1978

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H C. Kirton

Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Abstract
In general it is a difficult if not impossible task to find a latin square orthogonal to a given latin square. Because of a practical problem it was required to find a frequency square orthogonal to a given latin square. We describe a computer approach which was successful in finding a (4,23) frequency square orthogonal to a given 10 x 10 latin square.

Publication Details
Kirton, HC and Seberry, J, Generation of a frequency square orthogonal to a 10 x 10 Latin square, Combinatorial Mathematics: Proceedings of the International Conference, Canberra, August, 1977, 686, in Lecture Notes in Mathematics, Springer--Verlag, Berlin--Heidelberg--New York, 1978, 193-198.

GENERATION OF A FREQUENCY SQUARE ORTHOGONAL TO A 10×10 LATIN SQUARE

H. C. Kirton and Jennifer Seberry

N.S.W. Department of Agriculture, 150 Liverpool St., Sydney, 2000, and Department of Applied Mathematics, University of Sydney, N.S.W., 2006.

ABSTRACT

In general it is a difficult if not impossible task to find a latin square orthogonal to a given latin square. Because of a practical problem it was required to find a frequency square orthogonal to a given latin square. We describe a computer approach which was successful in finding a $(4,2^3)$ frequency square orthogonal to a given 10×10 latin square.

1. INTRODUCTION

In 1966 a Grapefruit Variety and Rootstock experiment was planted at the Horticultural Research Station, Dareton (in south western N.S.W.) to compare the effects of various rootstocks on grapefruit scions with particular regard to yield and quality under the fast growing conditions present at that locality.

The design used was a 10×10 latin square, with each of the five rootstock by two variety combinations being present as a single tree plot in each row and column. The source of the particular design used is unknown, and the person who proposed its use remained anonymous. In May 1977, the horticultural research officer currently responsible for this experiment indicated he would like to superimpose some new treatments that should improve the quality of the fruit by making them stay longer on the trees. These new treatments were to be various rates of application of a hormone spray, and it was felt that any superimposed design should leave about 40% of the trees untreated by this growth regulatory substance.

The problem was thus to find a frequency square $(4,2^3)$ that was orthogonal to rows, columns and existing treatments in the latin square that had already been used for 11 years:-

D	J	I	F	B	H	E	G	A	C
C	H	B	D	G	E	J	A	I	F
F	A	C	J	E	D	I	H	B	G
H	D	E	B	I	C	A	F	G	J
J	B	A	I	C	F	G	D	H	E
I	E	H	G	J	A	B	C	F	D
E	F	G	C	A	I	H	J	D	B
B	I	J	A	F	G	D	E	C	H
G	C	D	E	H	B	F	I	J	A
A	G	F	H	D	J	C	B	E	I

2. DEFINITIONS

Hedayat [2] and Hedayat and Seiden [3] have defined an F-square as follows:

DEFINITION. Let $A = (a_{ij})$ be an $n \times n$ matrix and let $\Sigma = (c_1, \ldots, c_m)$ be an ordered set of m distinct elements or symbols of A. In addition, suppose that for each $k = 1,2,\ldots,m$, c_k appears λ_k times ($\lambda_k \geq 1$) in each row and in each column of A. Then A will be called a *frequency square* or, more concisely, an *F-square* on Σ of order n and frequency vector $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and will be denoted by $F(n;\lambda_1,\ldots,\lambda_m)$. Note that $\lambda_1+\lambda_2+\cdots+\lambda_m = n$ and that where $\lambda_k = 1$ for every k and $m = n$, a latin square results.

As with latin squares, one may consider orthogonality of a pair or a set of F-squares of the same order. The above cited authors give the following definition covering these cases:

DEFINITION. Given an F-square $F_1(n;\lambda_1,\lambda_2,\ldots,\lambda_k)$ on a set $\Sigma = (a_1,a_2,\ldots,a_k)$ and an F-square $F_2(n;\mu_1,\mu_2,\ldots,\mu_t)$ on a set $\Omega = (b_1,b_2,\ldots,b_t)$ we say F_2 is an *orthogonal* mate for F_1 (and write $F_2 \perp F_1$) if on superposition of F_2 on F_1, a_i appears $\lambda_i\mu_j$ times with b_j.

Federer [1] has recently written a most interesting paper indicating how Hadamard matrices can be used to obtain $(4t-1)^2$ mutually orthogonal $F(4t;2t,2t)$-squares (a complete set).

3. THE METHOD AND RESULTS

A computer program was written to provide a sufficient search of possible frequency squares involving t treatments that are orthogonal to a given latin square. Each cell of the latin square has associated with it the following parameters:-

(a) The original treatment from the latin square design.

(b) A vector whose i^{th} element denotes the number of free choices of frequency square treatments remaining for this cell at level i of the design generation process. The program caters for values of i up to 44, with the problem being declared "too big" if i exceeds 44.

(c) A matrix whose (i,j) element takes values as follows (k is a positive integer $\leq t$):

(i) $(i,1) = k$ means the cell has new treatment k,

and (ii) $(i,j) = -k$ where $1 < j \leq t$ means the cell cannot have new treatment k, at level i of the design generation.

The process for design generation consists of choosing the cell with the fewest free choices, and placing the smallest available k as the frequency square treatment for that cell at the current level of choice if no freedom exists for choice of k, and at the next level of choice otherwise. The implications of this choice are then checked for other cells in the same column, row and latin treatment. When the $(i,1)^{th}$ element for any cell would be negative, the level of choice is decreased by one step, and the last chosen k is eliminated from the set of available values for the cell where the choice was made. The generation process terminates when (i) (i,1) element for each cell is positive (successful completion), (ii) i becomes less than one (all possibilities rejected), or (iii) the problem is "too big".

The frequency square orthogonal to the "de-randomized" 10×10 latin square was generated in two runs of the program. In the first run, each cell of the latin square that received a treatment coded F to J was assumed to have a new treatment number 99 (a dummy). The program then searched for a $(2,1^3)$ frequency square to be superimposed on the cells containing latin square treatments coded A to E.

The program thus searched for a $(2,1^3)$ frequency square orthogonal to the equivalent latin square

A	B	C	D	E	F	G	H	I	J
B	E	G	F	I	J	H	D	C	A
C	F	H	B	D	A	E	G	J	I
D	A	B	E	H	C	J	I	G	F
E	D	I	A	C	H	B	J	F	G
F	J	D	I	B	G	C	A	H	E
G	I	J	H	A	B	F	C	E	D
H	C	E	J	G	I	D	F	A	B
I	G	F	C	J	D	A	E	B	H
J	H	A	G	F	E	I	B	D	C

starting with (k = 1,2,3, or 4)

k	k	k	k	k	99	99	99	99	99
k	k	99	99	99	99	99	k	k	k
k	99	99	k	k	k	k	99	99	99
k	k	k	k	99	k	99	99	99	99
k	k	99	k	k	99	k	99	99	99
99	99	k	99	k	99	k	k	99	k
99	99	99	99	k	k	99	k	k	k
99	k	k	99	99	99	k	99	k	k
99	99	99	k	99	k	k	k	k	99
99	99	k	99	99	k	99	k	k	k

After 14 seconds (Univac 1108), it found a solution

1	1	2	3	4	99	99	99	99	99
4	1	99	99	99	99	99	1	3	2
3	99	99	1	1	4	2	99	99	99
2	3	4	1	99	1	99	99	99	99
1	4	99	2	1	99	3	99	99	99
99	99	1	99	2	99	1	4	99	3
99	99	99	99	3	2	99	1	4	1
99	2	3	99	99	99	4	99	1	1
99	99	99	4	99	3	1	2	1	99
99	99	1	99	99	1	99	3	2	4

This solution is such that each new treatment (1,2,3,4) occurs with appropriate frequencies in each row, column and twice as often with each of the latin square treatments A to E.

The second half of the design was generated in the second run by assigning a dummy new treatment (99) to each cell of the latin square that received a treatment A to E. After 4 seconds, this gave

99	99	99	99	99	1	1	2	3	4
99	99	3	1	4	2	1	99	99	99
99	3	4	99	99	99	99	1	1	2
99	99	99	99	1	99	2	1	4	3
99	99	1	99	99	4	99	3	2	1
1	4	99	2	99	3	99	99	1	99
2	1	1	3	99	99	4	99	99	99
3	99	99	1	2	1	99	4	99	99
4	1	2	99	3	99	99	99	99	1
1	2	99	4	1	99	3	99	99	99

The two runs of the program were thus able to generate the $(4,2^3)$ frequency square

1	1	2	3	4	1	1	2	3	4
4	1	3	1	4	2	1	1	3	2
3	3	4	1	1	4	2	1	1	2
2	3	4	1	1	1	2	1	4	3
1	4	1	2	1	4	3	3	2	1
1	4	1	2	2	3	1	4	1	3
2	1	1	3	3	2	4	1	4	1
3	2	3	1	2	1	4	4	1	1
4	1	2	4	3	3	1	2	1	1
1	2	1	4	1	1	3	3	2	4

By applying the same randomization to this frequency square as had been applied to the original latin square, a feasible design was generated.

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