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The KMS states of the twisted Toeplitz algebra of a higher-rank graph

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Declaration

I, Rhys McDonald, declare that this thesis submitted in partial fulfilment of the requirements for the conferral of the degree Master of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Chapter 1

Introduction

1.1 Background

Here we will establish where key concepts fit into the literature.

1.1.1 Graphs as a framework for Cuntz-Krieger algebras

In 1977 Cuntz [10] studied a class of C^*-algebras generated by n isometries S_i satisfying $S_1S_1^* + \cdots + S_nS_n^* = 1$. In 1980, Cuntz and Krieger [11] generalised this to a class of C^*-algebras arising from the study of $\{0,1\}$-matrices, now commonly known as Cuntz-Krieger algebras. More specifically, the authors associate to an $n \times n$ matrix $A = (a_{i,j})$, with no zero rows or columns, a C^*-algebra O_A generated by n nonzero partial isometries $S_1, \dots, S_n$ in a Hilbert space such that

$$\begin{aligned} &S_i^*S_j = 0 \text{ for all distinct } i,j, \text{ and} \\ &S_i^*S_i = \sum_{j=1}^{n} a_{i,j}S_jS_j^*. \end{aligned} \tag{1.1}$$

In the same year, finite directed graphs were used by Enomoto and Watatani [13] to provide a more natural framework for these Cuntz-Krieger algebras. This was achieved by considering the matrix A as the connectivity matrix of a directed graph E, and allows properties of O_A (equivalently O_E) to be described through concepts in graph theory. In 1997 Kumjian, Pask, Raeburn, and Renault [22] investigated Cuntz-Krieger algebras associated to infinite graphs using groupoids; to each locally finite directed graph E they associate a groupoid $\mathcal{G}$. Under certain conditions they show that the groupoid C^*-algebra $C^*(\mathcal{G})$ is the universal C^*-algebra generated by families of partial isometries satisfying the Cuntz-Krieger relations determined by E. That is, $C^*(\mathcal{G})$ is generated by a family of partial isometries $\{S_e : e \in E\}$ with orthogonal ranges satisfying

$$S_e^* S_e = \sum_{f \in E} a_{e,f} S_f S_f^*, \tag{1.2}$$

where $A = (a_{e,f})$ is the connectivity matrix associated to E. So, comparing (1.1) and (1.2) we see that $C^*(\mathcal{G})$ is the Cuntz-Krieger algebra of an infinite $\{0,1\}$-matrix defined by E. An alternative approach to Cuntz-Krieger algebras of infinite graphs was given by Exel and Laca in 2000 [14].

Robertson and Steger [36] then introduced higher-rank Cuntz-Krieger algebras in 1999. Starting with a set of r-dimensional words, based on a finite alphabet A, the authors define transition matrices satisfying certain conditions. The C^*-algebra $\mathcal{A}$ is then the unique C^*-algebra generated by a family of partial isometries satisfying Cuntz-Krieger type relations. In fact, if $r = 1$ then $\mathcal{A}$ is a Cuntz-Krieger algebra. One of the main reasons for the introduction of higher-rank graphs by Kumjian and Pask [21] in 2000 was to provide a more visual framework for the higher-rank Cuntz-Krieger algebras of Robertson and Steger, as had been achieved previously with simple Cuntz-Krieger algebras and directed graphs. Intuitively, we may regard the generalisation of directed graphs to higher-rank graphs (k-graphs) as a problem of increasing dimension. In a directed graph we have the notion of the length of a path, which is a one-dimensional measure of size. In a k-graph, we extend this to the notion of the degree of a path, which measures the length in each of k dimensions. As such, we have a degree functor which maps into $\mathbb{N}^k$, which is equipped with the usual lattice order $\leq$, defined for each $m, n \in \mathbb{N}^k$ as $m \leq n$ if $m_i \leq n_i$ for each $i = 1, \ldots, k$. Additionally, we write $m \vee n$ for the coordinate-wise maximum of m and n, and $m \wedge n$ for the coordinate-wise minimum. Kumjian and Pask define a k-graph to be a countable small category Λ along with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the factorisation property (2.1).

As usual with categories, we identify each object of Λ with its identity morphism, so following the usual abuse of notation we write $\lambda \in \Lambda$ instead of $\lambda \in \mathrm{Mor}(\Lambda)$. We refer to the morphisms of Λ as *paths* and to the objects/identity morphisms (which by the factorisation property are exactly the morphisms of degree 0) as *vertices*. Notationally we separate the two as follows: we denote paths by lower-case Greek letters $\lambda, \mu, \nu, \ldots$, and vertices follow the usual notation as in directed graphs, denoted by $u, v, w, \ldots$. When two paths are composable, we write $\lambda\mu$ for $\lambda \circ \mu$. By the factorisation property, we see that k-graphs are left cancellative; that is $\lambda\mu = \lambda\nu \implies \mu = \nu$, and similarly for right cancellation. A morphism between k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ which is compatible with the degree maps; that is $d_2(f(\lambda)) = d_1(\lambda)$.

Kumjian and Pask [21] consider row-finite k-graphs with no sources, and go on to construct a C^*-algebra from each k-graph in such a way that for $k = 1$ the associated C^*-algebra is the same as that of a directed graph. They define $C^*(\Lambda)$ to be the universal C^*-algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,

(ii) $s_{\lambda\mu} = s_\lambda s_\mu$ whenever $s(\lambda) = r(\mu)$,

(iii) $s_\lambda s_\lambda^* = s_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

(iv) $s_v = \displaystyle\sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Further, they establish that this framework does indeed capture the higher-rank Cuntz-Krieger algebras of Robertson and Steger, and that in the case $k = 1$ the C^*-algebra of a 1-graph is isomorphic to that of the corresponding directed graph. In 2004, Raeburn, Sims and Yeend [34] extended the work of Kumjian and Pask to the case of *finitely aligned* k-graphs. This class contains, in particular, all row-finite k-graphs.

Higher-rank graphs as product systems of Hilbert bimodules

A Hilbert bimodule is a right Hilbert A-module X with a left action of A on X as adjointable operators. In 1997 Pimsner [31] associated to every such Hilbert bimodule a C^*-algebra $\mathcal{O}_X$ now known as the Cuntz-Pimsner algebra. He showed, in particular, that every Cuntz-Krieger algebra may be realised as a Cuntz-Pimsner algebra $\mathcal{O}_X$ for a suitable Hilbert bimodule X. Then in 2002, Fowler [16] considered families $X = \{X_s : s \in P\}$ of Hilbert bimodules indexed by a semigroup P. He requires that the families admit an associative multiplication $X_s \times X_t \to X_{st}$, which in turn implements isomorphisms $X_s \otimes_A X_t \to X_{st}$. He calls such a family a *product system of Hilbert bimodules.* He follows Pimsner in associating to each such product system a generalised Cuntz-Pimsner algebra $\mathcal{O}_X$, and in turn studies its Toeplitz extension $\mathcal{T}_X$. It was observed by Fowler and Sims [17] later that year that higher-rank graphs may be realised as product systems of graphs over the semigroup $\mathbb{N}^k$. Building on this and using the product systems approach of Fowler [16], Raeburn and Sims [32] introduced the following Toeplitz-Cuntz-Krieger relations. Let E be a finitely aligned product system of graphs over a semigroup P. Partial isometries $\{s_\lambda : \lambda \in E^1\}$ in a C^*-algebra B form a *Toeplitz-Cuntz-Krieger E-family* if

(i) $\{s_v : v \in E^0\}$ are mutually orthogonal projections,

(ii) $s_\lambda s_\mu = s_{\lambda\mu}$ whenever $r(\lambda) = s(\mu)$,

(iii) $s_\lambda^* s_\lambda = s_{r(\lambda)}$ for all $\lambda \in E^1$,

(iv) for all $p \in P \backslash \{e\}$, $v \in E^0$, and every finite $F \subset s_p^{-1}(v)$, we have $s_v \geq \sum_{\lambda \in F} s_\lambda s_\lambda^*$, and

(v) for all $\mu, \nu \in E^1$ $s_\mu s_\mu^* s_\nu s_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} s_\lambda s_\lambda^*$,

where s_p is the source map at the element $p \in P$. The authors then define $\mathcal{T}C^*(E)$ to be the unique universal C^*-algebra generated by such a family of partial isometries.

In 2014 an Huef, Laca, Raeburn and Sims [4] follow this construction in defining a *Toeplitz-Cuntz-Krieger Λ-family* for a k-graph Λ. A family of partial isometries $\{T_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger Λ-family if

(i) $\{T_v : v \in \Lambda^0\}$ are mutually orthogonal projections,

(ii) $T_\lambda T_\mu = T_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$,

(iii) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$,

(iv) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $T_v \geq \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*$, and

(v) for all $\mu, \nu \in \Lambda$, $T_\mu T_\mu^* T_\nu T_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} T_\lambda T_\lambda^*$.

We note that these are essentially just the relations of Raeburn and Sims [32], with the technical difference that roles of the range and source maps have reversed due to the difference in composition of paths for directed graphs and k-graphs. The Toeplitz algebra $\mathcal{T}C^*(\Lambda)$ is generated by a universal Toeplitz-Cuntz-Krieger family $\{T_\lambda\}$, and this construction is equivalent to that of Raeburn and Sims [32].

1.1.2 Higher-rank graph cohomology and twisted C^*-algebras

Homology theory for k-graphs was introduced in 2012 by Kumjian, Pask and Sims [23], as well as the associated (cubical) cohomology and its applications to C^*-algebras. They show that $\mathbb{T}$-valued 2-cocycles on a k-graph can be used to twist the defining relations of its C^*-algebra. These ideas are extended further by the same authors in [24], where the structure of twisted k-graph C^*-algebras is analysed. In order to construct a groupoid cocycle from a k-graph cocycle the authors introduce another (categorical) cohomology theory. They also establish that the resulting cohomology groups are isomorphic to those given in [23].

The categorical cohomology is constructed in [24] as follows. Let Λ be a k-graph and A an additive abelian group. For all integers $r \geq 1$, let $\Lambda^{*r} := \{(\lambda_0, \dots, \lambda_r) : s(\lambda_i) = r(\lambda_{i+1}) \text{ for each } i\}$ be the collection of all composable r-tuples in Λ. For $r = 0$, $\Lambda^{*0} := \Lambda^0$. For $r > 0$, a function $f : \Lambda^{*r} \to A$ is an *r-cochain* if $f(\lambda_1, \dots, \lambda_r) = 0$ whenever $\lambda_i \in \Lambda^0$ for some $1 \leq i \leq r$. Every function $f : \Lambda^0 \to A$ is a 0-cochain. Let $\underline{C}^r(\Lambda, A)$ be the set of all r-cochains, which is a group under pointwise addition. The authors then define the maps $\underline{\delta}^r : \underline{C}^r(\Lambda, A) \to \underline{C}^{r+1}(\Lambda, A)$ as follows. For $r \geq 1$, and $f \in \underline{C}^r(\Lambda, A)$, define $\underline{\delta}^r f : \Lambda^{*(r+1)} \to A$ by

$$\begin{aligned}(\underline{\delta}^r f)(\lambda_0, \dots, \lambda_r) := {} & f(\lambda_1, \dots, \lambda_r) \\ & + \sum_{i=1}^{r} (-1)^i f(\lambda_0, \dots, \lambda_{i-2}, (\lambda_{i-1}\lambda_i), \lambda_{i+1}, \dots, \lambda_r) \\ & + (-1)^{r+1} f(\lambda_0, \dots, \lambda_{r-1}).\end{aligned}$$

For $r = 0$, $f \in \underline{C}^0(\Lambda, A)$, define $\underline{\delta}^0 f : \Lambda^{*1} \to A$ by

$$(\underline{\delta}^0 f)(\lambda) := f(s(\lambda)) - f(r(\lambda)).$$

Then the sequence

$$0 \to \underline{C}^0(\Lambda, A) \xrightarrow{\underline{\delta}^0} \underline{C}^1(\Lambda, A) \xrightarrow{\underline{\delta}^1} \underline{C}^2(\Lambda, A) \xrightarrow{\underline{\delta}^2} \dots$$

is a cochain complex (that is, $\underline{\delta}^{i+1} \circ \underline{\delta}^i = 0$ for each i). We write $\underline{B}^r(\Lambda, A)$ for the group $\operatorname{Im}(\underline{\delta}^{r-1})$ of *r-coboundaries*, and $\underline{Z}^r(\Lambda, A)$ for the group $\ker(\underline{\delta}^r)$ of *r-cocycles*. Then the authors define the *categorical cohomology* of Λ with coefficients in A as the cohomology of the above cochain complex. That is,

$$\underline{H}^r(\Lambda, A) := \underline{Z}^r(\Lambda, A) \Big/ \underline{B}^r(\Lambda, A),$$

for each r.

This construction generalises to an arbitrary small category Λ, so if the category also comes equipped with a topology and A is a locally compact abelian group, it is natural to require n-cochains to be continuous. To distinguish between this continuous cocycle cohomology from the preceding discrete counterpart, we denote the cochain groups $\tilde{C}^*(\Lambda, A)$, the coboundary groups $\tilde{B}^*(\Lambda, A)$, the cocycle groups $\tilde{Z}^*(\Lambda, A)$ and the cohomology groups $\tilde{H}^*(\Lambda, A)$. Given that groupoids may be considered as categories with inverses, this construction replicates Renault's continuous cocycle cohomology introduced in [35].

The authors also give the following characterisation of 0-, 1-, and 2-cocycles on a k-graph. A cochain $f_0 \in \underline{C}^0(\Lambda, A)$ is a categorical 0-cocycle if and only if it is constant on connected components, that is if and only if $f_0(s(\lambda)) = f_0(r(\lambda))$ for all λ; a cochain $f_1 \in \underline{C}^1(\Lambda, A)$ is a categorical 1-cocycle if and only if it is a functor, that is if and only if $f_1(\lambda_1) + f_1(\lambda_2) = f_1(\lambda_1\lambda_2)$ for all $(\lambda_1, \lambda_2) \in \Lambda^{*2}$; and a cochain $f_2 \in \underline{C}^2(\Lambda, A)$ is a categorical 2-cocycle if and only if it satisfies the *cocycle identity*

$$f_2(\lambda_1, \lambda_2) + f_2(\lambda_1\lambda_2, \lambda_3) = f_2(\lambda_2, \lambda_3) + f_2(\lambda_1, \lambda_2\lambda_3) \tag{1.3}$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^{*3}$.

Prior to the development of the categorical cohomology in [24], Kumjian, Pask and Sims developed a cubical cohomology for k-graphs in 2012 [23]. In [23] the authors show how to twist the relations defining the C^*-algebra of a k-graph by a $\mathbb{T}$-valued 2-cocycle. In [24] they show that the cubical and categorical cohomology groups are isomorphic for $i \leq 2$. Hence they are able to give an equivalent definition of a twisted C^*-algebra using their categorical cohomology. For a row-finite k-graph with no sources, and a 2-cocycle $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, they define a Cuntz-Krieger (Λ, c) family in a C^*-algebra to be a family of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ such that

(i) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections,

(ii) $t_\mu t_\nu = c(\mu, \nu)t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$,

(iii) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

(iv) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Then $C^*(\Lambda, c)$ is the universal C^*-algebra generated by a Cuntz-Krieger (Λ, c)-family. In [24] the authors show that this definition is indeed equivalent to their original definition arising from the cubical cohomology. Below we give two examples of twisted Cuntz-Krieger algebras of k-graphs.

Example 1.1.1

(i) (*Quantum tori*). Consider the 2-graph T_2 as given in Example 2.1.8(iii), and note that $T_2^{*2} = T_2 \times T_2$. Fix $\theta \in \mathbb{R}$, and consider the map $c_\theta : T_2 \times T_2 \to \mathbb{T}$ given by

$$c_\theta(m, n) := e^{2\pi i (m_2 n_1)\theta},$$

where $m = (m_1, m_2)$, $n = (n_1, n_2) \in T_2$. It is easy to check that c_θ satisfies (1.3), and therefore we see that c_θ is a 2-cocycle on T_2. Recall that the rotation algebra A_θ, which we will call the noncommutative (quantum) 2-torus, is the universal C^*-algebra generated by unitaries U, V such that

$$UV = e^{2\pi i\theta} VU.$$

Consider $C^*(T_2, c)$ as defined above. Let $\tilde{U} := t_{e_2}$, $\tilde{V} := t_{e_1}$. Then

$$\begin{aligned}
\tilde{U}\tilde{V} &= t_{e_2} t_{e_1} \\
&= c_\theta(e_2, e_1) t_{(1,1)} \\
&= e^{2\pi i\theta} t_{(1,1)} \\
&= e^{2\pi i\theta} \left(t_{e_1} t_{e_2}\right) \\
&= e^{2\pi i\theta} \tilde{V}\tilde{U}.
\end{aligned}$$

So, since $\{e_1, e_2\}$ generates T_2, we see that $C^*(T_2, c_\theta)$ and A_θ have the same universal property, and hence coincide.

This construction generalises to higher-dimensional rotation algebras, A_z, which we refer to as quantum k-tori. Fix $k \geq 2$, and $z_{i,j} \in \mathbb{T}$ for $1 \leq j < i \leq k$. The quantum k-torus A_z is the universal C^*-algebra generated by unitaries $U_1, \ldots, U_k$

such that

$$U_iU_j = z_{i,j}U_jU_i$$

for $1 \leq j < i \leq k$. We have a $\mathbb{T}$-valued 2-cocycle c_z defined on T_k by

$$c_z(m,n) := \prod_{1 \leq j < i \leq k} z_{i,j}^{m_in_j}.$$

So by a similar argument to the $k = 2$ case, we see that $A_z \cong C^*(T_k, c_z)$.

(ii) (*Heegaard-type quantum 3-spheres*). In order to understand the construction of Heegaard-type quantum 2-spheres, we first review the Heegaard-splitting of S^3 into two solid tori, as given in [6]. We have

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

Now consider the solid torus $T = S^1 \times D^2$, which is the cartesian product of the circle and the disc; that is, the solid torus is the usual notion of the torus, including its interior. Then

$$X = \{(t_1, t_2) \in \mathbb{C}^2 : (1 - |t_1|^2)(1 - |t_2|^2) = 0, \ |t_i| \leq 1\}$$

is the gluing of two solid tori along their boundaries. To see that these two spaces are indeed homeomorphic we observe that the two maps $f : X \to S^3$ and $g : S^3 \to X$ defined by

$$\begin{aligned} f(t_1, t_2) &:= (|t_1|^2 + |t_2|^2)^{-\frac{1}{2}}(t_1, t_2) \\ g(z_1, z_2) &:= \sqrt{2}(1 + ||z_1|^2 - |z_2|^2|)^{-\frac{1}{2}}(z_1, z_2) \end{aligned}$$

are continuous and mutually inverse. Given that the rotation algebras $A_\theta = C(S^1) \times_\theta \mathbb{Z}$ model the quantum torus, the Heegaard splitting of S^3 should give a good model for constructing a quantum 3-sphere by “gluing" two quantum solid tori in the appropriate manner. Hence, as given in [6], our Heegaard-type quantum 3-sphere $C^*(S^3_{pq\theta})$ is the universal C^*-algebra generated by two elements a, b which satisfy the sphere equation, the quantum-disc relations, and the quantum torus relations as follows:

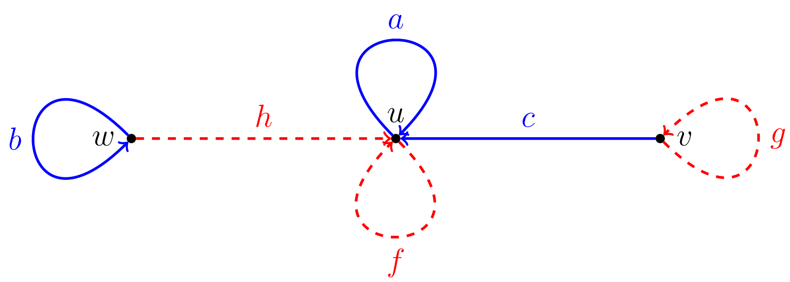
$$(1 - aa^*)(1 - bb^*) = 0, \tag{1.4}$$

$$a^*a = paa^* + 1 - p, \ \ b^*b = qbb^* + 1 - q, \tag{1.5}$$

$$ab = e^{2\pi i\theta}ba, \ \ ab^* = e^{-2\pi i\theta}b^*a. \tag{1.6}$$

It turns out that $C^*(S^3_{pq\theta}) \cong C^*(S^3_{00\theta})$ [6]. Due to this construction, we refer to $C^*(S^3_{00\theta})$ as the Heegaard-type quantum 3-spheres.

In [23] it was shown that these quantum spheres are twisted C^*-algebras of a finite 2-graph Λ with 1-skeleton E_Λ given below.



Let $\alpha = ah = hb$, $\beta = cg = fc$, $\gamma = af = fa$. For each $\theta \in [0, 1)$ the 2-cocycle on T_2 determined by $(1, 1) \mapsto e^{-2\pi i\theta}$ pulls back to a 2-cocycle c_θ on Λ satisfying $c_\theta(\alpha) = c_\theta(\beta) = c_\theta(\gamma) = e^{-2\pi i\theta}$. Fix $\theta \in [0, 1)$ and let $\{s_\lambda : \lambda \in E^1_\Lambda\}$ be the generators of $C^*(\Lambda, c_\theta)$. Define $S, T \in C^*(\Lambda, c_\theta)$ by

$$S := s_a + s_b + s_c \text{ and } T := s_f + s_g + s_h.$$

Then it is routine to check that S, T satisfy (1.4), (1.5) and (1.6). Hence by the universal property of $C^*(S^3_{00\theta})$ we have a map from the generators a, b of $C^*(S^3_{00\theta})$ to S, T which extends to a homomorhpism $C^*(S^3_{00\theta}) \to C^*(\Lambda, c_\theta)$. Similarly we may construct an inverse homomorphism $C^*(\Lambda, c_\theta) \to C^*(S^3_{00\theta})$ as follows. Let a, b be the generators of $C^*(S^3_{00\theta})$. Define

$$q_w = 1 - aa^*, \quad q_v = 1 - bb^*,$$

$$q_u = aa^*bb^*, \qquad t_\eta = q_{r(\eta)}aq_{s(\eta)},$$

for $\eta \in \Lambda^{e_1}$, and

$$t_\eta = q_{s(\eta)} b q_{r(\eta)}$$

for $\eta \in \Lambda^{e_2}$.

Again, checking the necessary relations is routine, so we obtain a homomorphism $C^*(\Lambda, c_\theta) \to C^*(S^3_{00\theta})$. It may be verified that the two homomorphisms are mutually inverse, and so $C^*(S^3_{00\theta}) \cong C^*(\Lambda, c_\theta)$.

1.1.3 KMS states of C^*-dynamical systems

Kubo-Martin-Schwinger (KMS) states provide a natural framework for the equilibrium of a C^*-dynamical system. Historically, the catalyst of their study lies in their application to quantum statistical mechanics. This approach offers a good intuition for the definition, however we also note that KMS states have been shown to be an interesting invariant in their own right. For example, Enomoto, Fujii and Watatani [12] showed that the unique KMS state for a Cuntz-Krieger algebra O_A encodes the topological entropy of the associated shift space. Additionally, Bost and Connes [7] showed that, given an appropriate C^*-dynamical system, the Riemann-Zeta function may be recovered from its KMS states. There are multiple characterisations of KMS states given, but the most useful and direct may be viewed as a quasi-commutativity condition as follows.

Let A be a C^*-algebra, let α be a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, and let $\beta \in (0, \infty)$. Then a state ϕ on A satisfies the $(\alpha, \beta) - $KMS condition if and only if for all α-analytic $a, b \in A$, we have $\phi(ab) = \phi(b\alpha_{i\beta}(a))$.

KMS states and measures on the unit space

In his study of groupoid C^*-algebras, Renault [35] gives a one-to-one correspondence between certain measures on the unit space $\mathcal{G}^{(0)}$ and KMS$_\beta$ states on $C^*(\mathcal{G}, \sigma)$. To see this correspondence we first need some background from [35].

Definition 1.1.2 Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. A *left Haar system* for $\mathcal{G}$ consists of measures $\{\lambda^u : u \in \mathcal{G}^{(0)}\}$ on $\mathcal{G}$ such that

(i) the support of λ^u is $\mathcal{G}^u$ for all $u \in \mathcal{G}^{(0)}$, and

(ii) for any $x \in \mathcal{G}$, and any $f \in C_c(\mathcal{G})$,

$$\int f(xy)d\lambda^{s(x)}(y) = \int f(y)d\lambda^{r(x)}(y).$$

The following proposition tells us that it is actually quite easy to check if a groupoid admits a left Haar system.

Proposition 1.1.3 *For a locally compact Hausdorff groupoid $\mathcal{G}$, the following properties are equivalent*

(i) $\mathcal{G}$ is r-discrete and admits a left Haar system,

(ii) the range map is a local homeomorphism,

(iii) the product map is a local homeomorphism, and

(iv) $\mathcal{G}$ has a base of open $\mathcal{G}$-sets.

Now suppose that we have a locally compact Hausdorff groupoid with left Haar system $\{\lambda^u\}$. Let $\lambda_u = (\lambda^u)^{-1}$ be the image of λ^u under the inverse map. Then $\{\lambda_u\}$ is a right Haar system. Given this groupoid structure, any measure μ on the unit space $\mathcal{G}^{(0)}$ induces a measure $\nu = \int \lambda^u d\mu(u)$ on $\mathcal{G}$ defined by

$$\int f(y)d\nu(y) = \int \left(\int f(y)d\lambda^u(y) \right) d\mu(u),$$

as well as its inverse image $\nu^{-1} = \int \lambda_u d\mu(u)$ defined by

$$\int f(y)d\nu^{-1}(y) = \int \left(\int f(y)d\lambda_u(y) \right) d\mu(u).$$

Definition 1.1.4 A measure μ on $\mathcal{G}^{(0)}$ is said to be *quasi-invariant* if the induced measure ν is equivalent to its inverse ν^{-1}.

It turns out that the Radon-Nikodym derivative of ν with respect to ν^{-1}, $D = \frac{d\nu}{d\nu^{-1}}$ is a positive 1-cocycle on $\mathcal{G}$ whose cohomology class is dependent only on the class of μ. We call D the *modular function* of μ.

Now we may define what it means for a measure on the unit space to satisfy the KMS condition. First, recall that $Z^1(\mathcal{G}, \mathbb{R})$ is the group of continuous homomorphisms of $\mathcal{G}$ into $\mathbb{R}$. For $c \in Z^1(\mathcal{G}, \mathbb{R})$ we have the following important subset of the unit space,

$$\mathrm{Min}(c) := \{u \in \mathcal{G}^{(0)} : c(\mathcal{G}_u) \subseteq [0, \infty)\}.$$

Definition 1.1.5 Let $c \in Z^1(\mathcal{G}, \mathbb{R})$, and $\beta \in [-\infty, \infty]$. We say that a measure μ on $\mathcal{G}^{(0)}$ satisfies the (c, β)- KMS condition if

(i) for $|\beta| < \infty$, μ is quasi-invariant and its modular function D is equal to $e^{-\beta c}$;

(ii) for $\beta = \pm\infty$, the support of μ is contained in $\mathrm{Min}(\pm c)$.

Renault associates a one-parameter automorphism group α of $C^*(\mathcal{G}, \sigma)$ to a given $c \in Z^1(\mathcal{G}, \mathbb{R})$ as follows. Define for each $t \in \mathbb{R}$ and each $f \in C_c(\mathcal{G}, \sigma)$,

$$\alpha_t(f)(x) := e^{itc(x)} f(x).$$

So we have a C^*-dynamical system $(C^*(\mathcal{G}, \sigma), \mathbb{R}, \alpha)$, and hence we have the usual notion of KMS states of the system. In particular, given a measure μ on $\mathcal{G}^{(0)}$, Renault defines a positive linear functional ϕ_μ on $C_c(\mathcal{G}, \sigma)$ by

$$\phi_\mu(f) := \int P(f) d\mu,$$

where $P : C_c(\mathcal{G}, \sigma) \to C_c(\mathcal{G}^{(0)})$ is the restriction map. If $\mathcal{G}$ is r-discrete and μ is a probability measure, then ϕ_μ is a state. The punchline is then given in the following proposition.

Proposition 1.1.6 *Let $\mathcal{G}$ be an r-discrete locally compact Hausdorff groupoid. Let $c \in Z^1(\mathcal{G}, \mathbb{R})$, $\beta \in [0, \infty]$, and let μ be a probability measure on $\mathcal{G}^{(0)}$. The automorphism group associated to c is denoted α, and the state associated to μ is denoted ϕ_μ. The following properties are equivalent.*

(i) The state ϕ_μ satisfies the (α, β)- KMS *condition.*

(ii) the measure μ satisfies the (c, β)- KMS *condition.*

Moreover, if $\mathcal{G}$ is principal and β is finite, any state ϕ which satisfies the (α, β)- KMS *condition arises from a measure μ on $\mathcal{G}^{(0)}$.*

This result tells us that we should be studying the KMS states of groupoid C^*-algebras via probability measures on the unit space of the underlying groupoid. However, the groupoid that we will construct from a k-graph is not principal so we will have to take a slightly different approach to that of Renault, which is due to Neshveyev [29] and Afsar and Sims [1] and is discussed in detail in Chapter 5.

1.2 Overview

This section details the content of this thesis. We give a description of the main content of each chapter, indicating links both to other work and to other components of the thesis.

Chapter 2. In this chapter we cover in detail the necessary background for the rest of the thesis. We define higher-rank graphs and provide the basic definitions and notation that we will use throughout the thesis. We also define the full path space W_Λ of a k-graph Λ which will be integral in constructing our groupoid model in Chapter 3. We take a similar approach with groupoids, giving the basic definitions and notation required. We also define groupoid cocycles, allowing us to define twisted groupoid C^*-algebras, which again become useful for our groupoid model in Chapter 3. We then define higher-rank graph cohomology, highlighting the difference between the original [23] and updated [24] definitions of Kumjian, Pask, and Sims. This allows us to define the twisted Toeplitz algebra of a higher-rank graph, upon which the results of this thesis are based. Finally, we define KMS states and discuss some of their properties that make it easier for us to check which states satisfy the KMS condition. In particular, we only need to check that the KMS condition holds for a dense subset of analytic elements in the algebra.

Chapter 3. In this chapter we develop a groupoid model for the twisted Toeplitz algebra of a higher-rank graph. This groupoid model is based upon the groupoid model for $C^*(\Lambda, c)$ given in [24] by Kumjian, Pask, and Sims, but we use the full path space to define the groupoid in place of the infinite path space. This makes for some more difficult calcultions, as we are mixing finite, semi-infite and infite paths into one space. This idea was used by an Huef, Kang, and Raeburn [2] to construct a groupoid model for the Toeplitz algebra $\mathcal{T}C^*(\Lambda)$. In order to define a cocycle on the k-graph groupoid $\mathcal{G}_\Lambda$ from a given cocycle on the k-graph, we adapt an argument from Kumjian, Pask, and Sims [24] that involves partitioning the groupoid by its cylinder sets. With this achieved we are able to check that our groupoid model indeed provides an isomorphism with the twisted Toeplitz algebra. We are able to then provide a basic example of the construction, and we learn that although this partition-cocycle construction gives a cohomologous cocycle to what we would expect, it tends to give a more complex formula.

Chapter 4. In this chapter we provide a method for defining an appropriate measure on the path space of a k-graph. This method is adapted from a similar argument by an Huef, Laca, Raeburn, and Sims [5]. We start with a brief recollection of Radon-Nykodym cocycles and quasi-invariant measures. Then we go into some background on inverse systems of outer measures, which ultimately allows us to define a measure on the product space $\prod_{n=0}^{\infty} \tilde{\Lambda}$, where $\tilde{\Lambda} = \Lambda \cup \{0\}$. We are then able to regard the path space as a subset of this product space, which gives us a measure on the path space. With a measure M defined on the appropriate space we then check that the measure has the desired properties. That is, we check that the measure is a Borel probability measure with Radon-Nykodym cocycle e^{-D} for the cocycle $D : \mathcal{G}_\Lambda \to \mathbb{R}$ given by $D(x, n, y) = n \cdot \ln(\rho(\Lambda))$. We also establish that M is the unique such measure, and for technical purposes later we check that certain sets have measure zero under M.

Chapter 5. In Chapter 5 we give the main result of the thesis, and discuss its implications. We use the groupoid model established in Chapter 3 and the measure on the path space defined in Chapter 4 to characterise the KMS states at the critical inverse temperature ($\beta = 1$ in this case) based on a result on Afsar and Sims [1]. The result states that the KMS_1 states of the twisted Toeplitz algebra at the critical inverse temperature factor through the twisted C^*-algebra. Together with a previous result that characterised the KMS states of the twisted Toeplitz algebra for large inverse temperatures (greater than the critical inverse temperature) this provides a full characterisation of the KMS states of the twisted Toeplitz algebra. We also note that a general result of Christensen [9, Theorems 5.1, 5.2] match with our result in the case of the twisted Toeplitz algebra. As he notes in [9, Remark 5.3], Christensen's formula (5.1) matches with our formula (5.4), and the subgroup B in [9, Theorem5.3] corresponds to the subgroup $\mathrm{Per}(\Lambda)$.

Chapter 6. Chapter 6 summarises the main result of my honours thesis [28], and then compares it to the main result of Chapter 5 in this thesis. The main result of the chapter uses a direct computation to determine the KMS states of $\mathcal{T}C^*(\Lambda, c)$ for 'large' inverse temperatures. We are able to compare this result both qualitatively and directly to the result of Chapter 5. Qualitatively we see that both results tell us that the KMS structure of the twisted Toeplitz algebra is dependent on the vertices of the k-graph. More directly, we are able to recover equation (6.7) from equation (5.2) under conditions of Chapter 5, with the exception that β must now be bigger than 1 (the critical inverse temperature for the preferred dynamics).

Chapter 2

Background

Here we will cover motivations for and definitions of key concepts needed throughout the rest of the thesis. First we will cover higher-rank graphs, including important subsets and properties and the path space. Then we will cover groupoids and their associated C^*-algebras. Following this we discuss the twisted Toeplitz algebra of a k-graph and the canonical gauge action on this algebra. Finally, we will discuss C^*-dynamical systems and KMS states on these systems.

2.1 Higher-rank Graphs

As we have talked about and as suggested by the name, higher-rank graphs generalise directed graphs to 'higher dimensions'; this is somewhat meaningless for now, but the generalisation essentially addresses the idea of the length of a path in a directed graph, and replaces this one-dimensional notion of size with a multi-dimensional notion of size. Higher-rank graphs are defined in terms of categories, so first we define what a category is and what a functor between categories looks like.

Definition 2.1.1 A *category* $\mathcal{C}$ is a sextuple $(\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$ consisting of

- a collection of *objects* $\mathrm{Ob}(\mathcal{C})$ and a collection of *morphisms* $\mathrm{Mor}(\mathcal{C})$,
- functions $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ called the *domain* and *codomain* maps,
- the *identity* function $\mathrm{id} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, and
- a composition function $\circ : \mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) := \{(f, g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) : \mathrm{dom}(f) = \mathrm{cod}(g)\}$ is the set of *composable*

pairs in $\mathrm{Mor}((C))$, satisfying:

- $\mathrm{dom}(\mathrm{id}(a)) = a = \mathrm{cod}(\mathrm{id}(a))$ for all $a \in \mathrm{Ob}((C))$,
- $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$ for all composable $f, g \in \mathrm{Mor}((C))$,
- the *unit law*: $\mathrm{id}(\mathrm{cod}(f)) = f = \mathrm{id}(\mathrm{dom}(f))$ for all $f \in \mathrm{Mor}((C))$, and
- *associativity*: $f \circ (g \circ h) = (f \circ g) \circ h$ for all composable $f, g, h \in \mathrm{Mor}((C))$.

We say that a category is *small* if the collections of objects and morphisms are both sets.

Definition 2.1.2 A *covariant functor* from a category (C) to a category (B) is a pair of functions (both labelled F): an *object function* $F : \mathrm{Ob}((C)) \to \mathrm{Ob}((B))$ and a *morphism function* $F : \mathrm{Mor}((C)) \to \mathrm{Mor}((B))$ satisfying

(i) $\mathrm{dom}(F(f)) = F(\mathrm{dom}(f))$ and $\mathrm{cod}(F(f)) = F(\mathrm{cod}(f))$ for all $f \in \mathrm{Mor}((C))$

(ii) $F(\mathrm{id}(a)) = \mathrm{id}(F(a))$ for all $a \in \mathrm{Ob}((C))$, and

(iii) $F(f) \circ F(g) = F(f \circ g)$ for all composable $f, g \in \mathrm{Mor}((C))$.

There is also a notion of contravariant functors, however this won't be required here so we just refer to covariant functors as functors.

We note that $E^* := \bigcup_{n=0}^{\infty} E^n$, the set of all paths in a directed graph E, forms a category with objects E^0 and composition defined by concatenation of paths as in Example 2.1.4. This gives motivation for the way that higher-rank graphs were defined by Kumjian and Pask in [21].

Definition 2.1.3 [21, Definitions 1.1]
A *k-graph (rank k graph, higher-rank graph)* (Λ, d) consists of a countable small category Λ and a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *factorisation property*:

$$\begin{aligned}&\text{for all } \lambda \in \mathrm{Mor}(\Lambda),\ m,\ n \in \mathbb{N}^k \text{ with } d(\lambda) = m + n,\\ &\text{there are unique elements } \mu, \nu \in \mathrm{Mor}(\Lambda) \text{ such that}\\ &\lambda = \mu \circ \nu, \text{ and } d(\mu) = m,\ d(\nu) = n.\end{aligned} \tag{2.1}$$

The first example of a higher-rank graph we give is the 1-graph defined by the set of paths of a directed graph.

Example 2.1.4 Let E be a directed graph. Consider $E^* := \bigcup_{n=0}^{\infty} E^n$, where each E^n is the set of paths in E of length n. We can see that E^* is a category with objects E^0, and composition defined by concatenation of paths. Defining a degree functor $d : E^* \to \mathbb{N}$ by $d(\mu) = n$ if and only if $\mu \in E^n$ shows that (E^*, d) is in fact a 1-graph.

Lemma 2.1.5 *Let (Λ, d) be a k-graph. The map $v \mapsto \mathrm{id}(v)$ is a bijection from the objects of Λ onto the morphisms of degree 0. That is, $\{\mathrm{id}(v) : v \in \mathrm{Ob}(\Lambda)\} = d^{-1}(0)$.*

Proof. Recall that d is a functor, and that $\mathbb{N}^k$ is a monoid and so only has one object, a_0. So for each $v \in \mathrm{Ob}(\Lambda)$

$$d(\mathrm{id}(v)) = \mathrm{id}(d(v)) = \mathrm{id}(a_0) = 0.$$

For the reverse containment, fix $\lambda \in \mathrm{Mor}(\Lambda)$ with $d(\lambda) = 0$, and let $v = \mathrm{cod}(\lambda)$. Since $0+0 = 0$, the factorisation property implies that there are unique paths $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $d(\mu) = d(\nu) = 0$ and $\lambda = \mu \circ \nu$. According to the unit law of categories, $\mu = \mathrm{id}(v)$, $\nu = \lambda$ and $\mu = \lambda$, $\nu = \mathrm{id}(\mathrm{dom}(\lambda))$ are two such factorisations. Hence the uniqueness of factorisations implies $\lambda = \mathrm{id}(v)$. □

In light of Lemma 2.1.5 we consider the objects of a k-graph (Λ, d) as a subset of the morphisms and from now on we will do so, writing $\lambda \in \Lambda$ instead of $\lambda \in \mathrm{Mor}(\Lambda)$. Since we are thinking of (Λ, d) as a type of graph, we write r and s for the codomain and domain maps respectively and call these the *range* and *source* maps. We refer to the morphisms of the k-graph as *paths* and the objects/morphisms of degree zero as *vertices*. Further, we refer to the morphisms of minimal non-zero degree, $d^{-1}(\{e_1, \ldots, e_k\})$ as *edges*, where $e_1, \ldots, e_k$ are the canonical generators of $\mathbb{N}^k$.

We now note that the multiplication in a k-graph is cancellative.

Lemma 2.1.6 *Let Λ be a k-graph. Then Λ is cancellative; that is, for each $\lambda, \mu, \nu \in \Lambda$ $\lambda\mu = \lambda\nu \implies \mu = \nu$ and $\mu\lambda = \nu\lambda \implies \mu = \nu$.*

Proof. For left cancellation, fix $\lambda, \mu, \nu \in \Lambda$ such that $\lambda\mu = \lambda\nu$. We have

$$d(\lambda\mu) = d(\lambda) + d(\mu), \text{ and} \tag{2.2}$$

$$d(\lambda\nu) = d(\lambda) + d(\nu). \tag{2.3}$$

Since the left-hand sides of (2.2) and (2.3) are equal, we see that $d(\mu) = d(\nu)$. Hence by the factorisation property we see that $\mu = \nu$. Right cancellation follows similarly. □

Next we work through some examples of k-graphs to get an understanding of their structure.

Examples 2.1.7

(i) Given a 1-graph Λ, we may construct a directed graph $E = (E^0, E^1, s_E, r_E)$ by defining by letting $E^0 = \Lambda^0$, $E^1 = \Lambda^1$, $s_E(\lambda) = r(\lambda)$ and $r_E(\lambda) = s(\lambda)$. Switching the range and source maps is due to a subtle difference in the composability of paths in directed graphs and k-graphs, and is of little consequence. Pairing this with Example 2.1.4 we have a way to go between directed graphs and 1-graphs.

(ii) For $n \geq 1$ we write B_n for the path category of the directed graph with a single vertex and n edges. The Cuntz algebras studied in [10] are isomorphic to the graph C^*-algebra $C^*(B_n)$ defined in [22].

It is frequently useful to visualise k-graphs via their 1-skeletons. The *1-skeleton* of a k-graph Λ is a directed graph with vertices Λ^0 and edges $\bigcup_{i=1}^k \Lambda^{e_i}$. Each edge is coloured according to its degree. In most of the following examples we will consider 2-graphs, with edges of degree $(1, 0)$ represented by blue, solid lines, and edges of degree $(0, 1)$ represented by red, dashed lines.

Examples 2.1.8

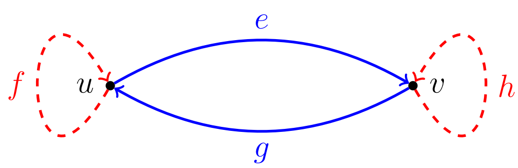
(i) This example is widely documented (see [37, 21]) as it offers a simple model for paths, which we will discuss further when we come across the path space of a k-graph. For $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, consider the category $\Omega_{k,m}$ defined by

$$\begin{aligned} \mathrm{Ob}(\Omega_{k,m}) &:= \{p \in \mathbb{N}^k : p_i \leq m_i \text{ for all } i \leq k\} \text{ and,} \\ \mathrm{Mor}(\Omega_{k,m}) &:= \{(p, q) : p, q \in \mathrm{Ob}(\Omega_{k,m}), p_i \leq q_i \text{ for all } i \leq k\}, \end{aligned}$$

where the range and source maps are given by $r(p, q) := p$ and $s(p, q) := q$, and composition is given by $(p, q)(q, t) := (p, t)$. Hence if we define $d : \Omega_{k,m} \to \mathbb{N}^k$ by $d(p, q) := q - p$, it is clear that d is a functor, and we may see that $(\Omega_{k,m}, d)$ is a k-graph by checking that the factorisation property holds. To do so, fix $\lambda = (p, q) \in \mathrm{Mor}(\Omega_{k,m})$ and fix $n_1, n_2 \in \mathbb{N}^k$ such that $d(\lambda) = n_1 + n_2$. That is, $q - p = n_1 + n_2$. Then $\mu = (p, p + n_1)$, $\nu = (p + n_1, q)$ satisfy the factorisation property: $\lambda = \mu\nu$ and $d(\mu) = n_1$, $d(\nu) = q - p - n_1 = n_2$. If $m = (\infty)^k$ then we write Ω_k for $\Omega_{k,m}$. Consider the case $k = 2, m = (2, 2)$. Then $\Omega_{k,m}$ has the following 1-skeleton.

$$\begin{tikzcd}
(0,2) \arrow[d, dashed] & (1,2) \arrow[l] \arrow[d, dashed] & (2,2) \arrow[l] \arrow[d, dashed] \\
(0,1) \arrow[d, dashed] & (1,1) \arrow[l] \arrow[d, dashed] & (2,1) \arrow[l] \arrow[d, dashed] \\
(0,0) & (1,0) \arrow[l] & (2,0) \arrow[l]
\end{tikzcd}$$

(ii) As noted by Raeburn, Sims and Yeend in [33], the 1-skeleton of a k-graph does not always determine a k-graph due to the factorisation property; we must also specify how the factorisation property fits into the composition of edges. The authors consider the 2-graph with the following 1-skeleton.



(2.4)

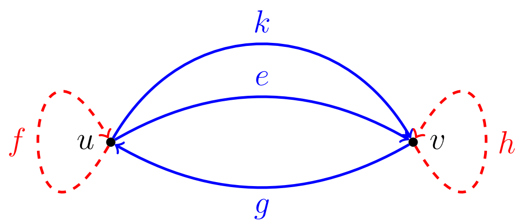
In this case, the unique path of degree (3,1), λ, with $r(\lambda) = u$ and $s(\lambda) = v$ is

$$\begin{tikzcd}
\bullet \arrow[d, dashed, "f"'] & \bullet \arrow[l, "g"'] \arrow[d, dashed, "h"'] & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] & \bullet \arrow[l, "g"'] \arrow[d, dashed, "h"] \\
\bullet & \bullet \arrow[l, "g"] & \bullet \arrow[l, "e"] & \bullet \arrow[l, "g"]
\end{tikzcd}$$

which admits the factorisations $\lambda = fgeg = gheg = gefg = gegh$. For $k = 2$ it suffices to specify the factorisations of all paths ef of length 2 in the 1-skeleton such that e, f have different degree. Any collection of squares S which contains each such bicoloured path uniquely determines a 2-graph Λ with the given 1-skeleton, and $\Lambda^{(1,1)} = S$. Such a collection may not exist, or there may be many choices. For the 1-skeleton (2.4), the factorisation property implies that $\Lambda^{(1,1)}$ consists of the two squares

$$\begin{tikzcd} \bullet \arrow[d, dashed, "h"'] & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] \\ \bullet & \bullet \arrow[l, "e"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[d, dashed, "f"'] & \bullet \arrow[l, "g"'] \arrow[d, dashed, "h"] \\ \bullet & \bullet \arrow[l, "g"] \end{tikzcd}$$

So there is exactly one 2-graph with this 1-skeleton, since the 1-skeleton completely determines these squares. This, however, is not always the case. For example, adding an extra edge forces us to make a choice in determining the 2-graph. As in [33], consider adding an extra edge to the 1-skeleton (2.4). We obtain the following 1-skeleton.



There are four bi-coloured paths from u to v (he, ef, kf, hk) and we must decide how to pair them through the factorisation property. The path he can be paired either with ef or with kf, so either

$$\begin{tikzcd} \bullet \arrow[d, dashed, "h"'] & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] \\ \bullet & \bullet \arrow[l, "e"] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[d, dashed, "h"'] & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] \\ \bullet & \bullet \arrow[l, "k"] \end{tikzcd}$$

is a path of degree (1,1), and once we have decided which, the other pair is determined by the factorisation property; for example if he is paired with ef, then kf is paired with hk.

(iii) Following [21], for $k \geq 1$ let T_k be the semigroup $\mathbb{N}^k$ viewed as a small category. That is,

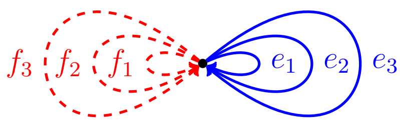
$$\mathrm{Ob}(T_k) = \{0\},$$
$$\mathrm{Mor}(T_k) = \mathbb{N}^k,$$

and composition is given by addition in the semigroup. Then if $d : T_k \to \mathbb{N}^k$ is the identity map, we can see that (T_k, d) is a k-graph.

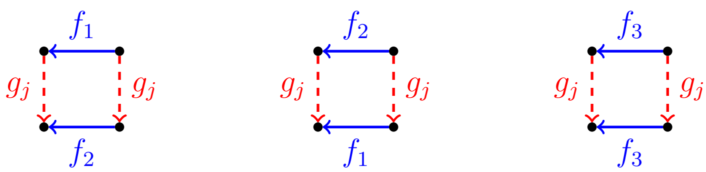
(iv) The following description of 2-graphs with a single vertex is due to Yang [40]. Intuitively, a 2-graph on a single vertex has n edges of degree (1,0) and m edges of degree (0,1). Formally, a 2-graph with a single vertex is a semigroup $\mathbb{F}_\theta^+$ which is generated by $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$. There are no relations among the e_i's, so they generate a copy of the free semigroup $\mathbb{F}_n^+$, and similarly the f_i's generate a copy of $\mathbb{F}_m^+$. There are, however, commutation relations between the e_i's and f_i's given by a permutation $\theta \in S_{n\times m}$:

$$e_i f_j = f_{j'} e_{i'}, \text{ where } \theta(i,j) = (i', j').$$

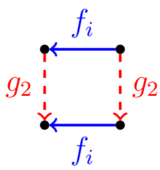
Any word $\omega \in \mathbb{F}_\theta^+$ has a fixed number of e's and f's irrespective of factorisation. Hence we may define $d(\omega) = (k,l)$, if ω contains k e's and l f's. Moreover, as long as $d(\omega) = (k,l)$, ω may be written uniquely according to any given pattern of e's and f's; that is, we can write ω with all e's first, all f's first, or alternating e's and f's if $d(\omega) = (k,k)$. In particular, consider the 2-graph Γ with the following 1-skeleton



where the factorisation rules are given by the following collection of squares



for $j = 1, 3$, and



for $i = 1, 2, 3$.

Then $\Gamma = \mathbb{F}_\theta^+$, where θ is given by

$$\theta(2,j) = (1,j),\ \theta(1,j) = (2,j),\ \theta(3,j) = (3,j),\ \text{for } j = 1,3 \text{ and}$$
$$\theta(i,2) = (2,i),\ \text{for } i = 1,2,3.$$

As with any other mathematical object, the structure-preserving maps between k-graphs play an important role in their analysis. To preserve the structure between two k-graphs we must preserve the categorical structure, while respecting the degree maps. Our definition of k-graph morphisms follows Kumjian and Pask [21].

Definition 2.1.9 [21, Definitions 1.1]
A *k-graph morphism* f between two k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ that is compatible with the degree functors; that is, $d_2(f(\lambda)) = d_1(\lambda)$.

At different stages we will be working with different classes of higher-rank graphs, all of which are defined through the size of certain important subsets of the k-graph. Here we define these subsets.

Definition 2.1.10 Let Λ be a k-graph. For each $n \in \mathbb{N}^k$, each $\lambda \in \Lambda$, and each $S \subseteq \Lambda$ we define

$$\Lambda^n := d^{-1}(\{n\}) = \{\lambda \in \Lambda : d(\lambda) = n\}, \tag{2.5}$$
$$\lambda S := \{\lambda\mu : \mu \in S\}, \tag{2.6}$$
$$S\lambda := \{\mu\lambda : \mu \in S\}. \tag{2.7}$$

By Lemma 2.1.5, Λ^0 is the set of vertices of a given k-graph Λ.

Two important properties of k-graphs rely on the size of the sets $v\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = v, d(\lambda) = n\}$ for each $v \in \Lambda^0$ and each $n \in \mathbb{N}^k$. The following definition of these properties is due to Kumjian and Pask [21].

Definition 2.1.11 Let Λ be a k-graph. We say that Λ has *no sources* if for each $v \in \Lambda^0$ and each $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is non-empty. Similarly we say that Λ is *row-finite* if for each $v \in \Lambda^0$ and each $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is finite.

Another important property of k-graphs generalises the idea of row-finiteness, and to define it we first need to consider the following important subsets of a given k-graph. The following comes from [32].

Definition 2.1.12 Let Λ be a k-graph. For each $\mu, \nu \in \Lambda$ we define the set of *minimal common extensions* of μ and ν to be

$$\begin{aligned} \mathrm{MCE}(\mu,\nu) &:= \{\lambda \in \Lambda^{d(\mu)\vee d(\nu)} : \lambda = \mu\alpha = \nu\beta \text{ for some } \alpha, \beta \in \Lambda\} \\ &= \mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}. \end{aligned} \tag{2.8}$$

There is another set related to minimal common extensions of two given paths in a k-graph, defined as follows.

Definition 2.1.13 Let Λ be a k-graph. For each $\mu, \nu \in \Lambda$ we write

$$\Lambda^{\min}(\mu,\nu) := \{(\alpha,\beta) : \alpha, \beta \in \Lambda, \mu\alpha = \nu\beta, \text{ and } d(\mu\alpha) = d(\mu) \vee d(\nu)\} \tag{2.9}$$

for the pairs which give minimal common extensions of μ and ν.

These two minimal common extension sets are related as follows.

Lemma 2.1.14 *Let Λ be a k-graph. For each $\mu, \nu \in \Lambda$, there is a bijection*

$$\begin{aligned} \Psi : \Lambda^{\min}(\mu,\nu) \to \mathrm{MCE}(\mu,\nu), &\text{ given by} \\ (\alpha,\beta) \mapsto \mu\alpha. \end{aligned}$$

Hence $|\mathrm{MCE}(\mu,\nu)| = |\Lambda^{\min}(\mu,\nu)|$ *for all* $\mu, \nu \in \Lambda$.

Proof. Suppose that $(\alpha,\beta), (\alpha',\beta') \in \Lambda^{\min}(\mu,\nu)$ such that $\Psi(\alpha,\beta) = \Psi(\alpha',\beta')$. That is, $\mu\alpha = \mu\alpha'$. Then $\nu\beta = \mu\alpha = \mu\alpha' = \nu\beta'$. So by Lemma 2.1.6 $(\alpha,\beta) = (\alpha',\beta')$, and we see that Ψ is injective.

Now suppose that $\lambda \in \mathrm{MCE}(\mu,\nu)$. Then $\lambda = \mu\alpha = \nu\beta$ for some $\alpha, \beta \in \Lambda$, and $d(\lambda) = d(\mu) \vee d(\nu)$. Hence $\Psi(\alpha,\beta) = \mu\alpha = \lambda$, and Ψ is surjective. $\square$

The notion of row-finiteness of k-graphs was generalised by Raeburn, Sims and Yeend [34] to the notion of finitely aligned k-graphs as follows.

Definition 2.1.15 Let Λ be a k-graph. We say that Λ is *finitely aligned* if for each $\mu, \nu \in \Lambda$ the set of minimal common extensions $\mathrm{MCE}(\mu,\nu)$ is finite. Equivalently, due to Lemma 2.1.14, Λ is finitely aligned if each $\Lambda^{\min}$ is finite.

The following lemma shows that finitely aligned k-graphs are a more general class than row-finite k-graphs.

Lemma 2.1.16 *Let Λ be a k-graph. If Λ is row-finite, then it is finitely aligned.*

Proof. For each $\mu, \nu \in \Lambda$ we have

$$\begin{aligned} |\operatorname{MCE}(\mu,\nu)| &= |\mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}| \\ &\leq |\mu\Lambda^{d(\mu)\vee d(\nu)}| \\ &\leq |s(\mu)\Lambda^{d(\mu)\vee d(\nu)}|. \end{aligned}$$

So if Λ is row-finite, then each $\operatorname{MCE}(\mu,\nu)$ is finite, and hence Λ is finitely aligned. □

As in [38], if $E \subset \Lambda$ and $\mu \in \Lambda$, then we write $\operatorname{Ext}(\mu; E)$ for the set

$$\operatorname{Ext}(\mu; E) := \{\alpha \in s(\mu)\Lambda : (\alpha,\beta) \in \Lambda^{\min}(\mu,\nu) \text{ for some } \nu \in E\}$$

of extensions of μ with respect to E. Let $v \in \Lambda^0$. We say that $E \subset \Lambda$ is *exhaustive* if $\operatorname{Ext}(\lambda; E)$ is nonempty for all $\lambda \in v\Lambda$. We write $FE(\Lambda)$ for the set of all finite exhaustive subsets of Λ.

$$FE(\Lambda) := \bigcup_{v\in\Lambda^0} \{E \subset v\Lambda \setminus \{v\} : E \text{ is finite and exhaustive}\}.$$

For $E \in FE(\Lambda)$ we write $r(E)$ for the vertex $v \in \Lambda^0$ such that $E \subset v\Lambda$.

We will also require the following notation in the following chapter.

Definition 2.1.17 Let Λ be a k-graph. We denote by $\Lambda *_s \Lambda$ the set of all pairs in Λ with equal sources. That is,

$$\Lambda *_s \Lambda := \{(\mu,\nu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}.$$

We now want to define the unimodular Perron-Frobenius eigenvector of a given k-graph Λ, following an Huef, Laca, Raeburn and Sims [5]. First we define the coordinate matrices of Λ as in [5].

Definition 2.1.18 Let Λ be a finite k-graph. For $1 \leq i \leq k$, let A_i be the matrix with entries $A_i(v,w) = |v\Lambda^{e_i}w|$ for $v, w \in \Lambda^0$. We call the A_i the *coordinate matrices* or *vertex matrices* of Λ.

We now give the result of an Huef, Laca, Raeburn and Sims [5]. Note that $\rho(M)$ is the spectral radius of a matrix M.

Proposition 2.1.19 *Let Λ be a strongly connected finite k-graph. For $1 \leq i \leq k$, let $A_i \in M_{\Lambda^0}([0,\infty))$ be the coordinate matrices of Λ.*

(a) Each $\rho(A_i) > 0$, and for $n \in \mathbb{N}^k$, we have $\rho(A^n) = \prod_i \rho(A_i)^{n_i} > 0$.

(b) There exists a unique non-negative vector $x^\Lambda \in [0,\infty)^{\Lambda^0}$ with unit 1-norm such that $A_i x^\Lambda = \rho(A_i) x^\Lambda$ for all $1 \leq i \leq k$. Moreover, $x^\Lambda > 0$ in the sense that $x^\Lambda_v > 0$ for all $v \in \Lambda^0$.

(c) If $z \in \mathbb{C}^{\Lambda^0}$ and $A_i z = \rho(A_i) z$ for all $1 \leq i \leq k$, then $z \in \mathbb{C}x^\Lambda$.

(d) If $y \in [0,\infty)^{\Lambda^0}$ has unit 1-norm and $A_i y \leq \rho(A_i) y$ for all $1 \leq i \leq k$, then $y = x^\Lambda$.

The following definition is also due to an Huef, Laca, Raeburn and Sims [5].

Definition 2.1.20 Let Λ be a strongly connected finite k-graph. We call the vector x^Λ of Proposition 2.1.19 the *unimodular Perron-Frobenius eigenvetor* of Λ. We write $\rho(\Lambda)$ for the vector $(\rho(A_i)) \in [0,\infty)^k$, and $\ln \rho(\Lambda)$ for the vector $(\ln \rho(A_i)) \in [-\infty,\infty)^k$. For $n \in \mathbb{N}^k$ we have $A^n x^\Lambda = \rho(\Lambda)^n x^\Lambda$, where $\rho(\Lambda)^n := \prod_{i=1}^k \rho(A_i)^{n_i}$ is defined using multi-index notation.

As in Remark 4.5 of [5], we note that since the individual A_i need not be irreducible, it does not make sense to discuss "the unique unimodular Perron-Frobenius eigenvectors of the A_i." However, Proposition 2.1.19 does assert that there is a unique non-negative unimodular eigenvector x^Λ common to all the A_i, and that the spectral radius of each A^n is achieved at x^Λ.

2.1.1 The full path space of a higher-rank graph

Here we brielfy digress into topological k-graphs so we we may refer to the results of Yeend [41], however the reader should keep in mind the case where the k-graph is countable and the topology is discrete.

Intuitively the path space of a k-graph is exactly what it sounds like, however it is defined rather abstractly. Here we go through some background in order to understand how to define the full path space of a k-graph. The following definitions are due to Yeend [41].

Definition 2.1.21 A k-graph Λ is a *topological k-graph* if

(i) $\mathrm{Ob}(\Lambda)$ and $\mathrm{Mor}(\Lambda)$ are second-countable, locally compact Hausdorff space,

(ii) $r, s : \mathrm{Mor}(\Lambda) \to \mathrm{Ob}(\Lambda)$ are continuous and s is a local homeomorphism,

(iii) composition is continuous and open,

(iv) the degree functor is continuous, where $\mathbb{N}^k$ has the discrete topology.

There is a topological notion equivalent to that of finitely aligned k-graphs.

Definition 2.1.22 A topological k-graph Λ is *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and all compact sets $U \subset \Lambda^p$, $V \subset \Lambda^q$, the set $U \vee V := U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$ is compact. As finiteness is equivalent to compactness for discrete topologies, we see that a discrete k-graph is compactly aligned if and only if it is compactly aligned.

Definition 2.1.23 Let Λ be a topological k-graph. We define

$$W_\Lambda := \bigcup_{m \in (\mathbb{N}\cup\{\infty\})^k} \{x : \Omega_{k,m} \to \Lambda : x \text{ is a } k\text{-graph morphism}\}$$

to be the *full path space* of Λ. We extend the range map to W_Λ by $r(x) = x(0)$, and we extend the degree map to $x : \Omega_{k,m} \to \Lambda \in W_\Lambda$ by $d(x) = m$.

By the factorisation property, for each $\lambda \in \Lambda$ there is a unique graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ such that $x_\lambda(0, d(\lambda)) = \lambda$, so that we may view Λ as a subset of W_Λ, and refer to the elements of W_Λ as paths. We must carefully consider the composition of elements of the underlying k-graph with elements of the path space. The following result of Yeend [41] tells us that this composition is well defined.

Lemma 2.1.24 *Let Λ be a topological k-graph. For each $x \in W_\Lambda$ and $m \in \mathbb{N}^k$ with $m \leq d(x)$ and $\lambda \in \Lambda$ with $s(\lambda) = r(x)$, there exist unique paths λx and $\sigma^m x$ in W_Λ satisfying $d(\lambda x) = d(\lambda) + d(x)$, $d(\sigma^m x) = d(x) - m$,*

$$(\lambda x)(0,p) = \begin{cases} \lambda(0,p), & \textit{if } p \leq d(\lambda), \\ x(0, p - d(\lambda)), & \textit{if } d(\lambda) \leq p \leq d(\lambda x), \end{cases}$$

and

$$(\sigma^m x)(0,p) = x(m, m+p) \textit{ for all } p \leq d(\sigma^m x).$$

2.2 Groupoids

Groupoids are algebraic objects that behave similarly to groups, with the characterising exception that the multiplication is only partially defined. Specifically, for a groupoid $\mathcal{G}$ there is a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ on which multiplication is defined. Thanks to this property groupoids generally have multiple units, as opposed to the unique unit in a group. We denote the space of units in $\mathcal{G}$ by $\mathcal{G}^{(0)}$. Frequently, groupoids are defined as countable small categories with inverses. This gives some insight as to why we are interested in them, since k-graphs are also countable small categories with additional structure. Below we give a more rigorous definition from [18], however we note that the two definitions coincide.

Definition 2.2.1 A *groupoid* is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication map $(\alpha, \beta) \mapsto \alpha\beta : \mathcal{G}^{(2)} \to \mathcal{G}$, and an inverse map $\gamma \mapsto \gamma^{-1} : \mathcal{G} \to \mathcal{G}$ such that the following properties hold.

(G1) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$,

(G2) if $(\alpha, \beta), (\beta, \gamma) \in \mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma), (\alpha, \beta\gamma) \in \mathcal{G}^{(2)}$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, and

(G3) $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$, and for all $(\gamma, \eta) \in \mathcal{G}^{(2)}$ we have $\gamma^{-1}(\gamma\eta) = \eta$, and $(\gamma\eta)\eta^{-1} = \gamma$.

Definition 2.2.2 Given a groupoid $\mathcal{G}$, we define the *unit space* $\mathcal{G}^{(0)}$ to be

$$\mathcal{G}^{(0)} := \{\gamma^{-1}\gamma : \gamma \in \mathcal{G}\},$$

and we say that each $x \in \mathcal{G}^{(0)}$ is a *unit*. Note that by (G1) we also have

$$\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}.$$

Definition 2.2.3 Given a groupoid $\mathcal{G}$, we define the *range* and *source* maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ as

$$r(\gamma) := \gamma\gamma^{-1}, \; s(\gamma) := \gamma^{-1}\gamma$$

for all $\gamma \in \mathcal{G}$.

By (G2) and (G3) we see that groupoids are cancellative, and hence we may check that $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $s(\alpha) = r(\beta)$. That is, the composable pairs are determined by the range and source maps, and hence we have

$$\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} : s(\alpha) = r(\beta)\}.$$

As with groups, we also have a notion of topology on groupoids. The idea is that whatever topology is given on the set $\mathcal{G}$ must also play nicely with the groupoid structure. A particularly nice type of topological groupoid is an étale groupoid.

Definition 2.2.4 A groupoid $\mathcal{G}$ is a *topological groupoid* if it has a locally compact topology under which $\mathcal{G}^{(0)}$ is Hausdorff, the maps r, s and $\gamma \to \gamma^{-1}$ are continuous, and the map $(\alpha, \beta) \to \alpha\beta$ is continuous with respect to the relative topology of $\mathcal{G}^{(2)}$ as a subset of $\mathcal{G} \times \mathcal{G}$. Further, a topological groupoid is *étale* if the range map $r : \mathcal{G} \to \mathcal{G}$ is a local homeomorphism, and we say the a locally compact groupoid is *r-discrete* if its unit space is an open subset.

The following example proves to be very useful in defining groupoids of k-graphs in order to build models for their C^*-algebras.

Examples 2.2.5

(i) **Deaconu-Renault groupoids.**

Let X be a set, and Γ an (additive) abelian group. Suppose that S is a subsemigroup of Γ containing 0. Suppose that S acts on X in the sense that there are maps $x \mapsto s \cdot x$ from $X \to X$ which satisfy $s \cdot (t \cdot x) = (s + t) \cdot x$, and $0 \cdot x = x$ for all $x \in X$ and all $s, t \in S$. We define

$$\begin{aligned} \mathcal{G} &:= \{(x, s - t, y) \in X \times \Gamma \times X : s \cdot x = t \cdot y\}, \text{ and} \\ \mathcal{G}^{(0)} &:= \{(x, 0, x) : x \in X\}. \end{aligned}$$

We identify $\mathcal{G}^{(0)}$ with X in the obvious way. We then define

$$\begin{aligned} r(x, s, y) &:= x, \\ s(x, s, y) &:= y, \text{ and} \\ (x, s, y)^{-1} &:= (y, -s, x). \end{aligned}$$

If $s_1 \cdot x = t_1 \cdot y$ and $s_t \cdot y = t_2 \cdot z$, then

$$\begin{aligned}(s_1 + s_2) \cdot x = s_2 \cdot (s_1 \cdot x) = s_2 \cdot (t_1 \cdot y) = (t_1 + s_2) \cdot y \\ = t_1 \cdot (s_2 \cdot y) = t_1 \cdot (t_2 \cdot z) = (t_1 + t_2) \cdot z.\end{aligned}$$

So we define

$$(x, s, y)(y, t, z) := (x, s + t, z).$$

Under these definitions it is routine to check that $\mathcal{G}$ is a groupoid. We call it the Deaconu-Renault groupoid of X and S. Further, if X is a second-countable, locally compact Hausdorff space, then $\mathcal{G}$ becomes a locally compact Hausdorff groupoid in the topology given by

$$Z(U, p, q, V) := \{(x, p - q, y) : x \in U, y \in V, p \cdot x = q \cdot y\},$$

where $U, V \subseteq X$ are open and $p, q \in S$. If the action of S on X is by local homeomorphisms, then $\mathcal{G}$ is in fact étale. For each $(x, p - q, y) \in \mathcal{G}$ such that $p \cdot x = q \cdot y$, we can choose open neighborhoods U of x and V of y such that $u \mapsto p \cdot u$ is a homeomorphism from U to $p \cdot U$, and similarly for V. Now consider their intersection, $W = p \cdot U \cap q \cdot V$, and let $U' = \{u \in U : p \cdot u \in W\}$ and $V' = \{v \in V : q \cdot v \in W\}$. Then $Z(U', p, q, V')$ is an open neighborhood of $(x, p-q, y)$ and the range map restricts to a homeomorphism of this neighborhood onto U'.

(ii) **The path groupoid of a directed graph.**

In 1997 Kumjian, Pask, Raeburn and Renault [22] constructed a groupoid from a directed graph as follows. Let $E = (E^0, E^1, r, s)$ be a directed graph. A *finite path* in E is a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of edges in E such that $s(\alpha_{j+1}) = r(\alpha_j)$ for each $1 \leq j \leq k - 1$. We write $F(E)$ for the set of all finite paths in E. We let $P(E)$ denote the corresponding set of infinite paths in E. We may compose finite and infinite paths in the obvious way via concatenation. The goal of the authors was to define a groupoid with unit space $P(E)$ that was associated to an equivalence relation on $P(E)$. In particular $x, y \in P(E)$ are *shift equivalent with* *lag* $k \in \mathbb{Z}$ (written $x \sim_k y$) if there exists $N \in \mathbb{N}$ such that $x_i = y_i + k$ for all $i \geq N$. They define the path groupoid to be $\mathcal{G} = \{(x, k, y) \in E \times \mathbb{Z} \times E : x \sim_k y\}$, where the multiplication is given by $(x, k, y)(y, \ell, z) = (x, k + \ell, z)$, and inversion is given by $(x, k, y)^{-1} = (y, -k, x)$.

2.2.1 Groupoid cocycles and the full groupoid C^*-algebra

We now examine how to associate a C^*-algebra to a given continuous groupoid 2-cocycle, σ, and a given étale groupoid, $\mathcal{G}$ through the convolution algebra $C_c(\mathcal{G}, \sigma)$ of compactly supported continuous functions on $\mathcal{G}$. This algebra has two natural C^*-completions, the full C^*-algebra $C^*(\mathcal{G}, \sigma)$, and the reduced C^*-algebra $C^*_r(\mathcal{G}, \sigma)$. Here, we are interested in the full C^*-algebra $C^*(\mathcal{G}, \sigma)$. First we give a brief definition of groupoid cocycles as in [35]. We note that the similarities between groupoid cocycles and k-graph cocycles motivate the use of a groupoid model for the twisted Toeplitz algebra.

Definition 2.2.6 Let $\mathcal{G}$ be a second-countable, locally compact Hausdorff étale groupoid. We say that a continuous function $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ is a *normalised 2-cocycle* (or just *2-cocycle* when the context is clear) if

(i) $\sigma(\lambda, \mu)\sigma(\lambda\mu, \nu) = \sigma(\mu, \nu)\sigma(\lambda, \mu\nu)$ for all composable $\lambda, \mu, \nu \in \mathcal{G}$, and

(ii) $\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$ for all $\gamma \in \mathcal{G}$.

To distinguish between cocycles on groupoids and cocycles on k-graphs, we will use tilde notation rather than underline notation. That is, for cocycles defined on groupoids, we write $\tilde{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$ for the collection of 2-cocycles, $\tilde{B}(\mathcal{G}, \mathbb{T})$ for the collection of coboundaries, and $\tilde{H}^2(\mathcal{G}, \mathbb{T})$ for the continuous cocycle cohomology.

The complex vector space $C_c(\mathcal{G}, \sigma)$ is a *-algebra with multiplication and involution given by

$$\begin{aligned}(f * g)(\gamma) &= \sum_{\alpha\beta=\gamma} \sigma(\alpha, \beta) f(\alpha) g(\beta) \\ &= \sum_{\alpha \in \mathcal{G}^{r(\gamma)}} \sigma(\alpha, \alpha^{-1}\gamma) f(\alpha) g(\alpha^{-1}\gamma), \text{and} \\ f^*(\gamma) &= \overline{\sigma(\gamma^{-1}, \gamma) f(\gamma^{-1})}.\end{aligned}$$

There exist a C^*-algebra $C^*(\mathcal{G},\sigma)$ and a *-homomorphism $\pi_{\max} : C_c(\mathcal{G},\sigma) \to C^*(\mathcal{G},\sigma)$ such that $\pi_{\max}(C_c(\mathcal{G},\sigma))$ is dense in $C^*(\mathcal{G},\sigma)$, and such that for every *-representation $\pi : C_c(\mathcal{G},\sigma) \to \mathcal{B}(\mathcal{H})$, there is a representation φ of $C^*(\mathcal{G},\sigma)$ such that $\varphi \circ \pi_{\max} = \pi$. The norm on $C^*(\mathcal{G},\sigma)$ satisfies

$$||\pi_{\max}(f)|| = \sup\{||\pi(f)|| : \pi \text{ is a *-representation of } C_c(\mathcal{G},\sigma)\}$$

for all $f \in C_c(\mathcal{G},\sigma)$. Essentially, with the above norm property in mind, we think of $C^*(\mathcal{G},\sigma)$ as the largest C^*-algebra that encompasses $C_c(\mathcal{G},\sigma)$.

2.3 The twisted Toeplitz algebra

In 2014 Sims, Whitehead and Whittaker [39] introduced the twisted Toeplitz algebra of a k-graph. Below is the definition they give.

Definition 2.3.1 For a finitely aligned k-graph and a $\mathbb{T}$-valued 2-cocycle $c \in \underline{Z}^2(\Lambda,\mathbb{T})$, they define a Toeplitz-Cuntz-Krieger (Λ, c)-family (abbreviated to TCK (Λ, c)-family) in a C^*-algebra A to be a family $\{t_\lambda : \lambda \in \Lambda\} \subseteq A$ which satisfies

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections,

(TCK2) $t_\mu t_\nu = c(\mu,\nu) t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$,

(TCK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

(TCK4) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$, where empty sums are zero.

The first thing we notice is that due to (TCK4), we require either that each of these sums converges, or more simply that each is a finite sum which is ensured if Λ is finitely aligned. We also note that it is (unusually) not specified that each t_λ is a partial isometry. This is because the relation (TCK3) ensures this property, as we may see by left multiplication by t_λ, and an application of (TCK2). The authors also establish that $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\mathrm{span}}\{t_\mu t_\nu^* : (\mu,\nu) \in \Lambda *_s \Lambda\}$. The following result based on [39, Lemma 3.2] details how these properties and more follow directly from the definition.

Proposition 2.3.2 *Let Λ be a finitely aligned k-graph, $c \in \underline{Z}^2(\Lambda,\mathbb{T})$ and let $\{t_\lambda : \lambda \in \Lambda\}$ be a TCK (Λ, c)-family. Then* *(i) for each* $\lambda \in \Lambda$, t_λ *is a partial isometry,*

(ii) $\{t_\mu t_\mu^* : \mu \in \Lambda\}$ *is a collection of pairwise commuting projections,*

(iii) $\{t_\mu t_\mu^* : \mu \in \Lambda^n\}$ *is a collection of mutually orthogonal projections for each* $n \in \mathbb{N}^k$,

(iv) for all $\nu, \eta \in \Lambda$

$$t_\nu^* t_\eta = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^*,$$

(v) for all $\mu, \nu, \eta, \zeta \in \Lambda$

$$t_\mu t_\nu^* t_\eta t_\zeta^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)} c(\eta,\beta)\overline{c(\zeta,\beta)} t_{\mu\alpha} t_{\zeta,\beta}^*,$$

(vi) $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : (\mu,\nu) \in \Lambda *_s \Lambda\}$.

Proof.

(i) Let $\lambda \in \Lambda$. By (TCK3) we have $t_\lambda^* t_\lambda = t_{s(\lambda)}$. Multiplying on the left by t_λ and applying (TCK2) we have $t_\lambda t_\lambda^* t_\lambda = t_\lambda t_{s(\lambda)} = c(\lambda, s(\lambda)) t_\lambda = t_\lambda$.

(ii) Let $\mu, \nu \in \Lambda$. Since $\mathrm{MCE}(\mu,\nu) = \mathrm{MCE}(\nu,\mu)$, then by (TCK4) we have

$$\begin{aligned} t_\mu t_\mu^* t_\nu t_\nu^* &= \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^* \\ &= \sum_{\lambda\in\mathrm{MCE}(\nu,\mu)} t_\lambda t_\lambda^* \\ &= t_\nu t_\nu^* t_\mu t_\mu^*. \end{aligned}$$

(iii) Let $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu)$ and $\mu \neq \nu$. By the factorisation property, $\mu\Lambda \cap \nu\Lambda = \emptyset$. In particular we see that $\mathrm{MCE}(\mu,\nu) = \emptyset$, and hence by (TCK4) $t_\mu t_\mu^* t_\nu t_\nu^* = 0$.

(iv) Let $\nu, \eta \in \Lambda$. By (i) t_ν, t_η are partial isometries. So applying (TCK4), (TCK2), and (TCK3) we calculate

$$\begin{aligned}
t_\nu^* t_\eta &= (t_\nu^* t_\nu t_\nu^*)(t_\eta t_\eta^* t_\eta) \\
&= t_\nu^* (t_\nu t_\nu^* t_\eta t_\eta^*) t_\eta \\
&= t_\nu^* \left(\sum_{\lambda \in \mathrm{MCE}(\nu,\eta)} t_\lambda t_\lambda^* \right) t_\eta \\
&= t_\nu^* \left(\sum_{\nu\alpha=\mu\beta \in \mathrm{MCE}(\nu,\eta)} t_{\nu\alpha} t_{\mu\beta}^* \right) t_\eta \\
&= \sum_{\nu\alpha=\mu\beta \in \mathrm{MCE}(\nu,\eta)} t_\nu^* t_{\nu\alpha} t_{\mu\beta}^* t_\eta \\
&= \sum_{\nu\alpha=\mu\beta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\nu^* t_\nu t_\alpha t_\beta^* t_\eta^* t_\eta \\
&= \sum_{\nu\alpha=\mu\beta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_{s(\nu)} t_\alpha t_\beta^* t_{s(\eta)}.
\end{aligned}$$

Now since $\nu\alpha = \mu\beta \in \mathrm{MCE}(\nu,\eta)$, we have $r(\alpha) = s(\nu)$ and $r(\beta) = s(\eta)$. So we have $t_{s(\nu)} t_\alpha = c(s(\nu),\alpha) t_{s(\nu)\alpha} = t_\alpha$, and similarly $t_\beta^* t_{s(\eta)} = t_\beta^*$. Hence the above becomes

$$\begin{aligned}
t_\nu^* t_\eta &= \sum_{\nu\alpha=\mu\beta \in \mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^* \\
&= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^*.
\end{aligned}$$

(v) Let $\mu, \nu, \eta, \zeta \in \Lambda$. Multiplying the result from (iv) on the left by t_μ and on the right by t_ζ^* and applying (TCK2), we obtain

$$\begin{aligned}
t_\mu t_\nu^* t_\eta t_\zeta^* &= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\mu t_\alpha t_\beta^* t_\zeta^* \\
&= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\nu,\eta)} c(\mu,\alpha) \overline{c(\nu,\alpha)} c(\eta,\beta) \overline{c(\zeta,\beta)} t_{\mu\alpha} t_{\zeta\beta}^*.
\end{aligned}$$

(vi) From (v) it follows that $\overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\}$ is closed under multiplication, and hence is equal to $C^*(\{t_\lambda : \lambda \in \Lambda\})$. We now observe that if $s(\mu) \neq s(\nu)$, then by (i), (TCK3), and (TCK1) we have $t_\mu t_\nu^* = t_\mu t_\mu^* t_\mu t_\nu^* t_\nu t_\nu^* = t_\mu t_{s(\mu)} t_{s(\nu)} t_\nu^* = 0$. Hence $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : s(\mu) = s(\nu)\}$. $\square$

In [39], the authors define $\mathcal{T}C^*(\Lambda, c)$ to be the unique universal C^*-algebra generated by a Toeplitz-Cuntz-Krieger (Λ, c)-family. The following result says that such an algebra exists and clarifies what we mean by 'unique' and 'universal'. We spend the rest of this section proving this result.

Theorem 2.3.3 *For a given finitely aligned k-graph Λ and cocycle $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, there exists a C^*-algebra $\mathcal{T}C^*(\Lambda, c)$ generated by a TCK (Λ, c)-family $\{t_\lambda : \lambda \in \Lambda\}$ that is* universal *in the sense that for any TCK (Λ, c)-family $\{T_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, there exists a *-homomorphism $\pi_{\mathcal{T}} : \mathcal{T}C^*(\Lambda, c) \to B$ such that $\pi_{\mathcal{T}}(t_\lambda) = T_\lambda$ for all $\lambda \in \Lambda$.*

The algebra $\mathcal{T}C^*(\Lambda, c)$ is unique *in the sense that if B is a C^*-algebra generated by a TCK (Λ, c)-family $\{T_\lambda : \lambda \in \Lambda\}$ which satisfies the universal property above, then there is an isomorphism of $\mathcal{T}C^*(\Lambda, c)$ onto B such that $t_\lambda \mapsto T_\lambda$ for all $\lambda \in \Lambda$.*

We call $\mathcal{T}C^*(\Lambda, c)$ the twisted Toeplitz algebra of Λ.

We now give an example of a Toeplitz-Cuntz-Krieger (Λ, c)-family for an arbitrary finitely-aligned k-graph Λ and cocycle c.

Example 2.3.4 Let Λ be a finitely-aligned k-graph, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. For $\lambda \in \Lambda$ let $\delta_\lambda : \Lambda \to \mathbb{C}$ be the point mass function at λ. Consider the Hilbert space $\ell^2(\Lambda) = \overline{\text{span}}\{\delta_\lambda : \lambda \in \Lambda\}$ with the usual inner product $\langle \delta_\mu, \delta_\nu \rangle = 1$ if $\mu = \nu$ and 0 otherwise. There is a Toeplitz-Cuntz-Krieger (Λ, c)-family $\{T_\lambda\} \subset B(\ell^2(\Lambda))$ such that

$$T_\lambda \delta_\mu = \begin{cases} c(\lambda, \mu)\delta_{\lambda\mu}, & s(\lambda) = r(\mu), \\ 0, & \text{otherwise.} \end{cases}$$

It is routine to see that the adjoint of T_λ is given by

$$T_\lambda^* \delta_\mu = \begin{cases} \overline{c(\lambda, \mu')}\delta_{\mu'}, & \mu = \lambda\mu', \\ 0, & \text{otherwise.} \end{cases}$$

So we verify (TCK1)-(TCK4). For (TCK1), let $\mu \in \Lambda$, $v, w \in \Lambda^0$ such that $v \neq w$.

Then

$$\begin{aligned}
T_v^2\delta_\mu &= \begin{cases} c(v,\mu)T_v\delta_{v\mu}, & v = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} T_v\delta_\mu, & v = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= T_v\delta_\mu, \text{ and} \\
T_v^*\delta_\mu &= \begin{cases} \overline{c(v,\mu')}\delta_{\mu'}, & \mu = v\mu', \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \delta_\mu, & v = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= T_v\delta_\mu.
\end{aligned}$$

So $T_v = T_v^* = T_v^2$, and hence each T_v is a projection. To see that they are mutually orthogonal, we calculate

$$\begin{aligned}
T_wT_v\delta_\mu &= \begin{cases} c(v,\mu)T_w\delta_\mu, & v = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \delta_\mu, & w = v = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= 0,
\end{aligned}$$

since $v \neq w$. Now, for (TCK2), let $\lambda, \nu, \mu \in \Lambda$ such that $s(\lambda) = r(\nu)$. Then

$$\begin{aligned}
T_\lambda T_\nu\delta_\mu &= \begin{cases} c(\nu,\mu)T_\lambda\delta_{\nu\mu}, & s(\nu) = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} c(\nu,\mu)c(\lambda,\nu\mu)\delta_{\lambda\nu\mu}, & s(\nu) = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} c(\lambda,\nu)c(\lambda\nu,\mu)\delta_{\lambda\nu\mu}, & s(\lambda\nu) = r(\mu), \\ 0, & \text{otherwise,} \end{cases} \\
&= c(\lambda,\nu)T_{\lambda\nu}\delta_\mu.
\end{aligned}$$

For (TCK3) fix $\lambda\mu \in \Lambda$. Then

$$\begin{aligned}
T_\lambda^* T_\lambda \delta_\mu &= \begin{cases} c(\lambda,\mu) T_\lambda^* \delta_{\lambda\mu}, & s(\lambda) = r(\mu), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} c(\lambda,\mu)\overline{c(\lambda,\mu)}\delta_\mu, & s(\lambda) = r(\mu), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \delta_\mu, & s(\lambda) = r(\mu), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} c(s(\lambda),\mu)\delta_\mu, & s(\lambda) = r(\mu), \\ 0, & \text{otherwise}, \end{cases} \\
&= T_{s(\lambda)}\delta_\mu.
\end{aligned}$$

Finally, for (TCK4), first consider for $\lambda\alpha \in \Lambda$

$$\begin{aligned}
T_\lambda T_\lambda^* \delta_\alpha &= \begin{cases} \overline{c(\lambda,\alpha')} T_\lambda \delta_{\alpha'}, & \alpha = \lambda\alpha', \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} c(\lambda,\alpha')\overline{c(\lambda,\alpha')}\delta_{\lambda\alpha'}, & \alpha = \lambda\alpha', \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \delta_\alpha, & \alpha = \lambda\alpha', \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

Now fix $\mu, \nu, \alpha, \beta \in \Lambda$. Then by the above calculations we see that

$$\begin{aligned}
\big\langle T_\mu T_\mu^* T_\nu T_\nu^* \delta_\alpha, \delta_\beta \big\rangle &= \big\langle T_\nu T_\nu^* \delta_\alpha, T_\mu T_\mu^* \delta_\beta \big\rangle \\
&= \begin{cases} \langle \delta_\alpha, \delta_\beta \rangle, & \beta = \mu\beta',\ \alpha = \nu\alpha', \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1, & \beta = \mu\beta' = \alpha = \nu\alpha', \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} 1, & \beta = \alpha = \lambda\alpha'' \text{ for some } \lambda \in \operatorname{MCE}(\mu,\nu), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \langle \delta_\alpha, \delta_\beta \rangle, & \alpha = \lambda\alpha'' \text{ for some } \lambda \in \operatorname{MCE}(\mu,\nu), \\ 0, & \text{otherwise}. \end{cases} \qquad (2.10)
\end{aligned}$$

By the factorisation property, there is a unique $\lambda \in \mathrm{MCE}(\mu,\nu)$ such that $\alpha = \lambda\alpha''$, so $T_\eta T_\eta^* \delta\alpha = 0$ for each other $\eta \in \mathrm{MCE}(\mu,\nu)$. So we conclude that (2.10) is equal to

$$\left\langle \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} T_\lambda T_\lambda^* \delta_\alpha, \delta_\beta \right\rangle.$$

This holds for all $\alpha, \beta \in \Lambda$, establishing (TCK4).

We now prove some results about this particular TCK (Λ, c)-family that will allow us to prove Theorem 2.3.3.

Lemma 2.3.5 *Let Λ be a finitely aligned k-graph, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\{T_\lambda : \lambda \in \Lambda\}$ be the TCK (Λ, c)-family from Example 2.3.4. Then $\{T_\mu T_\nu^* : (\mu,\nu) \in \Lambda *_s \Lambda\}$ is a linearly independent set.*

Proof. Let $F \subseteq \Lambda *_s \Lambda$ be a finite set. We will show that $\{T_\mu T_\nu^* : (\mu,\nu) \in F\}$ is a linearly independent set. We proceed by induction on $|F|$. We suppose that

$$\sum_{(\mu,\nu)\in F} a_{\mu,\nu} T_\mu T_\nu^* \xi = 0$$

for all $\xi \in \ell^2(\Lambda)$. If $|F| = 1$ we have a unique $(\mu',\nu') \in F$. Then we have

$$\begin{aligned} 0 &= a_{\mu',\nu'} T_{\mu'} T_{\nu'}^* \delta_{\nu'} \\ &= a_{\mu',\nu'} \overline{c(\nu', s(\nu'))} T_{\mu'} \delta_{s(\nu')} \\ &= a_{\mu',\nu'} \delta_{\mu'}. \end{aligned}$$

Hence $a_{\mu',\nu'} = 0$. Now suppose that $j \in \mathbb{N}$ is such that $\{T_\mu T_\nu^* : (\mu,\nu) \in F\}$ is linearly independent whenever $|F| \leq j$. Suppose $|F| = j+1$. Fix $(\mu_0, \nu_0) \in F$ such that $\nu_0 \neq \beta\nu_0'$ for all $(\alpha,\beta) \in F \setminus \{(\mu_0,\nu_0)\}$. Since F is finite, such a pair exists. We have

$$\begin{aligned} 0 &= \sum_{(\mu,\nu)\in F} a_{\mu,\nu} T_\mu T_\nu^* \delta_{\nu_0} \\ &= \sum_{(\mu,\nu_0)\in F} a_{\mu,\nu_0} \overline{c(\nu_0, s(\nu_0))} T_\mu \delta_{\nu_0} \\ &= \sum_{(\mu,\nu_0)\in F} a_{\mu,\nu_0} c(\mu, s(\nu_0)) \delta_\mu \\ &= \sum_{(\mu,\nu_0)\in F} a_{\mu,\nu_0} \delta_\mu. \end{aligned}$$

Since the δ_μ are linearly independent, each $a_{\mu,\nu_0} = 0$. Now let $G = \{(\mu,\nu) \in F : \nu \neq \nu_0\}$. By the inductive hypothesis, $\{T_\mu T_\nu^* : (\mu,\nu) \in G\}$ is a linearly independent set. So

$$\begin{aligned} 0 &= \sum_{(\mu,\nu)\in F} a_{\mu,\nu} T_\mu T_\nu^* \\ &= \sum_{(\mu,\nu)\in G} a_{\mu,\nu} T_\mu T_\nu^* \end{aligned}$$

implies that $a_{\mu,\nu} = 0$ for all $(\mu,\nu) \in G$. Hence $a_{\mu,\nu} = 0$ for all $(\mu,\nu) \in F$, and by induction we have that $\{T_\mu T_\nu^* : (\mu,\nu) \in F\}$ is a linearly independent set for any finite set $F \subset \Lambda *_s \Lambda$. □

Lemma 2.3.6 *Let Λ be a finitely aligned k-graph, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $C_c(\Lambda *_s \Lambda)$ be the finitely supported functions on $\Lambda *_s \Lambda$. We note that $C_c(\Lambda *_s \Lambda)$ is a complex vector space under scalar multiplication and pointwise addition. For each $(\mu,\nu) \in \Lambda *_s \Lambda$, let $\delta_{\mu,\nu}$ be the point mass function at (μ,ν). There is a unique associative multiplication on $C_c(\Lambda *_s \Lambda)$ satisfying*

$$\delta_{\mu,\nu}\delta_{\eta,\xi} = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}\delta_{\mu\alpha,\xi\beta}, \tag{2.11}$$

*for all $(\mu,\nu),(\eta,\xi) \in \Lambda *_s \Lambda$. There is an involution $* : C_c(\Lambda *_s \Lambda) \to C_c(\Lambda *_s \Lambda)$ given by $f^*(\mu,\nu) = \overline{f(\nu,\mu)}$. Under these operations $C_c(\Lambda *_s \Lambda)$ is a complex *-algebra.*

Proof. Fix $f, g \in C_c(\Lambda *_s \Lambda)$, $(\mu,\nu) \in \Lambda *_s \Lambda$, and $\lambda \in \mathbb{C}$. We calculate

$$\begin{aligned} (f+\lambda g)^*(\mu,\nu) &= \overline{(f+\lambda g)(\nu,\mu)} \\ &= \overline{f(\nu,\mu)} + \overline{\lambda g(\nu,\mu)} \\ &= f^*(\mu,\nu) + \overline{\lambda} g^*(\mu,\nu) \\ &= (f^* + \overline{\lambda} g^*)(\mu,\nu), \end{aligned}$$

and

$$\begin{aligned} (f^*)^*(\mu,\nu) &= \overline{f^*(\nu,\mu)} \\ &= \overline{\overline{f(\mu,\nu)}} \\ &= f(\mu,\nu). \end{aligned}$$

Hence $*$ defines an involution. We must check that the formula (2.11) extends to all of $C_c(\Lambda *_s \Lambda)$, that it is associative, and that $(fg)* = g^* f^*$ for all $f, g \in C_c(\Lambda *_s \Lambda)$.

First we note that the $\delta_{\mu,\nu}$ are linearly independent and that they span $C_c(\Lambda *_s \Lambda)$, so that each f has the unique expansion

$$f = \sum_{(\eta,\xi)\in\Lambda *_s \Lambda} f_{\eta,\xi}\delta_{\eta,\xi}.$$

Hence we may write

$$\begin{aligned}
(fg)(\mu,\nu) &= \left(\left(\sum_{(\eta,\xi)\in\Lambda *_s\Lambda} f_{\eta,\xi}\delta_{\eta,\xi}\right)\left(\sum_{(\rho,\sigma)\in\Lambda *_s\Lambda} g_{\rho,\sigma}\delta_{\rho,\sigma}\right)\right)(\mu,\nu)\\
&= \sum_{(\eta,\xi),(\rho,\sigma)\in\Lambda *_s\Lambda} f_{\eta,\xi} g_{\rho,\sigma}(\delta_{\eta,\xi}\delta_{\rho,\sigma})(\mu,\nu)\\
&= \sum_{(\eta,\xi),(\rho,\sigma)\in\Lambda *_s\Lambda}\ \sum_{(\alpha,\beta)\in\Lambda^{\min}(\xi,\rho)} c(\eta,\alpha)\overline{c(\xi,\alpha)}c(\rho,\beta)\overline{c(\sigma,\beta)} f_{\eta,\xi} g_{\rho,\sigma}\delta_{\eta\alpha,\sigma\beta}(\mu,\nu)\\
&= \sum_{(\eta,\xi),(\rho,\sigma)\in\Lambda *_s\Lambda}\ \sum_{\substack{(\alpha,\beta)\in\Lambda^{\min}(\xi,\rho)\\ \mu=\eta\alpha,\nu=\sigma\beta}} c(\eta,\alpha)\overline{c(\xi,\alpha)}c(\rho,\beta)\overline{c(\sigma,\beta)} f(\eta,\xi) g(\rho,\sigma),
\end{aligned}$$

so that the multiplication extends to $C_c(\Lambda *_s \Lambda)$.

For associativity, we construct an injective, multiplication preserving linear map $\phi : C_c(\Lambda *_s \Lambda) \to \mathcal{B}(\ell^2(\Lambda))$ using the TCK (Λ, c)-family of Example 2.3.4, $\{T_\lambda : \lambda \in \Lambda\} \subset \mathcal{B}(\ell^2(\Lambda))$. We define ϕ by

$$\phi\left(\sum_{(\mu,\nu)\in\Lambda *_s\Lambda} f_{\mu,\nu}\delta_{\mu,\nu}\right) = \sum_{(\mu,\nu)\in\Lambda *_s\Lambda} f_{\mu,\nu} T_\mu T_\nu^*.$$

Since the $\delta_{\mu,\nu}$ are linearly independent this is well-defined, and by definition of the vector space axioms, ϕ is linear. Hence to show that ϕ is associative, it is sufficient to show that $\phi(\delta_{\mu,\nu}\delta_{\eta,\xi}) = \phi(\delta_{\mu,\nu})\phi(\delta_{\eta,\xi})$. So we calculate, using the fact that $\{T_\lambda : \lambda \in \Lambda\}$

is a TCK (Λ, c)-family

$$\begin{aligned}
\phi(\delta_{\mu,\nu}\delta_{\eta,\xi}) &= \phi\left(\sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}\delta_{\mu\alpha,\xi\beta}\right)\\
&= \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}T_{\mu\alpha}T^*_{\xi\beta}\\
&= T_\mu T^*_\nu T_\eta T^*_\xi\\
&= \phi(\delta_{\mu,\nu})\phi(\delta_{\eta,\xi}).
\end{aligned}$$

To see that ϕ is injective, fix $f = \sum f_{\mu,\nu}\delta_{\mu,\nu} \in C_c(\mu,\nu)$. Then

$$\phi\left(\sum_{(\mu,\nu)\in\Lambda *_s\Lambda} f_{\mu,\nu}\delta_{\mu,\nu}\right) = 0 \implies \sum_{(\mu,\nu)\in\Lambda *_s\Lambda} f_{\mu,\nu}T_\mu T^*_\nu = 0.$$

By Lemma 2.3.5 this implies that $f_{\mu,\nu} = 0$ for all $(\mu,\nu) \in \Lambda *_s \Lambda$. Hence $f = 0$. So $\ker(\phi) = \{0\}$. Hence ϕ is injective. Associativity follows from associativity of the multiplication in $\mathcal{B}(\ell^2(\Lambda))$.

Finally to see that $(fg)^* = g^*f^*$ it is sufficient to check that $(\delta_{\mu,\nu}\delta_{\eta,\xi})^* = \delta^*_{\eta,\xi}\delta^*_{\mu,\nu}$ for all $(\mu,\nu),(\eta,\xi) \in \Lambda *_s \Lambda$. We calculate

$$\begin{aligned}
(\delta_{\mu,\nu}\delta_{\eta,\xi})^* &= \left(\sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}\delta_{\mu\alpha,\xi\beta}\right)^*\\
&= \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\eta)} \overline{c(\mu,\alpha)}c(\nu,\alpha)\overline{c(\eta,\beta)}c(\xi,\beta)\delta_{\xi\beta,\mu\alpha}\\
&= \delta_{\xi,\eta}\delta_{\nu,\mu}\\
&= \delta^*_{\eta,\xi}\delta^*_{\mu,\nu}.
\end{aligned}$$
□

Proof of Theorem 2.3.3. First we will show that every TCK (Λ, c)-family admits a homomorphism of $C_c(\Lambda, c)$. So fix a TCK (Λ, c)-family $\{T_\lambda : \lambda \in \Lambda\}$, and define a map $\pi_T : C_c(\Lambda *_s \Lambda) \to C^*(\{T_\lambda\})$ by

$$\pi_T(f) = \sum_{(\mu,\nu)\in\Lambda *_s\Lambda} f(\mu,\nu)T_\mu T^*_\nu.$$

Clearly π_T is linear, so we check that it is a homomorphism. Fix $f, g \in C_c(\Lambda *_s \Lambda)$ such that

$$f = \sum_{(\mu,\nu)\in F} f_{\mu,\nu}\delta_{\mu,\nu}, \qquad \text{and} \qquad g = \sum_{(\eta,\xi)\in G} f_{\eta,\xi}\delta_{\eta,\xi},$$

for some finite $F, G \subseteq \Lambda *_s \Lambda$. So we have

$$\begin{aligned}
\pi_T(fg) &= \pi_T\left(\sum_{(\mu,\nu)\in F,(\eta,\xi)\in G}\ \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}f_{\mu,\nu}g_{\eta,\xi}\delta_{\mu\alpha,\xi\beta}\right)\\
&= \sum_{(\mu,\nu)\in F,(\eta,\xi)\in G}\ \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu,\eta)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\eta,\beta)\overline{c(\xi,\beta)}f_{\mu,\nu}g_{\eta,\xi}T_{\mu\alpha}T^*_{\xi\beta}\\
&= \sum_{(\mu,\nu)\in F,(\eta,\xi)\in G} f_{\mu,\nu}g_{\eta,\xi}T_\mu T^*_\nu T_\eta T^*_\xi\\
&= \left(\sum_{(\mu,\nu)\in F} f_{\mu,\nu}T_\mu T^*_\nu\right)\left(\sum_{(\eta,\xi)\in G} g_{\eta,\xi}T_\eta T^*_\xi\right)\\
&= \pi_T(f)\pi_T(g),
\end{aligned}$$

and

$$\begin{aligned}
\pi_T(f^*) &= \sum_{(\mu,\nu)\in F} \overline{f_{\mu,\nu}}T_\nu T^*_\mu\\
&= \pi_T(f)^*.
\end{aligned}$$

So π_T is a homomorphism.

We now seek an upper bound on the set

$$\{||\pi_T(f)|| : \ T \text{ is a TCK } (\Lambda, c)\text{-family}\}$$

for each $f \in C_c(\Lambda *_s \Lambda)$. For each $(\mu,\nu) \in \Lambda *_s \Lambda$ we have

$$\begin{aligned}
||\pi_T(\delta_{\mu,\nu})||^2 &= ||\pi_T(\delta_{\mu,\nu})^*\pi_T(\delta_{\mu,\nu})||\\
&= ||\pi_T(\delta_{\nu,\mu})\pi_T(\delta_{\mu,\nu})||\\
&= ||T_\nu T^*_\mu T_\mu T^*_\nu||\\
&= ||c(\nu,s(\mu))c(\mu,s(\mu))\overline{c(\nu,s(\nu))c(\mu,s(\mu))}T_{\nu s(\mu)}T^*_{\nu s(\mu)}||\\
&= ||T^*_\nu T_\nu||\\
&= ||T_{s(\nu)}||.
\end{aligned}$$

Since $T_{s(\nu)}$ is a projection, it has either norm 0 or 1, so by the above the same is true of $\pi_T(\delta_{\mu,\nu})$. Hence

$$\begin{aligned} ||\pi_T(f)|| &\leq \sum_{(\mu,\nu)\in F} |f(\mu,\nu)|||\pi_T(\delta_{\mu\nu})|| \\ &= \sum_{(\mu,\nu)\in F} |f(\mu,\nu)| \\ &< \infty, \end{aligned}$$

since F is finite. Hence for each $f \in C_c(\Lambda *_s \Lambda)$, the set $\{||\pi_T(f)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\}$ is bounded above. It is also non-empty as we saw in Example 2.3.4. Hence for each $f \in C_c(\Lambda *_s \Lambda)$ we may define

$$\rho(f) := \sup\{||\pi_T(f)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\}.$$

Now, let $I := \ker(\rho) = \bigcap\{\ker(\pi_T) : T \text{ is a TCK } (\Lambda, c)\text{-family}\}$, and define $||\cdot|| : C_c(\Lambda *_s \Lambda)/I \to [0,\infty)$ by $||f + I|| = \rho(f)$. To see that this is well-defined, suppose that $g \in I$. Then

$$\begin{aligned} \rho(f+g) &= \sup\{||\pi_T(f+g)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\} \\ &= \sup\{||\pi_T(f) + \pi_T(g)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\} \\ &= \sup\{||\pi_T(f)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\}. \end{aligned}$$

Now we calculate

$$\begin{aligned} ||(f+I)(g+I)|| &= ||fg + I|| \\ &= \sup\{||\pi_T(fg)|| : T \text{ is a TCK } (\Lambda, c)\text{-family}\} \\ &\leq \sup\{||\pi_T(f)||\} \sup\{||\pi_T(g)||\} \\ &= \rho(f)\rho(g) \\ &= ||f + I||||g + I||, \end{aligned}$$

and

$$\begin{aligned}||(f+I)^*(f+I)|| &= ||f^*f+I|| \\ &= \sup\{||\pi_T(f^*f)|| : \ T \text{ is a TCK } (\Lambda, c)\text{-family}\} \\ &= \sup\{||\pi_T(f)||^2 : \ T \text{ is a TCK } (\Lambda, c)\text{-family}\} \\ &= \rho(f)^2 \\ &= ||f+I||^2,\end{aligned}$$

to see that $||\cdot||$ is a Banach *-algebra norm which satisfies the C^* identity. Denote the completion of $C_c(\Lambda *_s \Lambda)/I$ by $\mathcal{T}C^*(\Lambda, c)$, which by construction is a C^*-algebra.

For $\lambda \in \Lambda$ we define $t_\lambda := \delta_{\lambda,s(\lambda)} + I \in \mathcal{T}C^*(\Lambda, c)$. We claim that $\{t_\lambda : \lambda \in \Lambda\}$ is a TCK (Λ, c)-family that generates $\mathcal{T}C^*(\Lambda, c)$. So let $v, w \in \Lambda^0$ such that $v \neq w$. Then

$$t_v^2 = \delta_{v,v}\delta_{v,v} + I = \delta_{v,v} + I = \delta_{v,v}^* + I = t_v^*,$$

and

$$t_v t_w = \delta_{v,v}\delta_{w,w} + I = 0,$$

establishing (TCK1). For (TCK2), let $\mu, \nu \in \Lambda$ such that $s(\mu) = r(\nu)$. Then

$$\begin{aligned}t_\mu t_\nu &= \delta_{\mu,s(\mu)}\delta_{\nu,s(\nu)} + I \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(s(\mu),\nu)} c(\mu,\alpha)\overline{c(s(\mu),\alpha)}c(\nu,\beta)\overline{c(s(\nu),\beta)}\delta_{\mu\alpha,s(\nu)\beta)} + I \\ &= c(\mu,\nu)\overline{c(s(\mu),\nu)}c(\nu,s(\nu))\overline{c(s(\nu),s(\nu))}\delta_{\mu\nu,s(\nu)s(\nu))} + I \\ &= c(\mu,\nu)\delta_{\mu\nu,s(\mu\nu))} + I \\ &= c(\mu,\nu)t_{\mu\nu}.\end{aligned}$$

For (TCK3), let $\lambda \in \Lambda$. Then

$$\begin{aligned}t_\lambda t_\lambda^* &= \delta_{\lambda,s(\lambda)}\delta_{\lambda,s(\lambda)} + I \\ &= \delta_{s(\lambda),\lambda}\delta_{\lambda,s(\lambda)} + I \\ &= \delta_{s(\lambda),s(\lambda)} + I \\ &= t_{s(\lambda)}.\end{aligned}$$

For (TCK4), fix $\mu, \nu \in \Lambda$. We calculate

$$\begin{aligned}
t_\mu t_\mu^* t_\nu t_\nu^* &= \delta_{\mu,s(\mu)}\delta_{s(\mu),\mu}\delta_{\nu,s(\nu)}\delta_{s(\nu),\nu} + I \\
&= \delta_{\mu,\mu}\delta_{\nu,\nu} + I \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} c(\mu,\alpha)\overline{c(\mu,\alpha)}c(\nu,\beta)\overline{c(\nu,\beta)}\delta_{\mu\alpha,\nu\beta} + I \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\mu,\nu)} \delta_{\mu\alpha,\nu\beta} + I \\
&= \sum_{\sigma\in\mathrm{MCE}(\mu,\nu)} \delta_{\sigma,\sigma} + I \\
&= \sum_{\sigma\in\mathrm{MCE}(\mu,\nu)} \delta_{\sigma,s(\sigma)}\delta^*_{\sigma,s(\sigma)} + I \\
&= \sum_{\sigma\in\mathrm{MCE}(\mu,\nu)} t_\sigma t_\sigma^*.
\end{aligned}$$

If $\mu, \nu \in \Lambda$ such that $s(\mu) = s(\nu)$, then

$$t_\mu t_\nu^* = \delta_{\mu,s(\mu)}\delta_{s(\nu),\nu} + I = \delta_{\mu,\nu} + I.$$

Thus, $\{t_\lambda\}$ generates $\mathcal{T}C^*(\Lambda, c)$.

Now we check that $\mathcal{T}C^*(\Lambda, c)$ has the desired universal property. Fix a TCK (Λ, c)-family $\{T_\lambda : \lambda \in \Lambda\}$. Note that we have

$$||\pi_T(f)|| \leq \sup\{||\pi_{T'}(f) : T' \text{ is a TCK } (\Lambda, c)\text{-family } \} = \rho(f). \tag{2.12}$$

We also have $I \subset \ker(\pi_T)$, so there is a well-defined linear map $\pi_\mathcal{T} : C_c(\Lambda *_s \Lambda)/I \to C^*(\{T_\lambda\})$ given by $\pi_\mathcal{T}(f+I) = \pi_T(f)$. Since π_T is a homomorphism, then by definition of the operations on the quotient space, we see that $\pi_\mathcal{T}$ is a homomorphism. By equation (2.12) we have

$$||\pi_\mathcal{T}(f + I)|| = ||\pi_T(f)|| \leq \rho(f) = ||f + I||.$$

Hence $\pi_\mathcal{T}$ is continuous, and hence extends to a homomorphism $\pi_\mathcal{T} : \mathcal{T}C^*(\Lambda, c) \to C^*(\{T_\lambda\})$.

For uniqueness, suppose that B is the C^*-algebra generated by a TCK (Λ, c)-family $\{T_\lambda : \lambda \in \Lambda\}$ which satisfies the universal property. Since $\mathcal{T}C^*(\Lambda, c)$ is universal, there is a homomorphism $\varphi : \mathcal{T}C^*(\Lambda, c) \to B$ such that $\varphi(t_\lambda) = T_\lambda$ for each $\lambda \in \Lambda$. Since B is universal, there is a homomorphism $\psi : B \to \mathcal{T}C^*(\Lambda, c)$ such that $\psi(T_\lambda) = t_\lambda$ for each $\lambda \in \Lambda$. Both $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on generators (of B and $\mathcal{T}C^*(\Lambda, c)$ respectively). Hence φ, ψ are mutually inverse, and thus isomorphisms. □

2.3.1 The gauge action of the k-torus

In this section we will examine the existence and importance of the gauge action. First we have the following notation definition for powers of k-tuples.

Definition 2.3.7 Let Λ be a k-graph. Let $\lambda \in \Lambda$ and $z \in \mathbb{T}^k$. We define $z^{d(\lambda)}$ to be the product

$$z^{d(\lambda)} := \prod_{i=1}^{k} z_i^{d(\lambda)_i} \in \mathbb{T}.$$

Lemma 2.3.8 *Let Λ be a finitely aligned k-graph, let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and fix $z \in \mathbb{T}^k$. There exists an automorphism $\gamma_z : \mathcal{T}C^*(\Lambda, c) \to \mathcal{T}C^*(\Lambda, c)$ such that $\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for each $\lambda \in \Lambda$.*

Proof. First we observe that (TCK1)-(TCK4) hold for the collection $\{z^{d(\lambda)} t_\lambda\}$ since $\{t_\lambda\}$ is a TCK (Λ, c)-family, and hence by Theorem 2.3.3 there exists a homomorphism $\gamma_z : \mathcal{T}C^*(\Lambda, c) \to \mathcal{T}C^*(\Lambda, c)$ such that $\gamma_z(t_\lambda) = z^{d(\lambda)} t_\lambda$ for all $\lambda \in \Lambda$. Now we observe that $\gamma_z \circ \gamma_{\overline{z}} = \gamma_{\overline{z}} \circ \gamma_z = \mathrm{id}$, and hence γ_z is an automorphism. □

Kumjian and Pask [21] remark that 'By the universal property $C^*(\Lambda)$ carries a canonical action of $\mathbb{T}^k \ldots$ called the gauge action.' This action has carried through the subsequently defined Toeplitz algebra of a k-graph, as well as the twisted versions of each, since all of these algebras have the same universal property. The reason why universality is important is that we define the gauge action on the generators of the algebra, and let the algebra do the rest of the work.

To see why we care about the gauge action for our applications, we now observe that via the gauge action we are able to define an automorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ which applies to the systems of interest to us.

Definition 2.3.9 Let Λ be a finitely aligned k-graph, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Fix $r \in (0, \infty)^k$. Define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ by

$$\alpha_t := \gamma_{e^{itr}}. \tag{2.13}$$

2.4 Kubo-Martin-Schwinger equilibrium states

Kubo-Martin-Schwinger (KMS) states provide a natural framework for the equilibrium of a C^*-dynamical system. As stated by Bratteli and Robinson [8] in 1997, C^*-algebras can be used to represent observables in quantum systems, so there is an innate interest in their equilibrium states. They also note that in order to makes sense of 'equilibrium', we must consider the dynamic law which governs the change over time of the observables. This is where C^*-dynamical systems come into the picture, providing a framework for this dynamics.

Definition 2.4.1 A *C^*-dynamical system* (A, G, α) consists of a C^*-algebra A, a locally compact Hausdorff group G, and a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ which is continuous with respect to the strong topology on $\mathrm{Aut}(A)$.

The Kubo-Martin-Schwinger (KMS) equilibrium condition originated in two papers; one by Kubo [20] in 1957 and one by Martin and Schwinger [27] in 1959, both of which studied the analytic properties of Green's functions. In the context of C^*-dynamical systems, Bratteli and Robinson [8] motivate the definition of KMS states via the Gibbs equilibrium state. For $\beta \in \mathbb{R}$ and operators A, B on a Hilbert space $\mathcal{H}$, the Gibbs equilibrium state, ω, satisfies the condition

$$\omega(A\tau_t(B))|_{t=i\beta} = \omega(BA)$$

with respect to the automorphism group

$$\tau_t(A) = e^{itH}Ae^{-itH},$$

where H is a self-adjoint operator. It is this twisted commutation under ω that the KMS state captures. With this example in mind, we give the definition of a KMS state, but first we need to know about analytic elements in a C^*-algebra.

Definition 2.4.2 Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system. Then $a \in A$ is *analytic* for α if the map $t \mapsto \alpha_t(a)$ is the restriction to $\mathbb{R}$ of an analytic function $z \mapsto \alpha_z(a)$ that is defined on all of $\mathbb{C}$. We write A_α for the sets of analytic elements of A.

Definition 2.4.3 ([30], 8.12.2)
Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let $\beta \in (0, \infty)$. We say that a state ϕ

satisfies the (α, β)-*KMS condition* if for any $a \in A_\alpha$ and $b \in A$, we have

$$\phi(\alpha_\zeta(a)b) = \phi(b\alpha_{\zeta+i\beta}(a)) \tag{2.14}$$

for all $\zeta \in \mathbb{C}$.

Pedersen also gives the following important property of KMS states, which allows us to calculate them much more readily.

Proposition 2.4.4 ([30], 8.12.3)
Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let $\beta \in (0, \infty)$. Then a state ϕ on A satisfies the (α, β)-KMS condition if and only if ϕ satisfies (2.14) *for a dense set of $a \in A_\alpha$.*

The following proposition gives us an alternative characterisation of KMS states which is easier to work with.

Proposition 2.4.5 *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let $\beta \in (0, \infty)$. Then a state ϕ on A satisfies the (α, β)-KMS condition if and only if for all $a, b \in A_\alpha$*

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)). \tag{2.15}$$

Proof. To see that this is a necessary condition, consider the case $\zeta = 0$ in (2.14). To see that it is sufficient takes a little more work; first for each $x \in A$ define

$$x_n := \sqrt{\frac{n}{\pi}} \int \alpha_t(x) e^{-nt^2} dt.$$

Then $x_n \to x$ as $n \to \infty$, and x_n is analytic for α with

$$\alpha_\xi(x_n) = \sqrt{\frac{n}{\pi}} \int \alpha_t(x) e^{-n(t-\xi)^2} dt$$

for all $\xi \in \mathbb{C}$, since for each $s \in \mathbb{R}$,

$$\begin{aligned} \alpha_s(x_n) &= \sqrt{\frac{n}{\pi}} \int \alpha_{t+s}(x) e^{-nt^2} dt \\ &= \sqrt{\frac{n}{\pi}} \int \alpha_t(x) e^{-n(t-s)^2} dt. \end{aligned}$$

Similarly, $\alpha_\xi(x_n)$ is analytic for α, with

$$\alpha_\mu(\alpha_\xi(x_n)) = \sqrt{\frac{n}{\pi}} \int \alpha_t(x) e^{-n(t-\xi-\mu)^2} dt = \alpha_{\mu+\xi}(x_n)$$

for all $\mu \in \mathbb{C}$. Taking limits as n approaches ∞ gives

$$\alpha_\mu(\alpha_\xi(x)) = \alpha_{\mu+\xi}(x)$$

for all $x \in A_\alpha$. Now fix $a \in A_\alpha$, $b \in A$ and $\xi \in \mathbb{C}$. Then we have

$$\begin{aligned}\phi(\alpha_\xi(a)b_n) &= \phi(b_n\alpha_{i\beta}(\alpha_\xi(a))) \\ &= \phi(b_n\alpha_{\xi+i\beta}(a)),\end{aligned}$$

and hence as n approaches ∞ we obtain

$$\phi(\alpha_\xi(a)b) = \phi(b\alpha_{\xi+i\beta}(a)). \qquad \square$$

We naturally obtain a C^*-dynamical system for each Cuntz-Krieger algebra via a gauge action of the circle which may be lifted to an action of the real line. In 1984 Enomoto, Fujii and Watatani [12] showed that there is a unique KMS$_\beta$ state for simple Cuntz-Krieger algebras O_A, ocurring only for $\beta = \ln(\rho(A))$, where $\rho(A)$ is the spectral radius of A. Then in 2003 Exel and Laca [15] extended this result to Cuntz-Krieger algebras of infinite matrices, and in 2013 it was extended to the graph algebras of finite graphs with sources by an Huef, Laca, Raeburn and Sims [3] and Kajiwara and Watatani [19]. Due to Exel and Laca [15], Laca and Neshveyev [25], and an Huef, Laca, Raeburn and Sims [3] we know that the Toeplitz-Cuntz-Krieger algebras of graphs and matrices have a greater abundance of KMS states, encoding more information about the underlying object. An important difference in the case of higher-rank algebras is the choice of dynamics. In the simple case the gauge action of the circle is lifted to $\mathbb{R}$ and it doesn't matter how this is done. However, in the case of k-graphs the gauge action is via $\mathbb{T}^k$. We can still lift this to an action of $\mathbb{R}$ through an embedding of $\mathbb{R}$ in $\mathbb{T}^k$, but it matters which embedding we choose. As noted by an Huef, Laca, Raeburn and Sims [4], there is a "preferred dynamics", determined by the spectral radii of the connectivity matrices of the k-graph, for which there is a critical inverse temperature β_c. At this critical inverse temperature there is a unique KMS state that factors through the Cuntz-Krieger algebra. Under this preferred dynamics, the authors go on to give a full characterisation of the KMS states of the Toeplitz-Cuntz-Krieger algebras of finite higher-rank graphs with no sources as follows. Suppose that Λ is a finite k-graph with no sources. Let $r \in (0, \infty)^k$, and let $\gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ be the gauge action. Define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda))$ by $\alpha_t = \gamma_{e^{itr}}$. Suppose that $\beta \in (0, \infty)$ satisfies $\beta r_i > \ln(\rho(A_i))$ for $1 \leq i \leq k$. Then a state ϕ on $\mathcal{T}C^*(\Lambda)$ is a KMS$_\beta$ state for α if and only if

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} \phi(t_{s(\mu)})$$

for all $\mu, \nu \in \Lambda$.

Chapter 3

A groupoid model for the twisted Toeplitz algebra

With all of that bakground out of the way, we can work on understanding the KMS states of the twisted Toeplitz algebra of a higher-rank graph. We aim to employ the results of Neshveyev [29] and Afsar and Sims [1] for the KMS states of groupoid C^*-algebras, so in this chapter we develop a groupoid model for $\mathcal{T}C^*(\Lambda, c)$. There are already groupoid models for both the untwisted Toeplitz algebra and the twisted Cuntz-Krieger algebra of a higher-rank graph; that is, we have groupoid models for $\mathcal{T}C^*(\Lambda)$ in [2] and $C^*(\Lambda, c)$ in [24]. What we aim to do here is to use the underlying groupoid from [2] and the construction of a cocycle on the groupoid from [24] to obtain a groupoid model for $\mathcal{T}C^*(\Lambda, c)$.

3.1 The graph groupoid

Given a higher-rank graph Λ and a 2-cocycle $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, Kumjian, Pask and Sims [24] constructed a groupoid model for $C^*(\Lambda, c)$. Their groupoid model depended on the infinite path space of the given k-graph to form the unit space of the groupoid. We aim to extend their ideas to construct a groupoid model for $\mathcal{T}C^*(\Lambda, c)$ by using the full path space to form the unit space of the groupoid. Firstly the shift map on W_Λ differs from the shift map on Λ^∞ only in the sense that it reduces the degree of finite paths.

Definition 3.1.1 For $x \in W_\Lambda$ and $n \leq d(x)$, define $\sigma^n(x) \in \Lambda^{d(x)-n}$ by

$$\sigma^n(x)(p, q) = x(p + n, q + n). \tag{3.1}$$

This gives a partially defined *shift map* σ on the path space.

Our groupoid model is the Deaconu-Renault groupoid for the above partially defined local homeomorphisms. The following definition of such a groupoid comes from [2] and was used to construct a groupoid model for the Toeplitz algebra (in the untwisted case). In our notation below, if we write $\sigma^m x = \sigma^n y$, we are implicitly asserting that $m \leq d(x)$ and $n \leq d(y)$.

Definition 3.1.2 Given a k-graph Λ, we define the *k-graph groupoid* as follows,

$$\mathcal{G}_\Lambda := \{(x, m-n, y) \in W_\Lambda \times \mathbb{Z}^k \times W_\Lambda : m, n \in \mathbb{N}^k \text{ and } \sigma^m x = \sigma^n y\}, \tag{3.2}$$

with the operations defined by

$$\begin{aligned} r(x,p,y) &:= (x,0,x) \\ s(x,p,y) &:= (y,0,y) \\ (x,p,y)(y,q,z) &:= (x,p+q,z) \\ (x,p,y)^{-1} &:= (y,-p,x), \end{aligned}$$

and unit space $\mathcal{G}_\Lambda^{(0)} := \{(x,0,x) : x \in W_\Lambda\}$ which we identify with W_Λ in the obvious way.

We also require a refinement of the cylinder sets used to define the topology on the groupoid of Kumjian and Pask (see, for example, Yeend [41]).

Definition 3.1.3 For $(\mu,\nu) \in \Lambda *_s \Lambda$ and $F \subset \Lambda$ finite, we define

$$Z(\mu,\nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) : x \in W_\Lambda, r(x) = s(\mu)\}, \tag{3.3}$$

and

$$Z((\mu,\nu)\backslash F) := Z(\mu,\nu) \Big\backslash \left(\bigcup_{\tau \in F} Z(\mu\tau, \nu\tau) \right). \tag{3.4}$$

Yeend's work on groupoid models [41] offers us the following insights into the properties of the cylinder sets of $\mathcal{G}_\Lambda$.

Proposition 3.1.4 [41, Proposition 3.6]

Let Λ be a topological k-graph. The sets $Z((\mu,\nu)\backslash F)$ indexed by $(\mu,\nu) \in \Lambda *_s \Lambda$ and $F \subset \Lambda$ finite form a basis for a second countable Hausdorff topology on $\mathcal{G}_\Lambda$.

Proposition 3.1.5 [41, Proposition 3.15]

Let Λ be a compactly aligned topological k-graph. Let $(\mu,\nu) \in \Lambda *_s \Lambda$. Then $Z(\mu,\nu)$ is compact.

The following uses Yeend's results in order to see that the cylinder sets $Z((\mu,\nu)\backslash F)$ form a basis for a locally compact second-countable Hausdorff topology on $\mathcal{G}_\Lambda$ under which it is an étale topologial groupoid.

Theorem 3.1.6 *Let Λ be a topological k-graph. The sets of the form $Z((\mu,\nu)\backslash F)$ form a basis for a locally compact second-countable Hausdorff topology on $\mathcal{G}_\Lambda$ under which it is an étale topological groupoid.*

Proof. Since Λ is row-finite, it is finitely aligned. As compactness is equivalent to finiteness for discrete topologies, a discrete k-graph is compactly aligned if and only if it is finitely aligned. So, as a topological k-graph with the discrete topology, Λ is compactly aligned. By Proposition 3.1.4, the sets $Z((\mu,\nu)\backslash F)$ form a basis for a second-countable Hausdorff topology on $\mathcal{G}_\Lambda$. Since Λ is compactly aligned, we apply Proposition 3.1.5 to see that the topology is locally compact. To see that $\mathcal{G}_\Lambda$ is étale under this topology, we observe that for each pair $(\mu,\nu) \in \Lambda *_s \Lambda$, $r|_{Z(\mu,\nu)}$ is a homeomorphism. □

3.2 Defining a groupoid cocycle from a graph cocycle

As in [24] we wish to construct a partition of $\mathcal{G}_\Lambda$ which should allow us to define a cocycle on the groupoid from a given cocycle on the k-graph. In order to show that the elements $1_{Z(\lambda,s(\lambda))}$ are a TCK(Λ,c)-family in the twisted groupoid C^*-algebra of the groupoid cocycle we will obtain from the partition we require that each $Z(\lambda,s(\lambda))$ is in the partition. The following lemma, adapted from [24, Lemma 6.6], says that such a partition always exists.

Lemma 3.2.1 *Suppose that Λ is a row-finite k-graph with no sources. Let $\mathcal{F}$ be the collection of all finite subsets of Λ. Then there exists $P \subset \Lambda *_s \Lambda \times \mathcal{F}$ such that $((\lambda,s(\lambda)),\emptyset) \in P$ for all $\lambda \in \Lambda$, and $\{Z((\mu,\nu)\backslash F) : ((\mu,\nu),F) \in P\}$ is a partition of $\mathcal{G}_\Lambda$.*

Proof. As in Lemma 6.6 of [24], we see that $Z(\lambda, s(\lambda)) \cap Z(\mu, \nu)$ is nonempty if and only if $\mu = \lambda\nu$, in which case $Z(\mu, \nu) \subseteq Z(\lambda, s(\lambda))$. Hence the sets $\{Z(\lambda, s(\lambda)) : \lambda \in \Lambda\}$ are a partition of the set

$$X := \bigsqcup_{\lambda \in \Lambda} Z(\lambda, s(\lambda)).$$

We aim to show that this set X is closed so that $\mathcal{G}_\Lambda \backslash X$ is open and therefore a union of the basic open sets $Z((\mu, \nu) \backslash F)$. Let $(x_n, p_n, y_n)_{n=1}^{\infty}$ be a sequence of elements in X which converges in $\mathcal{G}_\Lambda$. Since (x_n, p_n, y_n) is convergent, and $(x, p, y) \mapsto p$ is continuous, there exists $p \in \mathbb{Z}^k$ such that $p_n = p$ for sufficiently large n. So without loss of generality we may assume that $p_n = p$ for all n. Since $(x_n, p, y_n) \in Z(\lambda_n, s(\Lambda_n))$ for some $\lambda_n \in \Lambda$, we observe that $p = d(\lambda_n)$, and $x_n = \lambda_n y_n$. So we have $(x_n, p_n, y_n) = (\lambda_n y_n, d(\lambda_n), y_n)$. Since this is convergent, say $(x_n, p_n, y_n) \to (x, p, y)$, we have $\lambda_n y_n \in Z(x(0, p))$ for large n, so $\lambda := x(0, p)$ satisfies $\lambda_n = \lambda$ for large n. Again we may assume that $\lambda_n = \lambda$ for all n. So we have $(x_n, p_n, y_n) = (\lambda y_n, d(\lambda), y_n) \in Z(\lambda, s(\lambda))$ for all n. Since $Z(\lambda, s(\lambda))$ is compact we deduce that $(x, p, y) \in Z(\lambda, s(\lambda)) \subseteq X$. Thus X is closed.

It follows that $\mathcal{G}_\Lambda \backslash X$ is open so there is a countable collection $\mathcal{U}$ of basic open sets of the form $Z((\mu, \nu) \backslash F)$ whose union is $\mathcal{G}_\Lambda \backslash X$. We can list $\mathcal{U}$ as $\mathcal{U} = \{Z(\mu_i, \nu_i) \backslash F_i : i \in \mathbb{N}\}$.

It remains to show that there is a collection $\mathcal{V}$ of pairwise disjoint sets of the form $Z((\mu, \nu) \backslash F)$ such that

$$\bigsqcup \mathcal{V} = \bigcup \mathcal{U}.$$

To do this we will show that $Z(\mu_1, \nu_1 \backslash F_1) \backslash Z(\mu_2, \nu_2 \backslash F_2)$ may be expressed as a disjoint union of basic open sets. First we note the following from [24]

$$Z(\mu_1, \nu_1) = \bigsqcup_{d(\mu_1 \alpha) = d(\mu_1) \vee d(\nu_1)} Z(\mu_1 \alpha, \nu_1 \alpha), \tag{3.5}$$

and

$$Z(\mu_1, \nu_1) \cap Z(\mu_2, \nu_2) = \bigsqcup_{(\mu_1 \alpha, \nu_1 \alpha) \in (\mu_1, \nu_1) \wedge (\mu_2, \nu_2)} Z(\mu_1 \alpha, \nu_1 \alpha), \tag{3.6}$$

where

$$(\mu_1, \nu_1) \wedge (\mu_2, \nu_2) := \{(\mu_1 \alpha, \nu_1 \alpha) : \mu_1 \alpha \in \mathrm{MCE}(\mu_1, \mu_2), \nu_1 \alpha \in \mathrm{MCE}(\nu_1, \nu_2)\}. \tag{3.7}$$

Hence, as also noted in [24],

$$Z(\mu_1,\nu_1)\backslash Z(\mu_2,\nu_2) = \bigsqcup_{d(\mu_1\alpha)=d(\mu_1)\vee d(\nu_1)} \{Z(\mu_1\alpha,\nu_1\alpha) : (\mu_1\alpha,\nu_1\alpha) \notin (\mu_1,\nu_1)\wedge(\mu_2,\nu_2)\}. \tag{3.8}$$

We define

$$F' := \{\tau(d(\alpha),d(\tau)) \in F : \tau(0,d(\alpha)) = \alpha\}. \tag{3.9}$$

We now apply the definition of $Z((\mu,\nu)\backslash F)$ and (3.5) to calculate

$$\begin{aligned}
Z((\mu,\nu)\backslash F) &= Z((\mu,\nu)\backslash F) \Big\backslash \left(\bigcup_{\tau\in F} Z(\mu\tau,\nu\tau)\right) \\
&= \left(\bigsqcup_{d(\mu\alpha)=d(\mu)\vee d(\nu)} Z(\mu\alpha,\nu\alpha)\right) \Big\backslash \left(\bigcup_{\tau\in F} Z(\mu\tau,\nu\tau)\right) \\
&= \bigsqcup_{d(\mu\alpha)=d(\mu)\vee d(\nu)} \left(Z(\mu\alpha,\nu\alpha)\Big\backslash \left(\bigcup_{\tau\in F} Z(\mu\tau,\nu\tau)\right)\right) \\
&= \bigsqcup_{d(\mu\alpha)=d(\mu)\vee d(\nu)} Z(\mu\alpha,\nu\alpha\backslash F').
\end{aligned} \tag{3.10}$$

Let

$$\begin{aligned}
A &:= \bigcup_{\tau\in F_1} Z(\mu_1\tau,\nu_1\tau), \\
B &:= \bigcup_{\gamma\in F_2} Z(\mu_2\gamma,\nu_2\gamma).
\end{aligned}$$

Then by the definition of $Z((\mu,\nu)\backslash F)$ and by (3.6) we calculate

$$\begin{aligned}
Z((\mu_1,\nu_1)\backslash F_1) \cap Z((\mu_2,\nu_2)\backslash F_2) &= (Z(\mu_1,\nu_1)\backslash A) \cap (Z(\mu_2,\nu_2)\backslash B) \\
&= (Z(\mu_1,\nu_1)\cap Z(\mu_2,\nu_2))\,\backslash(A\cup B) \\
&= \left(\bigsqcup_{(\mu_1\alpha,\nu_1\alpha)\in(\mu_1,\nu_1)\wedge(\mu_2,\nu_2)} Z(\mu_1\alpha,\nu_1\alpha)\right)\Big\backslash (A\cup B) \\
&= \bigsqcup_{(\mu_1\alpha,\nu_1\alpha)\in(\mu_1,\nu_1)\wedge(\mu_2,\nu_2)} Z((\mu_1\alpha,\nu_1\alpha)\backslash F_1'\cup F_2'),
\end{aligned} \tag{3.11}$$

where F_1', F_2' are defined analogously to F' in (3.9). Finally, we obtain

$$\begin{aligned} &Z((\mu_1,\nu_1)\backslash F_1)\backslash Z((\mu_2,\nu_2)\backslash F_2) \\ &\qquad = \bigsqcup_{d(\mu_1\alpha)=d(\mu_1)\vee d(\nu_1)} \{Z((\mu_1\alpha,\nu_1\alpha)\backslash F_1') : (\mu_1\alpha,\nu_1\alpha) \notin (\mu_1,\nu_1)\wedge(\mu_2,\nu_2)\}. \end{aligned} \tag{3.12}$$

Recall that we may write

$$\begin{aligned} Z((\mu_1,\nu_1)\backslash F_1)\cup Z((\mu_2,\nu_2)\backslash F_2) = [Z((\mu_1,\nu_1)\backslash F_1)\cap Z((\mu_2,\nu_2)\backslash F_2)] \sqcup [Z((\mu_1,\nu_1)\backslash F_1)\backslash Z((\mu_2,\nu_2)\backslash F_2)] \\ \sqcup [Z((\mu_2,\nu_2)\backslash F_2)\backslash Z((\mu_1,\nu_1)\backslash F_1)]\,. \end{aligned}$$

We can now construct a collection $\mathcal{V}$ of pairwise disjoint basic open sets such that

$$\bigsqcup \mathcal{V} = \bigcup \mathcal{U} = G_\Lambda\backslash X,$$

as follows. Let $\mathcal{U} = \{U_1, U_2, U_3, \dots\}$. We will form $\mathcal{V}$ inductively. Let $\mathcal{V}_1 = \{U_1\}$. Suppose for $i \geq 1$ that $\mathcal{V}_i$ is a collection of pairwise disjoint basic open sets such that

$$\bigcup_{j=1}^{i} U_j = \bigsqcup \mathcal{V}_i.$$

Then by (3.12) we may write

$$U_{i+1}\backslash \bigsqcup \mathcal{V}_i = \bigsqcup \mathcal{W}_{i+1},$$

where $\mathcal{W}_{i+1}$ is a finite collection of pairwise disjoint basic open sets. We let $\mathcal{V}_{i+1} := \mathcal{V}_i \sqcup \mathcal{W}_{i+1}$. Then we have

$$\bigcup_{j=1}^{i+1} U_j = \bigsqcup \mathcal{V}_{i+1}.$$

By induction we obtain the desired family $\mathcal{V}$ of pairwise disjoint basic open sets such that $\bigsqcup \mathcal{V} = \bigcup \mathcal{U}$. Hence

$$\{Z(\lambda, s(\lambda)) : \lambda \in \Lambda\} \sqcup \mathcal{V}$$

is the desired partition of $\mathcal{G}_\Lambda$. □

We now utilize the partition $\mathcal{V}$ just constructed to define a continuous cocycle on $\mathcal{G}_\Lambda$. Since $\mathcal{V}$ is a partition of $\mathcal{G}_\Lambda$ by sets of the form $Z((\mu,\nu)\backslash F)$, for each $g \in \mathcal{G}_\Lambda$ there is a unique element $((\mu_g, \nu_g), F_g)$ such that $g \in Z((\mu_g, \nu_g)\backslash F_g)$. This allows us to pick elements $\mu_g, \nu_g \in \Lambda$ to apply the cocycle c to in order to define the new cocycle on $\mathcal{G}_\Lambda$. This idea comes directly from [24].

Lemma 3.2.2 *Let Λ be a row-finite k-graph with no sources, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that P is a subset of $(\Lambda *_s \Lambda) \times \mathcal{F}$ such that $\{Z((\mu, \nu)\backslash F) : ((\mu, \nu), F) \in P\}$ is a partition of $\mathcal{G}_\Lambda$. For each $g \in \mathcal{G}_\Lambda$, let $((\mu_g, \nu_g), F_g)$ be the unique element of P such that $g \in Z((\mu_g, \nu_g)\backslash F_g)$.*

(i) For each $(g, h) \in \mathcal{G}_\Lambda^{(2)}$, there exists $\alpha \in s(\mu_g)\Lambda$, $\beta \in s(\mu_h)\Lambda$, $\gamma \in s(\mu_{gh})\Lambda$, and $y \in W_\Lambda$ such that

(a) $r(y) = s(\alpha) = s(\beta) = s(\gamma)$,

(b) $\mu_g\alpha = \mu_{gh}\gamma$, $\nu_h\beta = \nu_{gh}\gamma$, and $\nu_g\alpha = \mu_h\beta$, and

(c)

$$\begin{aligned} g &= (\mu_g\alpha y, d(\mu_g) - d(\nu_g), \nu_g\alpha y), \\ h &= (\mu_h\beta y, d(\mu_h) - d(\nu_h), \nu_h\beta y), \text{and} \\ gh &= (\mu_{gh}\gamma y, d(\mu_{gh}) - d(\nu_{gh}), \nu_{gh}\gamma y). \end{aligned} \tag{3.13}$$

(ii) Fix $(g, h) \in \mathcal{G}_\Lambda^{(2)}$ and α, β, γ satisfying (3.13). Then the following formula is independent of α, β, γ

$$c(\mu_g, \alpha)\overline{c(\nu_g, \alpha)}c(\mu_h, \beta)\overline{c(\nu_h, \beta)c(\mu_{gh}, \gamma)}c(\nu_{gh}, \gamma). \tag{3.14}$$

(iii) For $(g, h) \in \mathcal{G}_\Lambda^{(2)}$ define

$$\sigma_c(g, h) := c(\mu_g, \alpha)\overline{c(\nu_g, \alpha)}c(\mu_h, \beta)\overline{c(\nu_h, \beta)c(\mu_{gh}, \gamma)}c(\nu_{gh}, \gamma) \tag{3.15}$$

for any α, β, γ satisfying (3.13). Then σ_c is a continuous groupoid 2-cocycle.

Proof. See [24, Lemma 6.3]. □

The groupoid cocycle $\sigma_c \in \tilde{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$ constructed in Lemma 3.2.2 from $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and the partitioning set P is dependent on both c and P. However, we will now show that the cohomology class $[\sigma_c]$ is independent of the choice of P. This is an important point, as $C^*(\mathcal{G}_\Lambda, \sigma_c)$ depends only on the cohomology class of σ_c, and therefore our choice of partition doesn't matter. The following proof follows closely [24, Theorem 6.5].

Lemma 3.2.3 *Let Λ be a row-finite k-graph with no sources. Suppose P is a subset of $(\Lambda *_s \Lambda) \times \mathcal{F}$ such that $\{Z((\mu,\nu)\backslash F) : ((\mu,\nu),F) \in P\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c \in \underline{Z}^2(\Lambda,\mathbb{T})$ and let $\sigma_c \in \tilde{Z}^2(\mathcal{G}_\Lambda,\mathbb{T})$ be the continuous cocycle constructed from c and P as defined in (3.15). The cohomology class $[\sigma_c]$ is independent of the choice of P and only depends on the cohomology class of c. Further, the map $[c] \mapsto [\sigma_c]$ is a homomorphism $\underline{H}^2(\Lambda,\mathbb{T}) \to \tilde{H}^2(\mathcal{G}_\Lambda,\mathbb{T})$.*

Proof. To see that $[c] \mapsto [\sigma_c]$ is a homomorphism, it suffices to observe that $c \mapsto \sigma_c$ is a homomorphism (by definition of σ_c) that maps 2-coboundaries to 2-coboundaries. The details of this argument may be found in [24, Theorem 6.5]. It then remains to show that $[\sigma_c]$ is not dependent on P. This argument is slightly different to that given in [24] as we have $P \subset (\Lambda *_s \Lambda) \times \mathcal{F}$ rather than $P \subset \Lambda *_s \Lambda$.

Fix subsets P and Q of $\Lambda *_s \Lambda \times \mathcal{F}$ yielding partitions of $\mathcal{G}_\Lambda$. Recall that

$$Z((\mu,\nu)\backslash F) \cap Z((\sigma,\tau)\backslash H) = \bigsqcup_{(\eta,\zeta)\in(\mu,\nu)\wedge(\sigma,\tau)} Z((\eta,\zeta)\backslash F' \cup H').$$

Define

$$P \vee Q := \bigcup_{\substack{((\mu,\nu),F)\in P \\ ((\sigma,\tau),H)\in Q}} \{((\eta,\zeta), F' \cup H') : (\eta,\zeta) \in (\mu,\nu) \wedge (\sigma,\tau)\}, \tag{3.16}$$

where F', H' are defined as in (3.9). Then $\{Z((\eta,\zeta)\backslash F' \cup H') : ((\eta,\zeta), F' \cup H') \in P \vee Q\}$ is a common refinement of $\{Z((\mu,\nu)\backslash F) : ((\mu,\nu),F) \in P\}$ and $\{Z((\sigma,\tau)\backslash H) : ((\sigma,\tau),H) \in Q\}$. Additionally, if $g \in G_\Lambda$ satisfies $g \in Z((\mu,\nu)\backslash F)$ for $((\mu,\nu),F) \in P$ and $g \in Z(\eta,\zeta\backslash F' \cup H')$ for $((\eta,\zeta), F' \cup H') \in P \vee Q$, then $\eta = \mu\lambda$ and $\zeta = \nu\lambda$ for some $\lambda \in \Lambda$. So, by replacing Q with $P \vee Q$, we may assume that $\{Z((\eta,\zeta)\backslash E) : ((\eta,\zeta),H) \in Q\}$ is a refinement of $\{Z((\mu,\nu)\backslash F) : ((\mu,\nu),F) \in P\}$ and that for each element $((\eta,\zeta),E) \in Q$ there is a unique element $((\mu,\nu),F) \in P$ and a unique $\lambda \in \Lambda$ such that $\eta = \mu\lambda$ and $\zeta = \nu\lambda$. For $g \in \mathcal{G}_\Lambda$, let $((\mu_g,\nu_g),F_g) \in P$ and $((\eta_g,\zeta_g),E_g)$ be the unique elements such that

$$g \in Z((\eta_g,\zeta_g)\backslash E_g) \subseteq Z((\mu_g,\nu_g)\backslash F_g),$$

and let λ_g be the unique path such that

$$(\eta_g,\zeta_g) = (\mu_g\lambda_g, \nu_g\lambda_g).$$

Fix $(g, h) \in \mathcal{G}_\Lambda^{(2)}$. By Lemma 3.2.2(i) we may fix α', β', γ', and y satisfying

$$\begin{aligned} g &= (\eta_g \alpha' y, d(\eta_g) - d(\zeta_g), \zeta_g \alpha' y), \\ h &= (\eta_h \beta' y, d(\eta_h) - d(\eta_h), \zeta_h \beta' y), \text{ and} \\ gh &= (\eta_{gh} \gamma' y, d(\eta_{gh}) - d(\zeta_{gh}), \zeta_{gh} \gamma' y). \end{aligned}$$

Then $\alpha = \lambda_g \alpha'$, $\beta = \lambda_h \beta'$, $\gamma = \lambda_{gh} \gamma'$ also satisfies (3.13).

Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let σ_c^P be the groupoid 2-cocycle constructed from P and c, and similarly for σ_c^Q. By the definition of σ_c^P and σ_c^Q (3.15), we have

$$\begin{aligned} \sigma_c^P(g, h) &= c(\mu_g, \alpha)\overline{c(\nu_g, \alpha)} c(\mu_h, \beta)\overline{c(\nu_h, \beta) c(\mu_{gh}, \gamma)} c(\nu_{gh}, \gamma) \\ &= c(\mu_g, \lambda_g \alpha')\overline{c(\nu_g, \lambda_g \alpha')} c(\mu_h, \lambda_h \beta')\overline{c(\nu_h, \lambda_h \beta') c(\mu_{gh}, \lambda_{gh} \gamma')} c(\nu_{gh}, \lambda_{gh} \gamma') \end{aligned}$$

and

$$\begin{aligned} \sigma_c^Q(g, h) &= c(\eta_g, \alpha')\overline{c(\zeta_g, \alpha')} c(\eta_h, \beta')\overline{c(\zeta_h, \beta') c(\eta_{gh}, \gamma')} c(\zeta_{gh}, \gamma') \\ &= c(\mu_g \lambda_g, \alpha')\overline{c(\nu_g \lambda_g, \alpha')} c(\mu_h \lambda_h, \beta')\overline{c(\nu_h \lambda_h, \beta') c(\mu_{gh} \lambda_{gh}, \gamma')} c(\nu_{gh} \lambda_{gh}, \gamma'). \end{aligned}$$

Define $b : \mathcal{G}_\Lambda \to \mathbb{T}$ by $b(g) = c(\mu_g, \lambda_g)\overline{c(\nu_g, \lambda_g)}$. Since b is locally constant it is continuous. If $g \in \mathcal{G}_\Lambda^{(0)}$ then $\mu_g = \nu_g$, and so $b(g) = 1$. So $b \in \tilde{C}^1(\mathcal{G}_\Lambda, \mathbb{T})$. By the 2-cocycle identity we have

$$c(\mu_g, \lambda_g \alpha')\overline{c(\nu_g, \lambda_g \alpha') c(\mu_g \lambda_g, \alpha')} c(\nu_g \lambda_g, \alpha') = c(\mu_g, \lambda_g)\overline{c(\lambda_g, \alpha') c(\nu_g, \lambda_g)} c(\lambda_g, \alpha') = b(g).$$

By symmetric calculations, we obtain

$$\sigma_c^P(g, h)\overline{\sigma_c^Q(g, h)} = b(g) b(h) \overline{b(gh)} = (\tilde{\delta}^1 b)(g, h).$$

Hence σ_c^P and σ_c^Q are cohomologous. □

3.3 Groupoid model isomorphism

In the preceding two sections we showed how to construct a groupoid and a continuous groupoid cocycle from a higher-rank graph endowed wth a categorical cocycle. In this section we show that the resulting twisted groupoid C^*-algebra is isomorphic to the twisted Toeplitz algebra of the k-graph. The following result follows Kumjian, Pask and Sims [24].

Theorem 3.3.1 *Let Λ be a row-finite k-graph with no sources. Let P be a subset of $(\Lambda *_s \Lambda) \times \mathcal{F}$ containing $\{((\lambda, s(\lambda)), \emptyset) : \lambda \in \Lambda\}$ such that $\{Z((\mu,\nu)\backslash F) : ((\mu,\nu), F) \in P\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and let $\sigma_c \in \tilde{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$ be the associated groupoid cocycle constructed from c and P. Then there is a *-isomorphism $\pi : \mathcal{T}C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \sigma_c)$ such that $\pi(t_\lambda) = 1_{Z(\lambda, s(\lambda))}$ for all $\lambda \in \Lambda$.*

Proof. By the universal property of $\mathcal{T}C^*(\Lambda, c)$ we may construct such a *-homomorphism by showing that $\{T_\lambda := 1_{Z(\lambda,s(\lambda))} : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger (Λ, c)-family in $C^*(\mathcal{G}_\Lambda, \sigma_c)$. So we check (TCK1)–(TCK4). For (TCK1) we first check that each T_v is a projection. Fix $v \in \Lambda^0$ and $g \in \mathcal{G}_\Lambda$. Then

$$T_v(g) = 1_{Z(v,v)}(g) = \begin{cases} 1, & g \in Z(v,v) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$T_v^*(g) = \overline{\sigma_c(g^{-1}, g) 1_{Z(v,v)}(g^{-1})} = \begin{cases} \overline{\sigma_c(g^{-1}, g)}, & g^{-1} \in Z(v,v) \\ 0, & \text{otherwise.} \end{cases}$$

We note that $g^{-1} \in Z(v,v)$ if and only if $g^{-1} = (x, 0, x)$ for some $x \in W_\Lambda$ such that $r(x) = v$. So we see that $g = (x, 0, x) \in Z(v,v)$. We also note that $g \in \mathcal{G}_\Lambda^{(0)}$, so $\sigma_c(g^{-1}, g) = 1$. Hence

$$\begin{aligned} T_v^*(g) &= \begin{cases} 1, & g \in Z(v,v) \\ 0, & \text{otherwise,} \end{cases} \\ &= T_v(g). \end{aligned}$$

Also,

$$T_v^2(g) = \sum_{h \in \mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g) 1_{Z(v,v)}(h) 1_{Z(v,v)}(h^{-1}g).$$

The only non-zero summands are those for which $h = (x, 0, x)$ for some $x \in W_\Lambda$ such that $r(x) = v$, and such that $h^{-1}g = (y, 0, y)$ for some $y \in W_\lambda$ such that $r(y) = v$. We obtain that $g = (x, 0, x)(y, 0, y)$, which is only defined if $x = y$. So there is only a non-zero term when $g = (x, 0, x) \in Z(v, v)$, given by $h = (x, 0, x)$. Hence

$$\begin{aligned} T_v^2(g) &= \begin{cases} \sigma_c\left((x,0,x),(x,0,x)\right), & g \in Z(v,v) \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & g \in Z(v,v) \\ 0, & \text{otherwise,} \end{cases} \\ &= T_v(g). \end{aligned}$$

So we have $T_v = T_v^* = T_v^2$ for each $v \in \Lambda^0$. Now to see that the T_v are mutually orthogonal, we fix $v \neq w \in \Lambda^0$ and $g \in \mathcal{G}_\Lambda$. We have

$$(T_v T_w)(g) = \sum_{h \in \mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g) 1_{Z(v,v)}(h) 1_{Z(w,w)}(h^{-1}g). \tag{3.17}$$

For non-zero summands we require that $h = (x, 0, x)$ for some $x \in W_\Lambda$ such that $r(x) = v$, and that $h^{-1}g = (y, 0, y)$ for some $y \in W_\Lambda$ such that $r(y) = w$. Then $g = (x, 0, x)(y, 0, y)$ is only defined if $x = y$. However $r(x) = v \neq w = r(y)$ implies that $x \neq y$. Hence there are no non-zero terms in (3.17), so $T_v T_w = 0$, and this establishes (TCK1). For (TCK2) we fix $\mu, \nu \in \Lambda$ such that $s(\mu) = r(\nu)$, and fix $g \in \mathcal{G}_\Lambda$. Then we have

$$(T_\mu T_\nu)(g) = \sum_{h \in \mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g) 1_{Z(\mu,s(\mu))}(h) 1_{Z(\nu,s(\nu))}(h^{-1}g).$$

For non-zero summands we require that $h = (\mu x, d(\mu), x)$ for some $x \in W_\Lambda$ such that $r(x) = s(\mu)$, and that $h^{-1}g = (\nu y, d(\nu), y)$ for some $y \in W_\Lambda$ such that $r(y) = s(\nu)$. Then $g = (\mu x, d(\mu), x)(\nu x, d(\nu), y)$ which is only defined if $x = \nu y$, in which case we obtain $g = (\mu\nu y, d(\mu\nu), y)$, and the non-zero term in the sum is given by $h = (\mu\nu y, d(\mu), \nu y)$. Hence

$$(T_\mu T_\nu)(g) = \begin{cases} \sigma_c\left((\mu\nu y, d(\mu), \nu y), (\nu y, d(\nu), y)\right), & g \in Z(\mu\nu, s(\mu\nu)) \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Let $a = (\mu\nu y, d(\mu), \nu y)$, $b = (\nu y, d(\nu), y)$; then $ab = (\mu\nu y, d(\mu\nu), y)$. Recall that $((\mu_a, \nu_a), F_a)$ is the unique element of the partitioning set P such that $a \in Z(\mu_a, \nu_a \backslash F_a)$. We observe that $a \in Z(\mu, s(\mu))$, $b \in Z(\nu, s(\nu))$ and $ab \in Z(\mu\nu, s(\mu\nu))$. So, since $((\lambda, s(\lambda)), \emptyset) \in P$ for each $\lambda \in \Lambda$, we have $\mu_a = \mu$, $\nu_a = s(\mu)$, $\mu_b = \nu$, $\nu_b = s(\nu)$, $\mu_{ab} = \mu\nu$ and $\nu_{ab} = s(\mu\nu)$. We also observe that $\alpha = \nu$, $\beta = \gamma = s(\nu)$ satisfy (3.13). Hence by (3.15) we obtain $\sigma_c(a, b) = c(\mu, \nu)$. Substituting back into (3.18) we obtain

$$\begin{aligned}(T_\mu T_\nu)(g) &= \begin{cases} c(\mu,\nu), & g \in Z(\mu\nu, s(\mu\nu)) \\ 0, & \text{otherwise.} \end{cases} \\ &= c(\mu,\nu) 1_{Z(\mu\nu, s(\mu\nu))}(g) \\ &= c(\mu,\nu) T_{\mu\nu}(g),\end{aligned}$$

which establishes (TCK2). For (TCK3), we fix $\lambda \in \Lambda$ and $g \in \mathcal{G}_\Lambda$. Then

$$\begin{aligned}(T_\lambda^* T_\lambda)(g) &= \sum_{h \in \mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g) 1^*_{Z(\lambda, s(\lambda))}(h) 1_{Z(\lambda, s(\lambda))}(h^{-1}g) \\ &= \sum_{h \in \mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g) \overline{\sigma_c(h^{-1}, h)} 1_{Z(\lambda, s(\lambda))}(h^{-1}) 1_{Z(\lambda, s(\lambda))}(h^{-1}g).\end{aligned}$$

For non-zero summands we require that $h^{-1} = (\lambda x, d(\lambda), x)$ for some $x \in W_\Lambda$ such that $r(x) = s(\lambda)$, and that $h^{-1}g = (\lambda y, d(\lambda), y)$ for some $y \in W_\Lambda$ such that $r(y) = s(\lambda)$. Then $g = (x, -d(\lambda), \lambda x)(\lambda y, d(\lambda), y)$ is only defined if $x = y$, in which case $g = (x, 0, x)$. Hence there is only one non-zero summand, and it occurs when $h = (x, -d(\lambda), \lambda x)$. Let $a' = (x, -d(\lambda), \lambda x)$, $b' = (\lambda x, d(\lambda), x)$. Then we have

$$(T_\lambda^* T_\lambda)(g) = \begin{cases} \sigma_c(a', b') \overline{\sigma_c(a'^{-1}, a')}, & g \in Z(s(\lambda), s(\lambda)) \\ 0, & \text{otherwise.} \end{cases}$$

Again by (3.13) we have $\mu_{a'} = s(\lambda)$, $\nu_{a'} = \lambda$, $\mu_{b'} = \lambda$, $\nu_{b'} = s(\lambda)$, $\mu_{a'b'} = \nu_{a'b'} = s(\lambda)$, and $\alpha' = \beta' = \gamma' = s(\lambda)$. Hence (3.15) gives $\sigma_c(a', b') = 1$. Similarly, $\sigma(a'^{-1}, a') = 1$. Hence

$$\begin{aligned}(T_\lambda^* T_\lambda)(g) &= \begin{cases} 1, & g \in Z(s(\lambda), s(\lambda)) \\ 0, & \text{otherwise.} \end{cases} \\ &= T_{s(\lambda)}(g).\end{aligned}$$

We note that, by [39, Lemma 3.2], the following statement is equivalent to (TCK4); for each $\mu, \nu \in \Lambda$ and $g \in \mathcal{G}_\Lambda$

$$(T_\mu^* T_\nu)(g) = \sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} \overline{c(\mu,\alpha)}c(\nu,\beta)(T_\alpha T_\beta^*)(g). \tag{3.19}$$

So, for (TCK4), we fix $\mu, \nu \in \Lambda$ and $g \in \mathcal{G}_\Lambda$. We will first calculate $T_\alpha T_\beta^*$, so fix $\alpha, \beta \in \Lambda$ such that $\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)$. Then

$$(T_\alpha T_\beta^*)(g) = \sum_{h\in\mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g)\overline{\sigma_c(g^{-1}h, h^{-1}g)} 1_{Z(\alpha,s(\alpha))}(h) 1_{Z(\beta,s(\beta))}(g^{-1}h).$$

For non-zero summands we require that $h = (\alpha x, d(\alpha), x)$ for some $x \in W_\Lambda$ such that $r(x) = s(\alpha) = s(\beta)$, and that $g^{-1}h = (\beta y, d(\beta), y)$ for some $y \in W_\Lambda$ such that $r(y) = s(\beta) = s(\alpha)$. So $g^{-1} = (\beta y, d(\beta), y)(x, -d(\alpha), \alpha x)$ is only defined if $x = y$, in which case $g = (\alpha x, d(\alpha) - d(\beta), \beta x)$, and there is one non-zero summand when $h = (\alpha x, d(\alpha), x)$. Again by (3.13) and (3.14), we obtain

$$(T_\alpha T_\beta^*)(g) = \begin{cases} 1, & g \in Z(\alpha,\beta), \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

Now we compute

$$(T_\mu^* T_\nu)(g) = \sum_{h\in\mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g)\overline{\sigma_c(h^{-1}, h)} 1_{Z(\mu,s(\mu))}(h^{-1}) 1_{Z(\nu,s(\nu))}(h^{-1}g).$$

For non-zero summands we require that $h^{-1} = (\mu x, d(\mu), x)$ for some $x \in W_\Lambda$ such that $r(x) = s(\mu)$, and that $h^{-1}g = (\nu y, d(\nu), y)$ for some $y \in W_\Lambda$ such that $r(y) = s(\nu)$. So $g = (x, -d(\mu), \mu x)(\nu y, d(\nu), y)$ is only defined if $\mu x = \nu y$. So for each $\alpha, \beta \in \Lambda$ such that $\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu,\nu)$, we have $x(d(\alpha), d(x)) = y(d(\beta), d(y))$. Defining $z := x(d(\alpha), d(x)) = y(d(\beta), d(y))$, we then have $\alpha z = x$, $\beta z = y$. So for each such α, β we obtain a non-zero term given by $g = (\alpha z, d(\alpha) - d(\beta), \beta z)$ and $h = (\alpha z, -d(\mu), \mu\alpha z)$. Writing $a'' = (\alpha z, -d(\mu), \mu\alpha z)$ and $b'' = (\nu\beta z, d(\nu), \beta z)$, we see that

$$(T_\mu^* T_\nu)(g) = \sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} \sigma_c(a'', b'')\overline{\sigma_c(a''^{-1}, a'')}(T_\alpha T_\beta^*)(g).$$

By (3.13) we have $\mu_{a''} = s(\mu)$, $\nu_{a''} = \mu$, $\mu_{b''} = \nu$, $\nu_{b''} = s(\nu)$, $\mu_{a''b''} = \alpha$, $\nu_{a''b''} = \beta$, and $\alpha'' = \alpha$, $\beta'' = \beta$, $\gamma'' = s(\alpha) = s(\beta)$. So by (3.15) we obtain $\sigma_c(a'', b'') = \overline{c(\mu, \alpha)}c(\nu, \beta)$. Similarly, $\sigma_c(a''^{-1}, a'') = 1$. Hence

$$(T_\mu^* T_\nu)(g) = \sum_{\mu\alpha=\nu\beta\in \mathrm{MCE}(\mu,\nu)} \overline{c(\mu, \alpha)}c(\nu, \beta)(T_\alpha T_\beta^*)(g),$$

as required in (3.19), establishing (TCK4).

For surjectivity we aim to see that the range of π is dense in $C^*(\mathcal{G}_\Lambda, \sigma_c)$. By (3.20) we have that $T_\tau T_\tau^* = 1_{Z(\tau,\tau)}$. For a finite subset $G \subset s(\lambda)\Lambda\backslash\{s(\lambda)\}$, we calculate

$$\prod_{\tau\in G} \left(T_{s(\lambda)} - T_\tau T_\tau^*\right)$$

by first observing that, for all such $\lambda, \tau \in \Lambda$ and all $g \in \mathcal{G}_\Lambda$ we have

$$\begin{aligned}\left(T_{s(\lambda)} - T_\tau T_\tau^*\right)(g) &= \left(1_{Z(s(\lambda),s(\lambda))} - 1_{Z(\tau,\tau)}\right)(g) \\ &= \begin{cases} 1, & g \in Z((s(\lambda), s(\lambda))\backslash\{\tau\}), \\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

noting that we cannot obtain -1 since $r(\tau) = s(\lambda)$. Now, let $G_1, G_2 \subset G \subset \mathcal{G}_\Lambda$, where G is finite, and fix $g \in \mathcal{G}_\Lambda$. Then we calculate

$$\left(1_{Z(s(\lambda),s(\lambda)\backslash G_1)} * 1_{Z(s(\lambda),s(\lambda)\backslash G_2)}\right)(g) = \sum_{h\in\mathcal{G}_\Lambda^{r(g)}} \sigma_c(h, h^{-1}g)1_{Z(s(\lambda),s(\lambda)\backslash G_1)}(h)1_{Z(s(\lambda),s(\lambda)\backslash G_2)}(h^{-1}g).$$

For non-zero summands we require that $h = (x, 0, x)$ for some $x \in W_\Lambda$ such that $r(x) = s(\lambda)$ and $x(0, d(\tau)) \neq \tau$ for all $\tau \in G_1$, and similarly that $h^{-1}g = (y, 0, y)$ for some $y \in W_\Lambda$ such that $r(y) = s(\lambda)$ and $y(0, d(\tau)) \neq \tau$ for all $\tau \in G_2$. So we see that $g = (x, 0, x)(y, 0, y)$ is only defined if $x = y$. Hence $g = (x, 0, x)$ where $r(x) = s(\lambda)$ and $x(0, d(\tau)) \neq \tau$ for all $\tau \in G_1 \cup G_2$. We also observe that the cocycle in the non-zero summand will be zero as each argument will be in the unit space. So we obtain

$$\left(1_{Z(s(\lambda),s(\lambda)\backslash G_1)} * 1_{Z(s(\lambda),s(\lambda)\backslash G_2)}\right) = 1_{Z(s(\lambda),s(\lambda)\backslash G_1\cup G_2)}.$$

Therefore by associativity of the product we obtain

$$\prod_{\tau \in G} \left(T_{s(\lambda)} - T_\tau T_\tau^*\right) = \prod_{\tau \in G} \left(1_{Z(s(\lambda), s(\lambda) \setminus \{\tau\})}\right) = 1_{Z(s(\lambda), s(\lambda) \setminus G)}. \tag{3.21}$$

It is then straightforward to see that, for $\lambda, \eta \in \Lambda *_s \Lambda$, we have

$$T_\lambda \left(\prod_{\tau \in G} \left[T_{s(\lambda)} - T_\tau T_\tau^*\right] \right) T_\eta^* = 1_{Z(\lambda, \eta \setminus G)}.$$

As in [21, Corollary 3.5] and [24, Theorem 6.7], surjectivity now follows from the observation that $C^*(\mathcal{G}_\Lambda, \sigma_c) = \overline{\operatorname{span}}\{1_{Z(\lambda, \eta \setminus G)}\}$.

For injectivity we appeal to an analogue of Coburn's theorem. By Lemma 3.15 of [39], π is injective if and only if $T_v \neq 0$ for all $v \in \Lambda^0$ and

$$\Delta(T)^E := \prod_{\lambda \in E} \left(T_{r(E)} - T_\lambda T_\lambda^*\right) \neq 0$$

for all $E \in \mathrm{FE}(\Lambda)$. Fix $v \in \Lambda^0$. Then $T_v = 1_{Z(v,v)} \neq 0$ since $Z(v, v) \neq \emptyset$. Fix $E \in \mathrm{FE}(\Lambda)$. Since E is finite, let $E = \{\mu_1, \ldots, \mu_n\}$. By (3.21) $\Delta(T)^E = 1_{Z(r(E), r(E) \setminus E)}$. We therefore need to check that $Z(r(E), r(E) \setminus E)$ is not empty. Let $\lambda \in r(E)\Lambda$ such that $d(\lambda) < d(\mu_i)$ for each $i = 1, \ldots, n$. Let x_λ be the canonical k-graph morhpism associated to λ. Then $(x_\lambda, 0, x_\lambda) \in Z(r(E), r(E) \setminus E)$. Hence $Z(r(E), r(E), \setminus E)$ is not empty, so $\Delta(T)^E \neq 0$. Hence π is injective. □

We finish the section with an illustrative example that demonstrates that even for simple k-graphs, the construction above can yield a fairly complication groupoid cocycle.

Example 3.3.2 Fix $k \geq 1$. Consider the k-graph $\Lambda = T_k$. Fix $\theta \in M_k([0, 1])$, and define a 2-cocycle $c_\theta : T_k \times T_k \to \mathbb{T}$ by

$$c_\theta(m, n) = e^{2\pi i m^\top \theta n}.$$

We recall that each $x \in W_\Lambda$ must satisfy $d(x(p, q)) = d(p, q) = q - p$. Since each Λ^{q-p} contains a unique element, namely $q - p \in \Lambda^{q-p}$, for each fixed $m \in (\mathbb{N} \cup \{\infty\})^k$ we obtain a unique map $x \colon \Omega_{k,m} \to T_k$ given by $x(p, q) = q - p$. Hence we identify W_Λ

with $(\mathbb{N} \cup \{\infty\})^k$. So we see that the graph groupoid is given by

$$\mathcal{G}_\Lambda = \{(x, m-n, y) : x, y \in (\mathbb{N} \cup \{\infty\})^k,\ m, n \in \mathbb{N}^k,\ x+n = y+m\}.$$

Now fix $g = (n+x, n-p, p+x) \in \mathcal{G}_\Lambda$. Define $n' := n - (n \wedge p)$ and $p' := p - (n \wedge p)$. Then for $y = (x + (n \wedge p)) \in W_\Lambda$, we have $g = (n' + y, n' - p', p' + y)$. Additionally we have $(n' - p')^+ = n'$ and $(n' - p')^- = p'$. Setting $m = n - p$ we therefore have $g \in Z(m^+, m^-)$. So for all $(x, m, y) \in \mathcal{G}_\Lambda$, we observe that $(x, m, y) \in Z(m^+, m^-)$. If $m \in \mathbb{N}^k$, then $m^+ = m$ and $m^- = 0$, so each $Z(\lambda, s(\lambda)) = Z(d(\lambda)^+, d(\lambda)^-)$. So, if the sets $Z(m^+, m^-)$ are mutually disjoint, then $P = \{(m^+, m^-, \emptyset), m \in \mathbb{Z}^k\}$ is a partition of $\mathcal{G}_\Lambda$ of the sort described in Lemma 3.2.1. To see that these sets are mutually disjoint, fix $m \in \mathbb{Z}^k$, $a \in \mathbb{N}^k$, and $x \geq a \in W_\Lambda$. Let $g = (m^+ + x, m, m^- + x)$. Since $x \geq a$, $y = x - a \in W_\Lambda$. Hence we can write $g = (m^+ + a + y, m, m^- + a + y)$, and observe that $g \in Z(m^+ + a, m^- + a)$. However if $(m^+ + a) \wedge (m^- + a) = 0$ (that is, if $(m^+ + a, m^- + a)$ is of the form (n^+, n^-) for some $n \in \mathbb{Z}$), then $a = 0$. Hence the sets $Z(m^+, m^-)$ are disjoint.

By Lemma 3.2.2, the cocycle c and the partition P determine a groupoid cocycle on $\mathcal{G}_\Lambda$ given by the formula (3.14) for any α, β, γ satisfying the conditions (3.13). Let $g = (x, m, y)$ and $h = (y, n, z)$ be an arbitrary composable pair in $\mathcal{G}_\Lambda$. Then $\mu_g = m^+$, $\nu_g = m^-$, $\mu_h = n^+$, $\nu_h = n^-$, $\mu_{gh} = (m+n)^+$, and $\nu_{gh} = (m+n)^-$ due to the structure of the partitioning set P. So we need to find α, β, γ that satisfy (3.13). Let $p = (m^+ + n^+) \wedge (m^- + n^-)$ (the coordinate-wise minimum), and note that $(m+n)^+ = m^+ + n^+ - p$, and $(m+n)^- = m^- + n^- - p$. Since the cocycle formula is independent of the choice of α, β, γ, we may choose $\gamma = p$. This forces $\alpha = n^+$ and $\beta = m^-$. After some rearranging, we obtain

$$\sigma_c(g, h) = c_\theta(m, n^+) c_\theta(n, m^-) \overline{c_\theta(m+n, p)}.$$

We check that on restriction to

$$\mathcal{G}_{\Lambda^\infty} = \{(x, m, y) : m \in \mathbb{Z}^2, x = y = (\infty, \infty)\} \simeq \mathbb{Z}^2,$$

this formula gives a cocycle cohomologous to the cocycle c_θ given by $c_\theta(m, n) :=$

$e^{2\pi i m_2 n_1 \theta}$. Let $\omega = \sigma_c|_{\mathcal{G}_{\Lambda^\infty}}$. Then, for any $m, n \in \mathbb{N}$, we calculate

$$\begin{aligned}
\omega^*\omega(m,n) &= \overline{\omega(n,m)}\omega(m,n) \\
&= \left[\overline{c_\theta(n,m^+)c_\theta(m,n^-)}c_\theta(m+n,p)\right]\left[c_\theta(m,n^+)c_\theta(n,m^-)\overline{c_\theta(m+n,p)}\right] \\
&= \overline{c_\theta(n,m)}c_\theta(m,n) \\
&= c_\theta^* c_\theta(m,n).
\end{aligned}$$

Since $\omega^*\omega$ and $c_\theta^* c_\theta$ are bicharacters, we deduce that they are equal. Therefore ω and c_θ are cohomologous. Note, however, that the cocycle σ_c is not, perhaps, the most natural representative of its cohomology class. The formula $\sigma((p, p-q, q), (a, a-b, b)) = c_\theta(p-q, a-b)$ also defines a cocycle on $\mathcal{G}_\Lambda$ whose restriction to $\mathcal{G}_{\Lambda^\infty}$ is c_θ, and an easy argument using [39, Lemma 3.15] shows that $C^*(\mathcal{G}_\Lambda, \sigma) \cong C^*(\Lambda, c)$ via an isomorphism such that $1_{Z(\lambda, s(\lambda))} \mapsto s_\lambda$. So we might have expected to obtain $\sigma_c = \sigma$ from our construction.

Chapter 4

A measure on the path space of a higher-rank graph

Now that we are equipped with a groupoid model for the twisted Toeplitz algebra of a higher-rank graph, we are able to follow Afsar and Sims [1] in studying measures on the unit space of the groupoid in order to understand the KMS states of the algebra. The main results of Neshveyev [29] and Afsar and Sims [1] tell us to look for measures that satisfy an invariance relation dictated by the cocycle on the groupoid that determines the dynamics on its C^*-algebra. To ensure these properties we follow an Huef, Laca, Raeburn and Sims [5] in studying inverse systems of outer measures in order to define such a measure. The first problem we encounter is in extending their construction of Λ^∞ as the inverse limit of the sets Λ^m, because in our situation, Λ^{m+n} need not surject onto Λ^m. Instead, we view W_Λ as a subset of the cartesian product $\prod_{n=0}^{\infty} \tilde{\Lambda}^n$ where each $\tilde{\Lambda}^n = \Lambda^n \cup \{0\}$. With this achieved we go on to define a measure on W_Λ and prove that it has the desired properties.

4.1 Radon-Nykodym cocycles

One of the aforementioned important properties of the measure on W_Λ that will lead to KMS states on $\mathcal{T}C^*(\Lambda, c)$ is that it has Radon-Nykodym cocycle equal to the exponentiation of the cocycle that defines the dynamics. This will be important when we are calculating KMS states. Here we give a brief definition of what a Radon-Nykodym cocycle is and how to calculate the Radon-Nykodym cocycle of a measure. First we recall the Radon-Nykodym theorem and the definition of Radon-Nykodym derivatives.

Theorem 4.1.1 *Let λ, μ be measures on a measurable space $(\Omega, \mathcal{F})$ such that λ is* absolutely continuous with respect to μ *(that is, $\mu(A) = 0$ implies $\lambda(A) = 0$ for all measurable A). There exists a Borel function f on Ω such that for all $A \in \mathcal{F}$*

$$\lambda(A) = \int_A f d\mu. \tag{4.1}$$

The function f is unique in the sense that if g satisfies (4.1) then $f = g$ a.e with respect to μ. Since the function f is uniquely determined, we define f to be the Radon-Nykodym derivative of λ with respect to μ. *We write this as $f = \frac{d\lambda}{d\mu}$.*

The following definition comes from Afsar and Sims [1].

Definition 4.1.2 Let $\mathcal{G}$ be a groupoid. A measure on $\mathcal{G}^{(0)}$ is called *quasi-invariant* if the measures ν, ν^{-1} on $\mathcal{G}$ defined by

$$\nu(f) := \int_{\mathcal{G}(0)} \sum_{\gamma \in \mathcal{G}^x} f(\gamma) d\mu, \text{ and}$$
$$\nu^{-1}(f) := \int_{\mathcal{G}(0)} \sum_{\gamma \in \mathcal{G}_x} f(\gamma) d\mu,$$

are equivalent. We write $\Delta_\mu = \frac{d\nu}{d\nu^{-1}}$ for the Radon-Nikodym derivative of ν with respect to ν^{-1}, and call Δ_μ the *Radon-Nikodym cocycle* of μ. To see that a measure μ is quasi-invariant with Radon-Nikodym cocycle Δ_μ it suffices to show that

$$\int_{r(U)} f(T_U(x)) d\mu(x) = \int_{s(U)} \Delta_\mu(x) f(x) d\mu(x)$$

for all bisections U and all $f : s(U) \to \mathbb{R}$, where $T_U : r(U) \to s(U)$ is defined by $T(x) = s(U^x)$.

4.2 Inverse systems of outer measures

We will use inverse systems of outer measures in order to define an approriate measure on the path space W_Λ of a k-graph. The following definitions are due to Mallory and Sion [26].

Definition 4.2.1 We say that a measure μ on a space X is an *outer measure* if for every measurable set $A \subseteq X$ there exists a measurable set $B \subseteq X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

Let Δ denote the symmetric difference operation on sets. That is, $A\Delta B = (A\backslash B)\cup(B\backslash A)$.

Definition 4.2.2 Let $(I, \leq)$ be a directed set, $(X_i)_{i\in I}$ a collection of non-empty sets, and for each $i \in I$ let μ_i be an outer measure on X_i. For each i we denote the collection of μ_i-measurable sets by $\mathcal{M}_i$. Suppose that we have maps $\pi_{ij} : X_j \to X_i$ indexed by pairs $i, j \in I$ with $i \leq j$.. Then $(X_i, \pi_{ij}, \mu_i, I)$ is an *inverse system of outer measures* if

(i) for every $A \in \mathcal{M}_i$, $\pi_{ij}^{-1}(A) \in \mathcal{M}_j$ (π_{ij} is measurable)

(ii) for each $i, j \in I$ with $i \leq j$, and each $A \in \mathcal{M}_i$,

$$\mu_j(\pi_{ij}^{-1}(A)) = \mu_i(A),$$

and

(iii) for each $i, j, k \in I$ with $i \leq j \leq k$ and each $A \in \mathcal{M}_i$,

$$\mu_k\left(\pi_{ik}^{-1}(A)\Delta(\pi_{ij}\circ\pi_{jk})^{-1}(A)\right) = 0.$$

We write $X := \prod_{i\in I} X_i$, and we denote the projection of X onto X_i by p_i.

Definition 4.2.3 Let $(X_i, \pi_{ij}, \mu_i, I)$ be an inverse system of outer measures. We say that an outer measure ν on X is

(a) an *inverse limit outer measure* if

(i) ν is carried by the *inverse limit set*

$$L := \{\gamma \in X : \pi_{ij}(\gamma_j) = \gamma_i \text{ for all } i, j \in I \text{ with } i \leq j\},$$

in the sense that, $\nu(X\backslash L) = 0$;

(ii) $p_i^{-1}(A) \in \mathcal{M}_\nu$ for every $i \in I$ and $A \in \mathcal{M}_i$; and

(iii) $\nu(p_i^{-1}(A)) = \mu_i(A)$ for every $i \in I$ and $A \in \mathcal{M}_i$,

(b) a *π-limit outer measure* if (a)(ii)-(iii) are satisfied, and

(i) for each $i, j \in I$ with $i \leq j$ and each $A \in \mathcal{M}_i$,

$$\nu\left(p_i^{-1}(A)\Delta(\pi_{ij}\circ p_j)^{-1}(A)\right) = 0.$$

Definition 4.2.4 Let $\mathcal{C}$ be a collection of sets. We say that $\mathcal{C}$ is $\aleph_0$-*compact* if $\emptyset \in \mathcal{C}$ and for every countable sub-collection $\mathcal{C}' \subset \mathcal{C}$ such that $\bigcap_{C \in \mathcal{C}'} C = \emptyset$, there exists a finite sub-collection $\mathcal{F} \subset \mathcal{C}'$ such that $\bigcap_{F \in \mathcal{F}} F = \emptyset$.

Definition 4.2.5 Let $\mathcal{C}$ be a collection of subsets of a space X, and let μ be an outer measure on X. We say that $\mathcal{C}$ is an *inner family* for μ if $\mathcal{C} \subset \mathcal{M}_\mu$ and for all $M \subset \mathcal{M}_\mu$,

$$\mu(M) = \sup\{\mu(C) : C \in \mathcal{C}, C \subset M\}.$$

Definition 4.2.6 Let $(X_i, \pi_{ij}, \mu_i, I)$ be an inverse system of outer measures. Let $\mathcal{C} = (\mathcal{C})_{i \in I}$ be a system of collections of subsets of X_i. We say that $(X_i, \pi_{ij}, \mu_i, I)$ is inner regular relative to $\mathcal{C}$ if

(i) for every $i \in I$, $\mathcal{C}_i$ is an $\aleph_0$-compact collection of sets which is an inner family for μ_i, and

(ii) for each $i, j \in I$ with $i \leq j$ and each $C \in \mathcal{C}_j$, μ_j is σ-finite on $\pi_{ij}(C)$. That is, each $\pi_{ij}(C)$ is a countable union of μ_j-measurable sets with finite measure.

Definition 4.2.7 Let $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ be a system of collections of subsets of spaces X_i. We define the set of rectangles in $X := \prod_i X_i$ relative to $\mathcal{C}$ to be

$$\begin{aligned}\mathrm{Rect}(\mathcal{C}) = \Big\{R \subset X : \quad R = \prod_j A_j, \text{ where } A_j = X_j \text{ for all but finitely many } j, \quad (4.2)\\ \text{and } A_j \in \mathcal{C}_j \text{ otherwise}\Big\}.\end{aligned}$$

We also define, for $Y \subset X$,

$$J_Y = \{j \in I : p_j(Y) \neq X_j\}. \tag{4.3}$$

Theorem 4.2.8 ([26], Theorem 2.7)
If $(X_i, \pi_{ij}, \mu_i, I)$ is an inverse system of outer measures which is inner regular relative to some $\mathcal{C}$, then there exists a π-limit outer measure μ on X generated by the set function $g : \mathrm{Rect}(\mathcal{M}) \to [0, \infty]$ defined by

$$g(R) = \mu_j \left(\bigcap_{i \in J_R} \pi_{i,j}^{-1}(p_i(R)) \right),$$

where j is any element of $\mathbb{N}^k$ such that $j \geq i$ for all $i \in J_R$.

4.3 The path space as a subset of the product space

What we have learned is that we are able to define a π-limit outer measure on a product space of measurable spaces. We must now construct such a product space that is compatible with the path space W_Λ in order to define a measure on the path space. We do this by noting that as in [5] Λ^∞ is the inverse limit of the sets Λ^m. Thus we may view Λ^∞ as a subset of $\prod_{n\in\mathbb{N}^k}\Lambda^n$. So we just need to account for (in order to make up the full path space W_Λ) the finite paths Λ^m. However we also see that each Λ^m (that is, each of the finite paths) may be viewed as a subset of $\prod_{n\in\mathbb{N}^k}\Lambda^n$. The following Lemma gives us an inverse system of outer measures with which we will be able to define a measure on W_Λ. In the following, 0 is used as a distinguished formal symbol that does not belong to Λ.

Lemma 4.3.1 *Let Λ be a strongly connected, finite k-graph. For each $m \in \mathbb{N}^k$, define $\tilde{\Lambda}^m := \Lambda^m \cup \{0\}$. For each $m \le n \in \mathbb{N}^k$, define $\pi_{m,n} : \tilde{\Lambda}^n \to \tilde{\Lambda}^m$ by*

$$\pi_{m,n} := \begin{cases} \lambda(0,m), & \lambda \in \Lambda^m, \\ 0, & \lambda = 0. \end{cases}$$

For each $\mu \in \Lambda \cup \{0\}$, define $\chi : \Lambda \cup \{0\} \to [0,1]$ by

$$\chi_\mu := \begin{cases} x^\Lambda_{s(\mu)}, & \mu \in \Lambda, \\ 0, & \mu = 0, \end{cases}$$

where x^Λ is the unique unimodular Perron-Frobenius eigenvector of Λ. Now we define measures M_m on $\tilde{\Lambda}^m$ by

$$M_m(S) := \rho(\Lambda^{-m}) \sum_{\mu\in S} \chi_\mu,$$

for each $S \subseteq \tilde{\Lambda}^m$. Then $(\tilde{\Lambda}, \pi_{m,n}, M_m, \mathbb{N}^k)$ is an inverse system of outer measures, which is inner regular relative to $\mathcal{M} = (\mathcal{M}_m)$, where $\mathcal{M}_m$ is the collection of M_m-measurable subsets of $\tilde{\Lambda}^m$.

Proof. We first note that each $\mathcal{M}_m$ is the power set of $\tilde{\Lambda}^m$. Hence each $\pi_{m,n}$ is trivially measurable. For $m \leq n \in \mathbb{N}^k$ and $\mu \in \Lambda^m$, Proposition 8.1 of [5] shows that $M_m \circ \pi_{m,n}^{-1}(\{\mu\}) = M_n(\{\mu\})$, and for $0 \in \tilde{\Lambda}^m$ it is clear that $M_m \circ \pi_{m,n}^{-1}(\{0\}) = 0 = M_n(\{0\})$. Hence for each $S \subseteq \tilde{\Lambda}^m$, we have $M_m \circ \pi_{m,n}^{-1}(\{S\}) = M_n(\{S\})$. Finally, since

$$\pi_{m,l} = \pi_{m,n} \circ \pi_{n,l}$$

for each $m \leq n \leq l$, we see that

$$M_l\left(\pi_{m,l}^{-1}(A)\Delta(\pi_{m,n} \circ \pi_{n,l})^{-1}(A)\right) = 0$$

for each $m \leq n \leq l$ and each $A \in \mathcal{M}_m$. Hence $(\tilde{\Lambda}^m, \pi_{m,n}, M_m, \mathbb{N}^k)$ is an inverse system of outer measures.

To see that $(\tilde{\Lambda}^m, \pi_{m,n}, M_m, \mathbb{N}^k)$ is inner regular relative to $(\mathcal{M}_m)$, we recall that each $\mathcal{M}_m$ is the power set of $\tilde{\Lambda}^m$. Hence each $\mathcal{M}_m$ is trivially $\aleph_0$-compact since each $\tilde{\Lambda}^m$ is finite. Additionally, each $\mathcal{M}_m$ is an inner family for M_m since the condition reduces to

$$M_m(H) = \sup\{M_m(C) : C \subset H\}$$

for each $H \subseteq \tilde{\Lambda}^m$, which is also trivially true. Finally, we need to check that for every $n \in \mathbb{N}^k$, we have $S \in \mathcal{M}_n$, and every $m \leq n$, the measure $\pi_{m,n}(S)$ is σ-finite. For each $n \in \mathbb{N}^k$,

$$M_n(\tilde{\Lambda}^n) = \rho(\Lambda)^{-n} \sum_{\lambda \in \Lambda^n} x^{\Lambda}_{s(\lambda)} < \infty$$

since each $x^{\Lambda}_{s(\lambda)} < \infty$ and each Λ^n is finite. Hence each $\pi_{m,n}(S)$ is σ-finite. $\square$

With this in mind, we may use Theorem 4.2.8 to define a measure on the path space W_Λ.

Theorem 4.3.2 *Suppose that Λ is a strongly connected finite k-graph and let $D : \mathcal{G}_\Lambda \to \mathbb{R}$ be the cocycle given by $D(x, n, y) := n \cdot \ln(\rho(\Lambda))$. Then*

(i) there exists a unique Borel probability measure M on W_Λ that satisfies

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^{\Lambda}_{s(\lambda)} \tag{4.4}$$

for all $\lambda \in \Lambda$,

(ii) the measure M is quasi-invariant with Radon-Nykodym cocycle e^{-D},

(iii) the measure M is unique in the sense that it is the only probability measure with Radon-Nykodym cocycle e^{-D},

(iv) the measure M satisfies $M(\{x \in W_\Lambda : \{x\} \times \mathrm{Per}(\Lambda) \times \{x\} \neq (\mathcal{G}_\Lambda)_x^x\}) = 0$, and

(v) the measure M is supported on Λ^∞.

Proof of (i)-(iii) and (v).

(i) By Lemma 4.3.1, $(\tilde{\Lambda}^m, \pi_{m,n}, M_m, \mathbb{N}^k)$ is an inverse system of outer measures which is inner regular relative to $\mathcal{M} = (\mathcal{M}_m)$. Hence by Theorem 4.2.8, there is a π-limit outer measure M' on $X := \prod_{n\in\mathbb{N}^k} \tilde{\Lambda}^n$. To define a measure on W_Λ we observe that there is a copy of W_Λ in X. To see this, define $\varphi : W_\Lambda \to X$ by

$$\varphi(x)_m = \begin{cases} x(0,m), & m \leq d(x), \\ 0, & \text{otherwise.} \end{cases}$$

We then define a measure M on W_Λ by $M := M' \circ \varphi$. Now we verify that M has the properties given above.

Let $\mu \in \tilde{\Lambda}^m$ for some fixed $m \in \mathbb{N}^k$. Then we have

$$\begin{aligned} \varphi(Z(\mu)) &= \{\gamma \in X : \gamma_n = \mu(0,n) \text{ for all } n \leq m \text{ and } \gamma_n = 0 \text{ for all } n > m\} \\ &\subset \{\gamma \in X : \gamma_m = \mu\} \\ &= p_m^{-1}(\{\mu\}). \end{aligned}$$

Since M' is a π-limit outer measure, for all $i \leq j \in \mathbb{N}^k$ and M_i-measurable A we have

$$M'\left(\{p_i^{-1}(A)\Delta(\pi_{ij} \circ p_j)^{-1}(A)\}\right) = 0.$$

In particular, setting $A = \{\xi\}$ for some $\xi \in \Lambda^i$, for each pair $i \leq j \in \mathbb{N}^k$ we have

$$M'\left(\{\gamma \in X : \gamma_i = \xi \text{ XOR } \gamma_j(0,i) = \xi\}\right) = 0,$$

and hence

$$M'\left(\{\gamma \in X : \gamma_i \neq \xi, \gamma_j(0,i) = \xi\}\right) = 0.$$

So, setting $j = m$, we see that

$$M'\left(p_m^{-1}(\{\mu\})\backslash\varphi(Z(\mu))\right) = 0.$$

Therefore, since $\varphi(Z(\mu)) \subset p_m^{-1}(\{\mu\})$, we obtain

$$M'\left(\varphi(Z(\mu))\right) = M'\left(p_m^{-1}(\{\mu\})\right).$$

Now, since $p_m^{-1}(\{\mu\}) \in \mathrm{Rect}(\mathcal{M})$, by Theorem 4.2.8 we have

$$\begin{aligned} M'\left(p_m^{-1}(\{\mu\})\right) &= g\left(p_m^{-1}(\{\mu\})\right) \\ &= M_m(\{\mu\}). \end{aligned}$$

Therefore

$$\begin{aligned} M(Z(\lambda)) &= M'(\varphi(Z(\lambda))) \\ &= M'\left(p_m^{-1}(\{\lambda\})\right) \\ &= M_m(\{\lambda\}) \\ &= \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)}, \end{aligned}$$

as required. We also see that M is a probability measure as in [5]: $M(W_\Lambda) = \sum_{v\in\Lambda^0} M(Z(v)) = \sum_{v\in\Lambda^0} x^\Lambda_v = 1$.

(ii) To see that the Radon-Nykodym cocycle of M is e^{-D}, we check that for all bisections $U \subseteq \mathcal{G}_\Lambda$ and all $f : s(U) \to \mathbb{R}$,

$$\int_{r(U)} f(T_U(x))dM(x) = \int_{s(U)} e^{-D(U^x)} f(x)dM(x),$$

where $U^x := U \cap r^{-1}(x)$ and $T_U : r(U) \to s(U)$ is given by $T_U(x) = s(U^x)$. It suffices to check the above for U of the form $Z(\mu, \nu)$. We get

$$\int_{r(U)} f(T_U(x))dM(x) = \int_{Z(\mu)} f(\nu\sigma^{d(\mu)}(x))dM(x)$$
$$= \int_{Z(\mu)} f \circ g(x)dM(x),$$

where $g(x) = \nu\sigma^{d(\mu)}(x)$. Now, for $Z(\lambda) \subseteq Z(\mu)$ we have $M \circ g^{-1}(Z(\lambda)) = M(\{y \in Z(\mu) : (\nu\sigma^{d(\mu)}(y))(0, d(\lambda)) = \lambda\}) = M(\{y \in Z(\mu) : y(d(\mu), d(\lambda) + d(\mu) - d(\nu)) = \lambda(d(\nu), d(\lambda))\}) = M(Z(\mu\lambda(d(\nu), d(\lambda)))) = \rho(\Lambda)^{d(\nu)-d(\mu)}M(Z(\lambda))$. So the above becomes

$$\int_{z(\mu)} f \circ g(x)dM(x) = \int_{Z(\nu)} f(x)\rho(\Lambda)^{d(\nu)-d(\mu)}dM(x)$$
$$= \int_{s(U)} e^{-D(U^x)}f(x)dM(x).$$

(iii) We follow the logic of [5, Lemma 12.1]. Suppose that μ is a quasi-invariant probability measure on W_Λ with Radon-Nykodym cocycle e^{-D}. Then for $\lambda \in \Lambda$, since $Z(\lambda, s(\lambda))$ is a bisection with $r(Z(\lambda, s(\lambda))) = Z(\lambda)$ and $s(Z(\lambda, s(\lambda))) = Z(s(\lambda))$, we have

$$\mu(Z(\lambda)) = e^{-d(\lambda)\cdot \ln(\rho(\Lambda))}\mu(Z(s(\lambda))) = \rho(\Lambda)^{-d(\lambda)}\mu(Z(s(\lambda))). \tag{4.5}$$

Hence, for $\lambda \in v\Lambda^{e_i}$, $\mu(Z(\lambda)) = \rho(A_i)^{-1}\mu(Z(s(\lambda)))$. So we see that

$$\mu(Z(v)) \geq \mu\left(\bigsqcup_{w\in\Lambda^0}\bigsqcup_{\lambda\in v\Lambda^{e_i}w} Z(\lambda)\right)$$
$$= \sum_{w\in\Lambda^0}\sum_{\lambda\in v\Lambda^{e_i}w} \mu(Z(\lambda))$$
$$= \rho(A_i)^{-1}\sum_{w\in\Lambda^0} A_i(v, w)\mu(Z(w)),$$

and rearranging we have

$$\rho(A_i)\mu(Z(v)) \geq \sum_{w\in\Lambda^0} A_i(v, w)\mu(Z(w)). \tag{4.6}$$

Considering the vector $m := (\mu(Z(v))) \in [0, \infty)^{\Lambda^0}$, equation (4.6) yields $\rho(A_i)m \geq$

$A_i m$. Since μ is a probability measure, $\sum_{v\in\Lambda^0} m_v = \sum_{v\in\Lambda^0}\mu(Z(v)) = \mu(W_\Lambda) = 1$. Therefore by [5, Corollary 4.2 (d)], m is the unique Perron-Frobenius eigenvector x^Λ of Λ. So (4.5) simplifies to $\mu(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)}x^\Lambda_{s(\lambda)}$, which is the same as $M(Z(\lambda))$. Since the $Z(\lambda)$ form a basis for W_Λ, we have $\mu = M$.

(v) Fix $1 \le i \le k$ and $\lambda \in \Lambda$ such that $d(\lambda)_i = n$ for some $n \in \mathbb{N}$. We have $M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)}$, and we calculate

$$\begin{aligned} M\left(\bigcup_{\alpha\in s(\lambda)\Lambda^{e_i}} Z(\lambda\alpha)\right) &= \sum_{\alpha\in s(\lambda)\Lambda^{e_i}} M(Z(\lambda\alpha)) \\ &= \sum_{\alpha\in s(\lambda)\Lambda^{e_i}} \rho(\Lambda)^{-d(\lambda)-e_i}x^\Lambda_{s(\alpha)} \\ &= \rho(\Lambda)^{-d(\lambda)-e_i}\sum_{w\in\Lambda^0}|s(\lambda)\Lambda^{e_i}w|x^\Lambda_w \\ &= \rho(\Lambda)^{-d(\lambda)-e_i}(A_i x^\Lambda)_{s(\lambda)}. \end{aligned}$$

Therefore by Proposition 2.1.19 we see that

$$M(Z(\lambda)) = M\left(\bigcup_{\alpha\in s(\lambda)\Lambda^{e_i}} Z(\lambda\alpha)\right),$$

since x^Λ is the Perron-Frobenius eigenvector of Λ. Hence

$$M\left(Z(\lambda)\setminus\bigcup_{\alpha\in\Lambda^{e_i}} Z(\lambda\alpha)\right) = 0.$$

Now we note that for all $x \in Z(\lambda)\setminus\bigcup_{\alpha\in\Lambda^{e_i}} Z(\lambda\alpha)$, we have $d(x)_i = d(\lambda)$. In particular, we see that $d(x)_i$ is finite. Therefore we have

$$W_\Lambda\setminus\Lambda^\infty = \bigcup_{1\le i\le k}\bigcup_{\lambda\in S_i}\left(Z(\lambda)\setminus\bigcup_{\alpha\in\Lambda^{e_i}} Z(\lambda\alpha)\right), \tag{4.7}$$

where $S_i := \{\mu\in\Lambda : d(\mu)_i = n \text{ for some } n\in\mathbb{N}\}$. Hence we have $M(W_\Lambda\setminus\Lambda^\infty) = 0$ since the right hand side of (4.7) is a countable union of measure-zero sets. □

To prove (iv), we first require the following Lemmas. The first Lemma follows the proof of [5, Lemma 8.4], however as usual we are working with the full path space instead of the infinite path space.

Lemma 4.3.3 *Let Λ be a strongly connected finite k-graph, and let M be the measure defined in equation* (4.4)*. Suppose that $g \in \mathbb{Z}^k \setminus \mathrm{Per}(\Lambda)$. There exist $a \in \mathbb{N}^k \setminus \{0\}$ and $0 < K < 1$ such that whenever $s(\mu) = s(\nu)$ and $d(\mu) - d(\nu) = g$, we have for all $j \in \mathbb{N}$*

$$M\Big(\bigcup_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda) \Big) \leq K^j M(Z(\mu)). \tag{4.8}$$

Proof. This proof follows exactly the proof of [5, Lemma 8.4] since their calculations rely on a measure defined identically to ours, and on manipulations of the sets $Z(\lambda) \subset \Lambda^\infty$ which behave roughly the same as $Z(\lambda) \subset W_\Lambda$. □

The approximation in Lemma 4.3.3 allows us to determine that certain sets have measure zero as follows. Here we follow [5, Proposition 8.2].

Lemma 4.3.4 *Let Λ be a strongly connected finite k-graph, and let M be the measure defined in equation* (4.4)*. For $m, n \in \mathbb{N}^k$ such that $m - n \notin \mathrm{Per}(\Lambda)$ we have*

$$M(\{x \in W_\Lambda : \sigma^m(x) = \sigma^n(x)\}) = 0. \tag{4.9}$$

Proof. We observe that, as in the proof of [5, Proposition 8.2], we may write

$$\{x \in W_\Lambda : \sigma^m(x) = \sigma^n(x)\} \subseteq \bigcup_{\substack{\mu \in \Lambda^m \\ \nu \in \Lambda^n s(\mu)}} \bigcup_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda).$$

Hence Lemma 4.3.3 implies that for all j

$$M(\{x \in W_\Lambda : \sigma^m(x) = \sigma^n(x)\}) \leq \sum_{\substack{\mu \in \Lambda^m \\ \nu \in \Lambda^n s(\mu)}} K^j M(Z(\mu)) \leq |\Lambda^m| \cdot |\Lambda^n| \cdot K^j.$$

Since $0 < K < 1$, the right hand side goes to zero as $j \to \infty$. □

We are now able to complete the proof of Theorem 4.3.2.

Proof of Theorem 4.3.2(iv). Following the proof of [5, Lemma 12.1], we observe that

$$\{x \in W_\Lambda : \{x\} \times \mathrm{Per}(\Lambda) \times \{x\} = (\mathcal{G}_\Lambda)_x^x\} = W_\Lambda \setminus \bigcup_{\substack{m,n \in \mathbb{N}^k, \\ m-n \notin \mathrm{Per}\,\Lambda}} \{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}.$$

By equation (4.9), each of the sets in this union has measure zero. Since $\{(m, n) \in$

$\mathbb{N}^k \times \mathbb{N}^k : m - n \notin \operatorname{Per} \Lambda\}$ is countable we have

$$M\left(\bigcup_{\substack{m,n\in\mathbb{N}^k,\\ m-n\notin\operatorname{Per}\Lambda}} \{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}\right) = 0,$$

and hence the result follows. $\square$

Chapter 5

The KMS states of the twisted Toeplitz algebra

Using the result of Afsar and Sims [1] we are able to classify the KMS_1 states of the twisted Toeplitz algebra of a higher-rank graph Λ. The classification puts the KMS_1 states in one-to-one correspondence with M-measurable fields of tracial states on $C^*(\operatorname{Per}\Lambda, \omega_c)$, where M is the measure on W_Λ described in Chapter 4. In this chapter we will talk about what M-measurable fields of tracial states are, and then dive into proving the main result.

5.1 Measurable fields of tracial states

Here we briefly recall what a tracial state on a C^*-algebra is, and define measurable fields of tracial states on a groupoid C^*-algebra.

Definition 5.1.1 Let A be a C^*-algebra. A state $\psi : A \to \mathbb{C}$ is a positive linear functional on A with norm 1. If A is unital, a linear functional ψ on A is a state if and only if $\psi(1) = 1$. We say that ψ is a *tracial state* on A if it is a state, and $\psi(ab) = \psi(ba)$ for all $a, b \in A$.

The original definition for measurable fields of states was given by Neshveyev [29] in the case of untwisted algebras. Here we give the definition in the case of twisted algebras from Afsar and Sims [1].

Definition 5.1.2 Let $\mathcal{G}$ be a groupoid. For each $x \in \mathcal{G}^{(0)}$, we write $\{W_u : u \in \mathcal{G}_x^x\}$ for the universal unitary representation of the group $\mathcal{G}_x^x$ in $C^*(\mathcal{G}_x^x)$. Let μ be a probability measure on $\mathcal{G}^{(0)}$ We say that a collection of states $\{\psi_x\}_{x\in\mathcal{G}^{(0)}}$ on $C^*(\mathcal{G},\sigma)$ is a *μ-measurable field of states* if for every $f \in C_c(\mathcal{G},\sigma)$ the function $x \mapsto \sum_{u\in\mathcal{G}_x^x} f(u)\psi_x(W_u)$ is μ-measurable.

5.2 KMS$_1$ states

We now give a characterisation of the KMS$_1$ states of the twisted Toeplitz algebra via the groupoid model developed in Chapter 3. First we state a result of Afsar and Sims [1] that we will use in the proof of our main result.

Theorem 5.2.1 [1, Corollary 4.2]
Let $\mathcal{G}$ be a locally compact second countable étale groupoid, and let $\sigma \in Z^2(\mathcal{G},\mathbb{T})$. Let D be a continuous $\mathbb{R}$-valued 1-cocycle on $\mathcal{G}$ and let $\tilde{\tau}$ be the dynamics on $C^*(\mathcal{G},\sigma)$ given by $\tilde{\tau}_t(f)(\gamma) = e^{itD(\gamma)}f(\gamma)$. Take $\beta \in \mathbb{R}$. There is a bijection between the simplex of KMS$_\beta$ *states of $(C^*(\mathcal{G},\tau),\tilde{\tau})$ and the pairs $(\mu,[\Psi]_\mu)$ consisting of a probability measure μ on $\mathcal{G}^{(0)}$ and a μ-equivalence class $[\Psi]_\mu$ of μ-measurable fields of tracial states on $C^*(\mathcal{G}_x^x,\sigma)$ such that*

(I) *μ is a quasi-invariant measure with Radon-Nykodym cocycle $e^{-\beta D}$,*

(II) *for each representative $\{\psi_x\}_{x\in\mathcal{G}^{(0)}} \in [\Psi]_\mu$ and for μ-almost every $x \in \mathcal{G}^{(0)}$, we have*

$$\psi_x(W_u) = \sigma(\eta u,\eta^{-1})\sigma(\eta,u)\overline{\sigma(\eta^{-1},\eta)}\psi_{r(\eta)}(W_{\eta u\eta^{-1}}) \tag{5.1}$$

for $u \in \mathcal{G}_x^x$ and $\eta \in \mathcal{G}_x$.

The state corresponding to the pair $(\mu,[\Psi]_\mu)$ is given by

$$\psi(f) = \int_{\mathcal{G}^{(0)}} \sum_{u\in\mathcal{G}_x^x} f(u)\psi_x(W_u)d\mu(x) \tag{5.2}$$

for all $f \in C_c(\mathcal{G},\sigma)$.

We require a final Lemma before we can prove the final result.

Lemma 5.2.2 *Let Λ be a finite k-graph, and let $c \in \underline{Z}^2(\Lambda,\mathbb{T})$. Let σ_c be the groupoid cocycle induced by c as defined in* (3.15)*. For each $x \in W_\Lambda$, define $\sigma_c^x \in \tilde{Z}^2(\mathrm{Per}(\Lambda),\mathbb{T})$ by $\sigma_c^x(p,q) := \sigma_c((x,p,x),(x,q,x))$. Then there exists a bicharacter ω_c on* $\mathrm{Per}(\Lambda)$ *that is cohomologous to σ_c^x for M-a.e. $x \in W_\Lambda$.*

Proof. We note that by Theorem 4.3.2(v), M-a.e. $x \in W_\Lambda$ is in the infinite path space Λ^∞. So consider $x \in \Lambda^\infty$ and apply [24, Lemma 3.3] to see that the cohomology class of σ_c^x is independent of x. Hence by Section 2.4 of [1] there is a bicharacter ω_c on $\mathrm{Per}(\Lambda)$ that is cohomologous to σ_c^x for all $x \in \Lambda^\infty$. $\square$

We now apply this result of Afsar and Sims [1] to the C^*-dynamical system $(C^*(\mathcal{G}_\Lambda, \sigma_c), \tau)$, where $C^*(\mathcal{G}_\Lambda, \sigma_c)$ is the groupoid model defined in Chapter 3 for a higher-rank graph Λ and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and τ is the preferred dynamics on $C^*(\mathcal{G}_\Lambda, \sigma_c)$.

Theorem 5.2.3 *Suppose Λ is a strongly connected finite k-graph. Let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Suppose that $\omega_c \in \underline{Z}^2(\mathrm{Per}\,\Lambda, \mathbb{T})$ is a bicharacter that is cohomologous to $\sigma_c^x(p,q) := \sigma_c((x,p,x),(x,q,x))$ for all $x \in \Lambda^\infty$. Let τ be the preferred dynamics on $C^*(\mathcal{G}_\Lambda, \sigma_c)$. Let M be the measure on W_Λ defined by (4.4). There is a bijection between the simplex of* KMS$_1$*-states of $(C^*(\mathcal{G}_\Lambda, \sigma_c), \tau)$ and the set of M-equivalence classes $[\psi]_M$ of tracial states $\{\psi_x\}_{x \in W_\Lambda}$ on $C^*(\mathrm{Per}\,\Lambda, \omega_c)$ such that for all $W_p \in C^*(\mathrm{Per}\,\Lambda, \omega_c)$ and $\eta := (y,m,x) \in (\mathcal{G}_\Lambda)_x$ we have*

$$\psi_x(W_p) = \sigma_c(\eta, (x,p,x))\sigma_c((y,m+p,x), \eta^{-1})\overline{\sigma_c(\eta^{-1}, \eta)}\psi_y(W_p). \tag{5.3}$$

The state corresponding to the class $[\psi]_M$ satisfies

$$\psi(f) = \int_{W_\Lambda} \sum_{p \in \mathrm{Per}\,\Lambda} f(x,p,x)\psi_x(W_p)dM(x) \tag{5.4}$$

for all $f \in C_c(\mathcal{G}_\Lambda, \sigma_c)$.

Proof. Since the measure M is supported only on the infinite paths, fix $x \in \Lambda^\infty$ such that $\{x\} \times \mathrm{Per}\,\Lambda \times \{x\} = (\mathcal{G}_\Lambda)_x^x$. By Theorem 4.3.2(iv), M-a.e. $x \in \Lambda^\infty$ satisfies this condition. Let $\delta^1 b$ be the 2-coboundary such that $\omega_c = \delta^1 b \sigma_c^x$. As in [1] we have an isomorphism $\Phi : C^*(\mathrm{Per}\,\Lambda, \omega_c) \to C^*((\mathcal{G}_\Lambda)_x^x, \sigma_c)$ defined by

$$\Phi(W_p) = b(p)W_{(x,p,x)}$$

for all $p \in \mathrm{Per}\,\Lambda$.

Since M is the only probability measure on W_Λ with Radon-Nikodym cocycle e^{-D}, by Theorem 5.2.1 it suffices to show that there is a bijection between the fields of tracial states satisfying (5.3) and the M-measurable fields of tracial states on $C^*((\mathcal{G}_\Lambda)_x^x, \sigma_c)$ satisfying Theorem 5.2.1 (II). To see this we follow [1, Corollary 5.1].

Let $\{\varphi_x\}_{x\in W_\Lambda}$ be an M-measurable field of tracial states on $C^*((\mathcal{G}_\Lambda)_x^x, \sigma_c)$ satisfying Theorem 5.2.1 (II). Then using the above isomorphism, clearly $\{\varphi_x \circ \Phi\}_{x\in\Lambda^\infty}$ is an M-measurable field of tracial states on $C^*(\operatorname{Per}\Lambda, \omega_c)$. We now verify that (5.3) holds; so let $\eta = (y, m, x)$. We obtain

$$\begin{aligned}(\varphi_x \circ \Phi)(W_p) &= \varphi_x(b(p)W_{(x,p,x)})\\ &= b(p)\sigma_c(\eta, (x, p, x))\sigma_c((y, m + p, x), \eta^{-1})\overline{\sigma_c(\eta^{-1}, \eta)}\varphi_y(W_{(y,p,y)})\\ &= \sigma_c(\eta, (x, p, x))\sigma_c((y, m + p, x), \eta^{-1})\overline{\sigma_c(\eta^{-1}, \eta)}(\varphi_y \circ \Phi)(W_p).\end{aligned}$$

Now, let $\{\phi_x\}_{x\in W_\Lambda}$ be a field of states on $C^*(\operatorname{Per}\Lambda, \omega_c)$ satisfying (5.3). Since M is a Borel measure, for each $f \in C_c(\mathcal{G}_\Lambda, \sigma_c)$, the function

$$x \mapsto \sum_{u\in(\mathcal{G}_\Lambda)_x^x} f(u)(\phi \circ \Phi^{-1})(W_u) = \sum_{p\in\operatorname{Per}\Lambda} f(x, p, x)\overline{b(p)}\phi_x(W_p)$$

is continuous and hence M-measurable. So $\{\phi_x \circ \Phi^{-1}\}_{x\in W_\Lambda}$ is an M-measurable field of tracial states on $C^*((\mathcal{G}_\Lambda)_x^x, \sigma_c)$. Finally, applying (5.3) to $\{\phi_x\}_{x\in W_\Lambda}$ we get

$$(\phi_x \circ \Phi^{-1})(W_u) = \sigma_c((y, m+p, x), (x, p, x))\sigma_c(\eta, (x, p, x))\overline{\sigma_c(\eta^{-1}, \eta)}(\phi_y \circ \Phi^{-1})(W_u). \ \square$$

When we consider the result of Afsar and Sims [1] that says the KMS_1 states of the non-Toeplitz algebra are given by the same formulae as above, we can see that our result says that the KMS structure at the critical inverse temperature is the same across both algebras. That is, at the critical inverse temperature ($\beta = 1$ here) the KMS states of the twisted Toeplitz algebra $\mathcal{T}C^*(\Lambda, c)$ factor through the twisted Cuntz-Krieger algebra $C^*(\Lambda, c)$. In the next Chapter we will couple this with a previous result of ours which computed the KMS states of the twisted Toeplitz algebra for large inverse temperatures (greater than the critical inverse temperature).

We may also compare our results with Theorems 5.1 and 5.2 of Christensen [9]. In the context of the twisted Toeplitz algebra, Christensen's formula (5.1) matches with our formula (5.4), and the subgroup B in Theorem 5.2 corresponds to the subgroup $\operatorname{Per}(\Lambda)$.

Chapter 6

A direct computation of the KMS states of the twisted Toeplitz algebra at high inverse temperatures

In this chapter we detail previous computations that yield a formula for the KMS_β states for β greater than the critical inverse temperature. We note that the characterisation of KMS states on $\mathcal{T}C^*(\Lambda, c)$ given here is an extension of the characterisation of KMS states on $\mathcal{T}C^*(\Lambda)$ given by an Huef, Laca, Raeburn and Sims in [4]. The goal of the chapter is to combine these results with the result of the previous chapter in order to give a more complete description of the KMS states of the twisted Toeplitz algebra and to reconcile the result obtained from direct computation with the result predicted by our results in earlier chapters.

As in [4] we first assume that the $r \in (0, \infty)$ used to define the automorphism group α_t has rationally independent coordinates; that is, no coordinate of r may be written as a linear combination of its other coordinates with rational coefficients. We then spend the rest of the chapter showing how to relax this constraint.

6.1 KMS states for automorphisms arising from rationally independent vectors

This section deals with the 'easy' case, where the automorphism $\alpha_t = \gamma_{e^{itr}}$ is defined by $r \in (0, \infty)^k$ such that the coordinates of r are rationally independent. We need the following technical Lemma from [4].

Lemma 6.1.1 ([4], Lemma 3.2)
Let $m, n, p, q \in \mathbb{N}^k$ such that $m+p = n+q$. Then $m+p = m \vee n$ if and only if $p \wedge q = 0$.

The following proposition follows closely Proposition 3.1 of an Huef, Laca, Raeburn and Sims [4]. The major difference of course is that we are dealing with $\mathcal{T}C^*(\Lambda, c)$ as opposed to $\mathcal{T}C^*(\Lambda)$.

Proposition 6.1.2 *Suppose that Λ is a finite k-graph with no sources. Let $r \in (0, \infty)^k$, let $\gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ be the gauge action, and define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ by $\alpha_t = \gamma_{e^{itr}}$. Suppose $\beta \in [0, \infty)$ and ϕ is a state on $\mathcal{T}C^*(\Lambda, c)$.*

(a) If ϕ is a KMS_β state of $(\mathcal{T}C^*(\Lambda, c), \alpha)$, then

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} \phi(t_{s(\mu)}) \tag{6.1}$$

for all $\mu, \nu \in \Lambda$ with $d(\mu) = d(\nu)$.

(b) If equation (6.1) holds for all $\mu, \nu \in \Lambda$ then ϕ is a KMS_β state of $(\mathcal{T}C^*(\Lambda, c), \alpha)$. Further, if r has rationally independent coordinates then ϕ is a KMS_β state of $(\mathcal{T}C^*(\Lambda, c), \alpha)$ if and only if equation (6.1) holds.

Proof.

(a) Let $\mu, \nu \in \Lambda$ such that $d(\mu) = d(\nu)$. Since ϕ is a KMS_β state, we have

$$\begin{aligned}
\phi(t_\mu t_\nu^*) &= \phi(t_\nu^* \alpha_{i\beta}(t_\mu)) \\
&= \phi(t_\nu^* \gamma_{e^{-\beta r}}(t_\mu)) \\
&= \phi(t_\nu^* e^{-\beta r \cdot d(\mu)} t_\mu) \\
&= e^{-\beta r \cdot d(\mu)} \phi(t_\nu^* t_\mu).
\end{aligned}$$

Since $d(\mu) = d(\nu)$ we have $t_\nu^* t_\mu = \delta_{\mu,\nu} t_{s(\mu)}$. Hence $\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r \cdot d(\mu)} \phi(t_{s(\mu)})$.

(b) The elements $\{t_\mu t_\nu^* : s(\mu) = s(\nu)\}$ are analytic and span a dense subspace of $\mathcal{T}C^*(\Lambda, c)$, so it suffices to show that $\phi(t_\mu t_\nu^* t_\sigma t_\tau^*) = \phi(t_\sigma t_\tau^* \alpha_{i\beta}(t_\mu t_\nu^*))$ whenever

$s(\mu) = s(\nu)$ and $s(\sigma) = s(\tau)$. So we calculate

$$\begin{aligned}\phi(t_\mu t_\nu^* t_\sigma t_\tau^*) &= \sum_{(\alpha,\eta)\in\Lambda^{\min}(\nu,\sigma)} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\sigma,\eta)\overline{c(\tau,\eta)}\phi(t_{\mu\alpha}t_{\tau\eta}^*)\\ &= \sum_{\substack{(\alpha,\eta)\in\Lambda^{\min}(\nu,\sigma),\\ \mu\alpha=\tau\eta}} c(\mu,\alpha)\overline{c(\nu,\alpha)}c(\sigma,\eta)\overline{c(\tau,\eta)}e^{-\beta r\cdot d(\mu\alpha)}\phi(t_{s(\alpha)}) \qquad (6.2)\end{aligned}$$

and similarly,

$$\phi(t_\sigma t_\tau^* t_\mu t_\nu^*) = \sum_{\substack{(\gamma,\xi)\in\Lambda^{\min}(\tau,\mu),\\ \sigma\gamma=\nu\xi}} c(\sigma,\gamma)\overline{c(\tau,\gamma)}c(\mu\xi)\overline{c(\nu,\xi)}e^{-\beta r\cdot d(\sigma\gamma)}\phi(t_{s(\gamma)}). \qquad (6.3)$$

Let $I_1 := \{(\alpha,\eta) \in \Lambda^{\min}(\nu,\sigma) : \mu\alpha = \tau\eta\}$, and $I_2 := \{(\gamma,\xi) \in \Lambda^{\min}(\tau,\mu) : \sigma\gamma = \nu\xi\}$. Suppose $(\alpha,\eta) \in I_1$. Since $I_1 \subset \Lambda^{\min}(\nu,\sigma)$, we have $d(\alpha) \wedge d(\eta) = 0$. Since $d(\mu)+d(\alpha) = d(\tau)+d(\eta)$, Lemma 6.1.1 says that $d(\mu)+d(\alpha) = d(\mu)\vee d(\tau)$. Hence $d(\tau) + d(\eta) = d(\tau) \vee d(\eta)$, and we see that $(\eta,\alpha) \in \Lambda^{\min}(\tau,\mu)$. By a symmetric argument, if $(\eta,\alpha) \in I_2$, then $(\alpha,\eta) \in I_1$. Hence $(\alpha,\eta) \mapsto (\eta,\alpha)$ is a bijection of I_1 onto I_2. Now fix $(\alpha,\eta) \in I_1$. Since $s(\alpha) = s(\eta)$, we have $t_{s(\alpha)} = t_{s(\eta)}$. Using (6.3) and that

$$d(\mu) - d(\nu) + d(\nu\alpha) = d(\mu) - d(\nu) + d(\nu) + d(\alpha) = d(\mu\alpha),$$

we calculate

$$\begin{aligned}&\phi(t_\sigma t_\tau^* \alpha_{i\beta}(t_\mu t_\nu^*))\\ &\quad= e^{-\beta r\cdot(d(\mu)-d(\nu))}\phi(t_\sigma t_\tau^* t_\mu t_\nu^*)\\ &\quad= \sum_{\substack{(\eta,\alpha)\in\Lambda^{\min}(\tau,\mu),\\ \sigma\eta=\nu\alpha}} c(\sigma,\eta)\overline{c(\tau,\eta)}c(\mu\alpha)\overline{c(\nu,\alpha)}e^{-\beta r\cdot(d(\mu)-d(\nu)+d(\sigma\eta))}\phi(t_{s(\eta)})\\ &\quad= \sum_{\substack{(\eta,\alpha)\in\Lambda^{\min}(\tau,\mu),\\ \sigma\eta=\nu\alpha}} c(\sigma,\eta)\overline{c(\tau,\eta)}c(\mu\alpha)\overline{c(\nu,\alpha)}e^{-\beta r\cdot(d(\mu\alpha))}\phi(t_{s(\alpha)})\\ &\quad= \phi(t_\mu t_\nu^* t_\sigma t_\tau^*),\end{aligned}$$

where the last equality comes from equation (6.2). Hence ϕ is a KMS$_\beta$ state.

Next suppose that r has rationally independent coordinates, and let ϕ be a KMS$_\beta$ state. Let $\mu,\nu \in \Lambda$ such that $s(\mu) = s(\nu)$. If $d(\mu) = d(\nu)$, then by (a) equation

(6.1) holds. So we consider the case when $d(\mu) \neq d(\nu)$. Applying the KMS condition twice, we obtain

$$\phi(t_\mu t_\nu^*) = e^{-\beta r \cdot d(\mu)} \phi(t_\nu^* t_\mu) = e^{-\beta r \cdot d(\mu)} e^{\beta r \cdot d(\nu)} \phi(t_\mu t_\nu^*). \tag{6.4}$$

Since the coordinates of r are rationally independent and $d(\mu) \neq d(\nu)$, we have $r \cdot d(\mu) \neq r \cdot d(\nu)$. Hence $e^{-\beta r \cdot d(\mu)} e^{\beta r \cdot d(\nu)} \neq 1$, and so equation (6.4) implies that $\phi(t_\mu t_\nu^*) = 0$. Hence ϕ satisfies equation (6.1). □

6.2 Removing the constraint of rational independence

The rational independence of the coordinates of r tells us that $d(\mu) \neq d(\nu)$ implies $r \cdot d(\mu) \neq r \cdot d(\nu)$. In order to remove the assumption of rational independence we must deal with the case $d(\mu) \neq d(\nu)$ but $r \cdot d(\mu) = r \cdot d(\nu)$. To do so we employ the following results, which were adapted from an Huef, Laca, Raeburn and Sims [4].

Lemma 6.2.1 *Let Λ be a finite k-graph with no sources. Suppose that ϕ is a* KMS$_\beta$ *state of $(\mathcal{T}C^*(\Lambda, c), \alpha)$ for some $\beta > 0$, and that $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu)$ and $r \cdot d(\mu) = r \cdot d(\nu)$ for a fixed $r \in (0, \infty)^k$. Then $\phi(t_\mu t_\mu^*) = \phi(t_\nu t_\nu^*)$ and $|\phi(t_\mu t_\nu^*)| \leq \phi(t_\mu t_\mu^*)$.*

Proof. Since $s(\mu) = s(\nu)$ we may apply (TCK3) to see that

$$\phi(t_\mu t_\mu^*) = \phi(t_\mu t_{s(\mu)} t_\mu^*) = \phi(t_\mu t_{s(\nu)} t_\mu^*) = \phi(t_\mu t_\nu^* t_\nu t_\mu^*).$$

Since ϕ is a KMS$_\beta$ state, we may apply the KMS condition to see that

$$\phi(t_\mu t_\nu^* t_\nu t_\mu^*) = e^{-\beta r \cdot (d(\mu) - d(\nu))} \phi(t_\nu t_\mu^* t_\mu t_\nu^*).$$

We apply (TCK3) again to see that, since $r \cdot d(\mu) = r \cdot d(\nu)$,

$$e^{-\beta r \cdot (d(\mu) - d(\nu))} \phi(t_\nu t_\mu^* t_\mu t_\nu^*) = \phi(t_\nu t_\mu^* t_\mu t_\nu^*) = \phi(t_\nu t_{s(\mu)} t_\nu^*) = \phi(t_\nu t_{s(\nu)} t_\nu^*) = \phi(t_\nu t_\nu^*),$$

and hence $\phi(t_\mu t_\mu^*) = \phi(t_\nu t_\nu^*)$.

For the second statement, we observe that the Cauchy-Schwarz inequality for the sesquilinear form $(a, b) \mapsto \phi(ab^*)$ gives

$$|\phi(t_\mu t_\nu^*)|^2 \leq \phi(t_\mu t_\mu^*) \phi(t_\nu t_\nu^*) = \phi(t_\mu t_\mu^*)^2,$$

and hence $|\phi(t_\mu t_\nu^*)| \leq \phi(t_\mu t_\mu^*)$. □

Lemma 6.2.2 *Let Λ be a finite k-graph with no sources. Suppose that ϕ is a* KMS_β *state of $(\mathcal{T}C^*(\Lambda, c), \alpha)$ for some $\beta > 0$, and that $\mu, \nu \in \Lambda$ satisfy $s(\mu) = s(\nu)$.*

(i) If $\lambda \in \Lambda$ satisfies $\Lambda^{\min}(\mu\lambda, \nu\lambda) = \emptyset$, then $\phi(t_{\mu\lambda} t_{\nu\lambda}^*) = 0$.

(ii) Let $n := (d(\mu) \vee d(\nu)) - d(\mu)$. For $j \in \mathbb{N}$ we have

$$\phi(t_\mu t_\nu^*) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} c(\mu, \lambda)\overline{c(\nu\lambda)}\phi(t_{\mu\lambda} t_{\nu\lambda}^*). \tag{6.5}$$

Proof.

(i) Since $t_{\nu\lambda}^* t_{\mu\lambda} = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu\lambda,\mu\lambda)} \overline{c(\nu\lambda, \alpha)} c(\mu\lambda, \beta) t_\alpha t_\beta^*$, we see that if $\Lambda^{\min}(\mu\lambda, \nu\lambda)$ is empty, then $t_{\nu\lambda}^* t_{\mu\lambda} = 0$. So the KMS condition gives

$$\phi(t_{\mu\lambda} t_{\nu\lambda}^*) = e^{-\beta r \cdot d(\mu\lambda)} \phi(t_{\nu\lambda}^* t_{\mu\lambda}) = 0.$$

(ii) We will prove this by induction. The statement is trivial for $j = 0$, giving a base case. Now suppose as an inductive hypothesis that equation (6.5) holds for some fixed $j \in \mathbb{N}$. We must prove that it holds for $j + 1$. By (i) we only need to consider the case $\Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset$. Using the KMS condition, we compute

$$\phi(t_{\nu\lambda} t_{\nu\lambda}^* t_{\mu\lambda} t_{\nu\lambda}^*) = e^{-\beta r \cdot (d(\nu\lambda) - d(\nu\lambda))} \phi(t_{\mu\lambda} t_{\nu\lambda}^* t_{\nu\lambda} t_{\nu\lambda}^*) = \phi(t_{\mu\lambda} t_{\nu\lambda}^*).$$

So we have

$$\begin{aligned}
\phi(t_{\mu\lambda} t_{\nu\lambda}^*) &= \phi(t_{\nu\lambda} t_{\nu\lambda}^* t_{\mu\lambda} t_{\nu\lambda}^*) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu\lambda,\mu\lambda)} c(\nu\lambda, \alpha)\overline{c(\nu\lambda, \alpha)} c(\mu\lambda, \beta)\overline{c(\nu\lambda, \beta)} \phi(t_{\nu\lambda\alpha} t_{\nu\lambda\beta}^*) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu\lambda,\mu\lambda)} c(\mu\lambda, \beta)\overline{c(\nu\lambda, \beta)} \phi(t_{\nu\lambda\alpha} t_{\nu\lambda\beta}^*) \\
&= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\nu\lambda,\mu\lambda)} c(\mu\lambda, \beta)\overline{c(\nu\lambda, \beta)} \phi(t_{\mu\lambda\beta} t_{\nu\lambda\beta}^*).
\end{aligned}$$

Now we use the inductive hypothesis to obtain

$$\begin{aligned}\phi(t_\mu t_\nu^*) &= \sum_{\lambda\in s(\mu)\Lambda^{jn}} c(\mu,\lambda)\overline{c(\nu\lambda)}\phi(t_{\mu\lambda}t_{\nu\lambda}^*)\\ &= \sum_{\lambda\in s(\mu)\Lambda^{jn}} \sum_{\substack{(\alpha,\beta)\in\\ \Lambda^{\min}(\nu\lambda,\mu\lambda)}} c(\mu,\lambda)\overline{c(\nu\lambda)}c(\nu\lambda,\beta)\overline{c(\mu\lambda,\beta)}\phi(t_{\mu\lambda\beta}t_{\nu\lambda\beta}^*). \qquad (6.6)\end{aligned}$$

In order to see that this completes the induction, we must first simplify this sum. First, we simplify the product of the cocycles, then we simplify the set over which we are summing. So, for $(\alpha,\beta)\in\Lambda^{\min}(\nu\lambda,\mu\lambda)$ we note that

$$d(\mu\lambda)+d(\beta)=d(\mu\lambda\beta)=d(\nu\lambda)\vee d(\mu\lambda)=(d(\mu)\vee d(\nu))+d(\lambda),$$

and hence $d(\beta)=(d(\mu)\vee d(\nu))-d(\mu)=n$. So $d(\lambda\beta)=(j+1)n$. Now let $\tau=\lambda\beta$. Then by the above calculation and since $\lambda\in s(\mu)\Lambda^{jn}$, we see that $\tau\in s(\mu)\Lambda^{(j+1)n}$. Now we may compute the product of the cocycles in (6.6) via the cocycle axioms. We see that

$$\begin{aligned}c(\mu,\lambda)\overline{c(\nu,\lambda)}c(\nu\lambda,\beta)\overline{c(\mu\lambda,\beta)} &= c(\mu,\lambda)\overline{c(\nu\lambda)c(\mu\lambda)}c(\lambda,\beta)c(\mu,\lambda\beta)c(\nu,\lambda)\overline{c(\lambda,\beta)c(\nu,\lambda\beta)}\\ &= c(\mu,\lambda\beta)\overline{c(\nu,\lambda\beta)}\\ &= c(\mu,\tau)\overline{c(\nu,\tau)}.\end{aligned}$$

To simplify the set over which we are summing, we observe that the map

$$s(\mu)\Lambda^{jn}\times\Lambda^{\min}(\nu\lambda,\mu\lambda)\ni(\lambda,(\alpha,\beta))\mapsto\lambda\beta=\tau\in\Lambda^{(j+1)n}$$

is a bijection. Hence equation (6.6) becomes

$$\phi(t_\mu t_\nu^*)=\sum_{\tau\in s(\mu)\Lambda^{(j+1)n}} c(\mu,\tau)\overline{c(\nu,\tau)}\phi(t_{\mu\tau}t_{\nu\tau}^*).$$

Therefore by induction we have the desired result. □

With the preceding two results, we are able to strengthen Proposition 6.1.2 to include all $r\in(0,\infty)^k$, albeit with a new constraint. We now require that β is 'large' in the sense made precise below.

Theorem 6.2.3 *Let Λ be a finite k-graph with no sources, and let A_i $(i=1,\dots,k)$ be* *the vertex matrices of* Λ. *Let* $r \in (0,\infty)^k$, *let* $\gamma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ *be the gauge action, and define* $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}C^*(\Lambda, c))$ *by* $\alpha_t = \gamma_{e^{itr}}$. *Suppose* $\beta \in (0,\infty)$ *satisfies* $\beta r_i > \ln(\rho(A_i))$ *for each* $1 \le i \le k$. *Then a state* ϕ *on* $\mathcal{T}C^*(\Lambda, c)$ *is a* KMS_β *state for* α *if and only if*

$$\phi(t_\mu t_\nu^*) = \delta_{\mu,\nu} e^{-\beta r\cdot d(\mu)} \phi(t_{s(\mu)}) \tag{6.7}$$

for all $\mu, \nu \in \Lambda$ *such that* $s(\mu) = s(\nu)$.

Proof. Let ϕ be a KMS_β state on $(\mathcal{T}C^*(\Lambda, c), \alpha)$, and take $\mu, \nu \in \Lambda$ such that $s(\mu) = s(\nu)$ and $d(\mu) \neq d(\nu)$. Proposition 6.1.2 has already done most of the hard work for us, so as noted earlier we only need to prove that $\phi(t_\mu t_\nu^*) = 0$ in the case that $r\cdot d(\mu) = r \cdot d(\nu)$. Since $d(\mu) \neq d(\nu)$, either $(d(\mu) \vee d(\nu)) - d(\mu)$ or $(d(\mu) \vee d(\nu)) - d(\nu)$ is nonzero (or both). Since $\phi(t_\mu t_\nu^*) = 0$ if and only if $\phi(t_\nu t_\mu^*) = 0$, we may assume that $n := (d(\mu) \vee d(\nu)) - d(\mu)$ is nonzero without loss of generality. Now let $j \in \mathbb{N}$. By Lemma 6.2.2 we have

$$\phi(t_\mu t_\nu^*) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} c(\mu, \lambda)\overline{c(\nu\lambda)} \phi(t_{\mu\lambda} t_{\nu\lambda}^*).$$

For all $\lambda \in s(\mu)\Lambda^{jn}$, we have $r \cdot d(\mu\lambda) = r \cdot d(\nu\lambda)$. Hence we apply the triangle inequality and Lemma 6.2.2 to see that

$$\begin{aligned} |\phi(t_\mu t_\nu^*)| &= \left| \sum_{\lambda \in s(\mu)\Lambda^{jn}} c(\mu, \lambda)\overline{c(\nu\lambda)} \phi(t_{\mu\lambda} t_{\nu\lambda}^*) \right| \\ &\le \sum_{\lambda \in s(\mu)\Lambda^{jn}} |c(\mu, \lambda)\overline{c(\nu\lambda)} \phi(t_{\mu\lambda} t_{\nu\lambda}^*)| \\ &\le \sum_{\lambda \in s(\mu)\Lambda^{jn}} |\phi(t_{\mu\lambda} t_{\nu\lambda}^*)| \\ &\le \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\mu\lambda}^*). \end{aligned} \tag{6.8}$$

Now we apply Proposition 6.1.2 to see that $\phi(t_{\mu\lambda} t_{\mu\lambda}^*) = e^{-\beta r \cdot d(\mu\lambda)} \phi(t_{s(\lambda)})$. Therefore equation (6.8) becomes

$$|\phi(t_\mu t_\nu^*)| \le \sum_{\lambda \in s(\mu)\Lambda^{jn}} \phi(t_{\mu\lambda} t_{\mu\lambda}^*) = \sum_{\lambda \in s(\mu)\Lambda^{jn}} e^{-\beta r\cdot d(\mu\lambda)} \phi(t_{s(\lambda)}) = \sum_{\omega \in \Lambda^0} |s(\mu)\Lambda^{jn}\omega| e^{-\beta r\cdot (d(\mu) + jn)} \phi(t_\omega).$$

Rewriting in terms of the vertex matrices A_i we obtain

$$|\phi(t_\mu t_\nu^*)| \leq e^{-\beta r\cdot d(\mu)} \sum_{\omega\in\Lambda^0} \left(\prod_{i=1}^k e^{-\beta r_i j n_i} A_i^{jn_i}\right)(s(\mu),\omega)\ \phi(t_\omega). \tag{6.9}$$

For each i such that $n_i > 0$, $e^{-\beta r_i j n_i} A_i^{jn_i}$ is the $(jn_i)^{\text{th}}$ term in $\sum e^{-\beta r_i m} a_i^m$. So the following argument from [4, Theorem 5.1] holds: since $\beta r_i > \ln(\rho(A_i))$, this series converges to $(1 - e^{-\beta r_i} A_i)^{-1}$. In particular, we must have $e^{-\beta r_i j n_i} A_i^{jn_i} \to 0$ as $j \to \infty$ for each such i. Since n is nonzero, there is at least one such i, and thus as $j \to \infty$ the right hand side of (6.9) converges to 0. Hence $\phi(t_\mu t_\nu^*) = 0$. □

We now discuss how this result is related to the measure-theoretic result obtained in Theorem 5.2.3. First we note the difference in the way we dealt with the dynamics on $\mathcal{T}C^*(\Lambda, c)$. In Theorem 5.2.3 we studied the preferred dynamics, which corresponds to using $r = \ln(\rho(\Lambda))$ to define the dynamics α which was used in Theorem 6.2.3. That is, Theorem 6.2.3 applies to a general class of dynamics, whereas Theorem 5.2.3 applies to a specific dynamics, namely the preferred dynamics. We also note that Theorem 5.2.3 deals only with the case $\beta = 1$, whereas Theorem 6.2.3 deals with the case $\beta > 1$.

We now explain how to recover Theorem 6.2.3 for the preferred dynamics using the results of Chapter 4. In order to obtain Theorem 5.2.3 we relied heavily on [1, Corollary 4.2]. So in order to generalise to the case $\beta > 1$, we examine the conditions on the measure given in [1, Corollary 4.2]. The measure μ is a quasi-invariant probability measure on $\mathcal{G}_\Lambda^{(0)} = W_\Lambda$ with Radon-Nykodym cocycle $e^{-\beta D}$. Consider a bisection $Z(\mu, \nu)$. Then the Radon-Nykodym cocycle condition says that for any measurable $V \subseteq Z(\nu)$, we have $\mu(r(Z(\mu,\nu)V)) = \rho(\Lambda)^{-\beta(d(\mu)-d(\nu))}\mu(V)$. Let $W_\Lambda^{\geq n}$ denote the subset of W_Λ with degree greater than or equal to $n \in (\mathbb{N}\cup\{\infty\})^k$. Now consider for $i \in \{1,\dots,k\}$ the vector x^i defined by $x^i := (\mu(W_\Lambda^{\geq\infty e_i} \cap Z(v)))_{v\in\Lambda^0}$. So we have

$$\begin{aligned}
x_v^i &= \mu(W_\Lambda^{\geq\infty e_i} \cap Z(v)) \\
&= \sum_{\alpha\in v\Lambda^{e_i}} \mu(W_\Lambda^{\geq\infty e_i} \cap Z(\alpha)) \\
&= \sum_{\alpha\in v\Lambda^{e_i}} \mu(r(Z(\alpha, s(\alpha))W_\Lambda^{\geq\infty e_i})) \\
&= \rho(\Lambda)^{-\beta e_i} \sum_{w\in\Lambda^0} \sum_{\alpha\in v\Lambda^{e_i} w} \mu(Z(w)W_\Lambda^{\geq\infty e_i}) \\
&= \rho(\Lambda)^{-\beta e_i} (A_i x^i)_v,
\end{aligned}$$

and we have $\rho(A_i)^\beta x^i = A_i x^i$. This contradicts Proposition 2.1.19 (b) since $\beta > 1$. Hence we must have $x^i = 0$. Hence the measure μ is supported only on the finite paths of the k-graph. That is, $\mu(W_\Lambda \setminus \Lambda) = 0$. Therefore, since Λ is a discrete subspace of W_Λ, we deduce that μ is a sum of point-mass measures. The Radon-Nykodym cocycle condition now implies that for all $\lambda \in \Lambda$, we have $\mu(\{\lambda\}) = \rho(\Lambda)^{-\beta d(\lambda)}\mu(\{s(\lambda)\})$. Comparing this to equation (6.7) we see that Theorems 5.2.3 and 6.2.3 agree with each other in the sense that the KMS structure of the algebra is determined by the vertex structure of the k-graph.

Indeed, we can actually recover equation (6.7) from equation (5.2) of Afsar and Sims [1, Corollary 4.2]. By Theorem 3.3.1 the generators t_λ of $\mathcal{T}C^*(\Lambda, c)$ are mapped to the generators $1_{Z(\lambda,s(\lambda))}$, and so we will calculate the value of the state at $f = 1_{Z(\tau,s(\tau))} * 1^*_{Z(\nu,s(\nu))}$. Following equation (5.2) we have

$$\begin{aligned}\psi(f) &= \int_{W_\Lambda} \sum_{u \in (\mathcal{G}_\Lambda)^x_x} f(u)\psi_x(W_u)d\mu(x)\\ &= \sum_{\lambda\in\Lambda} f(\lambda,0,\lambda)\psi_\lambda\left(W_{(\lambda,0,\lambda)}\right)\mu(\{\lambda\}),\end{aligned}$$

since μ must be a sum of point-mass measures on Λ as above, and since each $(\mathcal{G}_\Lambda)^\lambda_\lambda = \{(\lambda,0,\lambda)\}$. By equation (3.20), we have $f = 1_{Z(\tau,s(\tau))} * 1^*_{Z(\nu,s(\nu))} = 1_{Z(\tau,\nu)}$, and therefore we obtain

$$\begin{aligned}\psi(f) &= \sum_{\lambda\in\Lambda} 1_{Z(\tau,\nu)}(\lambda,0,\lambda)\psi_\lambda\left(W_{(\lambda,0,\lambda)}\right)\mu(\{\lambda\})\\ &= \delta_{\tau,\nu}\psi_\nu\left(W_{(\nu,0,\nu)}\right)\mu(\{\nu\}),\end{aligned}$$

since there is only one non-zero summand, namely when $\lambda = \tau = \nu$. Finally, by the Radon-Nykodym property of μ given above, and since the cocycles in equation (5.1) are all equal to 1, we obtain

$$\psi(f) = \rho(\Lambda)^{-\beta d(\mu)}\psi_{s(\nu)}\left(W_{(s(\nu),0,s(\nu))}\right)\mu(\{s(\nu)\}).$$

By a similar calculation, $\psi(1_{Z(s(\nu),s(\nu))}) = \psi_{s(\nu)}\left(W_{(s(\nu),0,s(\nu))}\right)\mu(\{s(\nu)\})$. Hence

$$\psi\left(1_{Z(\tau,s(\tau))} * 1^*_{Z(\nu,s(\nu))}\right) = \delta_{\tau,\nu}\rho(\Lambda)^{-\beta d(\mu)}\psi(1_{Z(s(\nu),s(\nu))}),$$

which matches equation (6.7).

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