On Bhaskar Rao designs of block size four

Warwick de Launey

Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Follow this and additional works at: https://ro.uow.edu.au/infopapers

Part of the Physical Sciences and Mathematics Commons

Recommended Citation
https://ro.uow.edu.au/infopapers/1015

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au
On Bhaskar Rao designs of block size four

Abstract
We show that Bhaskar Rao designs of type BRD(v, b, r, 4, 6) exist for $v = 0,1 \mod 5$ and of type BRD(v, b, r, 4,12) exist for all $v \geq 4$.

Disciplines
Physical Sciences and Mathematics

Publication Details

This conference paper is available at Research Online: https://ro.uow.edu.au/infopapers/1015
ON BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

By WARWICK DE LAUNEY and JENNIFER SEBERRY

Department of Applied Mathematics, University of Sydney, N.S.W., Australia

ABSTRACT. We show that Bhaskar Rao designs of type \( \text{BRD}(v, b, r, k, \lambda) \) exist for \( v = 0, 1 \) (mod 5) and of type \( \text{BRD}(v, b, r, 12) \) exist for all \( v \geq 4 \).

Let \( A, B \) and \( A + B \) be \( v \times b \) matrices with entries 0, 1. Then \( X = A - B \) is said to be a Bhaskar Rao design with parameters \( \text{BRD}(v, b, r, k, \lambda) \) when the following matrix equations are satisfied:

\[
\begin{align*}
XX^T &= rI \quad \ldots \ (1) \\
(A + B)(A + B)^T &= (r - \lambda)I + \lambda J \quad \ldots \ (2) \\
J(A + B) &= kJ \quad \ldots \ (3)
\end{align*}
\]

\( X \) is a \( v \times b \) matrix with entries 0, +1, -1 with row inner product zero and which, when the -1 elements are replaced by +1, becomes the incidence matrix of a \( \text{BIBD}(v, b, r, k, \lambda) \). For example the following matrix is a \( \text{BRD}(6, 15, 10, 4, 6) \):

**Example 1**: There exists a \( \text{BRD}(6, 15, 10, 4, 6) \). Write—for -1

\[
\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & - & - & - & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & - & - & 0 & 1 & - & - & 0 & 1 \\
1 & 0 & - & 0 & - & 0 & 1 & 1 & 0 & - & - & 0 & 1 & 1 \\
0 & 0 & 1 & - & 0 & 1 & 0 & - & 1 & 0 & - & 0 & - & 1 \\
0 & 1 & 0 & 0 & 1 & - & 0 & 1 & - & 0 & - & 1 & - & -
\end{array}
\]

These designs were first studied by Bhaskar Rao [1, 2] and may be used to obtain group divisible \( \text{PBIBD} \) with parameters

\[
\begin{align*}
v^* &= 2v, \\
b^* &= 2b, \\
r^* &= r, \\
k^* &= k, \\
\lambda_1 &= 0, \\
\lambda_2 &= \lambda/2, \\
m &= 2, \\
n &= v.
\end{align*}
\]
The necessary conditions for the existence of a BRD\((v, b, r, k, \lambda)\) are for \(k = 4\),
\[
\lambda(v-1) = r(k-1) \\
bk = vr
\]
and other restrictions on the parameters have been found (see [3, 4]) when \(k \neq 4\). They have also been studied by Vyas [5] and Singh [6]. We use the notation BRD\((v, k, \lambda)\) for BRD\((v, b, r, k, \lambda)\) as \(b\) and \(r\) are dependent on \(v, k, \lambda\).

In this paper we use the following known results (see [7]) restricted to the group \(Z_2\):

Theorem 1: Suppose there exists a BRD\((k, j, \lambda_\mu)\) and
(i) a BRD\((v, k, \lambda_\mu)\) then there exists a BRD\((v, j, \lambda_\mu\lambda_\nu)\);
(ii) a BIBD\((v, k, \lambda)\) then there exists a BRD\((v, j, \lambda\lambda_\nu)\).

Or, as is obtained in a similar fashion:

Corollary 2: Suppose there exists a pairwise balanced design \(B(K, \lambda, v)\) where \(K = \{k_1, \ldots, k_6\}\) and a BRD\((k_i, j, \mu)\) for each \(k_i \in K\) then there exists a BRD\((v, j, \lambda\lambda_\mu)\).

The next result is a slight improvement on the result of Lam and Seberry [7] where the existence of \(k-1\) mutually orthogonal latin squares was required. The result may be proved by adjusting the matrix in the proof of the original theorem.

Theorem 3: Suppose there exists a BRD\((u, k, \lambda)\) with a subdesign on \(w\) points (the values \(w = 0\) and \(1\) are allowed), a BRD\((v, k, \lambda)\) and \(k-2\) mutually orthogonal Latin squares then there exists a BRD\((v(u-w)+w, k, \lambda)\) with subdesigns on \(u, w\) and \(v\) points.

Remark 4: In this paper we are interested in the case \(k = 4\), so we only need a pair of orthogonal latin squares and hence \(u-w\) may take on any value other than 2 or 6.

Hanani's theorem stated on p. 250 of Hall [8] states

Theorem 5 (Hanani): Let \(u \equiv 0, 1 \pmod{5}\) then \(u \in B(K_5, 1)\) where \(K_5^6 = \{5, 6, 10, 11, 15, 16, 20, 35, 36, 40, 60, 71, 75, 76\}.

Remark: We now see that if \(u \equiv 0, 1 \pmod{5}\) and there exists a BRD\((k_i, 4, 6)\) for every \(k_i \in K_5^6\) then we have the existence of a BRD\((u, 4, 6)\) using either Theorem 1 or Corollary 2 with Theorem 5.

The main theorem: First we establish:

Theorem 6: Let \(p \geq 5\), odd, be a prime or prime power. Then there is a BRD\((p, \frac{1}{2}p(p-1), 2(p-1), 4, 6)\).
ON BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

Proof: Let $g$ be a generator of the multiplicative group, $G$, of $GF(p)$. Consider the initial sets, writing $g^i$ for $g^a$ with the non-identity element of $Z_a$ attached,

$$D_i = \{0, g^i, g^{i+1}, g^{i+2}\}$$

where $i = 0, 1, \ldots, \frac{1}{2}(p-3)$.

The differences from $D_i$ are

$$E_i = \{g^i, g^{i+1}, g^{i+2}, g^{i(p-1)+i'}, g^{i(p-1)+i'}, g^{i(p-1)+i}+g, g^{i(p-1)+i}+g, g^{i(p-1)+i}+g\}.$$

As $i$ runs through $0, 1, \ldots, \frac{1}{2}(p-3)$ the totality of elements from all $E_i$ including repetitions is 3 copies of $G$ and 3 copies of $G$ with the non-identity element attached, that is,

$$\sum_{i=0}^{\frac{1}{2}(p-3)} E_i = 3G + 3G,$$

giving the result.

Theorem 7: A BRD($v$, 4, 6) exists for $v \equiv 0, 1 \pmod{5}$.

Proof: By the remark after Theorem 5 it is merely necessary to show the existence of a BRD($u$, 4, 6) for $u \in K_k$. These are obtained as given in Table 1 by developing the indicated initial blocks. First we exhibit a BRD($8$, 4, 6):

Example 2: There is a BRD($8$, 28, 14, 4, 6).

\[
\begin{array}{cccccccccccccccccccc}
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & 1 & 0 & 0 & I & -I & -I & -I & J-I & B & B & B & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
0 & 0 & 1 & 1 & 0 & 0 & - & - & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & -
\end{array}
\]

ca-40
Before we proceed to the case $\lambda = 12$ we establish the existence of a few more $BRD(v, 4, 6)$:

**Theorem 8:** There exist $BRD(v, 4, 6)$ for $v \in \{12, 14, 18, 22, 24, 32, 33, 38\}$. 

### Table 1.

<table>
<thead>
<tr>
<th>treatments</th>
<th>construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>6</td>
<td>Example 1.</td>
</tr>
<tr>
<td>7</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>8</td>
<td>Example 2.</td>
</tr>
<tr>
<td>9</td>
<td>Theorem 6.</td>
</tr>
<tr>
<td>10</td>
<td>$(\infty, 0, 2, 4)$, $(\infty, 0, 1, 3)$, $(0, 1, 2, 5)$, $(0, 5, 3, 6)$, $(0, 1, 5, 8)$ mod 9.</td>
</tr>
<tr>
<td>11</td>
<td>Prime, Theorem 6.</td>
</tr>
<tr>
<td>12</td>
<td>Form a $BRD(7, 4, 6)$ by developing $(\bar{0}, 1, 2, 4)$ mod 7. Use this $BRD$ with a $BIBD(15, 7, 3)$ in Theorem 1 (ii).</td>
</tr>
<tr>
<td>13</td>
<td>$(\infty, 0, 2, 6)$, $(\infty, 0, 3, 8)$, $(\bar{5}, 4, 5, 7)$, $(\bar{5}, 6, 7, 11)$, $(\bar{5}, 1, 5, 7)$, $(\bar{5}, 2, 3, 9)$, $(\bar{5}, 1, 3, 6)$ mod 15;</td>
</tr>
<tr>
<td>14</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 13, 16)$, $(\bar{5}, 3, 5, 11)$, $(\bar{5}, 3, 10, 18)$, $(\bar{6}, 4, 8, 15)$, $(\bar{5}, 5, 12, 13)$, $(\bar{5}, 2^{14}, 2^{14}, 2^{14})$, $a \cong (1, 2, 3, 4)$ mod 19;</td>
</tr>
<tr>
<td>15</td>
<td>$35 = 5 \times 7$, Theorem 3.</td>
</tr>
<tr>
<td>16</td>
<td>$36 = 5(8-1)+1$, Theorem 3.</td>
</tr>
<tr>
<td>17</td>
<td>$40 = 8 \times 5$, Theorem 3.</td>
</tr>
<tr>
<td>18</td>
<td>$70 = 7 \times 10$, Theorem 3.</td>
</tr>
<tr>
<td>19</td>
<td>$71$ Prime, Theorem 6.</td>
</tr>
<tr>
<td>20</td>
<td>$76 = 6 \times 15$, Theorem 3.</td>
</tr>
<tr>
<td>21</td>
<td>$78 = 15(6-1)+1$, Theorem 3.</td>
</tr>
</tbody>
</table>
Proof: Use Table 2 developing the initial blocks indicated.

<table>
<thead>
<tr>
<th>number of treatments</th>
<th>construction</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>14</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>18</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>22</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>24</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>32</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
<tr>
<td>38</td>
<td>$(\infty, 0, 3, 6)$, $(\infty, 0, 4, 7)$, $(3, 4, 5, 9)$, $(1, 3, 4, 5)$, $(1, 3, 4, 6)$, $(1, 3, 4, 9)$ mod $11$;</td>
</tr>
</tbody>
</table>

We use Hanani's theorem quoted from Hall [8, p. 248] to establish the result for $BRD(v, 4, 12)$.

Theorem 9 (Hanani): Let $u \geq 4$ then $u \in B(K^2_8, 1)$ where

$$K^2_8 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}.$$

Now we can show

Theorem 10: A $BRD(v, 4, 12)$ exists for all $v \geq 4$.

Proof: We note any four distinct rows of an Hadamard matrix of order 12 gives the result for $v = 4$. Thus, just as in the remark after Theorem 5, it is merely necessary to show the existence of a $BRD(u, 4, 12)$ for $u \in K^2_8$.

Taking two copies of the $BRD(u, 4, 6)$ given above gives the result immediately.
References


