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Abstract

Higher-dimensional orthogonal designs of type (l,l)n are used to obtain higher-dimensional weighing matrices of type (q)n, side q+l and propriety (2,2,...,2) for q = l(mod 4) a prime power. Next, n-dimensional orthogonal designs of type (l,l,l,l)n, side 4 and propriety (2,2, ...,2) are constructed. These are then used to show that higher-dimensional Hadamard matrices of order (4t)t exist whenever t is the side of 4-Williamson matrices. This establishes the existence of higher-dimensional Hadamard matrices of order (4t) t for t odd, 1 ≤ t ≤ 33 and several infinite families, all of propriety (2,2,...,2). Finally, we establish that if there is an Hadamard matrix that can be obtained from a group difference set with parameters (4s 2, 2s2±s, s2±s) then there is a higher-dimensional Hadamard matrix of order (4s2)n.

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HIGHER-DIMENSIONAL ORTHOGONAL DESIGNS AND HADAMARD MATRICES II

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Abstract: Higher-dimensional orthogonal designs of type $(1,1)^n$ are used to obtain higher-dimensional weighing matrices of type $(q)^n$, side $q+1$ and propriety $(2,2,\ldots,2)$ for $q \equiv 1 \pmod 4$ a prime power. Next, n-dimensional orthogonal designs of type $(1,1,1,1)^n$, side 4 and propriety $(2,2,\ldots,2)$ are constructed. These are then used to show that higher-dimensional Hadamard matrices of order $(4t)^t$ exist whenever t is the side of 4-Williamson matrices. This establishes the existence of higher-dimensional Hadamard matrices of order $(4t)^t$ for t odd, $1 \le t \le 33$ and several infinite families, all of propriety $(2,2,\ldots,2)$. Finally, we establish that if there is an Hadamard matrix that can be obtained from a group difference set with parameters $(4s^2, 2s^2 \pm s, s^2 \pm s)$ then there is a higher-dimensional Hadamard matrix of order $(4s^2)^n$.

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Introduction

In [2] it is pointed out that it is possible to define orthogonality for higher dimensional matrices in many ways.

Intuitively we see that each two-dimensional matrix within the n-dimensional matrix could have orthogonal row vectors (we call this propriety (2,2,...,2)); or perhaps each pair of two-dimensional layers

$$A^j = \begin{bmatrix} \underset{\sim}{a}_1 \\ \underset{\sim}{a}_2 \\ \vdots \\ \underset{\sim}{a}_t \end{bmatrix} \quad \text{and} \quad B^j = \begin{bmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \\ \vdots \\ \underset{\sim}{b}_t \end{bmatrix}$$

could have $A \cdot B = tr(AB^T) = \underset{\sim}{a}_1 \cdot \underset{\sim}{b}_1 + \underset{\sim}{a}_2 \cdot \underset{\sim}{b}_2 + \ldots + \underset{\sim}{a}_t \cdot \underset{\sim}{b}_t = 0$ (note if the row vectors in this direction had been orthogonal we would have had $\underset{\sim}{a}_i \cdot \underset{\sim}{b}_i = 0$ for each i) (we call this propriety (...,3,...)); or perhaps each pair of three-dimensional layers

$$\alpha = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^t \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^t \end{bmatrix}$$

could have $\alpha \cdot \beta = A^1 \cdot B^1 + \ldots + A^t \cdot B^t = 0$ (note that if the 2-dimensional matrices had been orthogonal we would have had $A^j \cdot B^j = 0$ for each j); and so on.

We say an n-dimensional matrix is orthogonal of *propriety* $(d_1, \ldots, d_n)$ with $2 \le d_i \le n$ where d_i indicates that in the i^{th} direction (i.e the ith coordinate) the $d_i - 1^{st}$, $d_i{}^{th}$, $d_i + 1^{st}, \ldots, (n-1)^{st}$ dimensional layers are orthogonal but the $d_i - 2^{nd}$ layer is not orthogonal. $d_i = \infty$ means not even the $(n-1)^{st}$ layers are orthogonal.

The Paley cube of size $(q+1)^n$ constructed in [2] for $q \equiv 3 \pmod 4$ a prime power has propriety $(\infty,\infty,\ldots,\infty)$ but if the 2-dimensional layer of all ones is removed in one direction the remaining n-dimensional matrix has all 2-dimensional layers in that direction orthogonal.

An *n-cube orthogonal design*, $D = [d_{ijk\ldots}]$, of propriety $(d_1,d_2,\ldots,d_n)$, side d and type $(s_1,s_2,\ldots,s_t)^n$ on the commuting variables $x_1,x_2,\ldots,x_t$ has entries from the set $\{0,\pm x_1,\ldots,\pm x_1,\ldots,\pm x_t\}$ where $\pm x_i$ occurs s_i times in each row and column of each 2-dimensional layer and in which each e_j-dimensional layer, $d_i-1 \le e_j \le n-1$, in the i^{th} direction is orthogonal.

Shlicta [3] found n-dimensional Hadamard matrices of size $(2^t)^n$ and propriety $(2,2,\ldots,2)$. In [2] the concept of higher dimensional m-suitable matrices was introduced to show that if t is the side of 4-Williamson matrices there is a 3-dimensional Hadamard matrix of size $(4t)^3$ and propriety $(2,2,2)$. In [4] it is shown that there are n-dimensional orthogonal designs of type $(2^t,2^t)^n$, side 2^t, $t \ge 0$ and propriety $(2,2,\ldots,2)$.

The results of [4] and [2] can be combined to give

THEOREM 1. *When* $q \equiv 1 \pmod 4$ *is a prime power, there exist higher-dimensional orthogonal designs of order* $(q)^{\frac{1}{2}(q+1)}$ *and propriety* $(2,2,\ldots,2)$.

The equivalence relations

A $(1,1,1,1)^{k+1}$ design would be preserved under the following equivalence relations:

(i) each variable is replaced throughout by its negative;

(ii) rearrangement of the parallel k-dimensional hyper-planes (for the $(1,1,1,1)^2$ design this is the rows and/or columns, for the $(1,1,1,1)^3$ design this is the parallel planes);

(iii) multiplication of every variable of one entire k-dimensional hyper-plane by -1.

There are exactly two inequivalent $(1,1,1,1)^2$ designs of order 4 on the variables a,b,c,d. They are

$$\underset{\text{I}}{\begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}} \quad \text{and} \quad \underset{\text{II}}{\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}}$$

A little checking shows that a $(1,1,1,1)^3$ design can be obtained with either I or II as the front face. The design in the figure at the end of the paper has faces from equivalence class II.

The $(1,1,1,1)^n$ *design*

We proceed inductively. First define

$$\begin{aligned} \underset{\sim}{a}_1 &= (-d,-c,b,a) \\ \underset{\sim}{a}_2 &= (c,-d,a,-b) \\ \underset{\sim}{a}_3 &= (b,a,d,c) \\ \underset{\sim}{a}_4 &= (-a,b,c,-d) \end{aligned}$$

and note

$$\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j = 0 \quad \text{for all } i \neq j.$$

Now we can describe the faces of the $(1,1,1,1)^3$ design given in Figure 4a of Hammer and Seberry [2], in one direction as:

$$\underset{\sim}{b}_4^T = \begin{bmatrix} \underset{\sim}{a}_4 \\ \underset{\sim}{a}_3 \\ \underset{\sim}{a}_2 \\ \underset{\sim}{a}_1 \end{bmatrix}, \quad \underset{\sim}{b}_3^T = \begin{bmatrix} \underset{\sim}{a}_3 \\ -\underset{\sim}{a}_4 \\ \underset{\sim}{a}_1 \\ -\underset{\sim}{a}_2 \end{bmatrix}, \quad \underset{\sim}{b}_2^T = \begin{bmatrix} \underset{\sim}{a}_2 \\ -\underset{\sim}{a}_1 \\ -\underset{\sim}{a}_4 \\ \underset{\sim}{a}_3 \end{bmatrix}, \quad \underset{\sim}{b}_1^T = \begin{bmatrix} -\underset{\sim}{a}_1 \\ -\underset{\sim}{a}_2 \\ \underset{\sim}{a}_3 \\ \underset{\sim}{a}_4 \end{bmatrix}$$

or

$$\begin{bmatrix} \underset{\sim}{b}_4 \\ \underset{\sim}{b}_3 \\ \underset{\sim}{b}_2 \\ \underset{\sim}{b}_1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{a}_4 & \underset{\sim}{a}_3 & \underset{\sim}{a}_2 & \underset{\sim}{a}_1 \\ \underset{\sim}{a}_3 & -\underset{\sim}{a}_4 & \underset{\sim}{a}_1 & -\underset{\sim}{a}_2 \\ \underset{\sim}{a}_2 & -\underset{\sim}{a}_1 & -\underset{\sim}{a}_4 & \underset{\sim}{a}_3 \\ -\underset{\sim}{a}_1 & -\underset{\sim}{a}_2 & \underset{\sim}{a}_3 & \underset{\sim}{a}_4 \end{bmatrix}$$

Now there are two inequivalent 2-dimensional (1,1,1,1) orthogonal designs but both of these can be completed to give a $(1,1,1,1)^3$ design. Thus we have a $(1,1,1,1)^3$ design on the commuting variables $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3, \underset{\sim}{a}_4$ and hence a $(1,1,1,1)^4$ design on the variables a,b,c,d.

The orthogonality within the $(1,1,1,1)^3$ design is established by construction. The orthogonality of the $(1,1,1,1)^4$ design is obtained by using the extra property that $\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j = 0$ for all $i \neq j$.

To obtain the $(1,1,1,1)^{k+1}$ design we assume the existence of the $(1,1,1,1)^j$ design for every $j \le k$ made by the construction. Now we have by construction a $(1,1,1,1)^k$ design whose hyper-rows, c_i, $i = 1,2,3,4$ comprise objects which are the hyper-rows of the $(1,1,1,1)^{k-1}$ design. We now write down the hyper-rows of each of the four hyper-planes containing these rows as the columns of a 4×4 matrix, D. By the construction D is an orthogonal design of type (1,1,1,1) whose objects are the c_i. We now complete D to form a $(1,1,1,1)^3$ design, E, with objects c_i. Now E is a $(1,1,1,1)^{k+1}$ design whose orthogonality is guaranteed by the existence assumption.

In fact at all stages of the construction propriety was completely preserved. Hence we have shown

THEOREM 2. *There exist higher-dimensional orthogonal designs of type* $(1,1,1,1)^n$ *and side* 4 *for every dimension* $n \ge 0$ *with propriety* $(2,2,\dots,2)$

Using the results of [2], we can now say

COROLLARY 3. *Let* t *be the order of* 4*-Williamson matrices. Then there is a higher-dimensional Hadamard matrix of order* $(4t)^t$. *In particular there are higher-dimensional Hadamard matrices of order* $(4t)^t$ *for all odd* $t < 100$ *except possibly* 35, 39, 47, 53, 59, 65, 67, 71, 73, 77, 83, 89. *All these matrices have propriety* $(2,2,\dots 2)$

n-dimensional Hadamard matrices from difference sets

Let $H = (h_{ij})$ be any Hadamard matrix of side h which can be defined by a function of the form

$$h_{ij} = \gamma(\alpha(i) + \alpha(j)),$$

where α is 1:1 and onto, so that

$$\sum_i h_{ai}h_{bi} = h\delta_{ab} = \sum_i \gamma(\alpha(a)+\alpha(i))\ \gamma(\alpha(b)+\alpha(i))$$

$$= \sum_k \gamma(k)\gamma(\alpha(b)-\alpha(a)+k)$$

$$= 0.$$

Example: Any Hadamard matrix that can be obtained from abelian group difference set is of this form.

THEOREM 4. *Let* $H = (h_{ij})$ *of side* h *be an Hadamard matrix of the type described above. Define* $G = (g_{pqr\ldots s})$ *of side* h *and dimension* n *by*

$$g_{pqr\ldots ts} = h_{p+q+r+\ldots+t,s}.$$

Then G *is a proper Hadamard matrix of side* h *and dimension* n.

Proof. We need to show any 2-dimensional face is an Hadamard matrix. We let the cth coordinate have two fixed values a and b and let the dth coordinate take the values $i = 1,\ldots,h$. We now fix all other coordinates and consider

$$\sum_i g_{pqr\ldots a\ldots i\ldots ts}\, g_{pqr\ldots b\ldots i\ldots ts}$$

x = sum of all subscripts except cth, dth and last

$$= \sum_i h_{x+a+i,s}\, h_{x+b+i,s}$$

$$= \sum_i \gamma(\alpha(x)+\alpha(a)+\alpha(i)+\alpha(s))\ \gamma(\alpha(x)+\alpha(b)+\alpha(i)+\alpha(s))$$

$$= \sum_k \gamma(k)\gamma(\alpha(b)-\alpha(a)+k)$$

$$= 0.$$

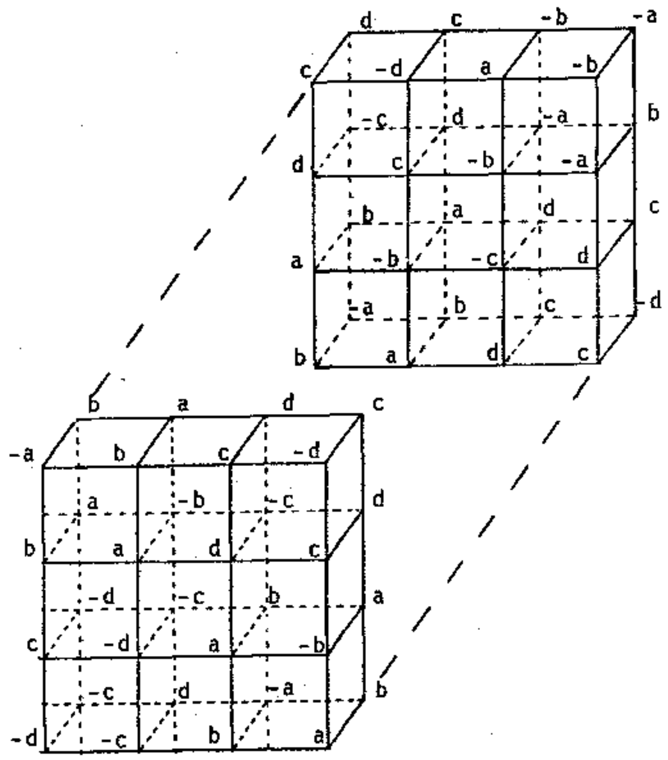
REFERENCES

1. A. V. Geramita and Jennifer Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York, 1979.

2. Joseph Hammer and Jennifer Seberry, Higher dimensional orthogonal designs and applications, *IEEE Transactions on Information Theory*, (to appear).

3. P. J. Shlicta, Higher dimensional Hadamard matrices, *IEEE Transactions on Information Theory*, (to appear).

4. Jennifer Seberry, Higher dimensional orthogonal designs and Hadamard matrices, *Combinatorics VII: Proc. Seventh Australian Conf, Lecture Notes in Mathematics*, Springer-Verlag (to appear).



The orthogonal design of type $(1,1,1,1)^3$