\pm 1, t) = k_{2m}(\pm 1, t) = 0$$ \hspace{1cm} (5.28)

the number of the conditions is $2m + 4$ on both boundaries which is the same as the highest order of the derivatives of the curvature.

**Lemma 33.** While a solution to the flow (5.2) with boundary condition (5.28), we have

$$\frac{d}{dt} L = - \int_{\gamma} k_{m+1} ds.$$ \hspace{1cm} (5.29)

**Proof.** The result follows from (5.27),

$$\frac{d}{dt} L = - \int_{\gamma} k_{m+1} ds + \sum_{j=0}^{m} (-1)^{m-j} k_{j} k_{2m+1-j} |_{\partial \gamma}.$$  \hspace{1cm} (5.29)

From above lemma, the boundary terms $\sum_{j=0}^{m} (-1)^{m-j} k_{j} k_{2m+1-j} |_{\partial \gamma} = 0$. Then we have the conclusion.
For special cases, when \( m = 0 \), the flow (5.2) is the classical curve diffusion flow:

\[
\frac{\partial}{\partial t} \gamma = F_2 \nu = -k_s \nu.
\]

The case \( m = 1 \) can be seen as the geometric triharmonic heat flow of curves \( \gamma \):

\[
\frac{\partial}{\partial t} \gamma = F_2 \nu = k_s^4 \nu.
\]

**Remark 3.** In fact, under the flow (5.2), from (5.29) the length \( L \) is strictly decreasing unless \( \gamma \) is a straight line segment.

Because in (5.29), it is easy to see that

\[
\frac{d}{dt} L = - \int_\gamma k_{s^m+1}^2 ds \leq 0,
\]

the length \( L \) is strictly decreasing except \( \frac{d}{dt} L = 0 \).

From \( \frac{d}{dt} L = 0 \), we get \( k_{s^m+1} \equiv 0 \), therefore the curvature of the curve is zero. And the two end points of the curve meet two parallel vertical lines \( \eta_{1,2} \) orthogonally from the Neumann boundary condition. Thus, the only smooth solutions of \( \frac{d}{dt} L = 0 \) are horizontal line segments.

In view of Lemma 33 and the distance \( |e| \) of two parallel lines \( \eta_{1,2} \), the length \( L \) of the evolving curve \( \gamma(\cdot, t) \) remains bounded above and below under the flow (5.2).

**Definition 9.** Let \( \gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2 \) be a family of smooth immersion whose ends meet the parallel lines \( \eta_{1,2} \) with \( m + 2 \) generalised Neumann boundary conditions (5.28). \( \gamma \) is said to move under \((2m + 4)\)th order curvature flow (5.2), if

\[
\begin{cases}
\frac{d}{dt} \gamma(s, t) = -F_2 \nu, & \text{for all } (s, t) \in [-1, 1] \times [0, T) \\
\gamma(\cdot, 0) = \gamma_0, \\
\langle \nu, \nu_{\eta_{1,2}} \rangle = k_s = \ldots = k_{s^{2m-1}} = k_{s^{2m+1}} = 0, & \text{for all } (s, t) \in \eta_{1,2} \times [0, T)
\end{cases}
\]

where \( F_2 = (-1)^{m+1}k_{s^{2m+2}}, \ m \in \mathbb{N} \cup \{0\} \), \( \nu \) and \( \nu_{\eta_{1,2}} \) are the unit normal fields to \( \gamma(\pm 1) \) and \( \eta_{1,2} \) respectively.

**Theorem 9.** Let \( \gamma : [-1, 1] \to \mathbb{R}^2 \) be a smooth embedded or immersed curve in Definition 9. If the initial curve \( \gamma_0 \) satisfies \( \omega = 0 \) and the curvature \( k(0) \) of \( \gamma_0 \) is sufficiently small in \( \mathcal{L}^2 \), that is

\[
L(0) \int_\gamma k^2(0) ds \leq \varepsilon \quad (5.30)
\]

where \( \varepsilon \) is positive and depends only on \( m \), then there exists a smooth solution \( \gamma : [0, \infty) \to \mathbb{R}^2 \) to (5.2) with \( \gamma(\cdot, 0) = \gamma_0 \). The solution \( \gamma \) converges to a horizontal line segment expo-
nentially and is unique up to parametrization. The distance between the limit curve and
the initial curve is finite.

The $\varepsilon$ in above theorem can be calculated from (5.34). The proof of this theorem
follows by the same argument as in the previous Section 5.1.

**Lemma 34.** The hypothesis of Theorem 9 implies that $\omega(t) = \omega(0) = 0$. The average
curvature $\bar{k}$ satisfies $\bar{k} := \frac{1}{L} \int_{\gamma} k ds \equiv 0$.

For the proof, see Lemma 17.

**Lemma 35.** All odd derivatives of the curvature equal to zero at the boundary,

$$k_{s2p+1}(\pm 1, t) = 0, \quad p = 0, 1, 2, \ldots$$

**Proof.** Here we do the first spatial derivative of the normal speed

$$F_2 = (-1)^{m+1} k_{2m+2},$$

then $(F_2)_s = (-1)^{m+1} k_{s2m+3} = 0$, yielding

$$k_{s2m+3}(\pm 1, t) = 0.$$

Calculate the third derivatives $(F_2)_s^3 = (-1)^{m+1} k_{s2m+5} = 0$, we obtain

$$k_{s2m+5}(\pm 1, t) = 0.$$

Let us give the induction argument, we assume that for $n = 2, 3, \ldots$

$$k_{s2n+2m-1}(\pm 1, t) = 0,$$

$$F_2 = (-1)^{m+1} k_{s2m+2},$$

\[
\partial_t k_{s^t} = F_{s^t+2} + \sum_{h=0}^{l} \partial_s^h \left( k k_{s^t-h} F_2 \right)
\]

\[
= (-1)^{m+1} k_{s2m+l+4} + \sum_{h=0}^{l} \partial_s^h \left[ k k_{s^t-h} (-1)^{m+1} k_{s2m+2} \right]
\]

\[
= (-1)^{m+1} k_{s2m+l+4} + (-1)^{m+1} \sum_{h=0}^{l} \sum_{q+r+u=h} c_{qru} k_{s^t-h+r} k_{s^t-h+u} k_{s2m+2+w},
\]

for constants $c_{qru} \in \mathbb{R}$ with $q, r, u, v, w \geq 0$. 
CHAPTER 5. HIGHER ORDER FLOWS OF CURVES

The inductive hypothesis implies that for \( l \) odd and less than or equal to \( 2n + 2m - 1 \), the derivatives \( k^l \) vanishes on the boundary. Here we take \( l = 2n - 3 \), then under the hypothesis \( k_{2n+2m-1}^2(\pm 1, t) = 0 \), we removed all terms with an odd number of derivatives of \( k \), we conclude

\[
k_{2n+2m+1} = 0, n = 2, 3, ...
\]

Together with boundary conditions (5.28), we get \( k_{2p+1}^3(\pm 1, t) = 0, p = 0, 1, 2, ... \)

5.2.1 Exponential decay

Next we show that \( \int_\gamma k^2 ds \) decays exponentially under the smallness assumption in Theorem 9.

**Proposition 16.** If \( \gamma_0 \) satisfies the conditions of Theorem 9, then under the flow (5.2), we have

\[
\int_\gamma k^2 ds \leq \int_\gamma k^2(0) ds \cdot e^{-\delta t}.
\]

where \( \delta > 0 \) depends on \( \varepsilon \) and \( L(0) \).

**Proof.** Under the flow (5.2), by using Lemma 33, the boundary conditions and integration, we obtain

\[
\frac{d}{dt} \int_\gamma k^2 ds = 2 \int_\gamma kk ds - \int_\gamma k^2 \cdot kF_2 ds
\]

\[
= 2 \int_\gamma k \left[ F_2 + kF_2 \right] ds - \int_\gamma k^3 F_2 ds
\]

\[
= 2 \int_\gamma k_{xx} F_2 ds + \int_\gamma k^3 F_2 ds
\]

\[
= 2 \int_\gamma k_{xx} (-1)^{m+1} k_{2m+2} ds + \int_\gamma k^3 (-1)^{m+1} k_{2m+2} ds
\]

\[
= 2(-1)^{2m+1} \int_\gamma k_{xx}^2 ds + (-1)^{2m+2} \int_\gamma (k^3)_{x^{m+1}} k_{x^{m+1}} ds
\]

\[
= -2 \int_\gamma k_{x^{m+2}} ds + \int_\gamma (k^3)_{x^{m+1}} k_{x^{m+1}} ds. \quad (5.31)
\]

The second term

\[
\int_\gamma (k^3)_{x^{m+1}} k_{x^{m+1}} ds = \int_\gamma p_{4}^{2m+2} (k) ds \leq \int_\gamma |p_{4}^{2m+2} (k)| ds.
\]

Since the highest order derivative in \( P_4^{2m+2}(k) \) is \( k_{x^{m+1}} \), we now estimate using the interpolation inequality in Proposition 4 in the first step and then PSW inequality (2.1) in
Proposition 1, as \( ||k||^2 \leq \frac{L^{2(m+3)}}{2^{2(m+3)}} ||k_{m+2}||^2 \), we have

\[
\int_\gamma |p_{2}^{2m+2}(k)| \, ds \leq cL^{1-2m-2-4} \frac{L^{2(m+3)}}{2^{2(m+3)}} ||k||^2 \frac{L^{2m+5}}{2^{2m+5}} \cdot ||k||^2 \frac{L^{2m+3}}{2^{2m+3}} \cdot ||k_{m+2}||^2 \frac{L^{2m+1}}{2^{2m+1}} \\
\leq c ||k||^2 \frac{L^{2m+3}}{2^{2m+3}} \cdot ||k_{m+2}||^2 \frac{L^{2m+1}}{2^{2m+1}} + cL^{1-2m-2-4} ||k||^4 \\
= c ||k||^2 ||k_{m+2}||^2 \frac{L^{2m+3}}{2^{2m+3}} + cL^{1-2m-2-4} ||k||^4 \\
\leq c ||k||^2 \left( \frac{L^{2}}{\pi^2} \right)^{m+2} ||k_{m+2}||^2 \frac{L^{2m+3}}{2^{2m+3}} \\
+ cL^{1-2m-2-4} ||k||^4 \\
= c(m)L ||k||^2 \frac{L^{2m+3}}{2^{2m+3}} \cdot ||k_{m+2}||^2 \frac{L^{2m+1}}{2^{2m+1}}.
\]

Combining above inequality with (5.31), we have

\[
\frac{d}{dt} \int_\gamma k^2 \, ds = -2 \int_\gamma k_{m+2}^2 \, ds + \int_\gamma (k^3)_{m+1} k_{m+1} \, ds \\
\leq - [2 - c(m)L ||k||^2] \cdot \int_\gamma k_{m+2}^2 \, ds.
\]

(5.32)

Suppose initially \( cL \int_\gamma k^2 \, ds \leq 2 - 2\delta \), for some \( \delta > 0 \). Then, at least for a short time, \( cL \int_\gamma k^2 \, ds \leq 2 - \delta \). We obtain

\[
\frac{d}{dt} \int_\gamma k^2 \, ds \leq -\delta \int_\gamma k_{m+2}^2 \, ds \leq -\delta \left( \frac{\pi^2}{L^2} \right)^{m+2} \int_\gamma k^2 \, ds \leq -\delta \left( \frac{\pi^2}{L(0)^2} \right)^{m+2} \int_\gamma k^2 \, ds
\]

where we have used again Lemma 33. The result follows

\[
\int_\gamma k^2 \, ds \leq \int_\gamma k^2(0) \, ds \cdot e^{-\delta t}.
\]

Next we show that all curvature derivatives remain bounded under the flow in \( L^2 \). This proof here is considerably more direct than the process for the flow (5.1).

**Proposition 17.** Suppose \( \gamma_0 \) satisfies the conditions of Theorem 9. Then, under the flow (5.2), we have for all \( l \in \mathbb{N} \cup \{0\} \),

\[
\int_\gamma k^2 \, ds \leq c_l,
\]

for constants \( c_l \).

**Proof.** Under the flow (5.2), by using integration by parts and all odd derivatives of the
curvature equal to zero, then for $l = 0, 1, 2, \ldots$, we have

$$
\frac{d}{dt} \int_{\gamma} k_m^2 ds
= 2 \int_{\gamma} k_o k_{m+l} ds - \int_{\gamma} k_o^2 \cdot kF_2 ds
$$

$$
= 2 \int_{\gamma} k_o \cdot (\gamma + 1)^{m+1} k_{2m+2} + 2 \int_{\gamma} k_o \cdot (\gamma + 1)^{m+1} \sum_{h=0}^{\infty} \sum_{r=0}^{\infty} c_{qru} k_{q} k_{r} k_{s} k_{m+2} ds
$$

$$
- \int_{\gamma} k_o^2 \cdot k \cdot (\gamma + 1)^{m+1} k_{2m+2} ds
$$

$$
= -2 \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds + 2 \cdot (\gamma + 1)^{m+1} \sum_{h=0}^{\infty} \sum_{r=0}^{\infty} c_{qru} k_{q} k_{r} k_{s} k_{m+1+u} ds
$$

$$
= -2 \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds + \int_{\gamma} p_{4}^{m+2l+2} (k) ds
$$

where the highest order of derivatives of $k$ in the second term $\int_{\gamma} p_{4}^{m+2l+2} (k) ds$ is $m + l + 1$. Using interpolation inequality in Proposition 4 we have for any $\varepsilon > 0$,

$$
\int_{\gamma} p_{4}^{m+2l+2} (k) ds \leq \varepsilon \int_{\gamma} k_m^2 ds + \phi (m, l, s) \left( \int_{\gamma} k_o^2 ds \right)^{2m+2l+5}.
$$

Combining above, we obtain by taking $\varepsilon < 2$

$$
\frac{d}{dt} \int_{\gamma} k_o^2 ds = -2 \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds + \int_{\gamma} p_{4}^{m+2l+2} (k) ds
\leq -2 \varepsilon \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds + \phi (m, l, s) \left( \int_{\gamma} k_o^2 ds \right)^{2m+2l+5}.
$$

As $\int_{\gamma} k_o^2 ds$ decays exponentially in Proposition 16, it is bounded by a constant. Using now PSW inequality (2.1) in Proposition 1, $- \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds \leq - \left( \frac{\pi^2}{L^2} \right)^{m+2} \int_{\gamma} k_o^2 ds$, then we obtain

$$
\frac{d}{dt} \int_{\gamma} k_o^2 ds \leq -2 \varepsilon \int_{\gamma} k_o^2 \cdot (\gamma + 1)^{m+1} k_{2m+2} ds + \phi (m, l, s) \left( \int_{\gamma} k_o^2 ds \right)^{2m+2l+5}
\leq -2 \varepsilon \left( \frac{\pi^2}{L^2(0)} \right)^{m+2} \int_{\gamma} k_o^2 ds + c_1
\leq -c_2 \int_{\gamma} k_o^2 ds + c_1,
$$
the result follows.

Remark 4. In view of above Proposition 17, the solution to (5.2) exist for all time, \( T = \infty \).

The proof of above remark follows the proof in Remark 2.

Proposition 16 and 17 imply via interpolation that all curvature derivatives decay exponentially in \( L^2 \) and in \( L^\infty \) via PSW inequalities (2.1) in Proposition 1 and (2.2) in Proposition 2. We have that \( \int \gamma k^2 ds \) decays exponentially. By induction, all derivatives of the curvature decay exponentially.

Proposition 18. Suppose \( \gamma_0 \) satisfies the conditions of Theorem 9. Then, while a solution to the flow (5.2) there exists constants \( c > 0 \) and \( \delta_l > 0 \), depending only on \( \varepsilon \) and \( L(0) \) such that, for all \( l \in \mathbb{N} \cup \{0\} \),

\[
\int \gamma k^2 ds \leq ce^{-\delta_l t}.
\]

The quantities \( \|k^l\|_\infty \) also decay exponentially for all \( l \).

Proof. The proof here is similar to Proposition 14. From Proposition 17, we know that all derivatives of the curvature are bounded. Thus, the curvature derivative decay in \( L^2 \) follows by standard integration by parts, we show the first two calculations:

\[
\int \gamma k^2 ds = - \int \gamma kk^s ds \leq \|k\|_2 \|k^s\|_2 \leq ce^{-\delta_l t}.
\]

From Proposition 17, we have \( \int \gamma k^2 ds \) is bounded, then Proposition 16 gives the exponential convergence of \( \int \gamma k^2 ds \).

We next compute

\[
\int \gamma k^2 ds = - \int \gamma kk^s ds \leq \|k\|_2 \|k^3\|_2 \leq ce^{-\delta_l t}.
\]

Generally,

\[
\int \gamma k^2 ds = - \int \gamma kk^s ds \leq \|k\|_2 \|k^s\|_2 \leq ce^{-\delta_l t}.
\]

Above \( c \) are all different constants. Continue doing this, we obtain the exponential decay of all curvature derivatives in \( L^2 \). Exponential convergence in \( L^\infty \) of \( \gamma \) to \( \gamma_\infty \) now follows, as \( \|k^l\|_\infty \leq \frac{L}{\pi} \|k^l+1\|_2^2 \) in Proposition 2, we have \( \int \gamma k^2 ds \) exponentially decays in \( L^\infty \).

From above proposition, we can also obtain uniform bounds for all derivatives of the evolving curve \( \gamma \).

Proposition 19. Suppose \( \gamma_0 : S^1 \to \mathbb{R}^2 \) solves (5.1) and satisfies the conditions of Theorem 8. Then for all \( l \in \mathbb{N}_0 \),

\[
\|\partial_{\new{u}^l} \gamma\|_\infty \leq c(l) + \sum_{p=0}^{l} \|\partial_{\new{u}^p} \gamma_0\|_\infty,
\]

where \( c(l) \) is a constant depending only on \( l \) and \( \varepsilon \).
where $c(l)$ is a constant only depending on $l$, $E(0)$, $L(0)$.

The proof of above proposition can refer to the proof of Proposition 15.

In Section 5.3, Lemma 37 implies there exists an immersion $\gamma_\infty : [-1, 1] \to \mathbb{R}^2$ satisfying the boundary conditions and a subsequence $t_i \to \infty$ such that $\gamma(\cdot, t_i) \to \gamma_\infty$ in $C^\infty([-1, 1], \mathbb{R}^2)$. Since $\|k\|_\infty \to 0$, Lemma 38 implies the curve $\gamma_\infty$ is a unique straight line segment. This completes the proof of Theorem 9.

### 5.3 The unique limit

**Lemma 36.** Assume $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$ satisfies flow (5.1) and conditions:

$$
c_1(m) L^{2m+1} \|k_{m\omega}\|_2^2 < 1 \quad \text{and} \quad c_2(m) L^{2m+1} \|k_{m\omega}\|_2^2 + c_3(m) L^{4m+2} \|k_{m\omega}\|_4^2 < 2. \quad (5.33)
$$

So there exists a subsequence $t_i \subset [0, \infty)$ such that $\gamma(\cdot, t_i) \to \gamma_\infty$ with $\gamma_\infty$ a unique straight line.

**Lemma 37.** Assume $\gamma : [-1, 1] \times [0, T) \to \mathbb{R}^2$ satisfies flow (5.2) and

$$
c(m) L \|k\|_2 < 2. \quad (5.34)
$$

So there exists a subsequence $t_i \subset [0, \infty)$ such that $\gamma(\cdot, t_i) \to \gamma_\infty$ with $\gamma_\infty$ a unique straight line.

For curves satisfying both curvature flows (5.1) and (5.2), we have the same proof:

**Proof.** Under different conditions, as we have proven that the curvature decays to zero in $L^2$ along the flow, $\|k\|_2^2 \leq e^{-cT} \|k\|_2^2 \big|_{t=0}$, So there exists a subsequence $t_i \subset [0, \infty)$ such that

$$
\int_\gamma k^2 ds \to 0, \text{as } t_i \to \infty.
$$

This implies there exist a limit curve $\gamma_\infty$ that

$$
\gamma(\cdot, t_i) \to \gamma_\infty,
$$

with $\gamma_\infty$ satisfying the boundary conditions. Then the flow converges to which satisfies curvature $k[\gamma_\infty] = 0$, that is, $\gamma_\infty$ is a straight line.

The only issue for Theorem 9 and Theorem 8 is that this does not uniquely determine the limit. The following lemma completes the proof under the decay in the curvature.
Lemma 38. Suppose the initial curves $\gamma_0$ satisfies the conditions of Theorem 9 and Theorem 8 respectively. Then, under the flows (5.2), (5.1), the limits of the curves are both unique.

Proof. In order to prove that $\gamma_\infty$ is the unique limit, we refer to [1, Theorem A.1], Suppose there exists a sequence $\{s_j\} \subset [0, \infty), s_j \to \infty$, such that $\gamma(\cdot, s_j) \to \gamma_\infty \neq \gamma_\infty$, in $C^\infty$.

Consider the functional
\[ G[\gamma] = \int_\gamma |\gamma - \gamma_\infty|^2 ds. \]

Since $\gamma_\infty$ and $\gamma_\infty$ are smooth, it follows that
\[ \lim_{s_j \to \infty} G[f(\cdot, s_j)] \neq 0. \] (5.35)

We estimate by using Lemma 1,
\[
\left| \frac{d}{dt} G \right| = \frac{d}{dt} \int_\gamma |\gamma - \gamma_\infty|^2 ds
\leq \left( \int_\gamma F^2 ds \right)^{\frac{1}{2}} \cdot \left[ \int_\gamma |\gamma - \gamma_\infty| (2 + k |\gamma - \gamma_\infty|) ds \right]^{\frac{1}{2}}
\leq \left( \int_\gamma F^2 ds \right)^{\frac{1}{2}} \cdot \left[ \int_\gamma |\gamma - \gamma_\infty|^2 (4 + k^2 |\gamma - \gamma_\infty|^2) ds \right]^{\frac{1}{2}}
\leq c \left( \int_\gamma F^2 ds \right)^{\frac{1}{2}} \cdot \left[ \int_\gamma |\gamma - \gamma_\infty|^2 + k^2 |\gamma - \gamma_\infty|^4 ds \right]^{\frac{1}{2}}
\leq c \left( \int_\gamma F^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_\gamma |\gamma|^2 ds + \int_\gamma |\gamma_\infty|^2 ds + \int_\gamma |k|^2 \cdot |\gamma|^4 ds + \int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds \right)^{\frac{1}{2}}.
\]

Now we show that $\int_\gamma |\gamma|^2 ds$, $\int_\gamma |\gamma_\infty|^2 ds$, $\int_\gamma |k|^2 \cdot |\gamma|^4 ds$ and $\int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds$ are uniformly bounded.

First, we can see that under the exponential decay of curvature and its derivatives in $C^\infty$, we obtain that
\[ \|\gamma(\cdot, t)\|^2 = \left( \int_0^t |F| dt \right)^2 \leq \tilde{c} \left( \int_0^t e^{-\delta t} dt \right)^2 \leq \frac{\tilde{c}}{\delta} e^{-\delta t} \leq c_1, \]
then
\[
\int_\gamma |\gamma|^2 ds \leq \|\gamma\|_2 \int_\gamma ds \leq c_1 L(0) \leq c_2,
\]
\[
\int_\gamma |k|^2 \cdot |\gamma|^4 ds \leq \|\gamma\|_4 \int_\gamma |k|^2 ds \leq c_3.
\]

From above two estimates, it is clear that \(\int_\gamma |\gamma_\infty|^2 ds\) and \(\int_\gamma |k|^2 \cdot |\gamma_\infty|^4 ds\) are bounded as well.

Thus,
\[
\left| \frac{d}{dt} G \right| \leq c \|F\|_2,
\]

By the exponential decay of curvature and its derivatives in \(L^2\), we have that
\[
G[f(\cdot, s_j)] \leq c \int_{s_j}^{\infty} \|F\|_2 dt \leq c \int_{s_j}^{\infty} e^{-\delta t} dt = ce^{-\delta s_j}
\]
it follows that
\[
\lim_{s_j \to \infty} G[f(\cdot, s_j)] = 0,
\]
which is in contradiction with (5.35).

This proves that there does not exist a sequence \(\{s_j\}\), the convergence of flow in \(L^2\) to a straight line segment is unique. We can obtain the exponential convergence in \(C^\infty\) to a unique line segment as the curvature and all its derivatives exponentially decay.

This finishes the proof.

\[\Box\]

**Remark 5.** Although we cannot find the precise height of the limiting straight horizontal line segment for both curves, that the flow speed decays exponentially shows that the solution curve remains in a bounded region of the initial curve: for any \(x\),
\[
|\gamma(x, t) - \gamma(x, 0)| \leq \int_0^t \left| \frac{\partial \gamma}{\partial t}(x, t) \right| dt \leq c \int_0^t e^{-\delta t} dt = \frac{c}{\delta} \left( 1 - e^{-\delta t} \right).
\]
Chapter 6

Length-constrained curve diffusion flow

6.1 Introduction

This chapter considers one-parameter families of immersed closed curves \( \gamma: \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 \). See Figure 6.1. We consider the energy functional

\[
L[\gamma] = \int_\gamma |\gamma_\nu| du.
\]

The curve diffusion flow has normal speed given by \( F \), that is

\[
\partial_t \gamma = F \nu,
\]

Under the evolution of the functional \( L \) which is also the first variation of the energy, a straightforward calculation yields
\[
\frac{d}{dt} \int_{\gamma} |\gamma_u| \, du = - \int_{\gamma} Fk |\gamma_u| \, du.
\]

For the flow to be the steepest descent gradient flow of length functional in \( H^{-1} \), we require

\[ F = -k_{ss}. \]

In this chapter, we study the length-constrained curve diffusion flow. To preserve length of the evolving curve \( \gamma(s,t) \), we take

\[
h(t) = -\frac{\int_{\gamma} k^2 ds}{2\pi \omega},
\]

where \( \omega \) denotes the winding number of \( \gamma \), which is defined in Chapter 2. Then we obtain

\[ F = -k_{ss}(s,t) + h(t). \]

Define the length-constrained curve diffusion flow \( \gamma : S^1 \times [0,T) \to \mathbb{R}^2 \) as follows,

\[
\begin{align*}
\partial_t \gamma &= -(k_{ss} - h(t))v, \\
\gamma(\cdot, 0) &= \gamma_0.
\end{align*}
\]

(6.1)

A curve moving under length-constrained curve diffusion flow fixes length and increases area. The area is enclosed by the closed plane curve. However, a curve satisfying regular curve diffusion flow fixes area and reduces length. We can say that the length-constrained curve diffusion flow is ‘dual’ to curve diffusion flow.

In this chapter, our goal is to show there is an immortal solution of (6.1) converging exponentially fast to a simple round circle. It is necessary to assume that initial data close to a round circle.

The short time existence for flow problem (6.1) is given in Theorem 4 in Chapter 3.

Our main theorem in this chapter is:

**Theorem 10.** Suppose \( \gamma_0 : S^1 \times [0,T) \to \mathbb{R}^2 \) is a regular smooth immersed closed curve with \( A(0) > 0 \) and \( \omega(0) = 1 \). Then there exists a constant \( K^* > 0 \) such that if

\[
K_{osc}(0) < K^*, I(0) < \frac{4\pi^2}{4\pi^2 - K^*},
\]

then the length-constrained curve diffusion flow (6.1) exists for all time and converges exponentially in \( C^\infty \) to a round circle with radius \( \frac{L_0}{2\pi} \).

From our calculation (6.7) in Proposition 21, we know that

\[ K^* \approx 0.05. \]
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The limiting curve is showed in Figure 6.2.

The structure of the proof of Theorem 10 is as follows. Firstly, we show that the length of curve satisfying (6.1) is constrained, and area is increasing. Under the condition that the oscillation and isoperimetric ratios of the curve are bounded in Theorem 10, we have in section 6.2 that $K_{osc}$ remains under control under (6.1) and all curvature derivatives in $L^2$ are bounded under the flow. Secondly, we move on to the analysis of the global behaviour of the flow, we prove the maximal time is infinite in section 6.3, and when time goes to infinity, the derivatives of curvature in $L^\infty$ decays exponentially in section 6.4. The oscillation goes to zero when time goes to infinity, thus the curve subconverges to a round circle. Finally, we show in section 6.5 this round circle is unique which is our conclusion. We also give the self-similar solutions for this length-constrained curve diffusion flow.

The short time existence is shown in Chapter 3, Theorem 4. From Lemma 4, we obtain the following Lemma.

Lemma 39. The hypothesis of Theorem 10 implies that $\omega(t) = \omega(0) = 1$.

Lemma 40. Suppose $\gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2$ solves (6.1) and satisfies the assumptions of Theorem 10, we have

$$L = L(0),$$

for all $t \in [0, T)$.

Proof. Using integration by parts, we obtain

$$0 = \frac{d}{dt}L = - \int_\gamma k(k_{ss} - h) ds$$

$$= - \int_\gamma k^2 ds - h \int_\gamma k ds$$

$$= - \int_\gamma k^2 ds - 2\pi \omega(t).$$
In Lemma 39, we have proofed that $\omega(t) = 1$, as $h(t) = \frac{\int_0^t k^2 \, ds}{2\pi} \leq 0$, thus under the small energy assumption we have the length is a constant.

**Lemma 41.** Suppose $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ solves (6.1). Under the assumption of Theorem 10,

$$A(t) \geq A(0),$$

for all $t \in [0, T)$.

**Proof.** Using integration by parts, we compute the evolution of the area functional,

$$\frac{d}{dt} A = \int_\gamma (k_{ss} - h) \, ds = -h(t)L(0) = \frac{\int_0^t k^2 \, d\mu}{2\pi} L(0) \geq 0.$$

Thus, we have the area is increasing under the small energy condition.

**Lemma 42.** Suppose $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ solves (6.1). The evolution of the $\ell$-th derivative of curvature

$$\partial_t k_\ell = k_{\ell+4} + \sum_{q+r+u=\ell} c_{qru} k_{q\ell} k_{r\ell} k_{u\ell} - h(t) \sum_{a+b=\ell} c_{ab} k_a k_b,$$

for constants $c_{qru}, c_{ab} \in \mathbb{R}$ with $a, b, q, r, u \geq 0$.

**Proof.** First, we calculate the evolution of the curvature

$$\partial_t k = F_{ss} + Fk^2$$

$$= (\kappa_{ss} + h(t))_{ss} + (\kappa_{ss} + h(t))k^2$$

$$= -\kappa_{ss} - k^2 + h(t)k$$

and the derivative of $k_s$ with respect to $t$,

$$\partial_t k_s = \partial_s \partial_t k = \partial_s \partial_t k + kFk_s$$

$$= \partial_s (-\kappa_{ss} - k^2 + h(t)k) + kk_s (-\kappa_{ss} + h(t))$$

$$= -\kappa_{ss} - k^2 - 2k_s k + 2h(t)k + k_s k - k_s k - h(t)k$$

$$= -\kappa_{ss} - k^2 + 3h(t)k,$$

then we calculate the derivative of $k_{ss}$ with respect to $t$,

$$\partial_t k_{ss} = \partial_t \partial_s k_s = \partial_s \partial_t k_s + kFk_{ss}$$

$$= \partial_s (\kappa_{ss} - \kappa^2 - 3k_{ss} k + 3h(t)k) + kk_{ss} (-\kappa_{ss} + h(t))$$

$$= -\kappa_{ss} - k^2 - 2k_s k - 3k_s k - 3k_{ss} k + 3h(t)k + 3h(t)k$$

$$+ 3h(t)k^2 - k_{ss} k + h(t)k$$

$$= -\kappa_{ss} - k^2 - 5k_{ss} k - 4k^2_{ss} k - 3k_{ss} k + 4h(t)k + 3h(t)k^2.$$  (6.3)
CHAPTER 6. LENGTH-CONSTRAINED CURVE DIFFUSION FLOW

It is straightforward to assume that when \( l \geq 2 \),

\[
\partial_t k_{l-1} = \partial_t \partial_t k_{l-2} = \partial_t \partial_t k_{l-2} + kFk_{l-1} \\
= \ -k_{l+3} + \sum_{q+r+u=1} c_{\gamma\nu_k} k_{q+2} k_{r} k_{s} + h(t) \sum_{a+b=l-1} c_{\nu_k} k_{a+b} k_{\phi},
\]

for constants \( c_{\gamma\nu_k}, c_{\nu_k} \in \mathbb{R} \) with \( a, b, q, r, u \geq 0 \).

Thus, we have

\[
\partial_t k_{l} = \partial_t \partial_t k_{l-1} = \partial_t \partial_t k_{l-1} + kFk_{l}
\]

\[
= \partial_t \left( -k_{l+3} + \sum_{q+r+u=1} c_{\gamma\nu_k} k_{q+2} k_{r} k_{s} + h(t) \sum_{a+b=l-1} c_{\nu_k} k_{a+b} k_{\phi} \right) \\
+ kFk_{l} (k_{ss} + h(t))
\]

\[
= \ -k_{l+4} + \sum_{q+r+u=l} c_{\gamma\nu_k} k_{q+2} k_{r} k_{s} + h(t) \sum_{a+b=l} c_{\nu_k} k_{a+b} k_{\phi},
\]

for constants \( c_{\gamma\nu_k}, c_{\nu_k} \in \mathbb{R} \) with \( a, b, q, r, u \geq 0 \), which is the conclusion of this Lemma. \( \square \)

The following evolution equation for various geometric quantities under the flow will be used in our analysis.

**Lemma 43.** Under the flow (6.1),

(i) \( \frac{d}{dt} L = 0; \)

(ii) \( \frac{d}{dt} A = -h(t)L(0); \)

(iii) \( \frac{d}{dt} \int_{\gamma} k^2 ds = -2 \int_{\gamma} k s_k \gamma ds + 3 \int_{\gamma} k^2 k_s^2 ds + h(t) \int_{\gamma} k^3 ds; \)

(iv) \( \frac{d}{dt} K_{asc} = -2L(0) \int_{\gamma} k_s^2 ds + 3L(0) \int_{\gamma} (k - \tilde{k})^2 k_s^2 ds + 6L(0) \int_{\gamma} (k - \tilde{k}) k_s^2 ds \\
+ 2\tilde{k}L(0) \int_{\gamma} k_s^2 ds + L(0) h(t) \left[ \int_{\gamma} (k - \tilde{k})^3 ds + 3\tilde{k} \int_{\gamma} (k - \tilde{k}) k_s^2 ds \right]; \)

(v) \( \frac{d}{dt} \int_{\gamma} k_s^2 ds = -2 \int_{\gamma} k_s^2 ds + 2 \int_{\gamma} k^2 k_s^2 ds + \frac{1}{2} \int_{\gamma} k^3 ds + 5h(t) \int_{\gamma} k k_s^2 ds; \)

(vi) \( \frac{d}{dt} \int_{\gamma} k_s^2 ds = -2 \int_{\gamma} k_s^2 ds + 2 \int_{\gamma} k^2 k_s^2 ds - 4 \int_{\gamma} k_s k_s k_s ds - 3 \int_{\gamma} k_s^2 k_s^2 ds + 7h \int_{\gamma} k_s^2 ds; \)

Moreover, for \( l \in \mathbb{N} \cup \{0\}, \)

(vii) \( \frac{d}{dt} \int_{\gamma} k_l^2 ds = -2 \int_{\gamma} k_{l+2}^2 ds + \int_{\gamma} k_{l-2} P_{l+2}^2 (k) ds + h(k) \int_{\gamma} k_{l-2} P_{l+2}^2 (k) ds. \)

Here \( L = L(0) \) is the constant length of the evolving curve \( \gamma. \)

**Proof.** (i) (ii) are proven in Lemma 40 and Lemma 41 respectively.

(iii) Calculate the evolution of \( \int_{\gamma} k^2 ds, \)

\[
\frac{d}{dt} \int_{\gamma} k^2 ds = \int_{\gamma} 2k \frac{d}{dt} k ds - \int_{\gamma} k^2 \frac{d}{dt} k ds = 2 \int_{\gamma} k (F_s + k S) ds - \int_{\gamma} k^2 kF ds \\
= -2 \int_{\gamma} kS F ds + 2 \int_{\gamma} k^3 F ds - \int_{\gamma} k^2 F ds \\
= 2 \int_{\gamma} kS F ds + \int_{\gamma} k^3 F ds = 2 \int_{\gamma} \left( k \frac{d}{dt} + \frac{1}{2} k^3 \right) F ds.
\]
For constrained curve diffusion flow, we can have

\[
\frac{d}{dt} \int_{\gamma} k^2 ds = 2 \int_{\gamma} \left( k_{ss} + \frac{1}{2} k^3 \right) (-k_{ss} + h(t)) ds
\]

\[
= -2 \int_{\gamma} k_{ss}^2 ds + 3 \int_{\gamma} k^2 k_{ss}^2 ds + 2h(t) \int_{\gamma} \left( k_{ss} + \frac{1}{2} k^3 \right) ds
\]

\[
= -2 \int_{\gamma} k_{ss}^2 ds + 3 \int_{\gamma} k^2 k_{ss}^2 ds + h(t) \int_{\gamma} k^3 ds.
\]

(iv) We compute the evolution of the oscillation,

\[
\frac{d}{dt} K_{osc} = \frac{d}{dt} L(0) \int_{\gamma} (k - \bar{k})^2 ds = L(0) \frac{d}{dt} \int_{\gamma} (k - \bar{k})^2 ds
\]

\[
= \int_{\gamma} 2(k - \bar{k}) (F_{ss} + k^2 F) ds - \int_{\gamma} (k - \bar{k})^2 kF ds
\]

\[
= -2L(0) \int_{\gamma} k_{ss}^2 ds + 3L(0) \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds + 6\bar{k}L(0) \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds
\]

\[
+ 2\bar{k}^2 L(0) \int_{\gamma} k_{ss}^2 ds + 2L(0)h(t) \int_{\gamma} (k - \bar{k})^2 ds - h(t)L(0) \int_{\gamma} (k - \bar{k})^2 kds
\]

\[
= -2L(0) \int_{\gamma} k_{ss}^2 ds + 3L(0) \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds + 6\bar{k}L(0) \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds
\]

\[
+ 2\bar{k}^2 L(0) \int_{\gamma} k_{ss}^2 ds + 2\bar{k}L(0)h(t) \int_{\gamma} (k - \bar{k})^2 ds + L(0)h(t) \int_{\gamma} (k - \bar{k})^3 ds
\]

\[
= -2L(0) \int_{\gamma} k_{ss}^2 ds + 3\bar{k}h(t)K_{osc} + 3L(0) \int_{\gamma} (k - \bar{k})^2 k_{ss}^2 ds
\]

\[
+ 6\bar{k}L(0) \int_{\gamma} (k - \bar{k})k_{ss}^2 ds + 2\bar{k}^2 L(0) \int_{\gamma} k_{ss}^2 ds + L(0)h(t) \int_{\gamma} (k - \bar{k})^3 ds.
\]

(v) Using integration by parts,

\[
\frac{d}{dt} \int_{\gamma} k^2 ds = 2 \int_{\gamma} k_s \cdot \frac{\partial k_s}{\partial t} ds - \int_{\gamma} k_s^2 \frac{\partial}{\partial t} ds
\]

\[
= \int_{\gamma} k_s \cdot \left( -2F_{s3} - 2F_s k^2 - 6F k_s k + k_s kF \right) ds
\]

\[
= \int_{\gamma} ( -2k_{ss} F_s - 2k_s k^2 F_s - 5k_s kF(ds
\]

\[
= \int_{\gamma} \left( -2k_{ss}^2 - 2k_s k_{ss} k^2 - 5k_{sss} k^2 k \right) ds + 5h \int_{\gamma} k_{ss}^2 ds
\]

\[
= -2 \int_{\gamma} k_{ss}^2 ds + 2 \int_{\gamma} k^2 k_{ss}^2 ds + \frac{1}{3} \int_{\gamma} k_{ss}^4 ds + 5h \int_{\gamma} k_{ss}^2 ds.
\]
(vi) By using (6.3) in Lemma 42, and

\[-10 \int \kappa_3 k_{ss} k ds = 5 \int \kappa_{ss}^2 k ds + 5 \int \kappa_s^2 k ds,
\]

\[-2 \int \kappa_3 k_{ss} k^2 ds = 2 \int \kappa_s^2 k^2 ds - 2 \int \kappa_{ss} k ds - 2 \int \kappa_s^2 k ds.
\]

We obtain

\[\frac{d}{dt} \int \kappa_{ss}^2 ds = 2 \int \kappa_{ss} \frac{\partial}{\partial t} \kappa_{ss} ds - \int \kappa_{ss} k F ds = 2 \int \kappa_{ss} (-k_5^6 - k_{s5} k^2 - 5k_{s3} k_3 k - 4k_{s2}^2 k - 3k_{ss} k_s^2 + 4h(t) k_{ss} k + 3h(t) k_s^2) ds - \int \kappa_{ss} k (-k_{ss} + h(t)) ds \]

\[= -2 \int \kappa_s^2 ds - 2 \int \kappa_{ss} k_{ss} k^2 ds + 10 \int \kappa_3 k_{ss} k ds - 7 \int \kappa_{ss}^2 k ds + 6h(t) \int \kappa_{ss} k^2 ds + 7h(t) \int \kappa_s^2 k ds \]

\[= -2 \int \kappa_3 k ds + \int \kappa_{ss} k_{ss} k^2 ds - 2 \int \kappa_s^2 k ds + 2 \int \kappa_{ss}^2 k ds - 4 \int \kappa_{ss} k ds + 6h(t) \int \kappa_{ss} k^2 ds + 7h(t) \int \kappa_s^2 k ds.
\]

(vii) Here we use (6.2) in Lemma 42, for \( l \in \mathbb{N} \),

\[\frac{d}{dt} \int \kappa_{ss}^2 ds = 2 \int \kappa_s \frac{\partial}{\partial t} \kappa_{ss} ds - \int \kappa F k_{ss}^2 ds \]

\[= 2 \int \kappa_s \left( -k_{s5}^6 + \sum_{q+r+u=0} \sum_{a+b+l} c_{qru} k_{sq} k_{s}^{r+2} k_{s}^{u} + h(t) \sum_{a+b=l} c_{ab} k_{s}^{a} k_{s}^{b} \right) ds - \int \kappa k_{ss}^2 (k_{ss} - h(t)) ds \]

\[= -2 \int \kappa_{ss}^2 ds + \frac{1}{2} \int \kappa_{s}^{l+2}(k) ds + h(t) \int \kappa_{s}^{l}(k) ds.
\]

We finish the proof here.
Lemma 44. For \( n \in \mathbb{N} \),

\[
\int_{\gamma} k_{n+1}^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{n}} \left( \int_{\gamma} k_{n}^2 ds \right)^{\frac{n-1}{n}} .
\]

Proof. The result is obvious for \( n = 1 \). By induction, we assume that

\[ \int_{\gamma} k_{i-1}^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{i}{n}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{i-1}{n}} . \]  

(6.4)

and use this to show

\[ \int_{\gamma} k_i^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{i+1}{n}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{i}{n}} . \]

By integration by parts and the Hölder inequality, we have

\[
\int_{\gamma} k_i^2 ds = - \int_{\gamma} k_{i+1} k_{i-1} ds \leq \left( \int_{\gamma} k_{i-1}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{1}{2}} .
\]

Inserting above on the right-hand side of (6.4), we get

\[ \int_{\gamma} k_i^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{i+1}{2n}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{i}{2n}} . \]

In other words

\[
\left( \int_{\gamma} k_i^2 ds \right)^{\frac{i+1}{2n}} \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{2n}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{1}{2n}} ,
\]

which implies

\[ \int_{\gamma} k_i^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{n}} \left( \int_{\gamma} k_{i+1}^2 ds \right)^{\frac{1}{n}} . \]

as required. \( \Box \)

Lemma 45. For each \( n \in \mathbb{N} \), the global term \( h(t) \) may be estimated as

\[ |h(t)| \leq \frac{1}{2\pi} \left( \int_{\gamma} k^2 ds \right)^{1-\frac{1}{n}} \left( \int_{\gamma} k_{n}^2 ds \right)^{\frac{1}{n}} . \]

Proof. Here we again use induction argument. As \( h(t) = -\frac{\int_{\gamma} k_i^2 ds}{2\pi w} \), it is obvious for \( n = 1 \).
Assume that
\[ |h(t)| \leq \frac{1}{2\pi} \left( \int_{\gamma} k^2 ds \right)^{1 - \frac{1}{r}} \left( \int_{\gamma} k^2_{\gamma} ds \right)^{\frac{1}{r}} \]  
and it shows
\[ |h(t)| \leq \frac{1}{2\pi} \left( \int_{\gamma} k^2 ds \right)^{1 - \frac{1}{r}} \left( \int_{\gamma} k_{\gamma+1}^2 ds \right)^{\frac{1}{r}}. \]

From Lemma 44, we have
\[ \int_{\gamma} k_{\gamma}^2 ds \leq \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{r}} \left( \int_{\gamma} k_{\gamma+1}^2 ds \right)^{\frac{1}{r}}. \]

Substituting this into (6.5) we obtain
\[ |h(t)| \leq \frac{1}{2\pi} \left( \int_{\gamma} k^2 ds \right)^{1 - \frac{1}{r}} \left[ \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{r}} \left( \int_{\gamma} k_{\gamma+1}^2 ds \right)^{\frac{1}{r}} \right]^{\frac{1}{r}} \]
which simplifies to the required expression.

From Lemma 4 in Chapter 2, we can have the following lemma.

**Lemma 46.** Suppose \( \gamma : S^1 \times [0,T) \rightarrow \mathbb{R}^2 \) solves (6.1) and
\[ \omega(0) = \frac{1}{2\pi} \int_{\gamma} k ds = 1, \]
then
\[ \omega(t) = \omega(0) = 1. \]

By previous lemma, we get that the average curvature \( \bar{k} \) satisfies
\[ \bar{k} := \frac{1}{L} \int_{\gamma} k ds = \frac{2\pi}{L(0)}. \]

### 6.2 Controlling the geometry of the flow

Here we show that \( K_{osc} \) is a \( L^1 \) function in time:

**Lemma 47.** Suppose \( \gamma : S^1 \times [0,T) \rightarrow \mathbb{R}^2 \) solves our flow (6.1). Then,
\[ \|K_{osc}\|_1 \leq \frac{L^2(0)}{2\pi} \left[ \frac{L(0)}{4\pi} - A(0) \right] \]
where \( A(0) \) denotes the signed enclosed area of the initial curve.
Proof. From Lemma 43 (ii), we have
\[ K_{osc} = L(0) \int_\gamma (k - \bar{k})^2 \, ds \leq \frac{L^2(0)}{4\pi^2} L(0) \|k_s\|_2^2 \leq \frac{L^2(0)}{2\pi} \frac{d}{dt} A, \]
then
\[ \|K_{osc}\|_1 \leq \frac{L^2(0)}{2\pi} [A(t) - A(0)] \leq \frac{L^2(0)}{4\pi} \left[ \frac{L^2(0)}{4\pi} - A(0) \right]. \]

Next, we show that \(K_{osc}\) remains bounded under (6.1) if initially \(K_{osc}\) is sufficiently small and the isoperimetric ratio \(I(0) = \frac{L^2(0)}{4\pi A(0)}\) is sufficiently close to 1.

**Proposition 20.** Suppose \(\gamma : S^1 \times [0, T) \to \mathbb{R}^2\) solves our flow (6.1) and satisfies \(\int_\gamma k \, ds = 2\pi\). Then
\[ K_{osc}(0) < K^*, I(0) < \frac{4\pi^2}{4\pi^2 - K^*}, \]
implies \(K_{osc} \leq 2K^*\), for all \(t \in [0, T)\).

**Proof.** Suppose we establish a contradiction these exists a first time \(T^* < T\) for which \(K_{osc} = 2K^*\). In view of Proposition 21,
\[ K_{osc} \leq K_{osc}(0) + \frac{16\pi^3 \omega^3}{L^2(0)} [A(t) - A(0)] \]
Since
\[ A(t) - A(0) \leq \frac{L^2(0)}{4\pi} - A(0) = \frac{L^2(0)}{4\pi} \left[ 1 - \frac{1}{I(0)} \right], \]
we have
\[ K_{osc} \leq K_{osc}(0) + 4\pi^2 \left[ 1 - \frac{1}{I(0)} \right] < \frac{3}{2} K^* \]
which is a contradiction. We obtain the conclusion that \(K_{osc} < 2K^*\) for all \(t \in [0, T)\).

As \(I(t) = \frac{L^2(0)}{4\pi A(t)}\), we know that \(A(t)\) is increasing, \(A(t) \geq A(0)\), then \(I(t)\) is decreasing when \(t \to \infty\). \(\square\)

**Proposition 21.** Suppose \(\gamma : S^1 \times [0, T) \to \mathbb{R}^2\) solves (6.1). If there exists a \(T^*\) such that for \(t \in [0, T^*)\), we have
\[ K_{osc} \leq 2K^*, \]
then during this time we have
\[ K_{osc}(t) \leq K_{osc}(0) + \frac{16\pi^3 \omega^3}{L^2(0)} [A(t) - A(0)] \]
Proof. We start with the evolution equation of Lemma 43 (iv),
\[
\frac{d}{dt} K_{osc} = -2L(0) \int k_s^2 ds + 3L(0) \int (k - \bar{k})^2 k_s^2 ds + 6L(0)\bar{k} \int (k - \bar{k}) k_s^2 ds \\
+ 2\bar{k}^2 L(0) \int k_s^2 ds + L(0) h(t) \left[ \int (k - \bar{k})^3 ds + 3\bar{k} \int (k - \bar{k})^2 ds \right].
\]
We estimate using PSW inequality (2.4) in Proposition 4, \(\|k_s\|_\infty^2 \leq \frac{L(0)}{2\pi} \|k_{ss}\|_2^2\),
\[
3L(0) \int (k - \bar{k})^2 k_s^2 ds \leq \frac{3L(0)}{2\pi} K_{osc} \|k_{ss}\|_2^2
\]
and
\[
6\bar{k}L(0) \int (k - \bar{k})^2 k_s^2 ds \leq 6\omega L(0) \sqrt{K_{osc}} \|k_{ss}\|_2^2.
\]
Although we can neglect the negative term \(h(t) \int (k - \bar{k})^2 ds\), we need to estimate the other \(h(t)\) term as follows:
\[
\int k_s^2 ds = -\int kk_{ss} ds = -\int (k - \bar{k}) k_{ss} ds \leq \left( \int (k - \bar{k})^2 ds \right)^{1/2} \left( \int k_s^2 ds \right)^{1/2},
\]
and
\[
\int (k - \bar{k})^3 ds \leq \|k - \bar{k}\|_\infty \int (k - \bar{k})^2 ds,
\]
thus
\[
L(0) h(t) \int (k - \bar{k})^3 ds \leq \frac{L(0)}{2\pi\omega} \left[ \int (k - \bar{k})^3 ds \right]^{3/2} \|k - \bar{k}\|_\infty \|k_{ss}\|_2.
\]
Using Lemma 43 (ii), we have
\[
\frac{d}{dt} K_{osc} + L(0) \left( 2 - \frac{1}{4\pi^2\sqrt{2\pi}\omega} K_{osc}^3 - \frac{3}{2\pi} K_{osc} - 6\omega K_{osc}^{1/2} \right) \int k_s^2 ds \leq \frac{16\pi^3\omega^3}{L^2(0)} \frac{dA}{dt}.
\] (6.6)
Here we take \(2K^*\) as the smallest positive solution of
\[
2 - \frac{1}{4\pi^2\sqrt{2\pi}\omega} K_{osc}^3 - \frac{3}{2\pi} K_{osc} - 6\omega K_{osc}^{1/2} = 0,
\] (6.7)
the coefficient of \(\int k_s^2 ds\) remains positive on the interval \([0, T^*)\). Then by integration in time, the result follows. \(\square\)
When $\omega = 1$, we can estimate (6.7) to get $2K^* \approx 0.1$.

Moving on to proof the curvature derivatives in $L^2$ are bounded under the flow. We start with the $L^2$ norm of first curvature derivative.

**Proposition 22.** Suppose $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ solves (6.1) and satisfies the assumptions of Theorem 10, then there exist a constant $c$ depending only on $\gamma_0$ such that

$$\|k_s\|_2^2 \leq c.$$

**Proof.** Using Lemma 43 (v),

$$\frac{d}{dt} \int_\gamma k_s^2 ds = -2 \int_\gamma k_s^2 ds + 2 \int_\gamma k_s k_{ss}^2 ds + \frac{1}{3} \int_\gamma k_s^3 ds + 5h \int_\gamma k_s^2 ds$$

$$= -2 \int_\gamma k_s^2 ds + \frac{5}{3} \int_\gamma k_s^3 ds + 2 \int_\gamma k_s k_{ss}^2 ds + 4 \int_\gamma k_s^2 k_{ss} ds + 5h \int_\gamma k_s^2 ds.$$ (6.8)

For the $h(t)$ term we estimate using integration by parts

$$5h \int_\gamma k_s^2 ds = \frac{5}{2} (-h) \int_\gamma k_{ss} k^2 ds$$

and

$$\int_\gamma k_{ss} k^2 ds = \int_\gamma k_{ss} (k - \bar{k})^2 ds + 2\bar{k} \int_\gamma k_{ss} k ds$$

$$= \int_\gamma k_{ss} (k - \bar{k})^2 ds + 2\bar{k} \int_\gamma k_{ss} (k - \bar{k}) ds$$

$$\leq \frac{1}{4} \left\| k - \bar{k} \right\|_2^2 + \epsilon \int_\gamma (k - \bar{k})^2 k_{ss}^2 ds - \frac{4\pi}{L(0)} \int_\gamma k_s^2 ds$$

$$\leq \frac{1}{4\epsilon L(0)} K_{osc} + \frac{\epsilon K_{osc}}{2\pi} \left\| k_s \right\|_2^2 - \frac{4\pi}{L(0)} \left\| k_s \right\|_2^2$$

$$= \frac{1}{4\epsilon L(0)} K_{osc} + \frac{\epsilon K_{osc}}{2\pi} \left\| k_s \right\|_2^2.$$

It follows that

$$5h \int_\gamma k_s^2 ds = \frac{5}{2} (-h) \int_\gamma k_{ss} k^2 ds$$

$$\leq \frac{5}{2} (-h) \frac{1}{4L(0)} K_{osc} + \frac{5}{2} (-h) \frac{K_{osc}}{2\pi} \left\| k_s \right\|_2^2$$

$$= \frac{5K_{osc} (-\epsilon h)}{4\pi} \left\| k_s \right\|_2^2 + \frac{5(-h)}{8\epsilon L(0)} K_{osc}.$$

Since $\int_\gamma k_s^2 ds = -3 \int_\gamma k_s k_{ss} k ds \leq \frac{1}{2} \int_\gamma k_s^2 ds + \frac{9}{2} \int_\gamma k_s k_{ss}^2 ds$, we have
where
\[
\int_\gamma k_s^2 ds = \int_\gamma (k - \bar{k})^2 k_s^2 ds + 2\bar{k} \int_\gamma k_s^2 ds - \bar{k}^2 \int_\gamma k_s^2 ds
\]
\[
= \int_\gamma (k - \bar{k})^2 k_s^2 ds + 2\bar{k} \int_\gamma (k - \bar{k}) k_s^2 ds + \bar{k}^2 \int_\gamma k_s^2 ds
\]
\[
= 2 \int_\gamma (k - \bar{k})^2 k_s^2 ds + 2\bar{k} \int_\gamma k_s^2 ds
\]
\[
\leq \frac{K_{osc}}{\pi} \|k_s\|_2^2 + 2\bar{k}^2 \|k_s\|_2 \|k_s\|_2
\]
\[
\leq \frac{K_{osc}}{\pi} \|k_s\|_2^2 + \frac{1}{108} \|k_s\|_2^2 + 108\bar{k}^4 \|k_s\|_2^2
\]
\[
\leq \frac{K_{osc}}{\pi} \|k_s\|_2^2 + \frac{1}{108} \|k_s\|_2^2 + \frac{54\bar{k}^4 \sqrt{L(0)} K_{osc}}{\pi} \|k_s\|_2^2
\]
\[
\leq \frac{K_{osc}}{\pi} \|k_s\|_2^2 + \frac{1}{108} \|k_s\|_2^2 + \frac{1}{108} \|k_s\|_2^2 + \frac{54\bar{k}^4 L(0) K_{osc}}{2\pi^2}
\]
\[
\leq \left( \frac{K_{osc}}{\pi} + \frac{1}{54} \right) \|k_s\|_2^2 + \frac{54\bar{k}^4 L(0) K_{osc}}{2\pi^2}.
\]

Substituting these estimates into (6.8), we obtain
\[
\frac{d}{dt} \int_\gamma k_s^2 ds = -2\|k_s\|_2^2 + \frac{5}{3} \int_\gamma k_s^2 ds + 2 \int_\gamma k_s^2 ds + 4 \int_\gamma k_s^2 ds + 5h \int_\gamma k_s^2 ds
\]
\[
= -2\|k_s\|_2^2 + 27 \int_\gamma k_s^2 ds + 5h \int_\gamma k_s^2 ds
\]
\[
\leq -2\|k_s\|_2^2 + 27 \left[ \left( \frac{K_{osc}}{\pi} + \frac{1}{54} \right) \|k_s\|_2^2 + \frac{54\bar{k}^4 L(0) K_{osc}}{2\pi^2} \right]
\]
\[
+ \frac{5K_{osc}(-h)}{4\pi} \|k_s\|_2^2 + \frac{5(-h)}{8L(0)} K_{osc}
\]
\[
= \left[ -2 + 27 \left( \frac{K_{osc}}{\pi} + \frac{1}{54} \right) + \frac{5K_{osc}(-h)}{4\pi} \right] \|k_s\|_2^2
\]
\[
+ \frac{54\bar{k}^4 \kappa^2 L(0)}{\pi^2} + \frac{5(-h)}{8L(0)} K_{osc}.
\]

Because \(h = \frac{\int_s k_s^2 d\mu}{2\pi \omega} \leq 0\), then
\[
-h = \frac{\int_s k_s^2 d\mu}{2\pi \omega} = \frac{2\pi \omega dA}{L(0) dt} \frac{1}{2\pi \omega} = \frac{1}{L(0) dt}
\]
and
\[
-L(0) h = \frac{dA}{dt}.
\]

\[-L(0)[h(t) - h(0)] = A(t) - A(0) \leq \frac{L^2(0)}{4\pi} - A(0) = \frac{L^2(0)}{4\pi} \left[ 1 - \frac{1}{I(0)} \right] \leq \frac{L^2(0)K^*}{32\pi^3},\]
\[-h(t) \leq -h(0) + \frac{L(0)K^*}{32\pi^3}. \quad (6.9)\]

In order to get
\[-2 + 27 \left( \frac{K_{\text{osc}}}{\pi} + \frac{1}{54} \right) + \frac{5K_{\text{osc}}(-\varepsilon h)}{4\pi} \leq 0,\]
we need to have
\[K_{\text{osc}} \leq \frac{6\pi}{108 - 5\varepsilon h},\]
here there is a small \(\varepsilon\) satisfies \(\frac{6\pi}{108 - 5\varepsilon h} > K^*\) (in Theorem 10), i.e. \(\varepsilon(-h) < \frac{6\pi/K^* - 108}{5}\).

Then we only need
\[K_{\text{osc}} \leq K^*.\]

If so, we find
\[\frac{d}{dt} \int_{\gamma} k^2 s ds + \|k_s\|_2^2 \leq \left[ \frac{54^2 27^2 \bar{k}^8 L(0)}{\pi^2} + \frac{5(-h)}{8L(0)} \right] K_{\text{osc}},\]
and
\[\frac{d}{dt} \int_{\gamma} k^2 s^2 ds \leq -\frac{16\pi^4}{L^2(0)} \|k_s\|_2^2 + \left[ \frac{54^2 27^2 \bar{k}^8 L(0)}{\pi^2} + \frac{5(-h)}{8\varepsilon L(0)} \right] K_{\text{osc}}.\]

Assume there exists \(\delta_0 \geq 0\), such that \(\int_{\gamma} k^2 s ds \geq \bar{c}\) for \(t \in [\delta_0, \delta_1]\), (here \(\delta_0\) can be 0), here \(\bar{c}\) is a very small constant.

Since \(-h\) is bounded by (6.9) and \(K_{\text{osc}} \in L^1\), so
\[\int_{\gamma} \left[ \frac{54^2 27^2 \bar{k}^8 L(0)}{\pi^2} + \frac{5(-h)}{8\varepsilon L(0)} \right] \|k_s\|_2^2 \leq \int_{\gamma} \left[ \frac{54^2 27^2 \bar{k}^8 L(0)}{\pi^2} + \frac{5(-h)}{8\varepsilon L(0)} \right] K_{\text{osc}} \leq c\|K_{\text{osc}}\|_1 \leq c,\]
and
\[\int_{\gamma} k^2 s ds \leq c \cdot e^{-\frac{16\pi^4}{L^2(0)}(t-\delta_0)} = c_1\]
Then when \(t \notin [\delta_0, \delta_1]\), we take \(c_1 = \bar{c}\), then we still have \(\int_{\gamma} k^2 s^2 ds < \bar{c} = c_1.\)

Also we know that after enough time, \(\|k\|^2_\infty \leq 2 \left( \frac{2\omega \pi}{L(0)} \right)^2 \leq c\).

Now we have that all curvature derivatives in \(L^2\) are bounded under the flow. These bounds are independent of time implies solutions exist for all time.

First, we have the following lemma,

**Lemma 48.** Suppose \(F = -k_{ss} + h(t)\), then for \(m \geq 0\) the derivatives of the curvature \(k_{sm}\) satisfy:
\[\frac{\partial}{\partial t} k_{sm} + k_{sm+4} = P_{3}^{m+2}(k) + h(t) P_{2}^{m}(k).\]
Proof. For $m = 0$ this follows from Proposition (3) (ii).

$$k_t = F_{ss} + k^2 F = -k_{ss} - k_{ss}k^2 + h(t)k^2.$$ 

For $m > 1$, we obtain using Proposition (3) (v),

$$\frac{\partial}{\partial t} k_m = F_{s}^{m+2} + \sum_{j=0}^{m} \partial_{s}^{m} (k_{l-j}F) = -k_{s}^{m+2} + P_{3}^{m+2}(k) + h(t)P_{2}^{m}(k).$$

The result follows. $\square$

Lemma 49. Suppose $\gamma : \mathbb{S}^1 \times [0,T) \rightarrow \mathbb{R}^2$ solves (6.1) and satisfies the assumptions of Theorem 10, there exists absolute constants $c_m > 0$ such that for any $m \in \mathbb{N} \cup \{0\}$,

$$\|k_m\|_{2}^2 \leq c_m.$$

It follows that $\|k_m\|_{\infty}^2$ is bounded as well.

Proof. Lemma 48 above yields for $m \geq 0$

$$\frac{d}{dt} \int_{\gamma} k_m^2 ds = 2 \int_{\gamma} k_m \frac{\partial}{\partial t} k_m ds + \int_{\gamma} k_m^2 kF ds = 2 \int_{\gamma} k_m \cdot \left[ -k_{s}^{m+4} + P_{3}^{m+2}(k) + h(t)P_{2}^{m}(k) \right] ds + \int_{\gamma} k_m^2 \cdot k \cdot \left[ -k_{ss} + h(t) \right] ds = -2 \int_{\gamma} k_{s}^{m+2} ds + \int_{\gamma} k_m P_{3}^{m+2}(k) ds + h(t) \int_{\gamma} k_m P_{2}^{m}(k) ds.$$

From interpolation inequality and by doing integration by parts, we achieve the estimates of last two terms on the right-hand side,

$$\int_{\gamma} k_m P_{3}^{m+2}(k) ds \leq \varepsilon_1 \int_{\gamma} k_{s}^{m+2} ds + c_1 \left( \int_{\gamma} k^2 ds \right)^{2m+5},$$

where $c_1$ is a constant depending only on $\varepsilon_1, m$.

$$h(t) \int_{\gamma} k_m P_{2}^{m}(k) ds \leq -h(t)\varepsilon_2 \int_{\gamma} k_{s}^{m+1} ds - h(t)c_2 \left( \int_{\gamma} k^2 ds \right)^{2m+2} \leq -h(t)\varepsilon_2 \cdot \frac{L^2(0)}{4\pi^2} \int_{\gamma} k_{s}^{m+2} ds - h(t)c_2 \left( \int_{\gamma} k^2 ds \right)^{2m+2}$$

where constant $c_2$ depends only on $\varepsilon_2, m$. 
Then we have
\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 ds = -2 \int_{\gamma} k_{\gamma}^{m+2} ds + \left( \varepsilon_1 - \varepsilon_2 \cdot \frac{h(t)L^2(0)}{4\pi^2} \right) \int_{\gamma} k_{\gamma}^2 ds \\
+ c_1 \left( \int_{\gamma} k^2 ds \right)^{2m+5} - h(t)c_2 \left( \int_{\gamma} k^2 ds \right)^{2m+2}
\]
which yields if we let \( \varepsilon_1 - \varepsilon_2 \cdot \frac{h(t)L^2(0)}{4\pi^2} = 1 \),
\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 ds + \int_{\gamma} k_{\gamma}^{m+2} ds \leq c_1 \left( \int_{\gamma} k^2 ds \right)^{2m+5} - h(t)c_2 \left( \int_{\gamma} k^2 ds \right)^{2m+2}. \tag{6.10}
\]
Here
\[
\int_{\gamma} k^2 ds = \int_{\gamma} (k - \bar{k}) ds + 2\bar{k} \int_{\gamma} k ds - \bar{k}^2 \int_{\gamma} ds = \frac{K_{osc}}{L(0)} + 2 \cdot 2\pi \cdot 2\pi - \left( \frac{2\pi}{L(0)} \right)^2 \cdot L(0) = \frac{K_{osc}}{L(0)} + \frac{4\pi^2}{L(0)},
\]
as \( K_{osc} \leq 2K^* \) in Proposition 21, \( \int_{\gamma} k^2 ds \) is bounded. Also \(-h(t)\) is bounded by (6.9), therefore on the right-hand side of (6.10), \( c_1 \left( \int_{\gamma} k^2 ds \right)^{2m+5} - h(t)c_2 \left( \int_{\gamma} k^2 ds \right)^{2m+2} \) is bounded.
So we have
\[
\frac{d}{dt} \int_{\gamma} k_{\gamma}^2 ds \leq -\int_{\gamma} k_{\gamma}^{m+2} ds + c.
\]
it follows
\[
\|k_{\gamma}\|_2^2 \leq c_m,
\]
and the \( L_\infty \) estimates follow immediately by Proposition 2.

The following proposition shows that there is an upper bound of time for which the curvature of a solution of (6.1) is not strictly positive.

**Proposition 23.** Suppose \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) solves (6.1) and satisfies the assumptions of Theorem 10, Then
\[
\mathcal{L} \{ t \in [0, \infty) : k(\cdot, t) \nless 0 \} \leq \frac{L^2(0)}{4\pi^3} \left( \frac{L^2(0)}{4\pi} - A(0) \right).
\]
In the above, \( k(\cdot, t) \nless 0 \) means that there exists a \( p \) such that \( k(p, t) \leq 0 \). This estimate is optimal in the sense that the right-hand side is zero for a simple circle.
Proof. Rearranging $\gamma$ in time if necessary, we may assume that for $t_0 > \frac{L^2(0)}{4\pi^3} \left( \frac{L^2(0)}{4\pi} - A(0) \right)$

$$k(\cdot, t) \neq 0, \text{ for all } t \in [0, t_0);$$

$$k(\cdot, t) > 0, \text{ for all } t \in [t_0, \infty).$$

We have that $k$ always has a zero when $t \in [0, t_0)$.

As $2\pi = \int_{\gamma} k ds \leq L^{1/2} \left( \int_{\gamma} k^2 ds \right)^{1/2}$, then

$$\int_{\gamma} k^2 ds \geq \frac{4\pi^2}{L(0)}$$

and

$$\frac{d}{dt} A = \frac{\int_{\gamma} k^2 ds}{2\pi} \cdot \frac{L(0)}{L(0)} \geq \frac{L(0)}{2\pi} \cdot \frac{4\pi^2}{L^2(0)} \int_{\gamma} k^2 ds$$

$$= \frac{L(0)}{2\pi} \cdot \frac{4\pi^2}{L^2(0)} \cdot \frac{4\pi^2}{L(0)} = \frac{8\pi^3}{L^2(0)}.$$  

For any $t \in [0, t_0)$,

$$A(t) - A(0) \geq \frac{8\pi^3}{L^2(0)} t,$$

As $t_0 > \frac{L^4(0)}{32\pi^3}$, then

$$A(t) \geq A(0) + \frac{8\pi^3}{L^2(0)} \cdot \frac{L^2(0)}{4\pi^3} \left( \frac{L^2(0)}{4\pi} - A(0) \right)$$

$$= A(0) + \frac{L^2(0)}{2\pi} - 2A(0)$$

$$\geq \frac{L^2(0)}{2\pi} - A(0).$$

As $A(0) < \frac{L^2(0)}{4\pi}$, then

$$A(t) > \frac{L^2(0)}{4\pi}.$$  

We establish a contradiction to the isoperimetric inequality. Because from the isoperimetric inequality, we know that $A(t) \leq \frac{L^2(0)}{4\pi}$. Thus, the result follows. \qed

6.3 Global existence

In this section, we prove that the flow exists globally. In order to prove that the solution of (6.1) exists for all time ($T = \infty$), we refer to [81, Corollary 4.1], see also [17, Theorem 3.1]. It is shown in the following Theorem 11.
Theorem 11. Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ be a maximal solution of (6.1). If $T < \infty$, then

$$\int_{\gamma} k^2 ds \geq c(T-t)^{-1/4}. $$

Proof. Let $m = 0$ in (6.10) in Lemma 48 we have

$$\frac{d}{dt} \int_{\gamma} k^2 ds + \int_{\gamma} k_{ss}^2 ds \leq c_1 \left( \int_{\gamma} k^2 ds \right)^5 - h(t) \cdot c_2 \left( \int_{\gamma} k^2 ds \right)^2, \quad (6.11)$$

where $h(t) = -\frac{\int_{\gamma} k^2 ds}{2\pi}$. Then we estimate two terms evolving $h(t)$ above as follows,

$$(-h) \cdot \left( \int_{\gamma} k^2 ds \right)^2 = \frac{1}{2\pi} \int_{\gamma} k^2 ds \left( \int_{\gamma} k^2 ds \right)^{-\frac{1}{2}} \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{2}}$$

$$\leq \varepsilon \left[ \int_{\gamma} k^2 ds \cdot \left( \int_{\gamma} k^2 ds \right)^{-\frac{1}{2}} \right]^2 + \frac{1}{16\pi^2 \varepsilon} \left( \int_{\gamma} k^2 ds \right)^5$$

$$= \varepsilon \left[ -\int_{\gamma} k k_{ss} ds \cdot \left( \int_{\gamma} k^2 ds \right)^{-\frac{1}{2}} \right]^2 + \frac{1}{16\pi^2 \varepsilon} \left( \int_{\gamma} k^2 ds \right)^5$$

$$\leq \varepsilon \left[ \left( \int_{\gamma} k^2 ds \right)^{-\frac{1}{2}} \cdot \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \cdot \left( \int_{\gamma} k^2 ds \right)^{-\frac{1}{2}} \right]^2$$

$$+ \frac{1}{16\pi^2 \varepsilon} \left( \int_{\gamma} k^2 ds \right)^5$$

$$= \varepsilon \int_{\gamma} k_{ss}^2 ds + \frac{1}{16\pi^2 \varepsilon} \left( \int_{\gamma} k^2 ds \right)^5. $$

Substituting above estimate into (6.11), we obtain

$$\frac{d}{dt} \int_{\gamma} k^2 ds + [1 - \varepsilon \cdot c_2] \int_{\gamma} k_{ss}^2 ds \leq \left( c_1 + \frac{c_2}{16\pi^2 \varepsilon} \right) \left( \int_{\gamma} k^2 ds \right)^5. $$

Let $\varepsilon$ small enough to have $1 - \varepsilon \cdot c_2 \geq 0$, then

$$\frac{d}{dt} \int_{\gamma} k^2 ds \leq c \left( \int_{\gamma} k^2 ds \right)^5,$$

here $c = c_1 + \frac{c_2}{16\pi^2 \varepsilon}$. By integrating above inequality on $[t, \tilde{t}]$, where $\tilde{t} \to T$, we have
Thus,
\[ \int_{\gamma} k_2^2 ds \geq \lim_{\tilde{t} \to T} c(\tilde{t} - t)^{-1/4}, \]
\[ \int_{\gamma} k_2^2 ds \geq c(T - t)^{-1/4}. \]
where \( c \) can be different from line to line. Hence the result.

**Corollary 1.** Suppose \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) solves (6.1) and satisfies the assumptions of Theorem 10. Then, \( T = \infty \).

**Proof.** Suppose on the contrary that \( \gamma \) satisfies the conditions of Proposition 20 and \( T < \infty \). Then by Theorem 11, \( \|k\|_2^2 \to \infty \) as \( t \to T \). However,
\[ K_{osc} = L(0) \int_{\gamma} (k - \bar{k})^2 ds = L(0)\|k\|_2^2 - 2\bar{k}2\pi + \bar{k}^2L^2(0) = L(0)\|k\|_2^2 - 2\pi^2, \]
then \( K_{osc} \to \infty \) as \( t \to T \). This is in contradiction with Proposition 20. We conclude that it must be the case that \( T = \infty \).

By Lemma 47, \( K_{osc} \in L^1([0, \infty)) \) we conclude \( K_{osc} \to 0 \) as \( t \to \infty \) and therefore limit curves are circles of circumference length \( L(0) \) under the conditions of Proposition 20.

### 6.4 Exponential Convergence

To complete the proof of Theorem 10, it remains to show that the limit circle is unique and convergence is exponential. Since we have convergence to circle of radius \( \frac{L(0)}{2\pi} \), we can be sure that \( k(s, t) \in \left[ \frac{\pi}{L(0)} \cdot \frac{3\pi}{L(0)} \right] \) say for all \( t \geq t_1 \). For such times we also have \( \|k\|_\infty \leq \frac{3\pi}{L(0)} \).

**Lemma 50.** Suppose \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) solves (6.1) and satisfies conditions of Theorem 10, we have \( T = \infty \) and there exists absolute constants \( c, \delta > 0 \) such that
\[ \|k_{ss}\|_2^2 \leq ce^{-\delta t}. \]

**Proof.** Using Lemma 43 (vi),
\[
\frac{d}{dt} \int_{\gamma} k_{ss}^2 ds = -2 \int_{\gamma} k_s^2 ds + 2 \int_{\gamma} k_s^2 k_s^2 ds - 4 \int_{\gamma} k_{ss}^2 k ds \\
-3 \int_{\gamma} k_s^2 k_s^2 ds + 7h \int_{\gamma} k_s^2 ds \\
= -2 \int_{\gamma} k_s^2 ds + 2 \int_{\gamma} k_s^2 k_s^2 ds - 4 \int_{\gamma} k_{ss}^2 k ds \\
-3 \int_{\gamma} k_s^2 k_s^2 ds + 7(-h) \int_{\gamma} k_s^2 k ds, \tag{6.12}
\]
then we calculate the second term in above,

\[
\int_{\gamma} k_{s3}^2 k^2 ds = \int_{\gamma} k_{s3}^2 (k - \bar{k})^2 ds - \bar{k}^2 \int_{\gamma} k_{s3}^2 ds + 2\bar{k} \int_{\gamma} k_{s3}^2 ds
\]

\[
\leq \frac{1}{2} \int_{\gamma} k_{s3}^2 ds + \bar{k} \int_{\gamma} k_{s3}^2 ds + \int_{\gamma} k_{s3}^2 (k - \bar{k})^2 ds,
\]

and

\[
\int_{\gamma} k_{s3}^2 ds \leq 2\bar{k}^2 \int_{\gamma} k_{s3}^2 ds + 2\int_{\gamma} k_{s3}^2 (k - \bar{k})^2 ds
\]

\[
= -2\bar{k}^2 \int_{\gamma} k_{s3} k_{ss} ds + 2\int_{\gamma} k_{s3}^2 (k - \bar{k})^2 ds
\]

\[
\leq 2\bar{k}^2 \left[ \sigma_1 \int_{\gamma} k_{s3}^2 ds + \frac{1}{4\sigma_1} \int_{\gamma} k_{s3}^2 ds \right] + 2\|k_{s3}\|^2 \frac{K_{osc}}{L(0)}
\]

\[
= 2\bar{k}^2 \left[ \sigma_1 \int_{\gamma} k_{s3}^2 ds + \frac{1}{4\sigma_1} \int_{\gamma} k_{s3}^2 (k - \bar{k}) ds \right] + 2\|k_{s3}\|^2 \frac{K_{osc}}{L(0)}
\]

\[
\leq 2\bar{k}^2 \left[ \sigma_1 \int_{\gamma} k_{s3}^2 ds + \frac{1}{4\sigma_1} \left( \sigma_2 \int_{\gamma} k_{s3}^2 ds + \frac{1}{4\sigma_2} K_{osc} L(0) \right) \right] + \frac{2}{\pi} K_{osc} \|k_{s3}\|^2
\]

\[
= 2\bar{k}^2 \left( \sigma_1 + \frac{\sigma_2}{4\sigma_1} \right) \int_{\gamma} k_{s3}^2 ds + 2\bar{k}^2 \frac{1}{16\sigma_1 \sigma_2} K_{osc} L(0) + \frac{2}{\pi} K_{osc} \|k_{s3}\|^2.
\]

Choosing \(\sigma_1 = \frac{\bar{k}}{2\bar{k}^2}\) and \(\sigma_2 = \frac{\bar{k}^2}{k}\), we have

\[
\int_{\gamma} k_{s3}^2 k^2 ds \leq 2 \left( \varepsilon_1 + \frac{K_{osc}}{\pi} \right) \int_{\gamma} k_{s3}^2 ds + \frac{\bar{k}^8}{4\varepsilon_1^4} K_{osc} L(0),
\]

now we compute the third term in (6.12),

\[
-4 \int_{\gamma} k_{s3}^3 k ds = 4 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds + 8 \int_{\gamma} k_{s3} k_{s3} k_{ss} k ds
\]

\[
\leq 8 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds + 4 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds,
\]

As we see several similar terms appear in above calculation, we put the second, third and fourth terms in (6.12) together,

\[
2 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds - 4 \int_{\gamma} k_{s3}^3 k ds - 3 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds
\]

\[
= 6 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds + 5 \int_{\gamma} k_{s3}^2 k_{s3}^2 ds.
\]
The following estimation

\[
5 \int_\gamma k_{ss}^2 \, ds \leq 5 \|k_s\|^2 \int_\gamma k_{ss}^2 \, ds
\]

\[
\leq \frac{5L(0)}{2\pi} \left( \int_\gamma k_{ss}^2 \, ds \right)^2 = \frac{5L(0)}{2\pi} \left[ \int_\gamma k_s (k - \bar{k}) \, ds \right]^2
\]

\[
\leq \frac{5K_{osc}}{2\pi} \int_\gamma k_{ss}^2 \, ds.
\]

yields

\[
2 \int_\gamma k_{ss}^2 \, ds - 4 \int_\gamma k_{ss}^3 \, ds - 3 \int_\gamma k_{ss}^2 \, ds
\]

\[
\leq 6 \left[ 2 \left( \varepsilon_1 + \frac{K_{osc}}{\pi} \right) \int_\gamma k_s^2 \, ds + \frac{\bar{k}^8 K_{osc}}{4\varepsilon_1^3 L(0)} \right] + \frac{5K_{osc}}{2\pi} \int_\gamma k_s^2 \, ds
\]

\[
= \left( 12\varepsilon_1 + \frac{29 K_{osc}}{\pi} \right) \int_\gamma k_s^2 \, ds + \frac{3\bar{k}^8 K_{osc}}{2\varepsilon_1^3 L(0)}
\]

Moving on to the last term in (6.12),

\[
7(-h) \int_\gamma k_s^3 \, ds \leq 7(-h) \frac{L(0)}{2\pi} \sqrt{\frac{L(0)}{2\pi}} \|k\|_\infty \int_\gamma k_{ss}^3 \, ds \cdot \int_\gamma k_s^2 \, ds
\]

\[
\leq \frac{1}{4\varepsilon_2} \int_\gamma k_{ss}^2 \, ds + \varepsilon_2 \cdot 49h^2 \left( \frac{L(0)}{2\pi} \sqrt{\frac{L(0)}{2\pi}} \right)^2 \|k\|_\infty^2 \int_\gamma k_{ss}^2 \, ds
\]

\[
\leq \frac{1}{4\varepsilon_2} \int_\gamma k_{ss}^2 \, ds + \frac{\varepsilon_2^2}{4} \int_\gamma k_s^2 \, ds + \varepsilon_2 \left( \frac{49h^2 L^3(0)}{8\pi^3} \right)^2 \|k\|_\infty^4 K_{osc}
\]

Thus, we get

\[
\frac{d}{dt} \int_\gamma k_{ss}^2 \, ds = -2 \int_\gamma k_{ss}^2 \, ds + 2 \int_\gamma k_{ss}^3 \, ds - 4 \int_\gamma k_{ss}^3 k_s \, ds - 3 \int_\gamma k_{ss}^2 \, ds
\]

\[
+ 7(-h) \int_\gamma k_s^3 \, ds
\]

\[
\leq \left( -2 + 12\varepsilon_1 + \frac{20 K_{osc}}{\pi} \right) \int_\gamma k_s^2 \, ds + \frac{3\bar{k}^8}{2\varepsilon_1^3 L(0)} K_{osc} + \frac{1}{4\varepsilon_2} \int_\gamma k_{ss}^2 \, ds
\]

\[
+ \frac{\varepsilon_2^2}{4} \int_\gamma k_s^2 \, ds + \varepsilon_2 \left( \frac{49h^2 L^3(0)}{8\pi^3} \right)^2 \|k\|_\infty^4 K_{osc}
\]

\[
\frac{d}{dt} \int_\gamma k_{ss}^2 \, ds \leq \left( -2 + 12\varepsilon_1 + \frac{29 K_{osc}}{\pi} + \frac{1}{4\varepsilon_2} + \frac{\varepsilon_2}{4} \right) \int_\gamma k_s^2 \, ds
\]

\[
+ \left[ \varepsilon_2 \left( \frac{49h^2 L^3(0)}{8\pi^3} \right)^2 \|k\|_\infty^4 + \frac{384}{\varepsilon_1^3 L^9(0)} \right] K_{osc}.
\]
When $K_{osc} \leq K^*$, we can have $-2 + 12\varepsilon_1 + \frac{29}{2} \frac{K_{osc}}{\pi} + \frac{1}{4\varepsilon_2} + \frac{\varepsilon_2}{4} = -c(L(0)) \leq 0$.

After enough time, we have $\|k\|_{\infty}^2 \leq 2 \left( \frac{2w\pi}{L(0)} \right)^2 \leq C$ and from Lemma (45), we know $h^2$ is bounded, thus,

$$\frac{d}{dt} \int_{\gamma} k_{ss}^2 ds \leq -c(L(0)) \frac{\pi}{L^4(0)} \|k_{ss}\|_2^2 + cK_{osc} = -\delta \|k_{ss}\|_2^2 + cK_{osc},$$

as $K_{osc} \in L^1([0,\infty))$, we apply Grönwall’s inequality in Proposition 7 to obtain

$$\int_{\gamma} k_{ss}^2 ds \leq ce^{-\delta t}$$

$$\Box$$

Exponential decay of the higher curvature derivatives follows by interpolation using the uniform bounds on $\int_{\gamma} k_{ss}^2 ds$ in Lemma 49.

**Lemma 51.** Suppose $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ solves (6.1) and satisfies conditions of Theorem 10, So $\|k_\ell\|_2$ and $\|k_\ell\|_\infty$ exponentially decays for any $l \in \mathbb{N}$.

**Proof.** As we have proved that when $t \to \infty$, $\int_{\gamma} k_{ss}^2 ds$ exponentially decays in Lemma 50, we can obtain

$$\int_{\gamma} k_{ss}^2 ds = -\int_{\gamma} k_{ss}k_{s4} ds \leq \left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_{s4}^2 ds \right)^{\frac{1}{2}}$$

as $\left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}} \leq \left( e^{-\delta t} \|k_{ss}\|_2 \right)_{t=0}^{\frac{1}{2}}$, and $\left( \int_{\gamma} k_{s4}^2 ds \right)^{\frac{1}{2}}$ is bounded in $L^2$ in Lemma 49, then $\int_{\gamma} k_{ss}^2 ds$ exponentially decays.

By induction argument, we assume that $\int_{\gamma} k_{ss}^2 ds$ exponentially decays,

$$\int_{\gamma} k_{ss}^2 ds = -\int_{\gamma} k_{s4}k_{s5} ds \leq \left( \int_{\gamma} k_{s4}^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_{s5}^2 ds \right)^{\frac{1}{2}}$$

as $\left( \int_{\gamma} k_{ss}^2 ds \right)^{\frac{1}{2}}$ is bounded in Lemma 49, then $\int_{\gamma} k_{s4}^2 ds$ exponentially decays.

Thus, we have $\int_{\gamma} k_{s4}^2 ds$ exponentially decays in $L^2$. As $\|k_\ell\|_\infty \leq \frac{L}{2\pi} \|k_{s4}\|_2$, we have in $L^\infty$ norm decays exponentially. $\Box$

From above proposition, we can also obtain uniform bounds for all derivatives of the evolving curve $\gamma$.

**Proposition 24.** Suppose $\gamma_0 : S^1 \to \mathbb{R}^2$ solves (6.1) and satisfies the conditions of Theorem 10. Then for all $l \in \mathbb{N}_0$, $\|\partial_\ell \gamma\|_\infty \leq c(l) + \sum_{p=0}^{l} \|\partial_\ell \gamma_0\|_\infty$. 


Proof. We claim that for \( l, p \in \mathbb{N}_0, l \geq 1, \)

\[
\partial_t \partial_j \gamma = v \sum_{p=0}^l \left( P_{1+l-p}^{2+p}(k) + P_{l-p}^p(k) \int_\gamma P_2^2(ds) \right) \\
+ \tau \sum_{p=0}^l \left( P_{1+l-p}^{2+p}(k) + P_{l-p}^p(k) \int_\gamma P_2^2(ds) \right). \tag{6.13}
\]

We prove this by induction. First, we have \( \partial_t \gamma = Fv = \left[ P_1^2(k) + \int_\gamma P_2^2(ds) \right] v \), we do the differentiation

\[
\partial_t \partial_j \gamma = kF \cdot \partial_j \partial_j \gamma + \partial_t (\partial_j \partial_j \gamma)
\]

\[
= kF \cdot \partial_j \tau + \partial_t \left[ v \sum_{p=0}^l \left( P_{1+l-p}^{2+p}(k) + P_{l-p}^p(k) \int_\gamma P_2^2(ds) \right) \right]
+ \partial_t \left[ \tau \sum_{p=0}^l \left( P_{1+l-p}^{2+p}(k) + P_{l-p}^p(k) \int_\gamma P_2^2(ds) \right) \right]
\]

\[
= kFv \sum_{p+q=l} P_{q}^p(k) + v \sum_{p=0}^l \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right)
+ kF \tau \sum_{p+q=l} P_{q}^p(k) + \tau \sum_{p=0}^l \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right)
\]

\[
= v \sum_{p+q=l} \left( P_{q+2}^{p+2}(k) + P_{q+1}^p(k) \int_\gamma P_2^2(ds) \right) + v \sum_{p=0}^l \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right)
+ \tau \sum_{p+q=l} \left( P_{q+2}^{p+2}(k) + P_{q+1}^p(k) \int_\gamma P_2^2(ds) \right)
+ \tau \sum_{p=0}^l \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right)
\]

\[
= v \sum_{p=0}^{l+1} \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right) + \tau \sum_{p=0}^{l+1} \left( P_{2+l-p}^{2+p}(k) + P_{1+l-p}^p(k) \int_\gamma P_2^2(ds) \right)
\]

as required. Integrating (6.13) and using Lemma 51, we find

\[
\|\partial_j \gamma\|_\infty \leq \|\partial_j \gamma_0\|_\infty + c \int_0^t e^{-\epsilon t'} dt' \leq \|\partial_j \gamma_0\|_\infty + \bar{c}(l).
\]

As \( u \) is the initial space parameter before reparameterization by arc-length, set \( v = |\partial_u \gamma| \).
Referring to the proof of Theorem 3.1 in [17], then for any function \( \Phi : S \to \mathbb{R} \), we have
\[
\partial^i_u \Phi = v \partial_s^i \Phi + P^i(v, ..., \partial_{u}^{i-1} v, \Phi, ..., \partial_{s}^{i-1} \Phi)
\]
where \( P^i \) is a polynomial. Then we obtain
\[
\| \partial^i_u \gamma \|_{\infty} \leq \| \partial^i_s \gamma_0 \|_{\infty} + c(I).
\]
as required.

Using Lemma 50 with Lemma 6.1 gives in turn exponential decay of \( \int_{\gamma} k_{2}^2 ds \), applying the PSW inequality (2.3) in Proposition 3 in the first step in the following estimate,
\[
\| k_{s} \|_{2}^{2} \leq \frac{L^2}{4\pi^2} \| k_{ss} \|_{2}^{2} \leq \frac{L^2(0)}{4\pi^2} e^{-\delta t} \| k_{ss} \|_{2}^{2} \big|_{t=0} \leq c e^{-\delta t},
\]
then as \( h = -\frac{\int_{\gamma} k_{2}^2 ds}{2\pi} \), hence we get the decay for \( h(t) \).

Also \( \int_{\gamma} (k - \bar{k})^2 ds \) decays exponentially as follows,
\[
\int_{\gamma} (k - \bar{k})^2 ds \leq \frac{L^2}{4\pi^2} \int_{\gamma} (k_{s}^2 ds \leq c e^{-\delta t},
\]
and the corresponding \( L^\infty \) norms,
\[
\| k - \bar{k} \|_{\infty}^{2} \leq \frac{L(0)}{2\pi} \| k - \bar{k} \|_{2}^{2} \leq c e^{-\delta t}.
\]

This implies subconvergence of the flow to circles with perimeter length \( L(0) \).

**Lemma 52.** Suppose \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) solves (6.1) and satisfies conditions of Theorem 10, there exist a subsequence of time \( t_j \) such that
\[
\gamma(\cdot, t_j) \to \gamma_{\infty}
\]
uniformly with \( \gamma_{\infty} \) a circle.

**Corollary 2.** Suppose \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) solves our flow and satisfies the assumptions of Theorem 10. then the curve diffusion flow exists globally \( (T = \infty) \) and converges exponentially fast to a round circle with radius \( \frac{L(0)}{2\pi} \).

### 6.5 The unique limiting image

**Lemma 53.** The image of limiting round circle is unique.

**Proof.** In order to prove that \( \gamma_{\infty} \) is the unique limit, we refer to [1, Theorem A.1]. Suppose there exists a sequence \( \{s_j\} \subset [0, \infty) \), \( s_j \to \infty \), such that \( \gamma(\cdot, s_j) \to \gamma_{\infty} = \gamma_{\infty} \), in \( C^\infty \).
Consider the functional
\[ G[\gamma] = \int_\gamma |\gamma - \gamma_\infty|^2 \, ds. \]

Since \( \bar{\gamma}_\infty \) and \( \gamma_\infty \) are smooth, it follows that
\[
\lim_{s_j \to \infty} G[f(\cdot, s_j)] \neq 0. \tag{6.14}
\]

We estimate by using Lemma 1,
\[
\left| \frac{d}{dt} G \right| = \frac{d}{dt} \int_\gamma |\gamma - \gamma_\infty|^2 \, ds = 2 \int_\gamma |\gamma - \gamma_\infty| \cdot \frac{d}{dt} |\gamma - \gamma_\infty| \, ds + \int_\gamma |\gamma - \gamma_\infty|^2 \cdot \frac{d}{dt} ds \\
= 2 \int_\gamma F \cdot |\gamma - \gamma_\infty| \, ds + \int_\gamma kF \cdot |\gamma - \gamma_\infty|^2 \, ds \\
= \int_\gamma F \cdot |\gamma - \gamma_\infty| \cdot (2 + k|\gamma - \gamma_\infty|) \, ds \\
\leq \left( \int_\gamma F^2 \, ds \right)^{\frac{1}{2}} \cdot \left[ \int_\gamma |\gamma - \gamma_\infty|^2 \left( 2 + k|\gamma - \gamma_\infty| \right)^2 \, ds \right]^{\frac{1}{2}} \\
\leq \left( \int_\gamma F^2 \, ds \right)^{\frac{1}{2}} \cdot \left[ \int_\gamma |\gamma - \gamma_\infty|^2 \left( 4 + k^2|\gamma - \gamma_\infty|^2 \right) \, ds \right]^{\frac{1}{2}} \\
\leq c \left( \int_\gamma F^2 \, ds \right)^{\frac{1}{2}} \cdot \left( 4 \int_\gamma |\gamma - \gamma_\infty|^2 + k^2|\gamma - \gamma_\infty|^4 \, ds \right)^{\frac{1}{2}} \\
\leq c \left( \int_\gamma F^2 \, ds \right)^{\frac{1}{2}} \cdot \left( \int_\gamma |\gamma|^2 \, ds + \int_\gamma |\gamma_\infty|^2 \, ds + \int_\gamma |k|^2 \cdot |\gamma|^4 \, ds + \int_\gamma |k|^2 \cdot |\gamma_\infty|^4 \, ds \right)^{\frac{1}{2}}
\]

Now we show that \( \int_\gamma |\gamma|^2 \, ds \), \( \int_\gamma |\gamma_\infty|^2 \, ds \), \( \int_\gamma |k|^2 \cdot |\gamma|^4 \, ds \) and \( \int_\gamma |k|^2 \cdot |\gamma_\infty|^4 \, ds \) are uniformly bounded.

First, we can see that under the exponential decay of curvature and its derivatives in \( C^\infty \), we obtain that
\[
||\gamma(\cdot, \bar{t})||^2_\infty = \left( \int_\bar{t}^\infty |F| \, dt \right)^2 \leq \bar{c} \left( \int_\bar{t}^\infty e^{-\delta t} \, dt \right)^2 \leq \frac{\bar{c}}{\delta} e^{-\delta \bar{t}} \leq c_1,
\]

then
\[
\int_\gamma |\gamma|^2 \, ds \leq ||\gamma||^2_\infty \int_\gamma ds \leq c_1 L(0) \leq c_2, \\
\int_\gamma |k|^2 \cdot |\gamma|^4 \, ds \leq ||\gamma||^4_\infty \int_\gamma |k|^2 \, ds \leq c_3.
\]

From above two estimates, it is clear that \( \int_\gamma |\gamma_\infty|^2 \, ds \) and \( \int_\gamma |k|^2 \cdot |\gamma_\infty|^4 \, ds \) are bounded as
Thus, 
\[ \left| \frac{d}{dt} G \right| \leq c \| F \|_2, \]

By the exponential decay of curvature and its derivatives in $L^2$, we have that
\[ G[f(\cdot, s_j)] \leq c \int_{s_j}^\infty \| F \|_2 dt \leq c \int_{s_j}^\infty e^{-\delta t} dt = ce^{-\delta s_j} \]

it follows that
\[ \lim_{s_j \to \infty} G[f(\cdot, s_j)] = 0, \]

which is in contradiction with (6.14).

This proves that there does not exist a sequence $\{s_j\}$, the convergence of flow in $L^2$ to a straight line segment is unique. We can obtain the exponential convergence in $C^\infty$ to a unique line segment as the curvature and all its derivatives exponentially decay.

This finishes the proof.

Therefore, we obtain the limiting curve image is unique.

Exponential decay of the speed allows us to compute the bounded distance between the limit solution and initial curve $\gamma_0$. For any $x \in [-1, 1]$, we have
\[ |\gamma(x, t) - \gamma(x, 0)| = \left| \int_0^t \frac{\partial \gamma}{\partial t}(x, \tau) \right| \leq \int_0^t |h(\tau) - \kappa_{ss}| d\tau \leq \frac{c}{\delta} \left( 1 - e^{-\delta t} \right) \leq \frac{c}{\delta} . \]

It follows that solution converge exponentially to a translate of the circle, with no correction for translations.

### 6.6 Self-similar solutions

A self-similar solution to a curvature flow equation such as (6.1) is a solution whose image maintains the same shape as it evolves. Its image changes in time only by translation, scaling and rotation. In our setting, the constrained length rules out expanding and contracting self-similar solutions, so we focus on stationary solutions, translators and rotators. Here we refer to [20].

**Lemma 54.** The only smooth, closed stationary solutions to (6.1) are multiply-covered circles.

**Proof.** Such solutions satisfy
\[ h(t) - k_{ss} \equiv 0, \]
which is \( \frac{1}{2\pi} \int_{\gamma} k_s^2 ds - k_{ss} \equiv 0 \).

Integrating this equation over \( \gamma \), we obtain

\[
\frac{L(0)}{2\pi} \int_{\gamma} k_s^2 ds \equiv 0,
\]

so these closed curves have \( k_s \equiv 0 \) and are circles, where the length is controlled by the initial length \( L(0) \).

A family of curves \( \gamma : S^1 \times [0,T) \to \mathbb{R}^2 \) evolving purely by translation satisfies

\[
\gamma(u,t) = \gamma_0(u) + Vt + \gamma_s(u,t)\phi(u,t)
\]

for some constant vectors \( V \) and smooth diffeomorphism \( \phi \). As \( \phi(u,t) \) is the tangential diffeomorphism, then

\[
\left( \gamma_s \frac{d}{dt} \phi, \nu \right) = 0.
\]

In this case, if \( \frac{\partial \gamma}{\partial t}(u,t) = Fv = [-k_{ss} + h(t)] \cdot \nu(u,t) \),

\[
\left( \frac{\partial \gamma}{\partial t}, \nu \right) = \langle V, \nu \rangle,
\]

then \( \gamma \) must satisfies

\[
-k_{ss} + h(t) \equiv \langle V, \nu \rangle.
\]

We call the solution \( \gamma \) a translator. If \( V = 0 \), then the solution \( \gamma \) is stationary, a trivial translator.

**Lemma 55.** Let \( \gamma : S^1 \to \mathbb{R}^2 \) be a smooth, closed, translating solution to (6.1). Then \( \gamma \) is trivial; that is, \( \gamma(S^1) \) is a standard round circle.

**Proof.** Integrating \( -k_{ss} + h(t) \equiv \langle V, \nu \rangle \) by \( k \) gives

\[
\int_{\gamma} [-k_{ss} + h(t)] ds = \int_{\gamma} \langle V, \nu \rangle ds.
\]

As \( V \) is a constant vector and the curve is closed, we have

\[
\int_{\gamma} \langle V, \nu \rangle ds = 0,
\]

and

\[
\int_{\gamma} -k_{ss} + h(t) ds = \int_{\gamma} h(t) ds = h(t) \int_{\gamma} ds = h(t)L(0),
\]

then we have

\[
h(t)L(0) = \int_{\gamma} \langle V, \nu \rangle ds = 0.
\]
As \( L(0) \neq 0 \), then \( h(t) = 0 \) and we have

\[-k_{ss} \equiv \langle V, v \rangle.\]

Integrating by parts, as \( \gamma_s = kv \), we obtain

\[
\int_{\gamma} k_s^2 ds = -\int_{\gamma} kk_{ss} ds = \int_{\gamma} \langle V, kv \rangle ds
= \int_{\gamma} \langle V, \gamma_s \rangle ds = \int_{\gamma} \partial_s \langle V, \gamma_s \rangle ds
= 0
\]
yields \( k_s \equiv 0 \) on \( \gamma \) which gives the curvature is constant and \( \gamma \) must be a round circle.

\[\square\]

**Lemma 56.** Let \( \gamma : \mathbb{S}^1 \to \mathbb{R}^2 \) be a smooth, closed, rotating solution to (6.1). Then \( \gamma(\mathbb{S}^1) \) is a standard round circle.

**Proof.** A family of curves \( \gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2 \) evolving purely by rotation satisfies

\[-k_{ss} + h(t) \equiv 2S(t)\langle \gamma_s, \gamma \rangle.\]

Since \( \gamma \) is closed, integrating \(-k_{ss} + h(t) \equiv 2S(t)\langle \gamma_s, \gamma \rangle \) by \( k \) gives

\[
\int_{\gamma} -k_{ss} + h(t) ds = \int_{\gamma} 2S(t)\langle \gamma_s, \gamma \rangle ds = 2S(t)\int_{\gamma} \langle \gamma_s, \gamma \rangle ds
\]

As \( \int_{\gamma} \langle \gamma_s, \gamma \rangle ds = \frac{1}{2} \int_{\gamma} |\gamma|^2 ds = 0 \) and \( \int_{\gamma} -k_{ss} + h(t) ds = \int_{\gamma} h(t) ds = h(t) \int_{\gamma} ds = h(t)L(0) \), we have

\[h(t)L(0) = 2S(t)\int_{\gamma} \langle \gamma_s, \gamma \rangle ds = 0.\]

As \( L(0) \neq 0 \), then \( h(t) = 0 \) and we have

\[-k_{ss} \equiv 2S(t)\langle \gamma_s, \gamma \rangle.\]

Integrating by parts, as \( \langle v, \gamma_s \rangle = 0 \), we obtain

\[
\int_{\gamma} k_s^2 ds = -\int_{\gamma} kk_{ss} ds = 2S(t)\int_{\gamma} \langle k\gamma_s, \gamma \rangle ds
= -2S(t)\int_{\gamma} \langle v_s, \gamma \rangle ds = 2S(t)\int_{\gamma} \langle v, \gamma_s \rangle ds
= 0
\]
yields \( k_s \equiv 0 \) on \( \gamma \), which gives us that the curvature is constant. Thus, we conclude that \( \gamma \) is a round circle. \[\square\]
6.7 Embeddedness

The result that solutions with sufficiently small oscillation of curvature remain embedded for all time applies by referring to [82, Theorem 1.6] (see also [51, Theorem 6]). This theorem is showed as follows.

**Theorem 12.** Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth immersed curve with winding number $w$ and let $m$ denote the maximum number of times $\gamma$ intersects itself in any one point; that is

$$\int_\gamma k ds = 2w\pi \quad \text{and} \quad m(\gamma) = \sup_{x \in \mathbb{R}^2} |\gamma^{-1}(x)|.$$

Then,

$$K_{osc}(\gamma) \geq 16m^2 - 4w^2\pi^2.$$

**Proposition 25.** Any solution of (6.1) with initial embedded curve $\gamma_0$ satisfying the assumption of Theorem 10 remains embedded for all time.

**Proof.** First we find the point $x_0$ where the curve $\gamma$ intersects itself the most. Let $m$ denote the maximum number of times $\gamma$ intersects itself in any one point, $m(\gamma) = \sup_{x \in \mathbb{R}^2} |\gamma^{-1}(x)|$.

We know that $\gamma^{-1}(x_0) = m$. When $m = 2$ the curve $\gamma$ has one intersection.

It is proven in Proposition 20 that $K_{osc} \leq 2K^*$. Let $m = 2$, by assumption $K_{osc}(0) < K^*$ in Theorem 10, we have $2K^* < 16 \cdot 2^2 - 4\omega^2\pi^2 = 64 - 4\pi^2$, which means that the curve does not have any intersection.

As the initial curve is embedded, thus the curve $\gamma$ is an embedded curve for each $t$. \qed
Chapter 7

Constrained ideal curve flow

Let us define the energy for the smooth closed planar curve \( \gamma : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2 \) by

\[
E[\gamma] = \frac{1}{2} \int_{\gamma} k_s^2 ds.
\]

We define the constrained ideal curve flow as follows,

\[
\begin{aligned}
\partial_t \gamma &= (k_s^4 + k_s^2 k_{ss} - \frac{1}{2}kk_s^2 + h(t)) \nu, \quad \text{for all } (s, t) \in \mathbb{S}^1 \times [0, T) \\
\gamma|_{t=0} &= \gamma_0, \quad \text{for all } s \in \mathbb{S}^1
\end{aligned}
\]

\hspace{1cm} (7.1)

Denote that \( \nu \) is the normal vector field of \( \gamma \). We let \( F := G + h(t) = k_s^4 + k_s^2 k_{ss} - \frac{1}{2}kk_s^2 + h(t) \). If let the curve move under the suitable six-order curvature flow, then we can control the length and the area of this closed curve. In this chapter, we will study two constrained ideal curve flow: the length-constrained ideal curve flow and area-preserving ideal curve flow.

7.1 Length-constrained ideal curve flow

Suppose the curve evolves by the sixth-order curvature flow

\[
\begin{aligned}
\partial_t \gamma &= (k_s^4 + k_s^2 k_{ss} - \frac{1}{2}kk_s^2 + h_1(t)) \nu, \quad \text{for all } (s, t) \in \mathbb{S}^1 \times [0, T) \\
\gamma|_{t=0} &= \gamma_0, \quad \text{for all } s \in \mathbb{S}^1
\end{aligned}
\]

\hspace{1cm} (7.1)

To preserve length of the evolving curve \( \gamma \) we take

\[
h_1(t) = \frac{1}{2\pi \omega} \left( - \int_{\gamma} k_s^2 ds + \frac{7}{2} \int_{\gamma} k_s^2 k_s^2 ds \right)
\]

\hspace{1cm} (7.2)

where \( \omega \) denotes the winding number of \( \gamma \).

Our main theorem is as follows:
CHAPTER 7. CONSTRAINED IDEAL CURVE FLOW

Theorem 13. Suppose $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}^2$ be a regular smooth immersed closed curve with fixed length $L(0)$ and $\omega(0) = 1$, for a small absolute constant $\varepsilon$, satisfies condition

$$E(0) \leq \varepsilon$$

the length-constrained ideal curve flow (7.1) with initial data $\gamma_0$ exists for all time and converges exponentially to a round circle with radius $\frac{L(0)}{2\pi}$.

The short time existence is shown in Chapter 3, Theorem 5. From Lemma 4, we can get that $\omega(t) = \omega(0) = 1$.

We show the curve remains embedded with the smallness condition in Theorem 13.

Proposition 26. Any solution of (7.1) with initial embedded curve $\gamma_0$ satisfying the assumptions of Theorem 13 remains embedded for all time.

Under the assumption, we have the $\varepsilon$ is small enough to satisfy the condition in [82, Theorem 1.6]. This theorem is showed as Theorem 12 in Chapter 6.

The structure of the proof of Theorem 13 is as follows. Firstly, in section 7.1.1 we focus on estimating the energy which appears to be decaying exponentially. With this result in hand, in section 7.1.2 we move on to the analysis of the global behaviour of the flow. We prove that the curve exists for all time and converges exponentially fast to a round circle under the stated conditions. Finally, we show this round circle is unique, then complete the proof.

Lemma 57. Under the flow (7.1), while a solution exists, it satisfies

$$L(t) = L(0).$$

Proof. We compute

$$\frac{d}{dt}L(t) = \int_{\gamma} kFds$$

$$= \int_{\gamma} kk_s ds + \int_{\gamma} k^3 k_s ds - \frac{1}{2} \int_{\gamma} kk_s^2 ds + h_1(t) \int_{\gamma} k ds$$

$$= \int_{\gamma} k^2 s ds - \frac{7}{2} \int_{\gamma} k_s^2 ds + 2\pi h_1(t)$$

As $h_1(t) = \frac{1}{2\pi} \left( -\int_{\gamma} k_s^2 ds + \frac{7}{2} \int_{\gamma} k^2 ds \right)$ in (7.2), the result follows.

Here, we establish a suitable bound on $h$ in terms of the $L^2$ norms of $k$.

Lemma 58. For a constant $c > 0$, the global term $h_1(t)$ may be estimated as

$$h_1(t) \leq c \left( \int_{\gamma} k^2 ds \right)^5.$$
Proof. By using interpolation inequality in Lemma 4, we estimate (7.2)

\[
    h_1(t) = \frac{1}{2\pi} \left( -\int_{\gamma} k''^2 ds + \frac{7}{2} \int_{\gamma} k^2 k'^2 ds \right)
    \leq \frac{1}{2\pi} \left[ -\int_{\gamma} k''^2 ds + \varepsilon \int_{\gamma} k''^2 ds + c \left( \int_{\gamma} k^2 ds \right)^5 \right]
    \leq c \left( \int_{\gamma} k^2 ds \right)^5
\]

\[\square\]

7.1.1 Exponential decay

Now we show a useful proposition.

Proposition 27. Let \( \gamma_0 : S \to \mathbb{R}^2 \) be a smooth immersed curve. Then there exist universal constants \( c \) and \( \varepsilon_1 = \varepsilon_1(L(0)) > 0 \) depending only on \( w \) such that

\[
    E(0) < \varepsilon_1 \quad \implies \quad \int_{\gamma} G^2 ds \geq cL^{-6}E.
\]

For the proof of above proposition see [1, Proposition 7.2]. We also put the proof in Appendix A.2.

Next we prove that the energy decays exponentially by using proposition 27.

Proposition 28. There exists an \( \varepsilon_2 = \varepsilon_2(L(0)) \) such that

\[
    E(0) < \varepsilon_2 \quad \implies \quad \frac{d}{dt} E(t) \leq -\frac{c}{2} \int_{\gamma} k^2 ds.
\]

Proof. From Lemma 57, we have \( h_1 = -\frac{1}{2\pi} \int_{\gamma} kG ds \), then

\[
    \frac{d}{dt} E = -\int_{\gamma} G^2 ds + \frac{1}{2\pi} \int_{\gamma} kG ds \int_{\gamma} G ds,
\]

Let us first note the estimate

\[
    \left| \int_{\gamma} G ds \right| \leq \frac{5}{2} \|k\|_{\infty} \int_{\gamma} k''^2 ds
    \leq 5E \left( \|k - \overline{k}\|_{\infty} + \frac{2\pi}{L} \right)
    \leq 5E \left( \frac{L^2(0)E}{2\pi^2} \right)^{1/2} + \frac{2\pi}{L(0)} := f_1(E, L(0)).
\]

Using this we find
\[
\frac{1}{2\pi} \int_{\gamma} G ds \int_{\gamma} k G ds \leq \frac{f_1(E,L(0))}{2\pi} \int_{\gamma} k G ds
\]
\[
\leq \frac{1}{2} \int_{\gamma} G^2 ds + \frac{[f_1(E,L(0))]^2}{8\pi^2} \int_{\gamma} k^2 ds
\]
\[
\leq \frac{1}{2} \int_{\gamma} G^2 ds + \frac{[f_1(E,L(0))]^2}{8\pi^2} \left( \frac{L^2(0)}{2\pi^2} - E + \frac{4\pi^2}{L(0)} \right).
\]

Now observe that the second expression satisfies
\[
\frac{[f_1(E,L(0))]^2}{8\pi^2} \left( \frac{L^2(0)}{2\pi^2} - E + \frac{4\pi^2}{L(0)} \right) \leq c_0 E^2 (1 + E^2).
\]

Recall the estimate
\[
\int_{\gamma} G^2 ds \geq c \int_{\gamma} k^2_\alpha ds - c_1 E^2 (1 + E^3)
\]
from the estimate (A.3) in the proof of Proposition 27, \(\varepsilon_2 \leq \varepsilon_1\). Combining these, we find
\[
\frac{d}{dt} E \leq -c \int_{\gamma} k^2_\alpha ds + c_2 E^2 (1 + E^3).
\]

Now observe that
\[
E \leq \frac{1}{2} \left( \frac{L(0)}{2\pi} \right)^6 \int_{\gamma} k^2_\alpha ds
\]
which implies
\[
\frac{d}{dt} E \leq \int_{\gamma} k^2_\alpha ds \left( c_3 E (1 + E^3) - c \right).
\]

Therefore, for \(E(0)\) small enough depending only on \(L(0)\) and \(\omega\), we have
\[
\frac{d}{dt} E \leq -c \int_{\gamma} k^2_\alpha ds,
\]
as required. \(\square\)

Proposition 28 implies both uniform \(L^1\)-control on \(\|k_\alpha\|^2\) as well as exponential decay of the ideal curve energy.

**Corollary 3.** Under the conditions of Proposition 28, we have
\[
E(t) \leq E(0) e^{-\delta t}
\]
where \(\delta = \delta(L(0))\) and
\[
\int_0^T \|k_\alpha\|^2(t) dt \leq \frac{2}{c} E(0).
\]
7.1.2 Global Existence

Let us now give two useful lemmas before showing uniform bounds for all derivatives of curvature.

**Lemma 59.** For any immersed curve $\gamma : \mathbb{S} \to \mathbb{R}^2$ we have the estimate

$$\| k \|_\infty \leq 2L(0)^{1/2}E(0) + \frac{2\pi}{L(0)} := C_0.$$  

where $C_0$ only depends on $L(0)$ and $E(0)$.

**Proof.** We calculate

$$k = k - \bar{k} + \bar{k} \leq \int_\gamma |k_s|ds + \frac{2\pi}{L}.$$  

Taking a supremum and using the Hölder inequality, we find

$$\| k \|_\infty \leq \frac{1}{L} \left( \sqrt{L^3 \| k_s \|_2^2 + 2\pi} \right) \leq 2L(0)^{1/2}E(0) + \frac{2\pi}{L(0)}.$$  

\[Q.E.D.\]

**Lemma 60.** Under the flow (7.1), for $E(0)$ sufficiently small, there is a corresponding constant such that

$$|h(t)| \leq c_h,$$

where $c_h$ only depends on $L(0)$ and $E(0)$.

**Proof.** We have from Lemma 3 (v) and Proposition 4

$$\frac{d}{dt} \int_\gamma k^2_s ds \leq -2 \int_\gamma k^2_s ds + 2 \int_\gamma k_s \left[ P^{l+4}_5(k) + P^{l+2}_5(k) \right] - h(t) \int_\gamma k^2_s ds$$

$$\leq (-2 + \varepsilon) \int_\gamma k^2_s ds + c(\varepsilon) \left( \int_\gamma k^2_s ds \right)^{2\varepsilon+7} - h(t) \int_\gamma k^2_s ds, \quad (7.3)$$

For $l = 2$, we have

$$\frac{d}{dt} \int_\gamma k^2_s ds \leq -(2 - \varepsilon) \int_\gamma k^2_s ds + c \left( \int_\gamma k^2_s ds \right)^{11} + \frac{1}{2\pi} \int_\gamma k^2_s ds \int_\gamma k^2_s ds$$

$$- \frac{7}{4\pi} \int_\gamma k^2_s ds \int_\gamma k^2_k ds$$

Using Lemma 59 and Hölder inequality, we estimate
\[
\frac{1}{2\pi} \int_\gamma k^2_{ss} ds \int_\gamma kk^2_{ss} ds \leq \frac{C_0}{2\pi} \left( \int_\gamma k^2_{ss} ds \right)^2 = \frac{C_0}{2\pi} \left( -\int_\gamma k_s k_{ss} ds \right)^2 \leq \frac{C_0}{\pi} E(t) \int_\gamma k^2_s ds \leq \frac{C_0 E(0)L^4(0)}{16\pi^5} \int_\gamma k^2_s ds
\]

\[
-\frac{7}{4\pi} \int_\gamma k^2_s ds \int_\gamma kk^2_{ss} ds \leq \frac{7C^3_0}{2\pi} E(t) \int_\gamma k^2_s ds \leq \frac{7C^3_0 E(0)L^6(0)}{128\pi^7} \int_\gamma k^2_s ds
\]

Hence

\[
\frac{d}{dt} \int_\gamma k^2_{ss} ds \leq - \left( 2 - \varepsilon - \frac{C_0 E(0)L^4(0)}{16\pi^5} - \frac{7C^3_0 E(0)L^6(0)}{128\pi^7} \right) \int_\gamma k^2_s ds + \left( \int_\gamma k^2_s ds \right)^{11}
\]

Assuming \( E(0) \) is small enough, we can get that \( \int_\gamma k^2_{ss} ds \) is bounded, \( \int_\gamma k^2_s ds \leq c_2 \), here \( c_2 \) only depends on \( L(0) \) and \( E(0) \).

We estimate from (7.2)

\[
|h(t)| \leq \frac{1}{2\pi} \int_\gamma k^2_s ds + \frac{7}{4\pi} \int_\gamma k^2_{ss} ds \leq \frac{1}{2\pi} \int_\gamma k^2_s ds + \frac{7}{4\pi} \|k\|_\infty^2 \int_\gamma k^2_s ds \leq \frac{1}{2\pi} c_2 + \frac{7}{4\pi} C_0 E(0) = c_h
\]

where \( c_h \) only depends on \( L(0) \) and \( E(0) \).

\[ \square \]

**Proposition 29.** Under the flow (7.1), for \( E(0) \) sufficiently small, there are corresponding constants such that

\[ \int_\gamma k^2_{ss} ds \leq c_1 \]

**Proof.** For \( \ell = 0 \) we have the uniform estimate via the inequality in Proposition 3 and Corollary 3:

\[ \int_\gamma k^2 ds = \int_\gamma (k - \tilde{k})^2 ds + \frac{4\pi^2}{L(0)} \leq \frac{L^2(0)}{2\pi^2} E(t) + \frac{4\pi^2}{L(0)} \]

For \( \ell = 1 \), it is the first estimate is in Corollary 3. For each \( \ell \geq 2 \) we have from (7.3) and using Proposition 4, Proposition 3 and Lemma 60,
Thus we get the conclusion \( \int k^2 ds \leq c_1. \)

**Proposition 30.** Under the flow (7.1), with \( E(0) < \varepsilon \), we have that there exist constants \( \delta, \ell \) depending only on \( L_0 \) and \( \omega \) such that

\[
\int k^2 ds \leq e^{-\delta t} \int k^2 ds \bigg|_{t=0},
\]

for \( \ell \geq 1, t \in [0, T] \). We can also have \( k_t \) exponentially decays in \( L^\infty \).

**Proof.** Note that \( \varepsilon \) satisfies the requirements in Proposition 28 and Lemma 60.

When \( t \to \infty \), \( \int k^2 ds \) exponentially decays in Corollary 3

\[
K_{osc} \leq \frac{L^3(0)}{4\pi^2} \int k^2 ds \leq ce^{-\delta t},
\]

then \( K_{osc} \) exponentially decays.

\[
\int k^2 ds = \int (k - \bar{k})k_t ds \leq \left[ \int (k - \bar{k})^2 ds \right]^{1/2} \left( \int k^2 ds \right)^{1/2},
\]

as \( \int k^2 ds \) is bounded in \( L^2 \) in Proposition 29, then \( \int k^2 ds \) exponentially decays.

We let \( l \geq 1, \)

\[
\int k^2 ds \leq \left( \int k^2 ds \right)^{1/2} \left( \int k^2 ds \right)^{1/2},
\]

as \( \int k^2 ds \) is bounded in \( L^2 \), then \( \int k^2 ds \) exponentially decays in \( L^2 \). Also \( \|k_t\|_\infty \leq \frac{L}{2\pi} \|k_{\ell+1}\|^2 \) so we have \( k_t \) exponentially decays in \( L^\infty \).

We have that all curvature derivatives remain bounded in \( L^2 \) and \( L^\infty \) from above proposition. That these bounds are independent of \( T \) implies \( T = \infty \), then we get the following corollary.

**Corollary 4.** Suppose \( \gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 \) solves (7.1) and satisfies the conditions of Theorem 13. Then \( T = \infty \).
From proposition 30, we can also obtain uniform bounds for all derivatives of the evolving curve $\gamma$.

**Proposition 31.** Suppose $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$ solves (7.1) and satisfies the conditions of Theorem 13. Then for all $l \in \mathbb{N}_0$,

$$\|\partial_l \gamma\|_{\infty} \leq c(l) + \sum_{p=0}^{l} \|\partial_{t^p} \gamma_0\|_{\infty},$$

where $c(l)$ is a constant only depending on $l$, $E(0)$, $L(0)$.

**Proof.** We claim that for $l, p \in \mathbb{N}_0$, $l \geq 1$,

$$\partial_t \partial_{s^l} \gamma = v \sum_{p=0}^{l} \left[ P_{1+1-p}^{4+p} (k) + P_{3+1-p}^{2+p} (k) + P_{1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right]$$

$$+ \tau \sum_{p=0}^{l} \left[ P_{1+1-p}^{4+p} (k) + P_{3+1-p}^{2+p} (k) + P_{1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right].$$

We prove this by induction. First, $\partial_t \gamma = Fv = \left\{ P_{1}^{4} (k) + P_{3}^{2} (k) + \int_{\gamma} [P_{2}^{4} (k) + P_{4}^{2} (k)] \, ds \right\} v$.

for $q \in \mathbb{N}_0$, we do the differentiation

$$\partial_t \partial_{s^{l+1}} \gamma = kF \cdot \partial_{s^{l+1}} \gamma + \partial_t (\partial_k \partial_{s^l} \gamma)$$

$$= kF \cdot \partial_{s^l} \tau$$

$$+ \partial_t \left[ v \sum_{p=0}^{l} \left( P_{1+1-p}^{4+p} (k) + P_{3+1-p}^{2+p} (k) + P_{1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right) \right]$$

$$+ \partial_t \left[ \tau \sum_{p=0}^{l} \left( P_{1+1-p}^{4+p} (k) + P_{3+1-p}^{2+p} (k) + P_{1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right) \right]$$

$$= kFv \sum_{p+q=l} P_{q}^{p} (k) + kF \tau \sum_{p+q=l} P_{q}^{p} (k)$$

$$+ v \sum_{p=0}^{l} \left[ P_{2+1-p}^{4+p} (k) + P_{4+1-p}^{2+p} (k) + P_{1+1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right]$$

$$+ v \sum_{p=0}^{l} \left[ P_{1+1-p}^{5+p} (k) + P_{3+1-p}^{3+p} (k) + P_{1-p}^{1+p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right]$$

$$+ \tau \sum_{p=0}^{l} \left[ P_{2+1-p}^{4+p} (k) + P_{4+1-p}^{2+p} (k) + P_{1+1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right]$$

$$+ \tau \sum_{p=0}^{l} \left[ P_{1+1-p}^{5+p} (k) + P_{3+1-p}^{3+p} (k) + P_{1-p}^{1+p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right]$$

$$+ \tau \sum_{p=0}^{l} \left[ P_{2+1-p}^{4+p} (k) + P_{4+1-p}^{2+p} (k) + P_{1+1-p}^{p} (k) \int_{\gamma} (P_{2}^{4} (k) + P_{4}^{2} (k)) \, ds \right].$$
\[ v \sum_{p+q=1} \left[ P_{q+2}^{p+4}(k) + P_{q+4}^{p+2}(k) + P_{q+1}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
+ v \sum_{p=0}^{l} \left[ P_{2+l-p}^{p+p}(k) + P_{4+l-p}^{p+p}(k) + P_{1+l-p}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
+ v \sum_{p=1}^{l+1} \left[ P_{2+l-p}^{p+p}(k) + P_{4+l-p}^{p+p}(k) + P_{1+l-p}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
+ \tau \sum_{p=0}^{l} \left[ P_{q+2}^{p+p}(k) + P_{q+4}^{p+p}(k) + P_{q+1}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
+ \tau \sum_{p=1}^{l+1} \left[ P_{2+l-p}^{p+p}(k) + P_{4+l-p}^{p+p}(k) + P_{1+l-p}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
= v \sum_{p=0}^{l+1} \left[ P_{2+l-p}^{p+p}(k) + P_{4+l-p}^{p+p}(k) + P_{l-p}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right] \\
+ \tau \sum_{p=0}^{l+1} \left[ P_{2+l-p}^{p+p}(k) + P_{4+l-p}^{p+p}(k) + P_{l-p}^{p}(k) \int_{\gamma} (P_{2}^{2}(k) + P_{2}^{2}(k)) \, ds \right]
\]

as required. Integrating (7.4) and using Proposition 30, we find

\[ \| \partial_{s} \gamma \|_{\infty} \leq \| \partial_{s} \gamma_{0} \|_{\infty} + c \int_{0}^{t} e^{-c't} \, dt \leq \| \partial_{s} \gamma_{0} \|_{\infty} + c(t). \]

As \( u \) is the initial space parameter before reparameterization by arc-length, set \( v = |\partial_{u} \gamma| \).

Referring to the proof of Theorem 3.1 in [17], then for any function \( \Phi : \mathbb{S} \rightarrow \mathbb{R} \), we have

\[ \partial_{u}^{l} \Phi = v \partial_{s}^{l} \Phi + P^{l}(v, ..., \partial_{u}^{l-1}v, \Phi, ..., \partial_{s}^{l-1} \Phi) \]

where \( P^{l} \) is a polynomial. Then we obtain

\[ \| \partial_{s} \gamma \|_{\infty} \leq \| \partial_{s} \gamma_{0} \|_{\infty} + c(t). \]

as required. \( \square \)

In order to complete the proof of Theorem 13 it remains to show that the limit circle is unique. Here we refer to [1, Theorem A.1] to conclude full convergence of the flow. This theorem is stated in Appendix A.4, Theorem 19.

Uniform boundedness of \( \gamma \) and all its derivatives in Proposition 31 imply the first hypothesis of [1, Theorem A.1] is satisfied. For the second hypothesis, using Proposition 30 we note that
\[ \int_{\gamma} F^2 ds \leq c_1 e^{-c_2 t}. \]

This implies
\[ \int_0^T \left( \int_{\gamma} F^2 ds \right)^{\frac{1}{2}} dt \leq c_1 \int_0^T e^{-\frac{c_2}{2} t} dt \leq \hat{c} \]
where \( \hat{c} \) is a constant depending only on \( w, E(0) \) and \( L \). For the third hypothesis, uniform boundedness of all derivatives of \( \gamma \) in Proposition 31 gives that for any sequence \( t_j \to \infty \), the \( C^\infty \)-norm of \( \gamma(\cdot, t_j) \) is uniformly bounded. We have the exponential decay of the energy, so \( E[\gamma(\cdot, t_j)] \to 0 \), which implies that a subsequence \( \gamma(\cdot, t_j) \) converges to a smooth circle in \( C^\infty \)-topology. Therefore, we conclude full convergence of the flow.

Exponential decay of the speed allows us to bound the region of the plane in which the solution lies relative to \( \gamma_0 \) via standard argument. We may bound the distance travelled by any point on the initial curve \( \gamma_0 \) as follows
\[
|\gamma(x,t) - \gamma(x,0)| = \left| \int_0^t \frac{\partial \gamma}{\partial t}(x,v) \right| \leq \int_0^t \left| k_s + k^2 k_{ss} - \frac{1}{2} kk_{ss} + h_1(t) \right| dv \leq \frac{c}{\sigma} \left( 1 - e^{-\alpha t} \right) \leq \frac{c}{\sigma}.
\]

### 7.2 Area-preserving ideal curve flow

For the smooth closed planar curve \( \gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2 \) by energy \( E[\gamma] = \frac{1}{2} \int_{\gamma} k^2 ds \). We suppose the curve evolves by the sixth-order curvature flow

\[
\left\{ \begin{array}{l}
\partial_t \gamma = (k_s + k^2 k_{ss} - \frac{1}{2} kk_{ss} + h_2(t)) v, \quad \text{for all } (s,t) \in \mathbb{S}^1 \times [0, T) \\
\gamma|_{t=0} = \gamma_0, \quad \text{for all } s \in \mathbb{S}^1
\end{array} \right. \quad (7.5)
\]

To preserve area of the evolving curve \( \gamma \) we take
\[
h_2(t) = \frac{5}{2L} \int_{\gamma} k^2 ds \quad (7.6)
\]
where \( L \) denotes the length of curve \( \gamma \).

The main theorem is as follows:

**Theorem 14.** Suppose \( \gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2 \) be a regular smooth immersed closed curve with constant area \( A(0) > 0 \) and \( \omega(0) = 1 \),

\[ E(0) \leq \varepsilon, \]

the area-preserving ideal curve flow \( (7.5) \) with initial data \( \gamma_0 \) exists for all time and con-
verges exponentially to a round circle with radius $\sqrt{A/\pi}$.

In above theorem, $\omega$ denotes the winding number of curve (7.5). From Lemma 4, we can get the winding number $\omega(t) = \omega(0) = 1$. In order to prove Theorem 14, we need to have short time existence of a solution to (7.5), see Theorem 6 in Chapter 3. We show the curve remains embedded with the smallness condition in Theorem 14.

**Proposition 32.** Any solution of (7.5) with initial embedded curve $\gamma_0$ satisfying the assumptions of Theorem 14 remains embedded for all time.

This section is organised as follows. In section 7.2.1, we show some preliminaries. In finite time, the energy decreases, the length is bounded. We also give the estimate for $h_2(t)$ term. With these results in hand, in section 7.2.3 we prove that the energy decays exponentially. Then, in section 7.2.2 we show that the curve exists for all time and converges exponentially fast to a round circle under the stated conditions, completing the proof of Theorem 14.

**Lemma 61.** Under the flow (7.5), while a solution exists, it satisfies

$$A(t) = A(0).$$

**Proof.** We compute

$$\frac{d}{dt} A = - \int_{\gamma} F ds = \frac{5}{2} \int_{\gamma} k^2 s ds - h_2(t)L = 0$$

As $h_2(t) = \frac{5}{2L} \int_{\gamma} k^2 ds$ in (7.6), the result follows.

**7.2.1 Preliminaries**

**Lemma 62.** Under the flow (7.5), $E$ is monotone decreasing.

**Proof.** We have $F = G + h_2(t) = k_x + k^2 k_s - \frac{1}{2} kk_s + h_2(t)$. As $h_2(t) = \frac{5}{2L} \int_{\gamma} k^2 s ds$, which is also $h_2(t) = \frac{1}{L} \int_{\gamma} G ds$. Because $\int_{\gamma} G ds \leq \left( \int_{\gamma} G^2 ds \right)^{\frac{1}{2}} \cdot L^{\frac{1}{2}}$, we have

$$\frac{d}{dt} \|K_s\|^2 = - \int_{\gamma} (G + h)G ds = - \int_{\gamma} G^2 ds - h \int_{\gamma} G ds$$

$$= - \int_{\gamma} G^2 ds + \frac{1}{L} \int_{\gamma} G ds \int_{\gamma} G ds$$

$$\leq - \int_{\gamma} G^2 ds + \frac{1}{L} \int_{\gamma} G^2 ds = 0$$
The \( f_x k^2 ds \) is monotone decreasing.

\[
\int_\gamma k^2 ds = \int_\gamma (k-\bar{k})^2 ds + 2\bar{k} \int_\gamma k ds - \bar{k}^2 ds
\]

\[
= \int_\gamma (k-\bar{k})^2 ds + \frac{8\pi^2}{L} - \frac{4\pi^2}{L} = \int_\gamma (k-\bar{k})^2 ds + \frac{4\pi^2}{L}
\]

\[
\leq \frac{L^2}{4\pi^2} \int_\gamma k^2 ds + \frac{4\pi^2}{L} \leq \frac{L^2}{4\pi^2} E_0 + \frac{4\pi^2}{L}
\]

We know from above that \( \int_\gamma k^2 ds \) is bounded for finite time. Also

\[
K_{osc} \leq \frac{L^3}{4\pi^2} \| k_s \|_2^2 \leq \frac{L^3}{4\pi^2} E_0.
\]

\[ \Box \]

**Lemma 63.** Under the flow (7.5), \( L \) is bounded in finite time.

**Proof.** As

\[
k = k - \bar{k} \leq |k_s| ds + \frac{2\pi}{L},
\]

by Hölder inequality, we have

\[
\| k \|_\infty \leq \sqrt{2LE} + \frac{2\pi}{L}.
\]

Then, we estimate the evolution of \( L \) by using above estimate,

\[
\frac{d}{dt} L = - \int_\gamma kF ds = - \int_\gamma k^2 ds + \int_\gamma k k_s ds + \frac{7}{2} \int_\gamma k^2 k_s ds - \frac{5\pi}{L} \int_\gamma k k_s^2 ds
\]

\[
\leq - \frac{4\pi^2}{L^2} \int_\gamma k^2 ds + \frac{7}{2} \| k \|_\infty \int_\gamma k^2 ds + \left( \frac{5\pi}{L} \right)^2 \int_\gamma k^2 k_s^2 ds + \frac{1}{4} \int_\gamma k_s^2 ds
\]

\[
\leq - \frac{4\pi^2}{L^2} \int_\gamma k^2 ds + \frac{7}{2} \| k \|_\infty \int_\gamma k^2 ds + \frac{25\pi^2}{L^2} \| k \|_\infty \int_\gamma k_s^2 ds + \frac{1}{4} \int_\gamma k_s^2 ds
\]

\[
= \left[ - \frac{4\pi^2}{L^2} + \frac{7}{2} \| k \|_\infty \frac{25\pi^2}{L^2} \left( \sqrt{L} \| k_s \|_2 + \frac{2\pi}{L} \right)^2 + \frac{1}{4} \right] \int_\gamma k_s^2 ds
\]

\[
\leq \left[ - \frac{4\pi^2}{L^2} + \left( \frac{7}{2} + \frac{25\pi^2}{L^2} \right) \left( \| k \|_\infty \frac{2}{L} + \frac{4\pi}{L} \right)^2 + \frac{1}{4} \right] \int_\gamma k_s^2 ds
\]

\[
\leq \left[ - \frac{4\pi^2}{L^2} + \left( \frac{7}{2} + \frac{25\pi^2}{L^2} \right) \left( L \| k_s \|_2^2 + \frac{4\pi^2}{L} \right)^2 + \frac{1}{4} \right] \int_\gamma k_s^2 ds
\]

\[
= -4\pi^2 \| k_s \|_2^2 L^{-2} + \frac{7}{2} \| k_s \|_2 \cdot L + 14\pi^2 \| k_s \|_2 L^{-2} + 14\pi \| k_s \|_2 \| k_s \|_2 L^{-\frac{1}{2}} + 25\pi^2 \| k_s \|_2^2 L^{-1}
\]

\[
+ 100\pi^4 \| k_s \|_2 L^{-4} + 100\pi^3 \| k_s \|_2 \cdot \| k_s \|_2^3 L^{-5} + \frac{1}{4} \| k_s \|_2^2
\]
≤ \frac{7}{2} \|k_s\|^4 \cdot L + 10\pi^2 \|k_s\|^3 L^{-2} + 14\pi \|k_s\|^2 L^{-1} + 25\pi^2 \|k_s\|^4 L^{-1} + 100\pi^4 \|k_s\|^2 L^{-4} \\
+ 100\pi^3 \|k_s\|^3 L^{-\frac{5}{2}} + \frac{1}{4} \|k_s\|^2 L
= \left( \frac{7}{2} \|k_s\|^4 + 10\pi^2 \|k_s\|^3 L^{-2} + 14\pi \|k_s\|^2 L^{-1} + 25\pi^2 \|k_s\|^4 L^{-2} + 100\pi^4 \|k_s\|^2 L^{-5} \\
+ 100\pi^3 \|k_s\|^3 L^{-\frac{7}{2}} + \frac{1}{4} \|k_s\|^2 L^{-1} \right) \cdot L \quad (7.7)

On the right-hand side of above estimate, we can see that the powers of \( L \) are all negative. As the evolving closed curve has preserved area \( A \), then the length is bounded below by \( \bar{L} := 2\sqrt{\pi A} \). Also the energy is monotone decreasing, we obtain

\[
\frac{d}{dt} L \leq c \cdot L \quad (7.8)
\]

where \( c \) only depending on \( E(0), \bar{L} \).

From (7.8), we estimate the evolution equation of \( \ln L \),

\[
\frac{d}{dt} \ln L = L^{-1} \frac{d}{dt} L \leq c.
\]

Integrating on both sides,

\[
\ln L(t) \leq \ln L(0) + ct
\]

which is

\[
L(t) \leq L(0)e^{ct}.
\]

Thus, we obtain that length is bounded in finite time. \( \square \)

In the following lemma, we estimate the global term \( h_2(t) \).

**Lemma 64.** For any constants \( c_1, c_2 > 0 \), the global term \( h_2(t) \) may be estimated as

\[
|h_2(t)| \leq c_1 L^{-\frac{35}{8}} \|k\|^3 + c_2 L^{-\frac{15}{8}} \|k\|^7 \|kss\|^\frac{5}{2}
\]

**Proof.** By using interpolation inequality,

\[
|h_2(t)| = \frac{5}{2L} \left| \int_{\gamma} k k^2 ds \right| = \frac{5}{2L} \left| \int_{\gamma} |P_3^2(k)| ds \right| \leq \frac{5}{2L} cL^{1-3-2} \cdot \|k\|^\frac{7}{2} \|k\|^\frac{5}{2}
\]

\[
= \frac{5}{2L} cL^{-\frac{35}{8}} \|k\|^\frac{7}{2} \left( L^{\frac{5}{2}} \|k\|^\frac{5}{2} + L^{\frac{25}{8}} \|kss\|^\frac{5}{2} \right)
\]

\[
= \frac{5}{2} cL^{-\frac{35}{8}} \|k\|^\frac{7}{2} + \frac{5}{2} cL^{-\frac{15}{8}} \|k\|^\frac{7}{2} \|kss\|^\frac{5}{2}
\]
Here we prove that $\|k_{ss}\|_2^2$ is bounded.

$$F_s^4 = -k_s^8 - 10k_s k_{ss} k_s^3 - \frac{15}{2} k_{ss}^2 k_s^4 - 5k_{ss}^2 k_s^4 - 10k k_{ss} k_s^4 - 3k_s k_s^5 + k^2 k_s^6,$$

from lemma 3 (v), we have

$$\frac{d}{dt} \int_{\gamma} k_{ss}^2 ds = 2 \int_{\gamma} k_{ss} \partial_t k_{ss} ds + \int_{\gamma} k_{ss}^2 k_F ds$$

$$= -2 \int_{\gamma} k_{ss} F_s ds - 2 \int_{\gamma} k_{ss} k^2 F_{ss} ds - 6 \int_{\gamma} k_{ss} k_F - 2 \int_{\gamma} k_{ss} k_s^2 F - 3 \int_{\gamma} k_{ss} F ds$$

$$= -2 \int_{\gamma} k_{ss}^2 ds - 4 \int_{\gamma} k_{ss}^2 ds + 3 \int_{\gamma} k_{ss} k_{ss} k_s^4 ds + 3 \int_{\gamma} k k_{ss} k_s^4 ds + 6 \int_{\gamma} k_{ss}^2 k_s^3$$

$$+ \frac{3}{2} \int_{\gamma} k^3 k_{ss} k_s^4 ds - \frac{13}{2} \int_{\gamma} k_{ss}^2 k_s^2 k_s^2 ds + \int_{\gamma} k_{ss}^3 k_{ss}^3 + 2 \int_{\gamma} k_{ss}^2 k_{ss} k_s^4 ds$$

$$+ 5 \int_{\gamma} k^2 k_{ss} k_s^4 ds + 2 \int_{\gamma} k^3 k_{ss} k_s^4 ds + 2h \int_{\gamma} k_{ss}^2 k_s^3 ds + 3h \int_{\gamma} k_{ss}^2 k_s^3 ds$$

$$\leq (-2 + 8\epsilon) \int_{\gamma} k_{ss}^2 ds + c + 2h \int_{\gamma} k_{ss} k_{ss}^2 ds + 3h \int_{\gamma} k_{ss} k_s^4 ds$$

$$= (-2 + 8\epsilon) \int_{\gamma} k_{ss}^2 ds + c + 3h \int_{\gamma} k_{ss} k_s^4 ds$$

As $h_2 = \frac{s}{2L} \int_{\gamma} k k_{ss}^2 ds$, we can get

$$3h_2 \int_{\gamma} k_{ss}^2 k_s^4 ds \leq \frac{15}{2L} \int_{\gamma} k k_{ss}^2 ds \cdot \int_{\gamma} k_{ss} k_s^4 ds \leq \frac{15}{2L} \|k\|_\infty^2 \int_{\gamma} k_{ss}^2 ds \int_{\gamma} k_{ss}^2 ds$$

$$\leq \frac{15}{2L} \left( \sqrt{E} + \frac{2\pi}{L} \right)^2 \cdot E \cdot \left( \frac{L^2}{4\pi^2} \right)^3 \int_{\gamma} k_{ss}^2 ds$$

From Lemma 62, we know that $E$ is decreasing. So when $t > \hat{t}$, we can have $E \leq \hat{e}$. Thus

$$3h_2 \int_{\gamma} k_{ss}^2 k_s^4 ds \leq \hat{e} c(L) \int_{\gamma} k_{ss}^2 ds = \epsilon \int_{\gamma} k_{ss}^2 ds$$
\frac{d}{dt} \int_{\gamma} k_{ss}^{2} ds \leq (\mathbf{-}2 + 8\epsilon) \int_{\gamma} k_{ss}^{2} ds + c + 3h_{2} \int_{\gamma} k_{ss}^{2} ds \\
\leq (\mathbf{-}2 + 9\epsilon) \int_{\gamma} k_{ss}^{2} ds + c(\|k\|_{2}^{2}) \\
\leq -c_{1}\|k_{ss}\|_{2}^{2} + c(\|k\|_{2}^{2})

From above we know that \|k_{ss}\|_{2}^{2} is bounded, then we have \(h_{2}(t)\) is bounded as the following.

\[ |h_{2}(t)| \leq c_{1}L^{-\frac{35}{8}}\|k\|_{3}^{3} + c_{2}L^{-\frac{15}{8}}\|k\|_{3}^{7}\|k_{ss}\|_{2}^{5}. \]

\[ \square \]

### 7.2.2 Global Existence

In above, we show that the length is bounded for finite time intervals, and the \(h(t)\) term is bounded as well. In this section, firstly, we give the boundness of curvature derivatives in \(L^{2}\) and \(L^{\infty}\). This proposition will be used to prove Theorem 15 and Proposition 40 later. Then we show that \(T = \infty\).

**Proposition 33.** Suppose \(\gamma_{0} : S^{1} \rightarrow \mathbb{R}^{2}\) solves (7.5). Then there exist constants \(c_{l}, \bar{c}_{l} > 0\) depending only on \(\gamma_{0}\) such that for all \(l \in \mathbb{N}_{0}\),

\[ \|k_{s}^{l}\|_{2}^{2} \leq c_{l} \text{ and } \|k_{s}^{l}\|_{\infty} \leq \bar{c}_{l}. \]

**Proof.** From Lemma 3 (v), we have

\[
\frac{d}{dt} \int_{\gamma} k_{s}^{2} ds \\
= -\int_{\gamma} 2k_{s}^{2} ds + \int_{\gamma} P_{4}^{2l+4}(k) ds + \int_{\gamma} P_{6}^{2l}(k) ds + \int_{\gamma} P_{5}^{2l+3}(k) ds + h_{2}(t) \int_{\gamma} k_{s} P_{2}^{l}(k) ds \\
\leq -\int_{\gamma} 2k_{s}^{2} ds + \epsilon \int_{\gamma} k_{s}^{2} ds + c\|k\|_{2}^{4l+14} + \epsilon \int_{\gamma} k_{s}^{2} ds + c\|k\|_{2}^{2l+10} + \epsilon \int_{\gamma} k_{s}^{2} ds \\
+ c\|k\|_{2}^{2l+10} + h_{2}(t) \int_{\gamma} k_{s} P_{2}^{l}(k) ds \\
\leq (\mathbf{-}2 + 3\epsilon) \int_{\gamma} k_{s}^{2} ds + c\|k\|_{2}^{4l+14} + c\|k\|_{2}^{2l+10} + c\|k\|_{2}^{4l+14} + h_{2}(t) \int_{\gamma} k_{s} P_{2}^{l}(k) ds
\]

From Lemma 64 we know that
\[ |h_2(t)| \leq c(L)\|k\|^3_2 + c(L)\|k\|^7_2 \]

and

\[
\int_{\gamma} k^p \kappa^2_s(k) ds \leq c(L) \left[ \left( \int_{\gamma} k^2 ds \right)^{1+\frac{5}{2(n-1)}} + \left( \int_{\gamma} k^2 ds \right)^{\frac{2(n-1)}{3}} \right],
\]

\[
\|k\|_2^3 = \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}} = \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}},
\]

\[
\|k\|_2^3 = \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}} = \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}}.
\]

By using Young’s inequality, we have

\[
h_2(t) \int_{\gamma} k^p \kappa^2_s(k) ds \leq \frac{c}{2} \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}} + \frac{2}{\varepsilon} \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}} + \varepsilon \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{2}}
\]

\[
= \varepsilon \int_{\gamma} k^p \kappa^2_s ds + c \left( \int_{\gamma} k^2 ds \right)^{\frac{3}{2}} + c \left( \int_{\gamma} k^2 ds \right)^{\frac{1}{2}}.
\]
Then we have
\[
\frac{d}{dt} \int_{\gamma} k^2 ds \leq (-1 + 3\epsilon) \int_{\gamma} k^2 ds + c\|k\|^{4l+14} + c\|\dot{k}\|^{2l+10} + c\|\ddot{k}\|^{4l+14} + h_2(t) \int_{\gamma} k^2 P_2 ds
\]
\[
\leq (-1 + 3\epsilon) \int_{\gamma} k^2 ds + c\|k\|^{4l+14} + c\|\dot{k}\|^{2l+10} + c\|\ddot{k}\|^{4l+14} + \epsilon \int_{\gamma} k^2 ds
\]
\[
+ c\|k\|^{16l+11} + c\|\dot{k}\|^{11} + c\|\ddot{k}\|^6 + c\|k\|^{19/2}
\]
\[
= (-1 + 4\epsilon) \int_{\gamma} k^2 ds + c\|k\|^{4l+14} + c\|\dot{k}\|^{2l+10} + c\|\ddot{k}\|^{4l+14} + c\|k\|^{16l+11} +
\]
\[
+ c\|k\|^{11} + c\|\dot{k}\|^6 + c\|k\|^{19/2}.
\]

We get that all derivatives of the curvature are bounded in $L^2$. As $\|k\|_2^2 \leq \frac{L}{2\pi} \|k_{l+1}\|_2^2$, all derivatives of the curvature are bounded in $L^\infty$ follows.

We show that if the maximal existence time is finite, then the curvature must blow up in $L^2$. For the similar proof, see Chapter 6.

**Theorem 15.** Suppose $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$ be a maximal solution of (7.5). If $T < \infty$ then
\[
\int_{\gamma} k^2 ds\bigg|_t \geq c(T - t)^{-\frac{1}{6}}
\]

**Proof.** Now we refer to [17, Theorem 3.1], assume $\|k\|_2$ is bounded for all $t < T$, then we can get that $\gamma$ can extend smoothly to $\mathbb{S}^1 \times [0, T]$ by short time existence, which contradicts the maximality of $T$ and therefore that $k$ cannot be uniformly bounded in $L^2$ on $[0, T)$.

For $l = 0$ in Proposition 33 we have
\[
\frac{d}{dt} \int_{\gamma} k^2 ds \leq c\|k\|^{14} + c\|\dot{k}\|^{10} + c\|\ddot{k}\|^{11} + c\|\dddot{k}\|^{12} + c\|k\|^{17} + c\|\dot{k}\|^6 + c\|k\|^{19/2}
\]

Using Young’s inequality yields
\[
\frac{d}{dt} \int_{\gamma} k^2 ds \leq c \left( \int_{\gamma} k^2 ds \right)^{7} + c \left( \int_{\gamma} k^2 ds \right)^{5} + c \left( \int_{\gamma} k^2 ds \right)^{14/9} + c \left( \int_{\gamma} k^2 ds \right)^{17/9}
\]
\[
+ c \left( \int_{\gamma} k^2 ds \right)^{3} + c \left( \int_{\gamma} k^2 ds \right)^{19/9}
\]

Since $\int_{\gamma} k^2 ds$ blows up as $t \to T$, all the others term can be absorbed by the power 7.
term, i.e. \( \frac{d}{dt} \int_{\gamma} k^2 ds \leq c \left( \int_{\gamma} k^2 ds \right)^7 \). Thus we obtain

\[
\int_{\gamma} k^2 ds \bigg|_t \geq c(T - t)^{-\frac{1}{7}}.
\]

Then, we get the \( T = \infty \) by a contradiction.

**Corollary 5.** Suppose \( \gamma_0 : S^1 \rightarrow \mathbb{R}^2 \) solves (7.5), then \( T = \infty \).

**Proof.** Suppose on the contrary that \( \gamma \) satisfies the conditions of Theorem 14 and \( T < \infty \). We know from Theorem 15 above that \( \|k\|^2 \rightarrow \infty \) as \( t \rightarrow T \).

\[
K_{osc} = L \int_{\gamma} (k - \bar{k})^2 ds = L\|k\|^2 - 2\pi^2 \rightarrow \infty
\]

this contradicts that \( K_{osc} \) is bounded in Lemma 62, then \( T = \infty \) and \( \|k\|^2 \) does not blow up. \( \square \)

### 7.2.3 Exponential decay

Before proving the energy decays exponentially, we show several useful propositions and a lemma which shows that the length is bounded for all time.

**Proposition 34.** For all time, if \( (L^3E)(t) \leq \varepsilon_0 \), we obtain

\[
\int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds \geq cL^{-6}E
\]

**Proof.**

\[
F - \bar{F} = (k^2 - \bar{k}^2) k_{ss} - \frac{1}{2}kk_s^2 + h_2(t),
\]

\[
\int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds = \int_{\gamma} F^2 ds - h_2 \int_{\gamma} G ds - h_2^2 L \tag{7.9}
\]

here \( h_2(t) = \frac{5}{2L} \int k k_s^2 ds \).

\[
\int_{\gamma} F^2 ds = \int_{\gamma} F^2 ds + 2 \int_{\gamma} F(F - \bar{F}) ds + \int_{\gamma} (F - \bar{F})^2 ds
\geq \frac{1}{2} \int_{\gamma} \bar{F}^2 ds - \int_{\gamma} (F - \bar{F})^2 ds
\geq c_1 \frac{1}{2} \int_{\gamma} k_s^2 ds - cL^{-3}E^2 - \int_{\gamma} (F - \bar{F})^2 ds \tag{7.10}
\]
Now $F - \bar{F} = (k^2 - \bar{k}^2)k_{ss} - \frac{1}{2}kk_s^2 + h_2(t)$, so

$$
\int_{\gamma} (F - \bar{F})^2 ds \leq 2 \int_{\gamma} (k^2 - \bar{k}^2)^2 k_{ss}^2 ds + \frac{1}{2} \int_{\gamma} k^2 k_s^4 ds + 2h \int_{\gamma} (k^2 - \bar{k}^2)k_{ss} - \frac{1}{2}kk_s^2 ds + h^2L \tag{7.11}
$$

Substituting (7.10) into (7.9), we have

$$
\int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds = \int_{\gamma} F^2 ds - h_2 \int_{\gamma} G ds - h^2L
\geq \frac{c_1}{2} \int_{\gamma} k_{ss}^2 ds - cL^{-3}E^2 - \int_{\gamma} (F - \bar{F})^2 ds + h_2 \int_{\gamma} G ds + h^2L \tag{7.12}
$$

From (7.11), we have

$$
\int_{\gamma} (F - \bar{F})^2 ds + h_2 \int_{\gamma} G ds + h^2L \\
\leq 2 \int_{\gamma} (k^2 - \bar{k}^2)^2 k_{ss}^2 ds + \frac{1}{2} \int_{\gamma} k^2 k_s^4 ds + 2h_2 \int_{\gamma} (k^2 - \bar{k}^2)k_{ss} - \frac{1}{2}kk_s^2 ds + h^2L + h_2 \int_{\gamma} (k_{ss}^2 + k^2k_{ss} - \frac{1}{2}kk_s^2) ds + h^2L
\leq 2 \int_{\gamma} (k^2 - \bar{k}^2)^2 k_{ss}^2 ds + \frac{1}{2} \int_{\gamma} k^2 k_s^4 ds + 3h_2 \int_{\gamma} (k^2k_{ss} - \frac{1}{2}kk_s^2) ds + 2h^2L \tag{7.13}
$$

For the first two terms in (7.13), we have

$$
2 \int_{\gamma} (k^2 - \bar{k}^2)^2 k_{ss}^2 ds + \frac{1}{2} \int_{\gamma} k^2 k_s^4 ds \\
\leq 2||k + \bar{k}||^2_{\infty} \left( \int_{\gamma} |k_s| ds \right)^2 \int_{\gamma} k_{ss}^2 ds + \frac{1}{2} ||k||^2_{\infty} \int_{\gamma} k_s^4 ds
\leq 4||k + \bar{k}||^2_{\infty} \cdot LE \cdot \int_{\gamma} k_{ss}^2 ds + \frac{1}{2} ||k||^2_{\infty} \int_{\gamma} k_s^2 ds + \frac{3L}{2\pi} \cdot E \cdot \int_{\gamma} |k_s| ds + \frac{3L}{2\pi} \cdot E \cdot \int_{\gamma} |k_s| ds
\leq \left( 4L||k + \bar{k}||^2_{\infty} \cdot \frac{3L}{4\pi} ||k||^2_{\infty} \right) \cdot E \int_{\gamma} k_{ss}^2 ds + \frac{3L}{4\pi} ||k||^2_{\infty} \cdot E \int_{\gamma} |k_s| ds \tag{7.14}
$$

as

$$
k = k - \bar{k} \leq \int_{\gamma} |k_s| ds + \frac{2\pi}{L},
$$

by Hölder inequality, we have

$$
||k||_{\infty} \leq \sqrt{2LE} + \frac{2\pi}{L},
$$
\[ \| k + \bar{k} \|_\infty = \sqrt{2LE + \frac{4\pi}{L}}. \]

Here \( h_2(t) = \frac{5}{2L} \int \kappa_2^2 ds, \)
So the third and fourth terms can be estimated as the following,

\[
3h_2 \int_{\gamma} \left( k^2 k_{ss} - \frac{1}{2} k k_s^2 \right) ds + 2h_2^2 L \]
\[
= \frac{15}{2L} \int_{\gamma} k k_s^2 ds \cdot \int_{\gamma} k^2 k_{ss} ds - \frac{15}{4L} \int_{\gamma} k k_s^2 ds \cdot \int_{\gamma} k s k_s^2 ds + 2L \left( \frac{5}{2L} \int_{\gamma} k s k_s^2 ds \right)^2 \]
\[
= -\frac{15}{L} \left( \int_{\gamma} k k_s^2 ds \right)^2 - \frac{15}{4L} \left( \int_{\gamma} k k_s^2 ds \right)^2 + \frac{25}{2L} \left( \int_{\gamma} k s k_s^2 ds \right)^2 \]
\[
= -\frac{25}{4L} \left( \int_{\gamma} k k_s^2 ds \right)^2 \]
\[
\leq 0 \quad (7.15) \]

Substituting (7.14) and (7.15) into (7.13), we have

\[
\int_{\gamma} (F - \bar{F})^2 ds + h_2 \int_{\gamma} G ds + h_2^2 L \]
\[
\leq \left( 4L \| k + \bar{k} \|_\infty^2 + \frac{3L^2}{4\pi} \| k \|_\infty^2 \right) \cdot E \int_{\gamma} k_s^2 ds + \frac{3L^2}{4\pi} \| k \|_\infty^2 \cdot E \int_{\gamma} |k_s k_s| ds \]
\[
= c_3 L^{-1} E \int_{\gamma} k_s^2 ds + c_4 L^{-1} E \int_{\gamma} |k_s k_s| ds \quad (7.16) \]

where \( c_3 = 4L^2 \| k + \bar{k} \|_\infty^2 + \frac{3L^2}{4\pi} \| k \|_\infty^2, \) \( c_3 \) and \( c_4 \) are bounded by function only involve \( \epsilon \) and \( L \) as follows.

\[
c_3 \leq 4L^{-1} \cdot L^{-2} \left( \sqrt{2\varepsilon E} + 4\pi \right)^2 + \frac{3}{4\pi L} L^{-2} \left( \sqrt{2LE} + 2\pi \right)^2 \]
\[
\leq 4L^{-3} \left( \sqrt{2\varepsilon} + 4\pi \right)^2 + \frac{3}{4\pi L} L^{-3} \left( \sqrt{2\varepsilon} + 2\pi \right)^2 \]

and

\[
c_4 \leq \frac{3}{4\pi} L^{-3} \left( \sqrt{2\varepsilon} + 2\pi \right)^2 \]

We have the following three inequalities

\[
c_3 L^{-1} E \int_{\gamma} k_s^2 ds \leq c c_3 L^{-1} E \left( \int_{\gamma} k_s^2 ds \right)^{\frac{3}{2}} \left( \int_{\gamma} k_s^2 ds \right)^{\frac{1}{2}} \leq c c_3 L^{-1} E (2E)^{\frac{3}{2}} \left( \int_{\gamma} k_s^2 ds \right)^{\frac{1}{2}} \]
\[
\leq \frac{c_1}{8} \int_{\gamma} k_s^2 ds + c c_3^3 L^{-\frac{5}{2}} E^{\frac{5}{2}} \quad (7.17) \]
\[ c_4 L^{-1} E \int_{\gamma} |k_s k_3| ds \leq c \cdot c_6 \frac{1}{2\pi} E \left( \int_{\gamma} k_s^2 ds \right)^{\frac{1}{2}} \left( \int_{\gamma} k_3^2 ds \right)^{\frac{1}{2}} \leq \frac{c_1}{8} \int_{\gamma} k_3^2 ds + c \cdot c_6^2 E^3 \]

Substituting above two inequalities into (7.16), we have

\[
\int_{\gamma} (F - \bar{F})^2 ds + h_2 \int_{\gamma} G ds + h_2^2 L \leq \frac{c_1}{4} \int_{\gamma} k_s^2 ds + c c_3^3 L^{-\frac{3}{2}} E^2 + c c_6^2 E^3 \quad (7.18)
\]

We substitute (7.18) into (7.12), we have

\[
\int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds \geq \frac{c L^3 E - 2^9 \omega^8 \pi^8 L^{-3} E^2 - c c_3^3 L^{-\frac{3}{2}} E^2 - c c_6^2 E^3}{L^9} \quad (7.19)
\]

Then, rearrange (7.19), we have

\[
L^9 \left( \int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds \right) \geq c L^3 E - 2^9 \omega^8 \pi^8 L^6 E^2 - c c_3^3 L^\frac{15}{2} E^2 - c c_6^2 L^9 E^3 \\
\geq c L^3 E - 2^9 \omega^8 \pi^8 (L^3 E)^2 - c c_3^3 (L^3 E)^{\frac{5}{2}} - c c_6^2 (L^3 E)^3
\]

As \( (L^3 E) (t) \leq \varepsilon_0 \) for all time, thus

\[
\int_{\gamma} G^2 ds + h_2 \int_{\gamma} G ds \geq c L^{-6} E
\]

holds for all time.

The following proposition can relax the condition \( (L^3 E) (t) \leq \varepsilon_0 \) in Proposition 34.
Proposition 35. Suppose $\gamma_0 : S^1 \to \mathbb{R}^2$ solves (7.5) and under the assumption

$$E(0) \leq \varepsilon,$$

we have for all $t$ in $[0, \infty)$,

$$(L^3 E)(t) \leq (L^3 E)(0).$$

Proof. From (7.7), we have

$$\frac{d}{dt} L = - \int_{\gamma} k F ds = - \int_{\gamma} k^2 s ds + \frac{7}{2} \int_{\gamma} k^2 k^2 s ds - \frac{5\pi}{L} \int_{\gamma} k s ds$$

$$\leq - \frac{4\pi^2}{L^2} \int_{\gamma} k^2 ds + \frac{7}{2} \|k\|_\infty \int_{\gamma} k^2 ds + \frac{1}{4} \int_{\gamma} k^2 ds$$

$$\leq - \frac{4\pi^2}{L^2} \int_{\gamma} k^2 ds + \frac{7}{2} \|k\|_\infty \int_{\gamma} k^2 ds + \frac{1}{4} \int_{\gamma} k^2 ds$$

As

$$\|k\|_\infty \leq \frac{1}{L} \left( \sqrt{2L^3 E} + 2\pi \right),$$

also in Proposition 34, for all time, we have $(L^3 E)(t) \leq \varepsilon_0$, then

$$\frac{d}{dt} L \leq \left[ \frac{9}{2} \left( \sqrt{2L^3 E} + 2\pi \right)^2 + \frac{9\pi^2}{4} \right] \cdot L^{-2} \int_{\gamma} k^2 ds$$

$$\leq \left[ \frac{9}{2} \left( \sqrt{2\varepsilon_0} + 2\pi \right)^2 + \frac{9\pi^2}{4} \right] \cdot L^{-2} \int_{\gamma} k^2 ds$$

$$= c(\varepsilon_0)L^{-2} \int_{\gamma} k^2 ds \ (7.20)$$

From Proposition 34, $\int_{\gamma} G^2 ds + h \int_{\gamma} G ds \geq c L^{-6} E$, and $- \int_{\gamma} G^2 ds - h \int_{\gamma} G ds \leq 0$ in Lemma 62, we do the time derivative of $L^3 E$ as follows,

$$\frac{d}{dt} L^3 E = L^3 \frac{d}{dt} E + 2L^2 E \frac{d}{dt} L$$

$$\leq -L^3 \left( \int_{\gamma} G^2 ds + h \int_{\gamma} G ds \right) + 2L^2 E \cdot \left( c_0 L^{-2} \int_{\gamma} k^2 ds \right)$$

$$\leq L^3 \left( \int_{\gamma} G^2 ds + h \int_{\gamma} G ds \right) \cdot \left[ -1 + c_0 L^3 E \right]$$

In above, if $(L^3 E)(0) \leq \min \left\{ \varepsilon_0, \frac{1}{c_0} - \varepsilon_2 \right\} = \varepsilon \cdot L^3(0) = \varepsilon_1$, i.e.

$$E(0) \leq \varepsilon,$$

then $\frac{d}{dt} L^3 E \leq -cL^3 \left( \int_{\gamma} G^2 ds + h \int_{\gamma} G ds \right) \leq 0$, thus $(L^3 E)(t) \leq \varepsilon_1 \leq \varepsilon_0$ holds for all time. \qed
By using Proposition 35, we can get the following proposition which is better than Proposition 34.

**Proposition 36.** Suppose \( \gamma_0 : S^1 \to \mathbb{R}^2 \) solves (7.5) and under the assumption

\[
E(0) \leq \epsilon,
\]

we have

\[
\int_\gamma G^2 ds + h \int_\gamma G ds \geq cL^{-6}E
\]

holds for all time.

Before proving the energy decays exponentially, we need to show that the length is bounded for all time.

**Lemma 65.** Suppose \( \gamma_0 : S^1 \to \mathbb{R}^2 \) solves (7.5) and under the assumption

\[
E(0) \leq \epsilon,
\]

the length is bounded for infinite time.

**Proof.** From Proposition 36 and (7.20) in Proposition 35, we can get

\[
\frac{d}{dt} \ln L \leq cL^{-3}E \leq cL^3 \left( \int_\gamma G^2 ds + h \int_\gamma G ds \right)
\]

In the end of the proof for Proposition 35, we have \( \frac{d}{dt}L^3E \leq -cL^3 \left( \int_\gamma G^2 ds + h \int_\gamma G ds \right) \leq 0 \). By applying this inequality, we get

\[
\frac{d}{dt} \ln L \leq -c \frac{d}{dt}L^3E,
\]

thus

\[
\ln L(t) \leq \ln L(0) + c \left( L^3E \right)(0).
\]

which gives an upper bound for length.

Above lemma shows the length is bounded under the smallness energy assumption, however, we can have a better result shown in the following proposition, where the smallness energy condition is not needed.

**Proposition 37.** Define \( \bar{E} = A^3E \). (Note that \( \bar{E} \leq \bar{E}_0 \) for any flow.) For any smooth embedded curve where \( k \) has a zero:

\[
L \leq \sqrt{A} \frac{4\bar{E}}{\pi}.
\]
If \( k \) does not have a zero (i.e. the curve is convex) then

\[
L \leq \sqrt{A} \left( 2\sqrt{\pi} + \frac{\tilde{E}}{\pi} + 2\sqrt{\frac{E^2}{\pi^2} + \frac{2}{\sqrt{\pi}}} \right).
\]

**Proof.** If the curve is embedded and star-shaped, then we claim the inequality

\[
L \leq 2A\|k\|_{\infty}
\]

holds. This is easy to see:

\[
L = \int |\gamma_s|^2 ds = \int k(-\langle \gamma, \nu \rangle) ds \leq 2A\|k\|_{\infty}.
\]

In order to upgrade this inequality to hold for all embedded curves, we flow \( \gamma \) by area-preserving curve shortening flow. Suppose that \( \|k\|_{\infty} < \frac{L}{2A} \). This curvature bound is preserved along the flow. However, the flow converges to a circle, which has of course \( \frac{L}{2A} = \frac{1}{r} = k \); this is a contradiction.

In order to obtain the two estimates, observe that if \( k \) has a zero then as it is periodic it must contain two zeros, then an integral splitting argument (see [82]) implies that

\[
\|k\|_{\infty}^2 \leq \frac{L}{2\pi} \int_\gamma k^2 ds \leq \frac{LE}{\pi}.
\]

Then

\[
L \leq A\frac{2\sqrt{L}}{\sqrt{\pi}} \sqrt{E}.
\]

Rearranging gives the first estimate. Then for the second estimate, \( k \) does not have a zero. We have

\[
\|k - \tilde{k}\|_{\infty}^2 \leq \frac{L}{\pi} E,
\]

as \( L^2 \geq 4\pi A \), we can get

\[
L \leq 2A\|k\|_{\infty} \leq 2A\|k - \tilde{k} + \tilde{k}\|_{\infty} \leq 2A (\|k - \tilde{k}\|_{\infty} + \|\tilde{k}\|_{\infty}) \leq 2A \sqrt{\frac{L}{\pi} E + \frac{2A \cdot 2\pi}{L}} \leq \frac{2A\sqrt{E}}{\sqrt{\pi}} \cdot \sqrt{L} + 2\sqrt{A\pi}.
\]

Rearranging this estimate gives the following

\[
\left( \sqrt{L} \right)^2 - \frac{2A\sqrt{E}}{\sqrt{\pi}} \cdot \sqrt{L} - 2\sqrt{A\pi} \leq 0.
\]
By solving this question, we get

$$\sqrt{L} \leq \frac{A\sqrt{E}}{\sqrt{\pi}} + \sqrt{\frac{A^2E}{\pi} + 2\sqrt{\pi}}$$

then

$$L \leq \sqrt{A} \left( 2\sqrt{\pi} + \frac{A^3E}{\pi} + 2\sqrt{\frac{A^3E^2}{\pi^2} + \frac{2A^3E}{\sqrt{\pi}}} \right)$$

$$\leq \sqrt{A} \left( 2\sqrt{\pi} + \frac{E}{\pi} + 2\sqrt{\frac{E^2}{\pi^2} + \frac{2}{\sqrt{\pi}}} \right)$$

which is the second estimate. \(\square\)

Note that in either case, the bound on the right-hand side is uniformly bounded under the flow. This means that so long as the curve remains embedded, length is well-controlled under the flow.

We show the curve remains embedded with the smallness condition in Theorem 14.

**Proposition 38.** Any solution of (7.5) with initial embedded curve \(\gamma_0\) satisfying the assumptions of Theorem 14 remains embedded for all time.

Under the assumption, we have the \(\epsilon\) is small enough to satisfy the condition in [82, Theorem 1.6]. This theorem is showed as Theorem 12 in Chapter 6.

Finally, we can get the exponential decaying of the energy by using above Lemma 65 and Proposition 36.

**Proposition 39.** Suppose \(\gamma_0 : S^1 \to \mathbb{R}^2\) solves (7.5) and under the assumption

$$E(0) \leq \epsilon,$$

there exist constants \(c, \delta\), we have

$$\int_{\gamma} k^2 ds \leq ce^{-\delta t}$$

for all time.

**Proof.** From Lemma 65 and Proposition 36, we get

$$\frac{d}{dt} \int_{\gamma} k^2 ds = - \int_{\gamma} G^2 ds - h \int_{\gamma} G ds \leq -c \int_{\gamma} k^2 ds,$$

which means \(\int_{\gamma} k^2 ds\) exponentially decays, where \(c\) depends on \(\epsilon\) and \(L\). \(\square\)
The following proposition shows that the curvature derivatives decay exponentially in $L^2$ and $L^\infty$.

**Proposition 40.** Suppose $\gamma_0 : S^1 \to \mathbb{R}^2$ solves (7.5) and satisfies the conditions of Theorem 14. Then there exists constants $c_1 > 0$ depending only on $\gamma_0$ such that for all $l \in \mathbb{N}_0$, $\parallel k_s \parallel_2^2 \leq c_1 e^{-\delta t}$ and $k_s$ exponentially decays in $L^\infty$.

**Proof.** When $t \to \infty$, 
\[
\int_\gamma k_s^2 ds = - \int_\gamma (k - \bar{k}) k_s ds \leq \left( \int_\gamma (k - \bar{k})^2 ds \right)^{\frac{1}{2}} \left( \int_\gamma k_{ss}^2 ds \right)^{\frac{1}{2}},
\]
as $\int_\gamma k_{ss}^2 ds$ is bounded in $L^2$ in Proposition 33 and $\int_\gamma k_s^2 ds \leq c e^{-\delta t}$ in Proposition 39, then $\int_\gamma k_s^2 ds$ exponentially decays.

We let $l \geq 1$, 
\[
\int_\gamma k_{s+l}^2 ds \leq \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \left( \int_\gamma k_{s+l}^2 ds \right)^{\frac{1}{2}},
\]
as $\int_\gamma k_{s+l}^2 ds$ is bounded in $L^2$ in Proposition 33, then $\int_\gamma k_s^2 ds$ exponentially decays.

We can obtain uniform bounds for all derivatives of the evolving curve $\gamma$ by using above Proposition 40.

**Proposition 41.** Suppose $\gamma_0 : S^1 \to \mathbb{R}^2$ solves (7.5) and satisfies the conditions of Theorem 14. Then for all $l \in \mathbb{N}_0$, $\parallel \partial_l \gamma \parallel_\infty \leq c(l) + \sum_{p=0}^l \parallel \partial_p \gamma_0 \parallel_\infty$, where $c(l)$ is a constant only depending on $l, E(0), L$.

The proof of above proposition is similar to Proposition 31.

For the proof of Theorem 14, it remains to show that the full convergence of the flow. Here again, we refer to [1, Theorem A.1]. There are three hypotheses in this theorem we need to check. Uniform boundedness of $\gamma$ and all its derivatives in Proposition 41 imply the first hypothesis is satisfied.
For the second hypothesis, using Proposition 40 we note that
\[ \int_{\gamma} F^2 ds \leq c_1 e^{-c_2 t}. \]
This implies
\[ \int_0^T \left( \int_{\gamma} F^2 ds \right)^{\frac{1}{2}} dt \leq c_1 \int_0^T e^{-\frac{c_2}{2} t} dt \leq \hat{c} \]
where  \( \hat{c} \) is a constant depending only on  \( w, E(0) \) and  \( L \).

For the third hypothesis, Proposition 41 gives that for any sequence  \( t_j \to \infty \), the  \( C^\infty \) norm of  \( \gamma(\cdot, t_j) \) is uniformly bounded. We have the exponential decay of the energy, so  \( E[\gamma(\cdot, t_j)] \to 0 \), which implies that a subsequence  \( \gamma(\cdot, t_j) \) converges to a smooth  \( w \)-circle in  \( C^\infty \)-topology. Therefore we apply [1, Theorem A.1] to conclude full convergence of the flow.

Exponential decay of the speed allows us to bound the region of the plane in which the solution lies relative to  \( \gamma_0 \) via standard argument. We may bound the distance travelled by any point on the initial curve  \( \gamma_0 \) as follows
\[ |\gamma(x,t) - \gamma(x,0)| = \left| \int_0^t \frac{d\gamma}{dt}(x,v) \right| \leq \int_0^t \left| k_s^4 + k^2 k_{ss} - \frac{1}{2} k k_s^2 + h_2(t) \right| dv \leq \frac{c}{\sigma} (1 - e^{-\sigma t}) \leq \frac{c}{\sigma}. \]
Bibliography


Appendix A

A.1 Proof of Theorem 4

For the proof of Theorem 4 in Chapter 3, we refer to [79, Theorem 2.7] and give the proof as the following.

Proof. We consider the family of initial value problem

$$\frac{\partial}{\partial t} \gamma_h = -(k_{ss} + \tilde{h})\nu,$$

where $\tilde{h} \in C^1([0,T])$ is a known function of time and $\gamma_0 = \gamma(\cdot,0) \in C^4(S^1)$. The processing arguments give short time existence for each $\gamma_h$. We will now show that at least one of the functions $\tilde{h}$ coincide with our given constrained function $h$, which is normal a ratio of integrals of curvature and not a priori known function of time. We prove that $h$ satisfies the initial condition $\frac{d^2}{dt^2} h(t) \leq c(\gamma_0)$, which we note forces some measure of regularity on the immersion $\gamma_0$, $\gamma_0 = \gamma(\cdot,0) \in C^7(S^1)$.

Let $S = C^1([0,T])$ for some $\sigma > 0$ which will be chosen. The theorem will be proved if we can apply Theorem 3 with the mapping $P : S \rightarrow S$ defined by

$$Ph = h.$$

Noting that $C^1([0,\sigma])$ is a compact, convex subset of the Banach space $C^1([0,T])$, we need to demonstrate that $P$ maps $S$ into itself and is continuous. Both of these follow from the assumption $\frac{d^2}{dt^2} h \leq c(\gamma_0)$. In particular, we have that $\frac{d^2}{dt^2} h \leq c(\gamma_0)$ and so $\frac{d}{dt} h$ is continuous on $[0,\sigma]$ for some $\sigma > 0$, and so

$$\frac{d}{dt} h = \frac{d}{dt} Ph \in C^1([0,\sigma]).$$

This also knows that $P'$ is bounded in the operator norm on $C^1([0,\sigma])$ and so $P$ is continuous. Therefore we apply Theorem 3 and deduce at least one of the functions $\tilde{h}$ coincides
with the given constraint function \( h \) on an interval \([0, \sigma] \subset [0, T)\). The initial condition \( \frac{d^2}{dt^2}h(t) \leq c(\gamma_0) \) will give a sequence \( \sigma_i \). The maximal time for length-constrained curve diffusion flow is \( T = \lim_{i \to \infty} \sigma_i \).

\[ \square \]

### A.2 Proof of Proposition 27

**Lemma 66.** Let

\[
\bar{F} = k_{ss} + \bar{k} k_{ss}.
\]

Then there exists constants \( \bar{c}_1, \bar{c}_2 \) such that

\[
\int \bar{F}^2 ds \geq \bar{c}_1 \int \gamma_{k_s^2} ds - \bar{c}_2 L^{-3} E^2.
\]

For the proof of this Lemma see [1, Lemma 7.1], we also show the proof as follows.

**Proof.** Consider the Fourier series for \( k \):

\[
k = \sum_p a_p \exp \left( \frac{2\pi}{L} p s \right).
\]

Then

\[
\kappa_0 = \sum_p a_p \left[ \left( \frac{4\pi^2}{L^2} p^2 \right)^2 - \frac{4\pi^2}{L^2} \right] \exp \left( \frac{2\pi}{L} p s \right)
= \sum_p a_p \left( \frac{4\pi^2}{L^2} \right)^2 p^2 \left( p^2 - \omega^2 \right) \exp \left( \frac{2\pi}{L} p s \right).
\]

This implies

\[
\int_\gamma \kappa_0^2 ds = \sum_p |a_p|^2 \left( \frac{4\pi^2}{L^2} \right)^4 p^4 \left( p^2 - \omega^2 \right)^2 L.
\]

We calculate

\[
a_{\pm w} = \frac{1}{L} \int_\gamma k \exp \left( \pm \frac{2\pi}{L} \omega s \right) ds = \frac{1}{L} \int_\gamma \left( k - \frac{2\pi\omega}{L} \right) \exp \left( \pm \frac{2\pi}{L} \omega s \right) ds.
\]

This implies

\[
|a_{\pm \omega}| \leq \frac{1}{L} \int_\gamma \left| k - \frac{2\pi\omega}{L} \right| \cdot \left| \exp \left( \pm \frac{2\pi}{L} \omega s \right) - \exp \left( \pm i \theta \right) \right| ds
+ \frac{1}{L} \int_\gamma \left| k - \frac{2\pi\omega}{L} \right| \exp \left( \pm i \theta \right) ds.
\]

In the above we have again used \( \theta \) to denote the tangential angle.
Noting that
\[ \int_{\gamma} \left( k - \frac{2\pi \omega}{L} \right) \exp(\pm i\theta) ds = \int_{\gamma} \theta_s \exp(\pm i\theta) ds - \frac{2\pi \omega}{L} \int_{\gamma} \tau ds = 0 \]
and
\[ \left| \frac{2\pi}{L} \omega s - \theta \right| \leq \int_0^s \left| \frac{d}{ds} \left( \frac{2\pi}{L} \omega s - \theta \right) \right| ds \]
\[ \leq \int_0^s \left| \frac{2\pi}{L} \omega - k \right| ds \]
\[ \leq L \int_{\gamma} |k_s| ds \]
\[ \leq \sqrt{2L^2} E^2 \]
we obtain
\[ |a_{\pm \omega}| \leq \frac{1}{L} \int_{\gamma} \left| k - \frac{2\pi \omega}{L} \right| \cdot \left| \exp \left( \pm \frac{2\pi}{L} \omega s \right) - \exp(\pm i\theta) \right| ds \]
\[ \leq \frac{1}{L} \int_{\gamma} |k_s| ds \int_{\gamma} \left| \left( \pm \frac{2\pi}{L} \omega s \right) - (\pm i\theta) \right| ds \]
\[ \leq L \left( \int_{\gamma} |k_s| ds \right)^2 \]
\[ \leq 2L^2 E. \]

Now
\[ \int_{\gamma} k_{s,0}^2 ds = \sum_p \left( \frac{4\pi^2}{L^2} \right)^4 p^8 |a_p|^2 L, \]
so we have
\[ \int_{\gamma} k_{s,0}^2 ds = \sum_p |a_p|^2 \left( \frac{4\pi^2}{L^2} \right)^4 p^4 \left( p^2 - \omega^2 \right)^2 L \]
\[ = \sum_{|p| \neq \omega} |a_p|^2 \left( \frac{4\pi^2}{L^2} \right)^4 p^4 \left( p^2 - \omega^2 \right)^2 L \]
\[ \geq \sum_{|p| \neq \omega, 0} |a_p|^2 \left( \frac{4\pi^2}{L^2} \right)^4 p^8 \left( 1 - \frac{\omega^2}{p^2} \right)^2 L \]

We define \( C_\omega \) by
\[ \left( 1 - \frac{\omega^2}{p^2} \right)^2 \geq \min \left\{ \left( 1 - \frac{\omega^2}{(\omega - 1)^2} \right)^2, \left( 1 - \frac{\omega^2}{(\omega + 1)^2} \right)^2 \right\} : = C_\omega. \]
Then
\[
\int_\gamma \kappa_0^2 ds \geq C_\omega \int_\gamma k_\gamma^2 ds - \left( \frac{4\pi^2}{L^2} \right)^4 \omega^8 \left( |a_+\omega|^2 + |a_-\omega|^2 \right) L
\]
\[
\geq C_\omega \int_\gamma k_\gamma^2 ds - 4\omega^8 \left( \frac{4\pi^2}{L^2} \right)^4 L^5 E^2
\]
\[
\geq C_\omega \int_\gamma k_\gamma^2 ds - 4^5 \omega^8 \pi^8 L^{-3} E^2,
\]
As for all time, \( \omega = 1 \) and \( L = L(0) \), then there exist constants \( \bar{c}_1, \bar{c}_2 \) such that
\[
\int_\gamma \bar{F}^2 ds \geq \bar{c}_1 \int_\gamma k_\gamma^2 ds - \bar{c}_2 L^{-3}(0)E^2.
\]
as required.

Secondly, we show the curvature bound here.

**Lemma 67.** For any immersed curve \( \gamma : \mathbb{S} \rightarrow \mathbb{R}^2 \) we have the estimate
\[
L\|k\|_\infty \leq \sqrt{L^3\|k_\gamma\|_2^2 + 2\omega\pi}.
\]
Here \( \omega \) is the winding number of \( \gamma \).

**Proof.** We calculate \( k = k - \bar{k} + \bar{k} \leq \int_\gamma |k_\gamma|ds + \frac{2\omega\pi}{L} \). Taking a supremum and using the Hölder inequality, we find
\[
\|k\|_{\infty} \leq \frac{1}{L} \left( \sqrt{L^3\|k_\gamma\|_2^2 + 2\omega\pi} \right).
\]

Now we show the proof of Proposition 27 by using above two lemmas.

**Proof.** Using above lemmas and \( \omega = 1 \), we have
\[
\int_\gamma G^2 ds = \int_\gamma F^2 ds + 2\int_\gamma F(G - \bar{F})ds + \int_\gamma (G - \bar{F})^2 ds
\]
\[
\geq \frac{1}{2} \int_\gamma \bar{F}^2 ds - \int_\gamma (G - \bar{F})^2 ds
\]
\[
\geq \frac{\bar{c}_1}{2} \int_\gamma k_\gamma^2 ds - 2^9 \pi^8 L^{-3}(0)E^2 - \int_\gamma (G - \bar{F})^2 ds \quad (A.1)
\]
Now \( G - \bar{F} = \left[ k^2 - \left( \frac{2\pi}{L} \right)^2 \right] k_\gamma - \frac{1}{2}k k_\gamma^2 \), so
\[
\int_\gamma (G - \bar{F})^2 ds \leq 2 \int_\gamma \left[ k^2 - \left( \frac{2\pi}{L} \right)^2 \right]^2 k_\gamma^2 ds + \frac{1}{2} \int_\gamma k^2 k_\gamma^4 ds.
\]
Now the curvature bound yields
\[
\int_\gamma (G - \bar{F})^2 ds \\
\leq 2 \left[ \left( \sqrt{2LE + \frac{2\pi}{L}} \right) + \left( \frac{2\pi}{L} \right) \right]^2 \left( \int_\gamma k_s ds \right)^2 \int_\gamma k_{ss}^2 ds \\
+ \frac{1}{2} \left( \sqrt{2LE + \frac{2\pi}{L}} \right)^2 \int_\gamma k_s^4 ds \\
\leq 4\sqrt{2} \left( \sqrt{2L^3E + 4\pi} \right)^2 E \int_\gamma k_{ss}^2 ds + \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 L^{-2} \int_\gamma k_s^4 ds \quad (A.2)
\]

Now the Gagliardo-Nirenberg Sobolev inequality yields universal constants \(c_3, c_4\) such that inequalities
\[
c_1 \int_\gamma k_{ss}^2 ds \leq c_1 c_3 \left( \int_\gamma k_s^2 ds \right)^{\frac{3}{2}} \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \leq c_1 c_3 (2E)^{\frac{3}{2}} \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \\
\leq \frac{Lc_1}{8E} \int_\gamma k_s^2 ds + cc_3 c_3 L^{-1} E^2
\]

and
\[
c_2 \int_\gamma k_s^4 ds \leq c_2 c_4 L^2 \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \leq c_2 c_4 L^2 (2E)^{\frac{1}{2}} \left( \int_\gamma k_s^2 ds \right)^{\frac{1}{2}} \\
\leq \frac{L^2 c_1}{8} \int_\gamma k_s^2 ds + cc_3 c_4 E^3 L^2
\]

hold. Using these with \(c_1 = 4\sqrt{2} \left( \sqrt{2L^3E + 4\pi} \right)^2\) and \(c_2 = \frac{1}{2} \left( \sqrt{2L^3E + 2\pi} \right)^2\) in combination with (A.2) above yields
\[
\int_\gamma (G - \bar{F})^2 ds \leq \frac{c_1}{4} \int_\gamma k_s^2 ds + c \left[ \left( L^3 E \right)^{\frac{3}{2}} + 1 \right] \left( L^{-\frac{1}{2}} E^2 \right) + \left( (L^3 E)^2 + 1 \right) E^3
\]

Plugging this into (A.1) we find
\[
\int_\gamma G^2 ds \\
\geq \frac{c_1}{2} \int_\gamma k_s^2 ds - 2^9 \pi^8 L^{-3} E^2 - \int_\gamma (G - \bar{F})^2 ds \\
\geq \frac{c_1}{4} \int_\gamma k_s^2 ds - c \left[ L^{-3} E^2 + \left( (L^3 E)^{\frac{3}{2}} + 1 \right) \left( L^{-\frac{3}{2}} E^2 \right) + \left( (L^3 E)^2 + 1 \right) E^3 \right] \\
\geq \frac{c_1}{4} \left( \frac{4\pi^2}{L^2} \right)^{\frac{3}{2}} E - c \left[ L^{-3} E^2 + \left( (L^3 E)^{\frac{3}{2}} + 1 \right) \left( L^{-\frac{3}{2}} E^2 \right) + \left( (L^3 E)^2 + 1 \right) E^3 \right] \quad (A.3)
\]
This implies
\[ L^9 \int \gamma F^2 \, ds \geq a(L^3 E) - b(L^2 E)^2 - g(L^3 E)^5 - d(L^3 E)^3 - e(L^3 E)^4 - f(L^3 E)^5 \]
where \( a, b, g, d, e, f \) are universal constants. Therefore, for \( L^3 E \) small enough, we have
\[ L^9 \int \gamma G^2 \, ds \geq \frac{a}{2} L^3 E, \]
as required.

\[ \Box \]

### A.3 Proof of Proposition 3

**Proof.** For the constrained problem, the associated Euler-Lagrange equation is
\[ L = f^2 + \lambda f_x^2, \]
with extremal functions satisfying
\[ \frac{\partial L}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial f_x} \right) = 2f - 2\lambda f_{xx} = 0. \]
That is to say
\[ f_{xx} - \frac{1}{\lambda} f = 0. \] (A.4)

This means that
\[ 0 \leq \int_0^P f_x^2 \, dx = -\int_0^P f f_{xx} \, dx = -\frac{1}{\lambda} \int_0^P f^2 \, dx, \]
which forces \( \lambda < 0 \). By standard arguments, we conclude from (A.4) that our extremal function is
\[ f(x) = A \cos \left( \frac{x}{\sqrt{|\lambda|}} \right) + B \sin \left( \frac{x}{\sqrt{|\lambda|}} \right). \] (A.5)

Here \( A, B \) are constants. The periodicity of \( f \) forces \( f(0) = f(P) \), so
\[ A = A \cos \left( \frac{P}{\sqrt{|\lambda|}} \right) + B \sin \left( \frac{P}{\sqrt{|\lambda|}} \right). \] (A.6)
Also, the requirement that \( \int_0^P f(x) \, dx = 0 \) forces
\[
A \sin \left( \frac{P}{\sqrt{|\lambda|}} \right) - B \cos \left( \frac{P}{\sqrt{|\lambda|}} \right) = -B. \tag{A.7}
\]
Combining (A.6) and (A.7),
\[
A^2 = A^2 \cos \left( \frac{P}{\sqrt{|\lambda|}} \right) + AB \sin \left( \frac{P}{\sqrt{|\lambda|}} \right)
\]
and
\[
B^2 = B^2 \cos \left( \frac{P}{\sqrt{|\lambda|}} \right) - AB \sin \left( \frac{P}{\sqrt{|\lambda|}} \right)
\]
meaning that
\[
A^2 + B^2 = (A^2 + B^2) \cos \left( \frac{P}{\sqrt{|\lambda|}} \right).
\]
We conclude
\[
\frac{P}{\sqrt{|\lambda|}} = 2n\pi
\]
for some \( n \in \mathbb{Z}/\{0\} \) to be determined. Hence
\[
f(x) = A \cos \left( \frac{2n\pi x}{P} \right) + B \sin \left( \frac{2n\pi x}{P} \right). \tag{A.8}
\]
A quick calculation yields
\[
\int_0^P f^2(x) \, dx = \left( \frac{A^2 + B^2}{2} \right) P,
\]
and
\[
\int_0^P f^2(x) \, dx = \left( \frac{2n\pi}{P} \right)^2 \left( \frac{A^2 + B^2}{2} \right) P.
\]
Hence for any of our extremal functions \( f \),
\[
\frac{\int_0^P f^2(x) \, dx}{\int_0^P f^2(x) \, dx} = \left( \frac{P}{2n\pi} \right)^2 \leq \frac{p^2}{4\pi^2},
\]
with equality if and only if \( n = 1 \). Thus our constrained function \( f \) that maximises the ratio \( \frac{\int_0^P f^2(x) \, dx}{\int_0^P f^2(x) \, dx} \) is given by
\[
f(x) = A \cos \left( \frac{2\pi x}{P} \right) + B \sin \left( \frac{2\pi x}{P} \right),
\]
with
\[ \int_0^P f^2 \, dx \leq \frac{P^2}{4\pi} \int_0^P f_x^2 \, ds \]
amongst all continuous and \( P \)-periodic function with \( \int_0^P f \, dx = 0 \).

Since \( \int_0^P f \, dx = 0 \) and \( f \) is \( P \)-periodic we conclude that there exists distinct \( 0 \leq p < q < P \) such that
\[ f(p) = f(q) = 0. \]

Next, the fundamental theorem of calculus tells us that for any \( x \in (0, P) \),
\[ \frac{1}{2} [f(x)]^2 = \int_p^x f_s \, dx = \int_q^x f_s \, dx. \]

Hence
\[ [f(x)]^2 = \int_p^x f_s \, dx - \int_x^q f_s \, dx \leq \int_x^q |f_s| \, dx \leq \int_0^P |f_s| \, dx \]
\[ \leq \left( \int_0^P f^2 \, dx \cdot \int_0^P f_x^2 \, dx \right)^{1/2} \leq \frac{P}{2\pi} \int_0^P f_x^2 \, dx. \]

\[ \square \]

### A.4 Referred Theorems

The general Theorem 4.4 in [69] is shown as below.

**Theorem 16.** Let \( T_1 > 0, \Omega \in \mathbb{R}^n, \phi \in H^\infty(\Omega) \) and \( h \in C^\infty([0, T_1]), H^\infty(\partial \Omega) \) satisfy the compatibility condition
\[ h^{(j)}(0) = \partial^j_B(t, u(t)) \bigg|_{t=0} \quad \text{on} \quad \partial \Omega, \quad j = 0, 1, 2, \ldots, \]
and normal boundary conditions. Assume that there are \( b \geq 0 \) and \( c_k > 0 \) and an open neighbourhood \( U \) of \( \phi \) in \( H^\infty(\Omega) \) so that any \( 0 < T \leq T_1 \) and
\[ u \in W = \{ w \in C^\infty([0, T], H^\infty(\Omega)) : w(t) \in U, t \in [0, T] \} \]
such that for any \( f(t) \in C^\infty([0, T], H^\infty(\Omega)) \) the linear problem
\[
\begin{cases}
  w_t(t) = \mathcal{F}_u(t, u(t))w(t) + f(t), & \text{in} \quad \Omega, t \in [0, T] \\
  \mathcal{B}_u(t, u(t))w(t) = 0, & \text{on} \quad \partial \Omega, t \in [0, T] \\
  w(0) = 0.
\end{cases}
\]

admits for any \( f \in C_0^\infty([0, T], H^\infty(\Omega)) \) a unique solution \( w \in C_0^\infty([0, T], H^\infty(\Omega)) \) satisfying
the estimates
\[ \|w\|_k \leq c_k [u; f]_{b,k}, \quad k = 0, 1, 2, \ldots \]
where \( c_k > 0 \) are suitable constants and \( b \geq 0 \) is an integer.

Then the nonlinear problem
\[
\begin{align*}
\{ & u_t = \mathcal{F}(t, u), \quad \text{in } \Omega, t \in [0, T] \\
& B(t, u) = h(t), \quad \text{on } \partial \Omega, t \in [0, T] \\
& u(0) = \phi.
\end{align*}
\]
has a unique solution \( u \in C^\infty_0 ([0, T], H^\infty(\Omega)) \).

For any integer \( k \geq 0 \), the Sobolev space \( H^k(\Omega) \) is equipped with its natural norms
\[
\|u\|_k = \left( \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u(x)|^2 ds \right)^{1/2}, \quad u \in H^k(\Omega).
\]

\( [u; f]_{b,k} \) is defined by the corresponding Sobolev norms of \( u, f \).
\( [u; f]_{b,k} = \sup \{ \|u\|_{b+i_1} \cdot \ldots \cdot \|u\|_{b+i_r} \cdot \|f\|_{b+j} \} \), where the "sup" is running over all integers \( i_1, \ldots, i_r, j \geq 0, i_1 + \ldots + i_r + j \leq k \), where \( 0 \leq r \leq k \). More properties of the terms \([ \ ]_{b,k}\) can be found in [70].

The normal boundary condition is defined as follows:
Let \( \{B_j\}_{j=1}^p \) be a set of differential operators \( B_j = B_j(u, \partial) \) of order \( m_j \) given by
\[
B_j = B_j(x, \partial) = \sum_{|\beta| \leq m_j} b_{j, \beta}(u) \partial^\beta, \quad j = 1, \ldots, p
\]
with \( b_{j, \beta} \in C^\infty(\eta_{1,2}) \). The set \( \{B_j\}_{j=1}^p \) is called normal if \( m_j \neq m_i \) for \( j \neq i \) and if for any \( x \in \eta_{1,2} \) we have \( B_j^p(u, \nu) \neq 0 \), \( j = 1, \ldots, p \) where \( \nu = \nu_{\eta_{1,2}} \) denotes the inward normal vector to \( \eta_{1,2} \) at \( u \) and \( B_j^p \) denotes the principal part of \( B_j \).

Here we give a simple definition for the principal part of an operator.

**Definition 10.** The principal part of a differential operator is the part which contains the highest order partial derivatives.

The classical results on linear parabolic boundary value problem ([53], Ch IV, 6.4) there is a unique solution \( w \in C^\infty_0 ([0, T], [-1, 1]) \) of the linearized problem. We state this theorem as follows.

**Theorem 17.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( \Gamma \) be the infinitely differentiable boundary of \( \Omega \).

\[
\begin{align*}
\frac{\partial w}{\partial t}(u, t) &= Aw + f(t), \quad \text{for all } (u, t) \in \Omega \times [0, T] \\
B_j w &= g_j, \quad \text{for all } (u, t) \in \Gamma \times [0, T] \\
w(\cdot, 0) &= w_0, \quad \text{for all } (u, t) \in \Omega \times \{t = 0\}
\end{align*}
\] (A.9)
where $A$ are given by $A\phi = \sum_{|p|, |q| \leq m} (-1)^{|p|} D_x^p (a_{pq} D_x^q \phi)$, the coefficients $a_{pq}$ satisfy $a_{pq} \in \mathcal{D}(\Omega \times [0,T])$. The boundary operators $B_j$, $0 \leq j \leq m - 1$, defined by $B_j \phi = \sum_{|h| \leq m} b_{jh} D_x^h \phi$, where the functions $b_{jh} = b_{jh}(x,t)$ satisfy $b_{jh} \in \mathcal{D}(\Omega \times \{t = 0\})$.

Let $r \geq 0$, $g_j, w_0, f$ be given with

$$g_j \in H^{2(r+1)m-m_j-1/2,(r+1)-(m_j+1/2)/2m}(\Gamma \times [0,T]), \ 0 \leq j \leq m - 1,$$

$$w_0 \in H^{2(r+1/2)m}(\Omega),$$

$$f \in H^{2rm,r}(\Omega \times [0,T])$$

and with the compatibility relation. Then problem (A.9) admits a unique solution in the space $H^{2(r+1)m,r+1}(\Omega \times [0,T])$.

The following theorem in [3, Main Theorem 5] give us that there is a unique solution for nonlinear parabolic problem.

**Theorem 18.** Let $E \times (0,w)$ be a vector bundle over $M \times (0,w)$, where $M$ a smooth closed manifold, and let $U$ be a section $\Gamma(E \times (0,w))$. Consider the following initial value problem:

$$P(U) := \partial_t U - F(x,t,U,\partial_x U,...,\partial_x^{2m} U) = 0 \text{ in } E \times (0,w),$$

$$U(M,0) = U_0,$$

with $U_0 \in C^{2m,1,\alpha}(E_w)$. The linearized operator of $P$ at $U_0$ in the direction $V$ is then given by

$$\partial P[U_0] V = \partial_t U + (-1)^m \sum_{|l| \leq 2m} A^l(x,t,U_0,\partial_x U_0,...,\partial_x^{2m} U_0) \partial_l V.$$ 

Suppose that the following conditions are satisfied:

1) The leading coefficient $A_{a_1j_1...l_mj_m}^{a_1j_1...l_mj_m}$ satisfies the symmetry condition $A_{a_1j_1...l_mj_m}^{a_1j_1...l_mj_m} = A_{a_1j_1...l_mj_m}^{a_1j_1...l_mj_m}$

2) The leading coefficient satisfies the Legendre-Hadamard condition with constant $\lambda$

3) There exists a uniform constant $\Lambda < \infty$ such that $\sum_{|l| \leq 2m} |A^l|_{a, E_w} \leq \Lambda$

4) $\tilde{F}$ is a continuous function of all its arguments

here $|\cdot|_{a,E_w}$ denotes the Hölder norm, $|u|_{a,E_w} = |u|_{a,E_w} := \sup_{X \neq Y \in E_w} \frac{|u(X) - u(Y)|}{d(X,Y)^a}$, also $\{u \in C^{2m,1}(P) : |u|_{2m,1,a,p<\infty} \}$. $\tilde{F}$ is the principal symbol of the linearized operator at the initial time.

Then there exists a unique solution $U \in C^{2m,1,\beta}(E_w)$, where $\beta < \alpha$, for some short time $t_e > 0$ to the above initial value problem. Furthermore, if $U_0$ and all the coefficients of the linearized operator are smooth, this solution is smooth.

**Definition 11.** If there exists a positive constant $\lambda \in \mathbb{R}$ such that the function $f$ satisfies $|f| \geq \lambda$. We say that function $f$ satisfies Legendre-Hadamard condition.
We state the Theorem A.1 in [1] as follows.

**Theorem 19.** Let \( (N^n, h) \) be an \( n \)-dimensional Riemannian manifold and \( M^m \) be an \( m \)-dimensional manifold with \( n > m \). Suppose \( \gamma : M^m \times [0, \infty) \to N^n \) is a one-parameter family of smooth isometric immersions satisfying

\[
\partial_t \gamma = F.
\]

Suppose furthermore that

1) (uniform bounds) We have the estimates

\[
\int_M |\gamma|^2 d\mu \leq c_1 \quad \text{and} \quad \int_M |k|^2 |\gamma|^4 d\mu \leq c_2
\]

for time-independent constants \( c_1 \) and \( c_2 \).

2) (\( L^1 - L^2 \) Velocity) The \( L^2 \)-norm of the velocity is uniformly \( L^1 \) in time, that is,

\[
\int_0^T \left( \int_M |F|^2 d\mu \right)^{1/2} dt \leq c_3
\]

for a constant \( c_3 \) that does not depend on \( T \).

3) (Subconvergence) There exists a smooth immersion \( \gamma_\infty : M^m \to N^n \) and a sequence \( \{t_j\} \subset [0, \infty) \to \infty \), such that \( \gamma(\cdot, t_j) \to^\infty \gamma_\infty \).

Then \( \gamma \) converges to \( \gamma_\infty \).