1984

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Abstract

We show that Bhaskar Rao designs of type BRD(v, b, r, 4, 6) exist for v = 0,1 (mod 5) and of type BRD (v, b, r, 4,12) exist for all v ≥ 4.

Disciplines

Physical Sciences and Mathematics

Publication Details

de Launey, W and Seberry, J, On Bhaskar Rao designs of block size four, Combinatorics and Applications, Proceedings of the Seminar on Combinatorics and its Applications in honour of Professor S.S.Shrikhande, (K.S.Vijayan and N.M.Singhi, (Eds.)), Indian Statistical Institute, Calcutta, 1984, 311-316.

Proceedings of the Seminar on
Combinatorics and Applications
in honour of Prof. S. S. Shrikhande on his 65th birthday
Indian Statistical Institute
December 14—17, 1982, pp. 311-316.

ON BHASKAR RAO DESIGNS OF BLOCK SIZE FOUR

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ABSTRACT. We show that Bhaskar Rao designs of type BRD(v, b, r, 4, 6) exist for $v \equiv 0, 1 \pmod 5$ and of type BRD(v, b, r, 4, 12) exist for all $v \geqslant 4$.

Let A, B and $A+B$ be $v \times b$ matrices with entries 0, 1. Then $X = A-B$ is said to be a *Bhaskar Rao design* with parameters BRD(v, b, r, k, λ) when the following matrix equations are satisfied :

$$XX^T = rI \qquad \dots \quad (1)$$

$$(A+B)(A+B)^T = (r-\lambda)I + \lambda J \qquad \dots \quad (2)$$

$$J(A+B) = kJ. \qquad \dots \quad (3)$$

X is a $v \times b$ matrix with entries 0, +1, −1 with row inner product zero and which, when the −1 elements are replaced by +1, becomes the incidence matrix of a BIBD(v, b, r, k, λ). For example the following matrix is a BRD(6, 15, 10, 4, 6) :

Example 1 : There exists a BRD(6, 15, 10, 4, 6). Write—for −1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & - & - & - & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & - & - & - & 0 & 1 & - & - & 0 & 1 \\ 1 & 0 & - & 0 & - & 0 & 1 & 1 & 0 & - & - & - & 0 & 1 & 1 \\ 0 & 0 & 1 & - & 0 & - & 1 & 0 & - & 1 & - & 0 & - & - & 1 \\ 0 & 1 & 0 & 0 & 1 & - & 0 & 1 & - & - & 0 & - & 1 & - & - \end{bmatrix}$$

These designs were first studied by Bhaskar Rao [1, 2] and may be used to obtain group divisible PBIBD with parameters

$$v^* = 2v, \qquad b^* = 2b, \qquad r^* = r, \qquad k^* = k,$$

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda/2, \qquad m = 2, \qquad n = v.$$

The necessary conditions for the existence of a BRD(v, b, r, k, λ) are for $k = 4$,

$$\lambda(v-1) = r(k-1)$$
$$bk = vr \qquad \dots \quad (4)$$

and other restrictions on the parameters have been found (see [3, 4]) when $k \neq 4$. They have also been studied by Vyas [5] and Singh [6]. We use the notation BRD(v, k, λ) for BRD(v, b, r, k, λ) as b and r are dependent on v, k, λ.

In this paper we use the following known results (see [7]) restricted to the group Z_2 :

Theorem 1 : *Suppose there exists a BRD(k, j, λ_B) and*

(i) *a BRD(v, k, λ_A) then there exists a BRD$(v, j, \lambda_A\lambda_B)$;*

(ii) *a BIBD(v, k, λ) then there exists a BRD$(v, j, \lambda\lambda_B)$.*

Or, as is obtained in a similar fashion :

Corollary 2 : *Suppose there exists a pairwise balanced design $B(K, \lambda, v)$ where $K = \{k_1, \dots, k_b\}$ and a BRD(k_i, j, μ) for each $k_i \in K$ then there exists a BRD$(v, j, \lambda\mu)$.*

The next result is a slight improvement on the result of Lam and Seberry [7] where the existence of $k-1$ mutually orthogonal latin squares was required. The result may be proved by adjusting the matrix in the proof of the original theorem.

Theorem 3 : *Suppose there exists a BRD(u, k, λ) with a subdesign on w points (the values $w = 0$ and 1 are allowed), a BRD(v, k, λ) and $k-2$ mutually orthogonal Latin squares then there exists a BRD$(v(u-w)+w, k, \lambda)$ with subdesigns on u, w and v points.*

Remark 4 : In this paper we are interested in the case $k = 4$, so we only need a pair of orthogonal latin squares and hence $u-w$ may take on any value other than 2 or 6.

Hanani's theorem stated on p. 250 of Hall [8] states

Theorem 5 (Hanani) : *Let $u \equiv 0, 1 \pmod 5$ then $u \in B(K_5^1, 1)$ where* $K_5^1 = \{5, 6, 10, 11, 15, 16, 20, 35, 36, 40, 70, 71, 75, 76\}$.

Remark : We now see that if $u \equiv 0, 1 \pmod 5$ and there exists a BRD$(k_i, 4, 6)$ for every $k_i \in K_5^1$ then we have the existence of a BRD$(u, 4, 6)$ using either Theorem 1 or Corollary 2 with Theorem 5.

The main theorem : First we establish :

Theorem 6 : *Let $p \geqslant 5$, odd, be a prime or prime power. Then there is a BRD$(p, \frac{1}{2}p(p-1), 2(p-1), 4, 6)$.*

Proof: Let g be a generator of the multiplicative group, G, of $GF(p)$. Consider the initial sets, writing $\bar{g}^a$ for g^a with the non-identity element of Z_2 attached,

$$D_i = (\bar{0}, g^i, g^{i+1}, g^{i+2}\} \text{ where } i = 0, 1, \ldots, \tfrac{1}{2}(p-3).$$

The differences from D_i are

$$\begin{aligned} E_i = \{&\bar{g}^i, \bar{g}^{i+1}, \bar{g}^{i+2}, \bar{g}^{\frac{1}{2}(p-1)+i}, \bar{g}^{\frac{1}{2}(p-1)+i+1}, \bar{g}^{\frac{1}{2}(p-1)+i+2}, g^i(g-1), \\ &g^i(g^2-1), g^{i+1}(g-1), g^{\frac{1}{2}(p-1)+i}(g-1), g^{\frac{1}{2}(p-1)+i}(g^1-1), \\ &g^{\frac{1}{2}(p-1)+i+1}(g-1)\}. \end{aligned}$$

As i runs through $0, 1, \ldots, \frac{1}{2}(p-3)$ the totality of elements from all E_i including repetitions is 3 copies of G and 3 copies of G with the non-identity element attached, that is,

$$\overset{\frac{1}{2}(p-3)}{\underset{i=0}{\delta}} E_i = 3G + 3\bar{G},$$

giving the result.

Theorem 7 : *A BRD(v, 4, 6) exists for $v \equiv 0, 1$ (mod 5).*

Proof: By the remark after Theorem 5 it is merely necessary to show the existence of a BRD(u, 4, 6) for $u \in K_5^1$. These are obtained as given in Table 1 by developing the indicated initial blocks. First we exhibit a BRD(8, 4, 6) :

Example 2: There is a BRD(8, 28, 14, 4, 6).

$$\begin{bmatrix} 1\,1\,0\,0 & 1\,1\,1 & 1\,1\,1 & 1\,1\,1 & 1\,1\,1 & 0\,0\,0 & 0\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 1\,1\,0\,0 & & & & & & & & \\ 1\,1\,0\,0 & I & -I & -I & -I & J-I & B & B & B \\ 1\,1\,0\,0 & & & & & & & & \\ 0\,0\,1\,1 & & & & & & & & \\ 0\,0\,1\,1 & B & I & -I & -B & B & J-I & I & I \\ 0\,0\,1\,1 & & & & & & & & \\ 0\,0\,1\,1 & 0\,0\,0 & -\,-\,- & 1\,1\,1 & 0\,0\,0 & 0\,0\,0 & 0\,0\,0 & -\,-\,- & 1\,1\,1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

TABLE 1.

treatments	construction
5	Prime, Theorem 6.
6	Example 1.
7	Prime, Theorem 6.
8	Example 2.
9	Theorem 6.
10	$(\infty, 0, 2, 4)$, $(\bar{\infty}, 0, 1, 3)$, $(0, \bar{1}, \bar{2}, 5)$, $(0, \bar{2}, 3, \bar{6})$, $(0, 1, \bar{2}, \bar{5})$ mod 9.
11	Prime, Theorem 6.
15	Form a BRD (7, 4, 2) by developing $(\bar{0}, 1, 2, 4)$ mod 7. Use this BRD with a BIBD (15, 7, 3) in Theorem 1 (ii).
16	$(\infty, 0, 2, 6)$, $(\bar{\infty}, 0, 3, 8)$, $(\bar{0}, 4, \bar{5}, 7)$, $(\bar{0}, 4, 5, \bar{7})$, $(\bar{0}, 6, 7, \overline{11})$, $(\bar{0}, 1, 5, 7)$, $(\bar{0}, 2, 3, 9)$, $(\bar{0}, 1, 3, 6)$ mod 15;
20	$(\infty, 0, 2, 6)$, $(\bar{\infty}, 0, 13, 16)$, $(\bar{0}, 3, \bar{5}, 11)$, $(\bar{0}, 3, \overline{10}, 16)$, $(\bar{0}, 4, 8, \overline{12})$, $(\bar{0}, 5, 12, 13)$, $(\bar{0}, 2^{5+a}, 2^{6+a}, 2^{7+a})$, $a \in (1, 2, 3, 4)$, mod 19;
35	$35 = 5 \times 7$, Theorem 3.
36	$36 = 5(8-1)+1$, Theorem 3.
40	$40 = 8 \times 5$, Theorem 3.
70	$70 = 7 \times 10$, Theorem 3.
71	Prime, Theorem 6.
75	$75 = 5 \times 15$, Theorem 3.
76	$76 = 15(6-1)+1$, Theorem 3.

Before we proceed to the case $\lambda = 12$ we establish the existence of a few more BRD$(v, 4, 6)$:

Theorem 8 : *There exist BRD$(v, 4, 6)$ for $v \in \{12, 14, 18, 22, 24, 32, 33, 38\}$.*

Proof: Use Table 2 developing the initial blocks indicated.

TABLE 2.

number of treatments	construction
12	$(\infty, 0, \bar{3}, 5)$, $(\infty, 0, \bar{4}, \bar{7})$, $(3, \bar{4}, 5, 9)$, $(1, 3, 4, 5)$, $(\bar{1}, 3, 4, \bar{8})$, $(1, \bar{3}, 4, \bar{9})$ mod 11 ;
14	$(\infty, 0, 1, 3)$, $(\overline{\infty}, 0, 2, 5)$, $(\bar{0}, 2, 4, 8)$, $(\bar{0}, 2, 3, 7)$, $(\bar{0}, 1, 2, \bar{6})$, $(\bar{0}, 1, 5, \bar{8})$, $(\bar{0}, 1, 4, \bar{7})$ mod 13 ;
18	$(\infty, 0, 3, 4)$, $(\overline{\infty}, 0, 5, 7)$, $(\bar{0}, 1, 5, \bar{8})$, $(0, \bar{6}, \bar{8}, 13)$, $(\bar{0}, 5, \bar{6}, 8)$ $(\bar{0}, 5^a, 5^{a+1}, 5^{a+2})$, $a \in \{2, 3, 4, 6\}$, mod 17 ;
22	$(\infty, 0, 2, 6)$, $(\overline{\infty}, 0, 9, 10)$, $(0, \bar{6}, \bar{7}, 16)$, $(0, \bar{4}, \bar{6}, 14)$, $(\bar{0}, 1, \bar{3}, 10)$, $(\bar{0}, 1, 4, 9)$, $(\bar{0}, 2, 5, 12)$, $(\bar{0}, 4, 8, 15)$, $(\bar{0}, 2, 3, 8)$, $(\bar{0}, 1, 3, 16)$, $(\bar{0}, 3, 7, 16)$, mod 21 ;
24	$(\infty, 0, 4, 12)$, $(\overline{\infty}, 0, 4, 10)$, $(\bar{0}, \bar{1}, 2, 5,)$, $(\bar{0}, 2, 5, 10)$, $(\bar{0}, 2, 4, 10)$, $(\bar{0}, 3, \bar{7}, 10)$, $(\bar{0}, 5^a, 5^{a+1}, 5^{a+2})$, $a \in \{5, 6, 7, 8, 10\}$, mod 23 ;
32	$(\infty, 0, 2, 6)$, $(\overline{\infty}, 0, 1, 8)$, $(\bar{0}, 3, \bar{4}, 15)$, $(\bar{0}, \bar{1}, 6, 14)$, $(\bar{0}, 3, \bar{7}, 9)$, $(\bar{0}, 9, 12, 27)$, $(0, 3^a, 3^{a+1}, 3^{a+2})$, $a \in \{2, 4, 5, 7, 8, 10, .., 14\}$, mod 31 ;
33	$33 = 8(5-1)+1$, Theorem 3.
38	$(\infty, 0, 2, 12)$, $(\overline{\infty}, 0, 1, 17)$, $(\bar{0}, 4, 17, \overline{33})$, $(\bar{0}, 10, \overline{12}, 13)$, $(\bar{0}, 3, \bar{8}, 16)$ $(\bar{0}, 4, 11, 12)$, $(\bar{0}, 2^a, 2^{a+1}, 2^{a+2})$, $a \in (1, 4, 5, 7, 8, 9, 11, .., 17\}$, mod s37.

We use Hanani's theorem quoted from Hall [8, p. 248] to establish the result for BRD(v, 4, 12).

Theorem 9 (Hanani) : *Let* $u \geqslant 4$ *then* $u \in B(K_4^2, 1)$ *where*

$$K_4^2 = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19, 22, 23\}.$$

Now we can show

Theorem 10 : *A BRD*(v, 4, 12) *exists for all* $v \geqslant 4$.

Proof: We note any four distinct rows of an Hadamard matrix of order 12 gives the result for $v = 4$. Thus, just as in the remark after Theorem 5, it is merely necessary to show the existence of a BRD(u, 4, 12) for $u \in K_4^2$.

Taking two copies of the BRD(u, 4, 6) given above gives the result immediately.

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