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Higher dimensional orthogonal designs and Hadamard matrices

Abstract

We construct n -dimensional orthogonal designs of type $(1, 1)_n$, side 2 and propriety $(2, 2, \dots, 2)$. These are then used to show that orthogonal designs of type $(2t, 2t)_n$, side $2t+1$ and propriety $(2, 2, \dots, 2)$ exist.

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We construct n -dimensional orthogonal designs of type $(1,1)^n$, side 2 and propriety $(2,2,\dots,2)$. These are then used to show that orthogonal designs of type $(2^t, 2^t)^n$, side 2^{t+1} and propriety $(2,2,\dots,2)$ exist.

1. INTRODUCTION

In [2] it is pointed out that it is possible to define orthogonality for higher dimensional matrices in many ways.

Intuitively we see that each two-dimensional matrix with the n -dimensional matrix could have orthogonal row vectors (we call this propriety $(2,2,\dots,2)$); or perhaps each pair of two-dimensional layers

$$A^j = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{pmatrix} \quad \text{and} \quad B^j = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_t \end{pmatrix}$$

could have $A \cdot B = \text{tr}(AB^T) = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_t \cdot b_t = 0$ (note if the row vectors in this direction had been orthogonal we would have had $a_i \cdot b_i = 0$ for each i) (we call this propriety $(\dots, 3, \dots)$); or perhaps each pair of three-dimensional layers

$$\alpha = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^t \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} B^1 \\ B^2 \\ \vdots \\ B^t \end{pmatrix}$$

could have $\alpha \cdot \beta = A^1 \cdot B^1 + \dots + A^t \cdot B^t = 0$ (note that if the 2-dimensional matrices had been orthogonal we would have had $A^j \cdot B^j = 0$ for each j); and so on.

We say an n -dimensional matrix is orthogonal of propriety $(d_1, \dots, d_n)$ with $2 \leq d_i \leq n$ where d_i indicates that in the i^{th} direction (i.e., the i^{th} coordinate) the $d_i - 1^{\text{st}}, d_i^{\text{th}}, d_i + 1^{\text{st}}, \dots, (n-1)^{\text{st}}$ dimensional layers are orthogonal but the $d_i - 2^{\text{nd}}$ layer is not orthogonal. $d_i = \infty$ means not even the $(n-1)^{\text{st}}$ layers are orthogonal.

The Paley cube of size $(q+1)^n$ constructed in [2] for $q \equiv 3 \pmod{4}$ a prime power has propriety $(\infty, \infty, \dots, \infty)$ but if the 2-dimensional layer of all ones is removed in one direction the remaining n -dimensional matrix has all 2-dimensional layers in

that direction orthogonal.

An n -cube orthogonal design, $D = [d_{ijk...}]$, of propriety $(d_1, d_2, \dots, d_n)$, side d and type $(s_1, s_2, \dots, s_t)^n$ on the commuting variables $x_1, x_2, \dots, x_t$ has entries from the set $\{0, \pm x_1, \dots, \pm x_t\}$ where $\pm x_i$ occurs s_i times in each row and column of each 2-dimensional layer and in which each e_j -dimensional layer, $d_i - 1 \leq e_j \leq n - 1$, in the i^{th} direction is orthogonal.

Shlichta [3] found n -dimensional Hadamard matrices of size $(2^t)^n$ and propriety $(2, 2, \dots, 2)$. In [2] the concept of higher dimensional m -suitable matrices was introduced to show that if t is the side of 4 Williamson matrices there is a 3-dimensional Hadamard matrix of size $(4t)^3$ and propriety $(2, 2, 2)$.

2. n -DIMENSIONAL ORTHOGONAL DESIGNS OF TYPE $(1, 1)^n$ AND SIDE 2

Theorem. *There exists an n -dimensional orthogonal design of type $(1, 1)^n$, side 2 and propriety $(2, 2, \dots, 2)$.*

Proof. Let a and b be commuting variables and $[h_{ijk...}]$ be the orthogonal design. Define $w = i + j + k + \dots + v$, the weight of the subscripts which can only assume the values 0 and 1 for side 2.

Now define

$$h_{ijk...v} = \begin{cases} (-1)^{\frac{1}{2}w+1}a & w \equiv 0 \pmod{2}, \\ (-1)^{\frac{1}{2}w-1}b & w \equiv 1 \pmod{2}. \end{cases}$$

In order to check the orthogonality we consider

$$h_{00x}h_{01x} + h_{10x}h_{11x} \quad (*)$$

and

$$h_{00x}h_{10x} + h_{01x}h_{11x} \quad (**)$$

where x is a constant vector of $n-2$ subscripts. For convenience we put the two varying constants first but of course we are really checking then in each of $n(n-1)$ positions. Suppose $v = \text{sum of the subscripts in } x$. Then we have four cases:

(1) $v \equiv 0 \pmod{4}$ then $(*)$ and $(**)$ both become

$$-ab + ba = 0;$$

(2) $v \equiv 1 \pmod{4}$ then $(*)$ and $(**)$ both become

$$ba + a(-b) = 0;$$

(3) $v \equiv 2 \pmod{4}$ then (*) and (**) both become

$$a(-b) + (-b)(-a) = 0;$$

(4) $v \equiv 3 \pmod{4}$ then (*) and (**) both become

$$(-b)(-a) + (-a)b = 0.$$

Hence each face of this orthogonal n-cube is a 2-dimensional orthogonal design and so we have a proper n-dimensional orthogonal design of type $(1,1)^n$.

Corollary. *There exist n-dimensional orthogonal designs of types $(2^t, 2^t)^n$, side 2^{t+1} and propriety $(2, 2, \dots, 2)$.*

Proof. Take the Kronecker product of Shlichta's proper n-dimensional Hadamard matrices of order $(2^t)^n$ with the orthogonal design established in the theorem.

This is illustrated in the Figure.

REFERENCES

- [1] A.V. Geramita and Jennifer Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices* (Marcel Dekker, New York, 1979).
- [2] Joseph Hammer and Jennifer Seberry, Higher dimensional orthogonal designs and applications, *IEEE Trans. Inform. Theory*, (to appear).
- [3] P.J. Shlichta, Higher dimensional Hadamard matrices, *IEEE Trans. Inform. Theory*, (to appear).

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$$A_1 \otimes A_2$$