All DBIBDs with block size four exist

Deborah J. Street

Jennifer Seberry

University of Wollongong, jennie@uow.edu.au

Follow this and additional works at: https://ro.uow.edu.au/infopapers

Part of the Physical Sciences and Mathematics Commons

Recommended Citation
Street, Deborah J. and Seberry, Jennifer: All DBIBDs with block size four exist 1980.
https://ro.uow.edu.au/infopapers/1003
All DBIBDs with block size four exist

Abstract
A directed balanced incomplete block design with parameters \((v,b,r,k,\lambda^*)\), is a balanced incomplete block design with parameters \((v,b,r,k,2\lambda^*)\), in which the blocks are regarded as ordered \(k\)-tuples and in which each ordered pair of elements occurs in \(\lambda^*\) blocks. By generalizing results of Hanani, we show that the necessary conditions for the existence of these designs, when \(k = 4\), are sufficient.

Disciplines
Physical Sciences and Mathematics

Publication Details
Street, DJ and Seberry, J, All DBIBDs with block size four exist, Utilitas Mathematica, 18, 1980, 27-34.

This journal article is available at Research Online: https://ro.uow.edu.au/infopapers/1003
A directed balanced incomplete block design (DBIBD), with parameters \((v,b,r,k,A^*)\), is a balanced incomplete block design (BIBD), with parameters \((v,b,r,k,2A^*)\), in which the blocks are regarded as ordered \(k\)-tuples and in which each ordered pair of elements occurs in \(2A^*\) blocks. By generalizing results of Hanani, we show that the necessary conditions for the existence of these designs, when \(k = 4\), are sufficient.

When \(k = 3\), Seberry and Skillicorn [3] have shown that the necessary conditions for the existence of a DBIBD are sufficient. In this paper, by generalizing results of Hanani [1], we show that for \(k = 4\) the necessary conditions are sufficient.
CD(k,λ,μ;ν) exists and DGD(k,λ,μ;ν) is the set of all ν such that a DGD(k,λ,μ;ν) exists.

As a consequence of the remarks above and the results in Hanani [1] we have the following.

**Lemma 1** (cf. [1, Lemma 4.10]).

\{(12,15) ∈ DGD(4,1*,3)\}.

**Lemma 2** (cf. [1, Lemma 5.11]). If \( ν \equiv 1 \) or \( 4 \) (mod 12), then \( ν ∈ DS(4,1*) \).

**Lemma 3** (cf. [1, Lemma 5.13]). If \( ν \equiv 0 \) or \( 1 \) (mod 4), then \( ν ∈ DS(4,3*) \).

**Lemma 4** (cf. [1, Lemma 2.26]). If \( n ∈ CD(3,1,R) \), \( mR+1 ∈ DB(k,λ*) \) and \( mS ∈ DGD(k,λ*,μ) \), then \( mnH \) \( DB(k,λ*) \).

**Proof.** Let the points of the design be \( \{1,2,\ldots,n\} × \{1,2,\ldots,m\} \) \( \cup \{\ast\} = (X × Y) \cup \{\ast\} \). Using the elements of \( Y \), construct a CD(cS,1,R;μ). For each group \( G \) of this design, \( |S| ∈ R \) and so we may construct a DBIBD(μ|G|+1,b,r,k,λ*) on \( X × Y \cup \{\ast\} \). For each block \( B \) of the CD, construct a DGD(k,λ*,μ;θ) on \( X × B \).

We now prove several lemmas which are used to construct a few initial designs from which the existence of all others may be deduced.

**Lemma 5** (cf. [1, Lemma 2.11]).

\( DGD(k,λ^{*},k-1) + 1 ∈ DS(k,λ*) \).

**Proof.** Adjoin a fixed additional point to each group of the DGD design. Then the additional blocks are the groups with fixed additional point written in some order, and the reverse of that order, each repeated \( λ* \) times.
LEMMA 6 (cf. [1, Lemma 2.16]). If \( n \in \text{DB}(K, \lambda^*) \) and \( mK \in \text{GD}(k, \lambda, m) \), then

\[ mn \in \text{GD}(k, \lambda, m) \].

Proof. Let the \( n \) groups of the \( \text{GD}(k, \lambda, m) \) be \( G_1, G_2, \ldots, G_n \) and use the set of symbols \( \{G_1, G_2, \ldots, G_n\} \) to construct a \( \text{DBIBD}(n, b, r, k, \lambda^*) \). For each block of this design use the groups in that block to construct a \( \text{GD}(k, \lambda, m) \) where, if \( G_i \) appears before \( G_j \) in the block, then elements from \( G_i \) appear before elements from \( G_j \) in the \( \text{GD}(k, \lambda, m) \).

COROLLARY 6.1. If \( n \in \text{DB}(K, \lambda^*) \) and \((k-1)K+1 \in B(k, 1)\) then \((k-1)n + 1 \in \text{DB}(k, \lambda^*)\).

Proof. Use Lemmas 5 and 6, and Lemma 2.12 of Hanani [1].

LEMMA 7 (cf. [1, Lemma 4.11]. Let \( q \) be a prime power, where \( q = 2f+1, f \) odd. Then \( q \in \text{DB}(4, 3^*) \).

Proof. Let \( C_0 = \{1, x^f\} \), where \( x \) is a generator of \( \text{GF}(q)^* \) (the multiplicative group of \( \text{GF}(q) \)) and let \( C_i = x^i \cdot 1 = (x^i, x^f) \), \( 0 \leq i \leq f-1 \). We will consider these sets as ordered pairs, writing \( D_i \) for the pair \( \{x^i, x^f\} \). It is then straightforward to check that the required blocks are \( C_0 \cup D_1 \cup D_2 \cup \ldots \cup D_{f-1} \cup D_0 \) (regarding these as ordered 4-tuples).

LEMMA 8 (cf. [1, Lemma 5.12]). If \( v \equiv 1 \pmod{3} \) then \( v \in \text{DB}(4, 1^*) \).

Proof. Let \( v = 3n+1 \), where \( n \) is a positive integer. By Hanani [1, Lemma 5.9], \( n \in \text{DB}(14, 5), 1, 1^* \), where \( M_4^* = \{1, 2, \ldots, 15, 26, 27\} \). By Lemmas 1 and 4 it suffices to show that \( 3M_4^* + 1 \in \text{DB}(4, 1^*) \). For \( n \in M_4^*, n \equiv 0 \) or \( 1 \pmod{4}, \) the result follows from Lemma 2; for the remaining values of \( v \) a solution is given in Table 1.
LEMMA 9 (cf. [1, Lemma 5.14]). For every integer \( v \geq 4, \)
\[ v \in DB(4, 3^k). \]

Proof. By Hanani [1, Lemma 5.10], it suffices to show that \( v \in DB(4, 3^k) \)
for all \( v \in \{4, 5, ..., 12, 14, 15, 18, 19, 23, 27\} \). If \( v \equiv 1 \pmod{3} \) then
Lemma 8 gives the result; if \( v \equiv 0 \) or \( 1 \pmod{4} \), then Lemma 3
gives the result; for the remaining values of \( v \) a solution is given
in Table II.

THEOREM. Let \( \lambda^k \) and \( v \geq 4 \) be given positive integers. A necessary
and sufficient condition for the existence of a DBIBD\((v, b, r, 4, \lambda^k)\)
is that
\[ \lambda^k(v-1) \equiv 0 \pmod{3} \quad \text{and} \quad \lambda^k v(v-1) \equiv 0 \pmod{6}. \]

Proof. That these conditions are necessary follows from the usual
counting arguments for BIBDs. We need only consider values of \( \lambda^k \)
which are factors of 3, as if \( \lambda^k | \lambda \), then \( DB(k, \lambda) = DB(k, \lambda^k) \). Thus we have
the following cases:

\[ \lambda^k = 1 \quad v \equiv 1 \pmod{3} \]
\[ \lambda^k = 3 \quad \text{all } v. \]

In Lemmas 8 and 9 we have established the existence of the required
designs.
### Table I.

<table>
<thead>
<tr>
<th>v</th>
<th>DBIBD(v, b, r, 4, 1*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Initial block (6, 0, 3, 5) developed mod 7 (in [3]).</td>
</tr>
<tr>
<td>10</td>
<td>Form the residual of the (16, 6, 1*) constructed from the initial block (a, b, c, d, ab, cd).</td>
</tr>
<tr>
<td>19</td>
<td>Initial blocks to be developed mod 19: (0, 3, 12, 1); (13, 1, 5, 0); (4, 9, 6, 0).</td>
</tr>
<tr>
<td>22</td>
<td>Use Corollary 6.1 and 7 ε DB(4, 1*) from above.</td>
</tr>
<tr>
<td>31</td>
<td>Initial blocks to be developed mod 31: (0, 1, 8, 11); (14, 11, 0, 2); (7, 13, 5, 0); (0, 15, 5, 9); (1, 17, 0, 13).</td>
</tr>
<tr>
<td>34</td>
<td>Use Corollary 6.1 with n&quot; = 11, K&quot; = 5, λ* = 1, and k = 4. (The initial block for the DBIBD(11, 11, 5, 5, 1*) is (3, 5, 1, 4, 9).)</td>
</tr>
<tr>
<td>43a</td>
<td>Initial blocks to be developed mod 43: (1, 0, 6, 36); (26, 0, 33, 27); (35, 0, 13, 38); (31, 0, 41, 14); (3, 0, 22, 18); (23, 9, 0, 11); (39, 19, 0, 28).</td>
</tr>
<tr>
<td>46b</td>
<td>We show 45 ε DGD(4, 1*, 3) and apply Lemma 5. Initial blocks to be developed mod (3; 3, 5): ((0; 0, 0), (0; 1, 0), (1; 2, 2), (1; 2, 3)); ((1; 2, 4), (0; 1, 0), (0; 0, 0), (1; 2, 2)); ((0; 2, 0), (2; 1, 3), (1; 2, 0), (0; 1, 0)); ((0; 2, 0), (0; 1, 0), (2; 1, 2), (1; 2, 0)); ((0; 2, 0), (0; 1, 2), (2; 2, 1), (1; 1, 0)); ((1; 1, 0), (0; 2, 0), (2; 2, 0), (0; 1, 3)); ((0; 0, 1), (0; 0, 0), (0; 1, 3), (0; 1, 2)).</td>
</tr>
<tr>
<td>75a</td>
<td>Initial blocks to be developed mod 79: (1, 0, 23, 55); (18, 0, 42, 19); (54, 0, 57, 47); (8, 0, 45, 26); (3, 6, 9, 7); (6, 0, 59, 14); (4, 0, 13, 62); (29, 0, 13, 35); (9, 49, 0, 21); (2, 0, 46, 31); (12, 0, 39, 28); (36, 0, 5, 38); (27, 68, 0, 63).</td>
</tr>
</tbody>
</table>
We show \( a \in D_{50}(4,1^*,3) \) and apply Lemma 5. Initial blocks to be developed \( \mod (3;2) \):

\[
\begin{align*}
(0;1), (0;2), (1;x), (1;2x), \quad & \\
(1;2x^2+x+1), (1;x^2+2x+2), (0;2x^2+2x+2), (0;x^2+x+1), \quad & \\
(1;x^2+x+1), (1;2x^2+2), (0;x^2+2x+2), (0;2x^2+2x+1), & \\
(0;2x^2+2), (0;2x^2+1), (1;2x^2), (0;2x^2+1). &
\end{align*}
\]

- \( \dots \)

\[
\begin{align*}
(1;x^2+2x+1), (0;2x^2+x+1), (1;x^2+2x+1), (0;x^2+2x+1), & \\
(1;2x^2+2x+1), (0;x^2+2x), (1;x^2+2x+1), (0;2x^2+2x), & \\
(0;x^2+2x), (1;2x^2+2x+1), (0;x^2+2x+1). &
\end{align*}
\]

a) Found by William H. Wilson using a backtrack algorithm written in Pascal.

b) We use the notation of Hanani [1] except that the numbers are elements of the field instead of exponents of elements of the field.
Table II.

<table>
<thead>
<tr>
<th>v</th>
<th>DBIBD(v,b,r,4,3*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Blocks: (1,2,3,4); (5,3,2,1); (1,2,3,6); (1,2,4,5); (6,4,2,1); (6,5,2,1); (5,4,3,1); (6,4,3,1); (1,3,5,6); (1,4,5,6); (3,2,4,5); (6,2,3,5); (5,4,2,6); (6,5,4,3); (3,4,2,6).</td>
</tr>
</tbody>
</table>

Use Lemma 7.

Initial blocks to be developed mod 13 (= is a fixed element): (0,1,3,9); (9,3,1,0); (0,1,3,9); (9,3,1,0); (l,0,3,9); (9,3,1,0); (=,5,2,6).

Initial blocks to be developed mod (3,5):

- (((1,1), (1,4), (2,2), (2,3)); ((2,3), (2,2), (1,4), (1,1));
- ((1,2), (1,3), (2,1), (2,4)); ((2,4), (2,1), (1,3), (1,2));
- ((0,0), (0,1), (1,0), (2,0)); ((2,0), (1,0), (0,0), (0,2));
- ((2,3), (1,1), (1,4), (2,2)).

Initial blocks to be developed mod 17 (= is a fixed element):

- (1,4,13,16); (16,13,4,1); (3,5,12,14); (14,12,5,3);
- (2,8,9,15); (15,9,8,2); (6,11,10,7); (=,16,15,11);
- (9,10,13,6).

Use Lemma 7.

Use Lemma 7.
REFERENCES


University of Sydney
N.S.W. 2006
Australia

Received February 20, 1980.