1982

The skew - weighing matrix conjecture

Jennifer Seberry

*University of Wollongong, jennie@uow.edu.au*

Follow this and additional works at: [https://ro.uow.edu.au/infopapers](https://ro.uow.edu.au/infopapers)

Part of the Physical Sciences and Mathematics Commons

**Recommended Citation**
https://ro.uow.edu.au/infopapers/1008

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au
The skew - weighing matrix conjecture

Abstract
We review the history of the skew-weighing matrix conjecture and show that there exist skew-symmetric weighing matrices W (21.2t, k) for all k=0,l,.....,21.2t - 1, t ≥ 4 a positive integer. Hence there exist orthogonal designs of type l(l,k) for all k=0,l,....., 21.2t - 1, t ≥ 4 a positive integer, in order 21.2t.

Disciplines
Physical Sciences and Mathematics

Publication Details
The Skew - Weighing Matrix Conjecture

Jennifer Seberry*

SUMMARY:

We review the history of the skew-weighing matrix conjecture and show that there exist skew-symmetric weighing matrices $W(21.2^t, k)$ for all $k=0, 1, \ldots, 21.2^t - 1$, $t \geq 4$ a positive integer. Hence there exist orthogonal designs of type $1(1,k)$ for all $k=0,1, \ldots, 21.2^t - 1$, $t \geq 4$ a positive integer, in order $21.2^t$.

1. Introduction:

An orthogonal design $X$ of order $n$ and type $(s_1, s_2, \ldots, s_t)$ on the commuting variables $0, X_1, \ldots, X_t$ is a square matrix with entries from $\{0, \pm X_1, \ldots, \pm X_t\}$ with the properties that

(i) the element $\pm X_i$ occurs $s_i$ times in each row and column,

(ii) the inner product of distinct rows is zero. Hence $X$ satisfies

$$XX^T = \sum_{i=1}^{t} s_i X_i^2 I_n$$

*This paper was written while visiting the Department of Statistics, University of Indore, Indore.

A weighing matrix $W=W(n,k)$ is a square matrix with entries 0, ±1 having $k$ non-zero entries per row and column and inner product of distinct rows zero. Hence $W$ satisfies $WW^T=kl_n$, and $W$ is equivalent to an orthogonal design of order $n$ and type $(k)$. The number $k$ is called the weight of $W$. Weighing matrices have long been studied because of their use in weighing experiments as first studied by Hotelling (1944) and latter by Raghavarao (1960) and others.

In Wallis (1972) it was conjectured that if $n \equiv 0 \pmod{4}$ then weighing matrices $W(n,k)$ exist for every $k=1, 2, \ldots, n$. Call this Conjecture A. This was proved true for orders $n = 2^t$, $t$ a positive integer in Geramita, Pullman, Wallis (1974). Later the conjecture was made stronger by Seberry until it appeared in the following form:

Conjecture B: There exist skew-symmetric weighing matrices $W(n,k)$ for every $k=1, 2, \ldots, n-1$ when $n \equiv 0 \pmod{8}$. Equivalently, every orthogonal design $(1,2, \ldots, n-1)$ exists when $n \equiv 0 \pmod{8}$.

This conjecture was established for $n = 2^t, 2^t+1, 2^t+3$ by Geramita and Wallis (1974, 1975), for $n = 2^t+7$ by Eades and Wallis (1976) and for $n = 2^t+15$ by Seberry (1980), $t \geq 3$, an integer. In this paper we will establish the conjecture for $n = 2^{t+1}+2$.

* This paper was written while visiting the Department of Statistics, University of Indore, Indore.
2. Preliminary Results And Notation:

We make extensive use of the book Geramita and Sebbery (1979). We quote the following theorems, giving their reference from the aforementioned book, that we use:

Lemma 1: (Lemma 4.11). If there exists an orthogonal design in order \( n \) of type \((s_1, s_2, \ldots, s_e)\) then there exists an orthogonal design in order \( 2n \) of type \((s_1, s_1, s_2, \ldots, s_e, \ldots, s_e)\), \( e = 1 \) or 2.

Lemma 2: (Lemma 4.4). If \( A \) is an orthogonal design of order \( n \) and type \((s_1, \ldots, s_e)\) on the variables \( x_1, \ldots, x_e \), then there is an orthogonal design of order \( n \) and types \((s_1, \ldots, s_{e-1}, s_e, \ldots, s_e)\) and \((s_1, \ldots, s_e, \ldots, s_1, s_{e-1})\) on the \( s-1 \) variables \( x_1, \ldots, x_{s-1} \).

Example:

\[
\begin{pmatrix}
  x & y & z & w \\
  -y & x & z & w \\
  -z & -w & x & y \\
  -w & -z & -y & x \\
\end{pmatrix}
\]

is an orthogonal design of type \((1, 1, 1, 1)\) in order 4. Using Lemma 2 we see

\[
\begin{pmatrix}
  x & y & z & w \\
  -y & x & z & w \\
  -z & -w & x & y \\
  -w & -z & -y & x \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  x & y & o & w \\
  -y & x & w & o \\
  o & -w & x & y \\
  -w & o & -y & x \\
\end{pmatrix}
\]

are orthogonal designs of types \((1,1,2)\) and \((1,1,1)\) respectively. Lemma 1 tells us there are orthogonal designs of types \((1,1,2,2,2)\), \((1,1,2,4)\), \((2,2,2,2)\) and \((2,2,2,2)\) in order 8.

Lemma 3 (Corollary 5.2): If all orthogonal designs \((1,k)\), \( k = 1, \ldots, n-1 \), exist in order \( n \), then all orthogonal designs \((1,j), j = 1, \ldots, 2n-1 \), exist in order \( 2n \).

Example: Using the orthogonal design of type \((1, 1, 1, 1)\) in order 4 with Lemma 2 we see that orthogonal designs of types \((1,1),(1,2)\) and \((1,3)\) exist in order 4. Lemma 3 is now used to see that orthogonal designs of types \((1,k), k = 1, 2, \ldots, 7\) exist in order 8. By repeated use of Lemma 3 we see that orthogonal designs of types \((1,j), j = 1, 2, \ldots, 2^{t-1}, \) \( t \) a positive integer, exist in every order \( 2^t \).

Theorem 4 (Theorems 2.19 and 2.20): suppose \( n \equiv 0 \) (mod 4). Then the existence of a \( W(n,n-1) \) implies the existence of a skew-symmetric \( W(n,n-1) \). The existence of a skew-symmetric \( W(n,k) \) is equivalent to the existence of an orthogonal design of types \((1,k)\) in order \( n \).

Theorem 5 (Proposition 3.54 and Theorem 2.20): An orthogonal design of type \((1,k)\) can only exist in order \( n \equiv 4 \) (mod 8) if \( k \) is the sum of three squares. An orthogonal design of type \((1,n-2)\) can only exist in order \( n \equiv 4 \) (mod 8) if \( n-2 \) is the sum of two squares.

Theorem 6: Orthogonal designs of types \((1,k)\) exist for \( k = 1, \ldots, n-1 \) in orders \( n=2t, 2^{t+1}, 3, 2^{t+2}, 5, 2^{t+3}, 7, 2^{t+4}, 9, 2^{t+4}, 15 \), \( t \geq 0 \) an integer.

Theorem 7: (Theorem 4.49) If there exist four circulant matrices \( A_1, A_2, A_3, A_4 \) of order \( n \) satisfying.
Theorem 8: (Theorems 4.124 and 4.41). Let \( q \) be a prime power then there is a circulant \( W(q^2+q+1, q^2) \). Let \( p \equiv 1 \pmod{4} \) then there are two circulant symmetric matrices \( R, S \) of order \( \frac{1}{2}(p+1) \) satisfying \( RR^T + SS^T = P \), and a symmetric matrix \( P \), with zero diagonal satisfying \( PP^T = p^2 p + 1 \) that is a symmetric \( W(p+1, p) \).

Corollary 9: There exists a circulant \( W(21, 16) \) and \( W(7, 4) \). There exist two circulant symmetric matrices \( R, S \) of order \( n=41, 21, 7 \) satisfying \( RR^T + SS^T = (2n-1) I_n \). There exist symmetric \( W(6, 5) = X \) and \( W(14, 13) = Y \) with zero diagonal.

We use the following notation:

1. \( I \) is the identity matrix with order taken from the context.
2. \( J \) is the matrix of ones with order taken from the context;
3. \( A \) is the back-circulant matrix with first row \( b \ a \ a \);
4. \( B \) is the back-circulant matrix with first row \( b \ a \ a \ a \ a \ a \ a \);
5. \( W_7 \) is the back-circulant matrix with first row \( 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \);
6. \( W_{21} \) is the back-circulant matrix with first row the same as the circulant \( W(21, 16) \);
7. \( R_n, S_n \) are circulant symmetric matrices of order \( n=7, 21, 41 \) satisfying \( S_n S_n^T + R_n R_n^T = (2n-1) I_n \);
8. \( X \) is the symmetric \( W(6, 5) \);
9. \( Y \) is the symmetric \( W(14, 13) \);
10. \( U, V, Z \) are the circulant symmetric matrices with first rows \( 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \) respectively, \( U^2 + V^2 + Z^2 = 19 I - J \).

3. Results In Orders Divisible By 21:

We recall from Theorem 5 that orthogonal designs of type \((1, k)\) can only exist in order 84 if \( k \) is the sum of three squares. We see \( 84 - 2 = 9^2 + 1^2 \) so the other condition is satisfied. Hence we have \((1, k)\) can not exist for \( k = 4a (8b + 7) \), i.e. \( k \in \{7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79\} \).

Theorem 10: Orthogonal designs of types \((1, k)\) exist in order 84 for \( k \in \{x | x = a^2 + b^2 + c^2, x = 42, 61, 62, 67, 70, 73-78, 82\} \).

Proof: Theorem 6 and Lemma 8.36 of Geramita and Seberry (1979) say that all orthogonal designs of type \((1, k)\) exist in order 40 and 44 for \( k \leq 38 \) the sum of three integer squares. We have by direct sum, \( A \oplus B \), of the OD of type \((1, k)\) in order 40, \( A \) and the OD of type \((1, k)\) in order 44, \( B \), the result in order 84.
Table H.4, Table H.2, Theorem 4.139 (xvi) and the proof of Theorem 4.139 of Geramita and Seberry (1979) give \((1, k)\) in order 84 for \(k \in \{40, 41, 43, 44, 48, 51, 52, 54, 58, 59, 66, 68\}\).

Using Table H.6 which gives 2-complementary sequences giving ODs of type \((4, 16)\) in orders \(2m, m \geq 10\) in Lemma 4.118 we obtain designs of types \((1, 64), (1, 80)\) and \((1, 81)\) in order 84. In particular, we obtain designs of types \((1, 64), (1, 80)\) and \((1, 81)\) in order 84.

From corollary 9 there is a circulant \(W(21, 16)\) and two circulant symmetric matrices, \(R, S\) satisfying

\[
(RRT + SS^T) = 4I.
\]

Hence \(x, y, z\) satisfy

\[
(x^2 + 16y^2 + 41z^2) = 1,
\]

and so by Theorem 7 we have an orthogonal design of type \((1, 64)\), and thus \((1, 57)\) in order 84.

We note from Geramita and Seberry (1979) that the following ODs exist: 

(i) \((1, 2, 3, 6)\) in order 12; 
(ii) \((1, 1, 9, 9)\) in 28; 
(iii) \((1, 1, 13, 13)\) in 28; 
(iv) \((1, 6, 18)\) in 28.

We replace the variables of these matrices by other matrices as follows: 

(i) \(a, b, c\), back circulant matrix with first row \(a, b, b, b\) to an \(OD\) of type \((1, 3, 6, 50)\) in order 84; 
(ii-a) \(a, b, c\), circulant matrix with first row \(a, b, c, c\) to get an \(OD\) of type \((1, 4, 45)\) and \((1, 4, 65)\); 
(ii-b) \(a, b, c\), the back-circulant matrix with first row \(a, b, c\) to get an \(OD\) of type \((1, 18, 54)\) in order 84.

These give ODs \((1, k)\) for \(k \in \{45, 46, 49, 50, 53, 56, 65, 69, 72\}\) in order 84.

This completes the result.

Theorem 11: There exist \(W(84, k)\) for all \(1 \leq k \leq 84\) except possibly \(k = 71, 79\).

Proof: Combining the results of the previous theorem, nothing that \((1, f)\) gives both \(W(84, j)\) and \(W(84, j+1)\), with the results of Lemma 8.46 of Geramita and Seberry gives the result for all values except \(W(84, 63)\) and \(W(84, 77)\).

We note that an \(OD\) of type \((7, 7, 14)\) exists in order 28 and replacing the variables by \(c\), the circulant matrix with first row \(a, c, c, c\) respectively, gives an \(OD\) of type \((7, 14, 63)\) in order 84 giving the \(W(84, 63)\) and \(W(84, 77)\) required.

Theorem 12: There exists an \(OD\) of type \((1, k)\) in order 168 for \(k \in \{1, \ldots, 154, 156, 157, 160, \ldots, 163, 165, \ldots, 167\}\).

Proof: Using Theorem 6 we have the result immediately for \((1, k)\) \(k \in \{1, \ldots , 71\}\) as 168 = 72 + 96 and \((1, k)\) exist in both these orders.

We use the existence of \(W = W(84, k)\), from Theorem 11, in \(\frac{aI - W}{W^T - aI}\) to obtain \((1, k)\) in 168 for \(k \in \{72, 78, 80, \ldots, 84\}\). Using Lemmas 1, 2 and Theorem 10
we get the result for \( k \in \{86, \ldots, 93, 96, \ldots, 109, 112, \ldots, 133, 136, \ldots, 139, 144, 145, 160, \ldots, 163, 166, 167\} \).

Suppose there exists an OD of type \((1, f)\) in order 28 then by Lemma 1 we have a design of type \((1, 1, j, f)\) in order 56.

Define \(K = 1-I\) and \(A\) to be the back-circulant matrix with first row \(b a a\). Then replacing the variables of the \((1, 1, j, f)\) design by (i) cI, dI, A, \(a (J-2I)\) and (ii) \(A, k, cK, c(J-2I)\), we obtain designs of types \((1, I, j, 5f)\) and \((1, 4, 5f)\) respectively in order 168. Now from Geramita and Seberry (1979), Appendix C, we have ODs of type \((1, n)\) in order 28 for \( j \in \{17, 18, 19, 22, 24, 25, 26, 27\} \) giving designs \((1, k)\) in 168 for \( k \in \{85, 94, 95, 110, 111, 120, 121, 124, 125, 126, 134, 135, 150, 151, 156, 157\} \).

There exist orthogonal designs of types \((1, 1, 25)\) and \((1, 3, 24)\) in order 28. Hence, using Lemma 1, ODs of types \((1, 1, 25, 25)\) and \((1, 3, 24, 24)\) exist in order 56. Replacing the variables by \(a, b, e, d, a (J-2I)\) and \(c, d, a, a (J-2I)\) respectively, we obtain ODs of types \((1, 1, 25, 125)\) and \((1, 3, 24, 120)\) in order 168. Hence ODs of types \((1, k)\) for \( k \in \{123, 127, 147, 152\} \) exist in order 168.

The following ODs exist in order 24 : \((1, 1, 1, 1, 1, 1, 19)\), \((1, 1, 1, 1, 10, 10)\), \((1, 1, 1, 1, 1, 5, 5, 9)\), \((1, 1, 1, 1, 1, 9, 9)\), \((1, 1, 1, 2, 3, 16)\) and \((1, 1, 1, 5, 15)\). We replace the variables by the matrices as follows :

(i) \(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (ii) \(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (iii) \(c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (iv) \(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (v) \(c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (vi) \(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\); (vii) \(c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\).

Hence we have orthogonal designs of types \((1, k)\) for \( k \in \{79, 122, 140, \ldots, 143, 146, 148, 149, 153, 154\} \) in order 168.

The proof of Lemma 8.46 of Geramita and Seberry (1979) gives a back-circulant matrix \(A_1\), and three circulant matrices, \(A_2, A_3, A_4\), all of order 21 which can be used to replace four of the variables of a \((1, 1, 1, 1, 1, 1, 1, 1, 1)\) in order 8 while the others are replaced by \(c_{R_{41}}, c_{S_{41}}, d_{R_{41}}, d_{S_{41}}\) to give an OD of type \((1, 83, 41, 41)\) in 168 that is a \((1,165)\) in 168. This completes the proof.

**Theorem 13** : All orthogonal designs of type \((1, k)\), \(0 \leq k \leq n-1\), exist in order \(n = n+1\), 21, \(t\) a positive integer.

**Proof** : From the previous theorem and Lemmas 1 and 2 we have all orthogonal designs of type \((1, k)\) in order 336 except for \( k \in \{310, 311, 316, 317, 318, 319, 328, 329\} \). Theorem 5.1 of Geramita and Seberry (1979) says that the existence of an orthogonal design of type \((a, b)\) in order \(n\) gives an OD of type \((a, a, 2a, b, b, 2b)\) in order \(4an\). Hence, using the \((1,26)\) and \((1,27)\) designs in order 28, there are ODs of type \((1, 1, 2, 26, 26, 52)\) and \((1, 1, 2, 27, 27, 54)\) in order 112. Replacing the variables by \(A, a (J-1), c, d, d (J-2I), d (J-2I)\) in both cases we get ODs of types \((1, 2, 4, 312)\) and \((1, 2, 4, 324)\) in order 336. So we have solutions for \( k \in \{316, 318, 328\} \).
Using the \((1, 1, 13, 13)\) and \((1, 27)\) designs in order 28 with Lemmas 1 and 2 we obtain ODs of types \((1, 1, 26, 26)\) and \((1, 1, 27, 27)\) in order 56. Replacing the variables by \(aI, bX, \) \(dI+eX, dI+eX\) and \(aI, bX, cI-dX, cI-dX\) respectively we obtain designs of types \((1, 2, 52, 52)\) and \((1, 2, 54, 54)\) giving \((1, 317)\), \((1, 319)\), \((1, 329)\) in order 336.

We use the theorem quoted earlier to show an OD of type \((1, 11)\) in order 12 may be used to construct an OD of type \((1, 1, 12, 11, 11)\) in order 48. Replacing the matrices by \(cI, dI, eI, a \) \([J-2I]\), \(B, B\) gives an OD of type \((1, 1, 23, 23)\) in order 336 giving \((1, 310)\) and \((1, 311)\).

Hence we have shown all orthogonal designs of type \((1, k)\), \(k=0, 1, \ldots, 335\) exist in order 336. We now use Lemma 3 to obtain the result of the theorem.

4. GENERAL RESULTS:

We note that orthogonal designs of types \((1, k)\), \(k \in \{1, \ldots, 71\}\) exist in order 72, 96 and 120. Hence orthogonal designs \((1, k)\) for \(k \in \{1, \ldots, 71\}\) exist in every order \(24n, n \geq 3\) and orthogonal designs of type \((1,f)\), \(f \in \{1, \ldots, 143\}\) exist in every order \(48n, n \geq 3\).

Since ODs of types \((1, k)\) for \(k \in \{1, \ldots, 95\}\) exist in orders 96, 120, 144, 168 they exist in every order \(24n, n \geq 4\) and designs of type \((1, j)\), \(j \in \{1, \ldots, 191\}\) exist in every order \(48n, n \geq 4\). Summarizing and using Seberry (1980) we have:

**Theorem 14:** There exist orthogonal designs of types \((1, k)\) in orders \(48n\) with
(a) \(n \geq 3, k \in \{1, \ldots, 143\}\),
(b) \(n \geq 4, k \in \{1, \ldots, 191\}\),
(c) \(n \geq 5, k \in \{1, \ldots, 211, 216, \ldots, 219, 222, 223, 226, 227, 231-239\}\).

**REFERENCES**


Department of Applied Mathematics,
University of Sydney, Australia.