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Generalized Bhaskar Rao designs with block size three

Abstract

We show that the necessary conditions $\lambda = 0 \pmod{IGI}$, $\lambda(v-1) \equiv 0 \pmod{2}$, $\lambda v(v-1) \equiv 0 \pmod{6}$ for IGI odd, $0 \pmod{24}$ for IGI even, are sufficient for the existence of a generalized Bhaskar Rao design $GBRD(v, b, r, 3, \lambda; G)$ for the elementary abelian group G , of each order IGI .

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GENERALIZED BHASKAR RAO DESIGNS OF BLOCK SIZE THREE

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Abstract: We show that the necessary conditions

$$\lambda \equiv 0 \pmod{|G|},$$

$$\lambda(v-1) \equiv 0 \pmod{2},$$

$$\lambda v(v-1) \equiv \begin{cases} 0 \pmod{6} & \text{for } |G| \text{ odd,} \\ 0 \pmod{24} & \text{for } |G| \text{ even,} \end{cases}$$

are sufficient for the existence of a generalized Bhaskar Rao design $\text{GBRD}(v, b, r, 3, \lambda; G)$ for the elementary abelian group G , of each order $|G|$.

AMS Subject Classifications: Primary 05B99; Secondary, 05B05, 05B30, 62K10.

Key words: Bhaskar Rao designs.

1. Introduction

For the definitions and notations used in this paper we refer the reader to Lam and Seberry (1984). Since the underlying structure for a $\text{GBRD}(v, 3, \lambda; G)$ is a $\text{BIBD}(v, 3, \lambda)$ we have the following necessary conditions for existence:

$$\lambda \equiv 0 \pmod{|G|}, \tag{1.1}$$

$$\lambda(v-1) \equiv 0 \pmod{2}, \tag{1.2}$$

$$\lambda v(v-1) \equiv 0 \pmod{6}. \tag{1.3}$$

In Lam and Seberry (1984) the extra condition

$$|G| \equiv 0 \pmod{2} \Rightarrow b \equiv 0 \pmod{4} \tag{1.4}$$

was established.

We also use the notation $\text{EA}(H) = \text{EA}(\prod_{i=1}^n p_i^{r_i})$ for the elementary abelian group $Z_{p_1} \times Z_{p_1} \times \cdots \times Z_{p_1} \times \cdots \times Z_{p_n} \times Z_{p_n} \times \cdots \times Z_{p_n}$ where Z_{p_i} occurs r_i times with $\prod_{i=1}^n p_i^{r_i}$ the prime decomposition of the order of the group H .

In this paper we establish that the necessary conditions are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; \text{EA}(H))$.

Lam and Seberry (1984), building on the results of Seberry (1982, 1984), proved:

Theorem 1.1. *The necessary conditions (1.1)–(1.4) are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; G)$ when*

- (i) $|G|$ is odd,
- (ii) $G = Z_2^r$,
- (iii) $G = Z_2^r \times H$ where $3 \nmid |H|$ and $r \geq 1$.

In this paper we establish existence for the remaining group orders.

2. Existence of GBRD on $Z_2 \times Z_3$

Theorem 2.1. *The necessary conditions*

$$\lambda \equiv 0 \pmod{6},$$

$$\lambda v(v-1) \equiv 0 \pmod{24}$$

are sufficient for the existence of a $\text{GBRD}(v, 3, \lambda; Z_2 \times Z_3)$.

Proof. From Section 1 these are clearly necessary conditions: in the case $\lambda \equiv 6 \pmod{12}$ they become $v(v-1) \equiv 0 \pmod{4}$ and for $\lambda \equiv 0 \pmod{12}$ there is no condition on v . Hence we consider these cases separately.

Case 1: $\lambda = 6$. The necessary condition becomes $v(v-1) \equiv 0 \pmod{4}$. We first establish the existence of $\text{GBRD}(v, 3, 6; Z_6 = Z_2 \times Z_3)$ for $v \in K_4^1 = \{4, 5, 8, 9, 12\}$. Using 0, 1, 2, 3, 4, 5 for the elements, additively, of Z_6 we have that

$$D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 2 & 3 & 4 & 5 & . & . & . & 0 & 0 & 0 \\ 0 & 2 & 1 & . & . & . & 3 & 4 & 5 & 3 & 2 & 4 \\ . & . & . & 3 & 2 & 1 & 5 & 4 & 0 & 1 & 5 & 3 \end{pmatrix}$$

is a $\text{GBRD}(4, 3, 6; Z_6)$. The initial blocks developed as

$$D_5 = \{(0_0, 1_0, 2_1), (0_0, 1_2, 2_5), (0_0, 1_4, 3_4), (0_0, 1_5, 3_2)\} \pmod{5, Z_6},$$

$$D_8 = \{(0_0, 1_0, 2_1), (0_0, 1_2, 2_5), (0_0, 1_4, 3_4), (0_0, 1_5, 3_2), (0_0, 2_2, 4_0), (\infty_0, 1_1, 4_2),$$

$$(\infty_0, 1_3, 4_0), (\infty_0, 1_4, 5_5)\} \pmod{7, Z_6},$$

$$D_{12} = \{(0_0, 1_1, 10_3), (0_0, 2_1, 9_3), (0_0, 3_1, 8_3), (0_0, 4_1, 7_3), (0_0, 1_0, 5_5), (0_0, 4_0, 5_4),$$

$$(0_0, 3_4, 5_0), (0_0, 3_0, 4_2), (0_0, 1_5, 3_5), (\infty_0, 0_5, 2_5), (\infty_0, 0_1, 5_4),$$

$$(\infty_0, 0_2, 5_3)\} \pmod{11, Z_6},$$

give the required designs for $u=5, 8, 12$. A $\text{GBRD}(9, 3, 6; \mathbb{Z}_6)$ with a subdesign on 4 points is

$$D_9 = \left(\begin{array}{c|c|c|c|c} e & e & e & e & e \\ e & e & e & e & e \\ e & e & e & e & e \\ e & e & e & e & e \\ e & e & e & e & e \\ \hline x & s & & & \\ y & & s & & \\ & y & & s & \\ & & y & & s \\ & & & y & & s \\ & t & x & & & \\ & & & t & x & \\ & & y & t & & \\ \hline & & & & & D_4 \end{array} \right)$$

where

$$e = (0, 0, 0),$$

$$x = (0, 1, 2), \quad s = (3, 4, 5), \quad y = (0, 2, 1), \quad t = (3, 5, 4),$$

so that

$$x \cdot y^{-1} = (0, 5, 1), \quad x \cdot t^{-1} = (3, 2, 4),$$

$$s \cdot t^{-1} = (0, 5, 1), \quad s \cdot y^{-1} = (3, 2, 4).$$

Now using Theorem 2.2 of Lam and Seberry (1984) with Hanani's theorem (Hall (1967), Lemma 15.5.1) we only need to establish the existence of $\text{GBRD}(v, 3, 6; \mathbb{Z}_6)$ for $v \in K_4^1 = \{4, 5, 8, 9, 12\}$ to establish the existence of all $\text{GBRD}(u, 3, 6; \mathbb{Z}_6)$ for $u \equiv 0$ or $1 \pmod{4}$, $u \geq 4$. Thus we have all these designs and by taking t copies (t odd) we have $\text{GBRD}(v, 3, 6t; \mathbb{Z}_6)$ for all $\lambda \equiv 6 \pmod{12}$.

Case 2: $\lambda = 12$. There are now no necessary conditions on v . We observe that if $v = 2p + 1$, p a positive integer, then $\text{GBRD}(2p + 1, 3, 12; \mathbb{Z}_6)$ are obtained by developing the blocks (modulo $2p + 1, \mathbb{Z}_6$)

$$(0_0, i_1, 2p + 1 - i_4), \quad i = 1, \dots, 2p,$$

$$(0_0, j_1, 2p + 1 - j_1), (0_0, j_2, 2p + 1 - j_2), \quad j = 1, \dots, p.$$

For $v = 2p + 2$, $p \geq 2$ integer, a $\text{GBRD}(2p + 2, 3, 12; \mathbb{Z}_6)$ can be obtained by developing the following blocks (modulo $2p + 1, \mathbb{Z}_6$):

$$(0_0, i_1, 2p + 1 - i_4), \quad i = 3, \dots, 2p,$$

$$(0_0, j_1, 2p + 1 - j_1), (0_0, j_2, 2p + 1 - j_2), \quad j = 1, \dots, p,$$

$$(\infty_0, 0_0, 2p - 1_3), (\infty_0, 0_1, 2p - 3_4), (\infty_0, 0_2, 2p_0),$$

$$(\infty_0, 0_1, 2p - 1_5), (\infty_0, 0_2, 1_3), (\infty_0, 0_4, 2_5).$$

The case for $v = 4$ is obtained by taking two copies of the $\text{GBRD}(4, 3, 6; \mathbb{Z}_6)$ given in case 1 above.

Hence we have constructed a $\text{GBRD}(v, 3, 12; \mathbb{Z}_6)$ for every v . Taking t copies gives us $\text{GBRD}(v, 3, 12t; \mathbb{Z}_6)$ for every $\lambda = 12t \equiv 0 \pmod{12}$. Hence we have the theorem. $\square$

3. Existence of GBRD on $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

Theorem 3.1. *The necessary condition $\lambda \equiv 0 \pmod{12}$ is sufficient for the existence of a $\text{GBRD}(v, 2, 12t; \text{EA}(12))$.*

Proof. For the group $G = \text{EA}(12)$ the only necessary condition for the existence of a $\text{GBRD}_G(v, 3, \lambda)$ is that $\lambda \equiv 0 \pmod{12}$.

To establish existence we use Theorem 2.2 of Lam and Seberry (1984) with Hanani's theorem (Hall (1967), Lemma 15.5.2). Thus we must establish the existence of $\text{GBRD}(v, 3, 12; \text{EA}(12))$ for $v \in K_4^2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}$ to establish existence for all $v \geq 4$.

From Lam and Seberry (1984) we see that $\text{GBRD}(v, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$, say A , exists for $v \equiv 0, 1 \pmod{3}$. Writing $W = (w_{ij})$ for

$$W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (3.1)$$

the generalized Hadamard matrix $\text{GH}(3, \mathbb{Z}_3)$, we have, replacing every A_{ij} of A by the vector of ordered pairs $[(A_{ij}, w_{i1}), (A_{ij}, w_{i2}), (A_{ij}, w_{i3})]$, the required design. Hence we have the existence of $\text{GBRD}(v, 3, 12; \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ for $v = 4, 6, 7, 9, 10, 12, 15, 18, 19, 22$.

Similarly, from Seberry (1982), we have a $\text{GBRD}(v, 3, 3; \mathbb{Z}_3)$ for all odd v . Thus, combining these designs with three rows of

$$H = \begin{pmatrix} e & e & e & e \\ e & a & b & ab \\ e & b & ab & a \\ e & ab & a & b \end{pmatrix}, \quad a^2 = b^2 = e, \quad ab = ba, \quad (3.2)$$

the generalized Hadamard matrix $\text{GH}(4, \mathbb{Z}_2 \times \mathbb{Z}_2)$, we have the existence of $\text{GBRD}(v, 3, 12; \text{EA}(12))$ for all odd v and in particular for $v = 5, 11$ and 23 .

A $\text{GBRD}(8, 3, 12; \text{EA}(12))$ is constructed by the use of Theorem 2.2 with the $\text{GBRD}(8, 4, 3; \mathbb{Z}_3)$ obtained by developing the initial blocks $(\infty_1, 1_1, 2_{\omega^2}, 4_{\omega})$ and $(0_1, 1_{\omega}, 2_{\omega}, 4_{\omega}) \pmod{7, \mathbb{Z}_3}$, and the $\text{GBRD}(4, 3, 4; \mathbb{Z}_2 \times \mathbb{Z}_2)$

$$\begin{pmatrix} e & e & e & e & e & e & . & . \\ e & a & b & ab & . & . & e & e \\ e & b & . & . & a & ab & a & b \\ . & . & e & ab & b & a & b & ab \end{pmatrix}, \quad a^2 = b^2 = e, \quad ab = ba.$$

A GBRD(14, 3, 12; EA(12)) is obtained by developing the following initial blocks (modulo 13, EA(12)):

$$\begin{aligned} & (0_1, i_a, 13 - i_{ab\omega^2}), (0_1, i_{a\omega}, 13 - i_{ab\omega}), (0_1, i_{\omega}, 13 - i_{\omega}), \quad i = 1, \dots, 6, \\ & (0_a, i_b, 13 - i_{\omega}), \quad i = 3, 4, 5, 6, \\ & (\infty_1, 1_1, 12_{ab}), (\infty_1, 1_{b\omega}, 12_{ab\omega^2}), (\infty_1, 2_{ab\omega}, 11_{a\omega^2}), \\ & (\infty_1, 0_a, 1_b), (\infty_1, 0_{\omega^2}, 2_{a\omega}), (\infty_1, 0_{b\omega^2}, 1_{\omega}), \end{aligned}$$

where $a^2 = b^2 = \omega^3 = 1$ and all elements commute.

Thus a GBRD($v, 3, 12$; EA(12)) exists for all $v \in K_4^2$ and hence, for all v . GBRD for $\lambda = 12t$ are obtained by taking t copies of the design for $\lambda = 12$, giving the result. $\square$

4. Existence of GBRD on $Z_2 \times Z_2 \times Z_2 \times Z_3$

Theorem 4.1. *The necessary condition $\lambda \equiv 0 \pmod{24}$, is sufficient for the existence of GBRD($v, 3, 24t$; EA(24)).*

Proof. For the group EA(24) the only necessary condition is $\lambda \equiv 0 \pmod{24}$. Hence, as in the previous section, it is only necessary to establish the existence of GBRD($v, 3, 24$; EA(24)) for $v \in K_4^2$.

From Seberry (1982) a GBRD($v, 3, 3; Z_3$) exists for all odd v and so we use these designs with three rows of the generalized Hadamard matrix $\text{GH}(8, Z_2^3) = \text{GBRD}(3, 3, 8; Z_2 \times Z_2 \times Z_2)$, say

$$\begin{pmatrix} e & e & e & e & e & e & e & e \\ e & a & b & ab & c & ac & bc & abc \\ e & b & bc & c & ac & abc & ab & a \end{pmatrix}, \quad a^2 = b^2 = c^2 = e, \quad (4.1)$$

where the elements commute, in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD($u, 3, 24$; EA(24)) for all odd u .

From Theorem 2.1 a GBRD($u, 3, 6; Z_6$) exists for $u \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and so may be used with three rows of (4.2), the $\text{GH}(4, Z_2^2) = \text{GBRD}(3, 3, 4; Z_2 \times Z_2)$ in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD($u, 3, 24$; EA(24)) for these u and in particular for $u \in \{4, 8, 12\}$. From Lam and Seberry (1984) we see a GBRD($u, 3, 8; Z_2 \times Z_2 \times Z_2$) exists for $u(u-1) \equiv 0 \pmod{3}$: these designs may be used with (3.1), the $\text{GH}(3, Z_3) = \text{GBRD}(3, 3, 3; Z_3)$ in Theorem 2.2 of Lam and Seberry (1984) to obtain GBRD($u, 3, 24$) for these u including $u \in \{6, 10, 18, 22\}$.

There exists a GBRD(13, 3, 24; EA(24)). It is obtained by developing the following blocks modulo 13:

$$\begin{aligned} & (0_1, i_{\omega}, 13 - i_{\omega}), (0_1, i_{a\omega}, 13 - i_{b\omega}), (0_1, i_{b\omega}, 13 - i_{c\omega}), (0_1, i_{ac\omega}, 13 - i_{a\omega}), \\ & (0_1, i_{ac\omega^2}, 13 - i_{abc\omega^2}), (0_1, i_{bc\omega}, 13 - i_{abc\omega}), (0_1, i_{ab\omega}, 13 - i_{bc\omega}), \\ & (0_1, i_{c\omega}, 13 - i_{ab\omega}), \quad i = 1, 2, 3, 4, 5, 6. \end{aligned}$$

A GBRD(14, 3, 24; EA(24)) can be obtained by developing the same blocks modulo 13 except that the last initial block, $(0_1, i_{c\omega}, 13 - i_{ab\omega})$, $i = 1, 2, \dots, 6$, should be replaced by

$$\begin{aligned} & (0_1, 5_{c\omega}, 8_{ab\omega}), (0_1, b_{c\omega}, 7_{ab\omega}), \\ & (\infty_1, 0_{ac\omega^2}, 1_a), (\infty_1, 0_b, 2_{bc\omega}), (\infty_1, 0_{ac}, 3_{a\omega}), (\infty_1, 0_{b\omega^2}, 4_{bc}), \\ & (\infty_1, 0_1, 1_{ab\omega^2}), (\infty_1, 0_{abc}, 2_{c\omega^2}), (\infty_1, 0_{a\omega^2}, 3_{b\omega}), (\infty_1, 0_{bc\omega^2}, 4_{ac\omega}), \\ & (\infty_1, 0_\omega, 2_{abc\omega}), (\infty_1, 0_{\omega^2}, 4_{abc\omega^2}), (\infty_1, 0_{ab\omega}, 6_{c\omega}), (\infty_1, 0_{ab}, 8_c), \end{aligned}$$

which should also be developed modulo 13.

Hence a GBRD($v, 3, 24$; EA(24)) exists for all $v \in K_4^2$ and hence, as in the proof of Theorem 3.1 for all v . GBRD for $\lambda = 24t$ are obtained by taking t copies of the design for $\lambda = 12$ giving the result. $\square$

5. Existence of Bhaskar Rao designs with block size three

Theorem 5.1. *The necessary conditions (1.1)–(1.4) are sufficient for the existence of a generalized Bhaskar Rao design GBRD($v, 3, \lambda; G$) on the elementary abelian group G for every order $|G|$.*

Proof. From Theorem 1.1 it is merely necessary to consider the cases $G = Z_2^r \times H$, $r \geq 1$ where $3 \nmid |H|$. We have three basic cases:

- (i) $|G| = 6p$, p odd, with necessary condition $pv(v-1) \equiv 0 \pmod{4}$,
- (ii) $|G| = 12p$, p odd, with no condition on v ,
- (iii) $|G| = 24p$, p odd, with no condition on v .

We use Theorems 2.1, 3.1 and 4.1 respectively to obtain the design for $|G| = 6$, 12 and 24 and makes repeated use of theorem 2.2 of Lam and Seberry (1984) to obtain the result. For orders $2^s p$, $s \geq 4$, use $|G| = 12$ or 24 and repeated use of GBRD(3, 3, 4; $Z_2 \times Z_2$) according as s is even or odd. $\square$

6. Application

Proceeding as in Lam and Seberry (1984) and Seberry (1982, 1984) we now have:

Theorem 6.1. *Whenever $\lambda \equiv 0 \pmod{g}$, $\lambda(v-1) \equiv 0 \pmod{2}$, $\lambda v(v-1) \equiv 0 \pmod{6}$ for g odd or $\lambda v(v-1) \equiv 0 \pmod{24}$ for g even there exists a regular group divisible design with parameters $(vg, bg, r, 3, \lambda_1 = 0, \lambda_2 = \lambda/g, m = v, n = g)$.*

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