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It is shown that Graeco-Latin block designs which have treatments totally balanced with respect to blocks in each set and which are pairwise orthogonal with respect to the other set can be constructed with parameters v1 = r2 = p + 1, r1 = v2 = p, b = 2p, k = (p+l)/2 for p a prime power. Moreoever they can be represented in a compact manner and previously ad hoc examples become part of the series.

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A NOTE ON ORTHOGONAL GRAECO-LATIN DESIGNS

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Abstract

It is shown that Graeco-Latin block designs which have treatments totally balanced with respect to blocks in each set and which are pairwise orthogonal with respect to the other set can be constructed with parameters

$$v_1 = r_2 = p + 1,\ r_1 = v_2 = p,\ b = 2p,$$

$$k = (p+1)/2$$

for p a prime power. Moreoever they can be represented in a compact manner and previously ad hoc examples become part of the series.

In the notation used by Hoblyn, Pearce and Freeman [4], a Graeco-Latin block design is of type T : TO if the treatments of each set are (a) "Totally balanced" with respect to blocks and (b) Orthogonal with respect to the treatments of the other set. Such a design is of particular statistical interest if it satisfies more restrictive conditions than (a) and (b), namely, if (a') the treatments of each set are disposed in a balanced incomplete block design and (b') each treatment of each set occurs exactly *once* with each treatment of the other set, and if additionally (c) the entire design possesses "overall" balance as defined below. Graeco-Latin block designs satisfying conditions (a'), (b') and (c) are reviewed in Section 6 of a paper by Preece [8]. Methods of constructing some such designs are given by Agrawal [1], [2] and a relevant proof is due to Raghavarao and Nageswararao [9]. Individual designs of the type in question are given by Potthoff [5], Preece [6] and Sterling and Wormald [10]. Others can be written down by omitting a factor from designs obtainable by method 2.6 of Agrawal and Mishra [3]. Potthoff indicates that some of the designs are standard movements in

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duplicate bridge tournaments.

For each of the designs, let the balanced incomplete block designs for the ith set of treatments $(i = 1,2)$ have v_i treatments each replicated r_i times, b blocks each of size k, and statistical efficiency factor $E_i = \{v_i(k-1)\}/\{k(v_i-1)\}$. (Then $v_1 = r_2$ and $v_2 = r_1$.) Let $\underset{\sim}{n}_i$ be the $v_i \times b$ incidence matrix whose (p,q)th element is 1 if the pth treatment of set i occurs in the qth block, and 0 otherwise. The design has the "overall" balance mentioned above, and can thus be called a "balanced Graeco-Latin block design", if

$$\underset{\sim}{n}_1 \underset{\sim}{n}_2' = kJ$$

where J is a matrix all of whose entries are 1 (Potthoff [5], Preece [7]), and the overall efficiency factor for the ith set of treatments is then E_i, i.e., the same as if the other set were absent.

We thank D.A. Preece for writing this introduction and now proceed to give our construction and some examples.

CONSTRUCTION. *Let* p *be a prime power and* x *be the generator of the multiplicative group of* $GF(p)$. *Then we may construct Graeco-Latin block designs as in table 2 of Preece [8] with parameters*

$$v_1 = r_2 = p + 1,\ r_1 = v_2 = p,\ b = 2p,\ k = (p+1)/2,$$

by circulating the initial blocks indicated.

(i) When $1 + x = x^{\text{even power}}$ *and* $p \equiv 3 \pmod 4$ *use the initial blocks*

$(00,\ x^2x^3,\ x^4x^5,\ \ldots,\ x^{p-1}x)$ *and* $(\infty 0,\ xx^3,\ x^3x^5,\ \ldots,\ x^{p-2}x)$.

(ii) When $1 + x = x^{\text{odd power}}$ *and* $p \equiv 3(\text{mod } 4)$ *or*

$1 + x = x^{\text{even power}}$ *and* $p \equiv 1(\text{mod } 4)$ *use the initial blocks*

$(00,\ x^3x^2,\ x^5x^4,\ \ldots,\ xx^{p-1})$ *and* $(\infty 0,\ xx^3,\ x^3x^5,\ \ldots,\ x^{p-2}x)$.

(iii) When $1 + x = x^{\text{odd power}}$ *and* $p \equiv 1(\text{mod } 4)$ *use the initial blocks*

$(00,\ xx^2,\ x^3x^4,\ \ldots,\ x^{p-2}x^{p-1})$ *and* $(\infty 0,\ xx^3,\ x^3x^5,\ \ldots,\ x^{p-2}x)$.

(The circulating process requires each element to be first mapped into the corresponding element of the additive abelian group of GF(p)*).*

Proof. We recall that when $p \equiv 3(\text{mod } 4)$ the initial blocks
$(0,\ x^{2+i},\ x^{4+i},\ \ldots,\ x^{p-1+i})$ and $(\infty,\ x,\ x^3,\ \ldots,\ x^{p-2})$ $i = 1$ or 2
$(0,\ x^{3+i},\ x^{5+i},\ \ldots,\ x^{1+i})$ and $(0,\ x^3,\ x^5,\ \ldots,\ x)$ $i = 1$ or 2 form BIBD with parameters $(p + 1,\ 2p,\ p,\ (p+1)/2,\ (p-1)/2)$ and $(p,\ 2p,\ p + 1,\ (p+1)/2,\ (p+1)/2)$ respectively.

Also if $p \equiv 1\ (\text{mod } 4)$ the initial blocks

$$(0,\ x^3,\ x^5,\ \ldots,\ x) \quad \text{and} \quad (\infty,\ x,\ x^3,\ \ldots,\ x^{p-2})$$

or

$$(0,\ x^2,\ x^4,\ \ldots,\ x^{p-2}) \quad \text{and} \quad (0,\ x,\ x^3,\ \ldots,\ x^{p-2})$$

form BIBD with the same parameters respectively.

Suppose we write A for the incidence matrix constructed from the quadratic residues and B for the incidence matrix constructed from the quadratic non-residues. Then

$$AJ = BJ = (p-1)/2, \qquad JJ = pJ,$$

$$AA^T + BB^T = (p+1)/2I + (p-3)/2J,$$

and

$$A + B + I = J, \qquad A^T = A \quad \text{for} \quad p \equiv 1 (\text{mod } 4),$$

$$A^T = -A \quad \text{for} \quad p \equiv 3 (\text{mod } 4).$$

In case (i) the incidence matrices of the two BIBD are

$$N' = \begin{bmatrix} A + I & B \\ 0 \ldots 0 & 1 \ldots 1 \end{bmatrix} \quad \text{and} \quad N^2 = [B + I \mid B + I]$$

Now $N'N^{2^T} = \begin{bmatrix} (A+I)(B+I)^T + B(B+I)^T \\ (p+1)/2 \; e \end{bmatrix}$ where e is $1 \times p$ matrix of 1's.

$$= \begin{bmatrix} (A+I+B)(B+I)^T \\ (p+1)/2 \; e \end{bmatrix}$$

$$= (p+1)/2 \; J.$$

In case (ii) and (iii), the incidence matrices of the two BIBD are

$$N' = \begin{bmatrix} B + I & B \\ 0 \ldots 0 & 1 \ldots 1 \end{bmatrix} \quad \text{and} \quad N^2 = [A + I \mid B + I]$$

respectively. Further

$$N'N^{2^T} = (p+1)/2 \; J$$

in both cases.

It remains to show that these initial elements give rise to all ordered pairs in the Graeco-Latin design.

(i) Now we wish to show that the initial elements

(a) $x^{2i}x^{2i+1}$ and $x^{2j}x^{2j+1}$

(b) $x^{2i}x^{2i+1}$ and 00

(c) $x^{2i}x^{2i+1}$ and $x^{2j+1}x^{2j+3}$

(d) 00 and $x^{2j+1}x^{2j+3}$

(e) $x^{2j+1}x^{2j+3}$ and $x^{2j+1}x^{2i+3}$

when $p^n \equiv 3 \pmod 4$ is a prime power with $1 + x = x^{\text{even power}}$ cannot give the same pair.

We note that $x^{2i+1} - x^{2i} = x^{2i}(x-1)$. Hence if x^{2i} is associated by cycling with 0 then x^{2i+1} will be associated with $x^{2i}(x-1)$, i.e.,

$$x^{2i}x^{2i+1} \to 0\ x^{2i}(x-1)$$

$$x^{2j}x^{2j+1} \to 0\ x^{2j}(x-1).$$

Clearly if $x^{2i} \to 0$ then $x^{2i+1} \not\to 0$ so cycling $x^{2i}x^{2i+1}$ never gives 00 and so (b) and similarly (d) do not give the same pair. Now we consider (a): it is not possible for $x^{2i}(x-1) = x^{2}(x-1)$, $x \neq 1$ unless $i = j$. Hence (a) does not give the same pair and similarly for (e). It now remains to consider (c). We consider if it is possible for

$$x^{2i}(x-1) = x^{2j+1}(1 - x^2).$$

Now $x \neq 1$ so we consider

$$x^{2i} = x^{2j+1}(1+x).$$

Clearly if $1 + x = x^{\text{even power}}$ this is not possible and so (c) does not give the same pair.

Hence we see that in cycling the initial elements $x^2x^3, \ldots, x^{p-1}x$, $00, xx^3, x^3x^5, \ldots, x^{p-1}x$ to find the element $0y$ arising from that initial element we see $0y$ is different for each initial element. But we have just considered $(p-1)/2 + 1 + (p-1)/2$ initial elements so we have p different pairs $0y$, that is, we have every pair $0y$.

Similarly we can consider any pair zy and find that for fixed z all p elements of $GF(p)$ will occur for y.

(ii) We proceed in similar fashion but instead of (c) we have (c'), i.e.,

$$x^{2i+1}x^{2i} \quad \text{and} \quad x^{2j+1}x^{2j+3}$$

should not give the same pair. We consider when

$$x^{2i}(x-1) = x^{2j+1}(1 - x^2).$$

Now $x \neq 1$ so we consider

$$x^{2i} = - x^{2j+1}(1+x). \qquad (*)$$

Since $-1 = x^{(p-1)/2}$ we have for $p \equiv 1 \pmod 4$ and $1 + x = x^{\text{even power}}$ or $p \equiv 3 \pmod 4$ and $1 + x = x^{\text{odd power}}$ that (*) cannot be solved and so (c') does not give the same pair.

Hence as before $0y$ can occur once and only once for each $y \in GF(p)$ and after circulating every pair occurs once and only once.

(iii) We proceed as in (i) and (ii) but instead of (c) we have (c"), i.e.,

$$x^{2i+1}x^{2i+2} \quad \text{and} \quad x^{2j+1}x^{2j+3}$$

should not give the same pair. We consider when

$$x^{2i}(1-x) = x^{2j+1}(1 - x^2)$$

or, since $x \neq 1$, when

$$x^{2i+1} = x^{2j+1}(1+x).$$

Clearly if $1 + x = x^{\text{odd power}}$ this is not possible and so (c") does not give the same pair.

Thus, as before, after circulating every pair occurs once and only once.

EXAMPLES.

(i) $p = 5$. Here $x = 2$ and $1 + x = 3 = 2^3$. So we use the initial blocks

$$(\infty 0,\ xx^2,\ x^3x^4) \quad \text{and} \quad (00,\ xx^3,\ x^3x).$$

We now associate each element with the corresponding element in the additive group, i.e., $x \to 2$, $x^2 \to 4$, $x^3 \to 3$, $x^4 \to 1$ and get the initial blocks

(∞0, 24, 31) and (00, 23, 32).

We now circulate each element in the additive group to get the Graeco-Latin design

∞0, 24, 31	00, 23, 32
∞1, 30, 42	11, 34, 43
∞2, 41, 03	22, 40, 04
∞3, 02, 14	33, 01, 10
∞4, 13, 20	44, 12, 21

(ii) $p = 11$. Here $x = 2$ and $1 + x = 3 = x^4$. So we use the initial blocks

$$(\infty 0,\ x^2x^3,\ x^4x^5,\ x^6x^7,\ x^8x^9,\ x^{10}x)$$

and

$$(00,\ xx^3,\ x^3x^5,\ x^5x^7,\ x^7x^9,\ x^9x).$$

We now associate each element with the corresponding element of the additive group, i.e., $x \to 2$, $x^2 \to 4$, $x^3 \to 8$, ..., $x^{10} \to 1$ and get the initial blocks

$$(\infty 0,\ 48\ 5\ \underline{10},\ 97,\ 36,\ 12)$$

and

$$(00,\ 28,\ 8\ \underline{10},\ \underline{10}\ 7,\ 76,\ 62).$$

The Graeco-Latin design is now obtained by circulating each element in the additive group, i.e., $i \to i + 1 \pmod{11}$.

(iii) $p = 9$. A primitive polynomial is $1 + x = x^2$. So we use the initial blocks

$$(\infty 0,\ x^3x^2,\ x^5x^4,\ x^7x^6,\ xx^8)$$

and

$$(00,\ xx^3,\ x^3x^5,\ x^5x^7,\ x^7x).$$

We now associate each element with the corresponding element of the additive group, i.e., $0 \to (0,0)$, $x \to (1,0)$, $x^2 = x + 1 \to (1,1)$, $x^3 = 2x + 1 \to (2,1)$, $x^4 = 2 \to (0,2)$, $x^5 = 2x \to (2,0)$, $x^6 = 2x + 2 \to (2,2)$, $x^7 = x + 2 \to (1,2)$ $x^8 = 1 \to (0,1)$ and get the initial blocks

$\{\infty(0,0),\ (2,1)(1,1),\ (2,0)(0,2),\ (1,2)(2,2),\ (1,0)(0,1)\}$ and
$\{(0,0)(0,0),\ (1,0)(2,1),\ (2,1)(2,0),\ (2,0)(1,2),\ (1,2)(1,0)\}$.

The Graeco-Latin design is now obtained by circulating each element in the additive group mod(3,3). e.g., the initial element (2,1)(1,1) when developed gives

(2,1)(1,1)

(2,2)(1,2)

(2,0)(1,0)

(0,1)(2,1)

(0,2)(2,2)

(0,0)(2,0)

(1,1)(0,1)

(1,2)(0,2)

(1,0)(0,0).

Note: This process then gives us Nos. 2, 5, 8, 9 and 14 of table 1 of Preece [8].

For 13:

(AA, IE, GD, LM, FJ, HK, CB) (NA, CI, IG, GL, LF, FH, HC) N fixed.

For 17:

(AA, KJ, FN, LP, OQ, HI, ME, EC, DB) (RA, DK, KF, FL, LO, OH, HM, ME, ED) R fixed.

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