1978

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Abstract

A computer has been used to list all known Hadamard matrices of order less than 40,000. If an Hadamard matrix is not known of order 4q (q odd) then the smallest t so that there is an Hadamard matrix of order 2tq is given. Hadamard matrices are not yet known for orders 268, 412, 428.

Disciplines

Physical Sciences and Mathematics

Publication Details

Seberry, J, A computer listing of Hadamard matrices, Combinatorial Mathematics: Proceedings of the International Conference, Canberra, August, 1977, 686, in Lecture Notes in Mathematics, Springer--Verlag, Berlin--Heidelberg--New York, 1978, 275-281.

A COMPUTER LISTING OF HADAMARD MATRICES

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ABSTRACT.

A computer has been used to list all known Hadamard matrices of order less than 40,000. If an Hadamard matrix is not known of order 4q (q odd) then the smallest t so that there is an Hadamard matrix of order $2^t q$ is given.

Hadamard matrices are not yet known for orders 268, 412, 428.

INTRODUCTION.

An *Hadamard matrix* of order n has every entry +1 or -1 and its distinct row vectors are orthogonal.

These were discussed by Sylvester [16] in 1867 and Hadamard [8] conjectured in 1893 that they exist for orders 1, 2 and 4t, for every natural number t. Hadamard proved that any complex $n \times n$ matrix $A = (a_{ij})$ with entries in the unit disc satisfies

$$(\det A)^2 \le \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 ,$$

and Hadamard matrices satisfy the equality of this inequality.

In 1933 Paley [10] produced a list showing that Hadamard matrices of orders 92, 116, 156, 172, 184 and 188 where the only unsolved cases of order ≤ 200.

The existence of the matrix of order 172 was settled by Williamson [27] in 1944.

This list induced L.D. Baumert, S.W. Golomb and Marshall Hall Jr to use sophisticated and exciting computer techniques with

Williamson's method to find the Hadamard matrices of orders 92 and 184 (in 1962 [2]), 116 (in 1966 [1]) and 156 (in 1965 [3]).

The case for 188 has been settled by R.J. Turyn using a technique of Goethals and Seidel [7] which generalized that of Williamson.

These results, and those of E. Spence, J. Cooper, J.S. Wallis and A.L. Whiteman have largely given Hadamard matrices of "low" order.

Four matrices W_1, W_2, W_3, W_4 of order w with entries +1 or -1 which satisfy

$$W_i W_j^T = W_j W_i^T \qquad i, j \in \{1, 2, 3, 4\},$$

$$\sum_{i=1}^{4} W_i W_i^T = 4wI_w,$$

are called *Williamson* matrices.

A square matrix A of order n with entries from the set of commuting variables $\{0, \pm x_1, \pm x_2, \ldots, \pm x_s\}$ will be called an *orthogonal design of order* n *and type* $(u_1, u_2, \ldots, u_s)$ if x_i occurs u_i times in each row and column of A and if the rows of A are formally orthogonal.

Then we can express the highly important result of Baumert and Hall (1965 [3]) as

BAUMERT-HALL THEOREM. *If there is an orthogonal design of order* $4t$ *and type* (t, t, t, t) *and Williamson matrices of order* w, *then there exists an Hadamard matrix of order* $4wt$.

Orthogonal designs, introduced by Geramita, Geramita and Wallis [6], were used by Wallis [22] to prove

THEOREM. *Let* q *be any odd natural number. Then there exists an integer* $t = [2 \log_2(q-3)]$ *such that there exists an Hadamard matrix of order* $2^s q$ *for every natural number* $s \geq t$.

q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t
		103	3													803	3		
		107	10															907	5
										509	8					809	12		
						311	14												
				213	3							613	3					913	3
										515	4								
																		917	3
				219	3			419	4	519	3			719	4			919	3
														721	4				
				223	3					523	3	623	3	723	3	823	3		
		127	3																
												631	13						
				233	4									733	18			933	10
						335	6												
										537	5			737	6				
				239	4									739	16	839	8		
																		941	6
								443	6			643	3						
								445	3										
						347	15					647	10					947	6
				249	3									749	11			949	3
		151	3	251	6									751	3				
												653	10			853	3	953	14
												655	3					955	3
												657	3	757	5	857	4		
						359	4					659	17			859	3	959	3
		163	3					463	3					763	3	863	4		
																865	3	965	3
67	5	167	4	267	3														
				269	8							669	3						
										571	3							971	6
						373	7			573	3			773	8				
		179	8					479	12	579	3					879	3	979	3
										581	3			781	3				
				283	3											883	3		
																		985	3
								487	5					787	5				
														789	3				
		191	13					491	15									991	3
										599	8	699	3						

If an odd number q does not appear in the list it means that there is an Hadamard matrix of order 4q . When q appears in the list the number t next to q indicates there is an Hadamard matrix of order $2^t q$ and no smaller power of 2.

q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t	q	t
						1301	14												
		1103	17			1303	3							1703	3			1903	3
				1205	3														
										1507	3							1907	10
		1109	19							1509	3							1909	3
				1211	3			1411	3										
				1213	16					1513	3			1713	4			1913	12
		1115	4			1315	3											1915	3
1019	8					1319	12					1619	4	1719	3				
																		1921	3
		1123	7	1223	8			1423	3					1723	3	1823	4		
				1227	3	1327	5	1427	15	1527	9								
																		1929	4
1031	6			1231	3							1631	5			1831	15		
		1133	4					1433	6			1633	3	1733	14				
						1335	3												
								1437	6									1937	3
1039	3							1439	12			1639	3						
				1241	3	1341	3	1441	3							1841	3		
1043	3			1243	3					1543	3							1943	6
														1745	3				
1047	3							1447	19					1747	5	1847	8		
						1349	3											1949	4
1051	3					1351	3	1451	6					1751	4			1951	3
														1753	11				
				1255	3														
		1157	3	1257	5					1557	3							1957	4
1059	3			1259	4	1359	3			1559	4								
												1661	4						
1063	3											1663	11					1963	4
		1165	3					1465	3										
						1367	4			1567	19	1667	19						
		1169	5					1469	3							1869	3	1969	9
								1471	13	1571	10	1671	3			1871	20		
				1273	6	1373	19	1473	3									1973	4
												1675	3						
		1177	5							1577	3								
1079	3					1379	3			1579	5	1679	3			1879	3	1979	4
						1381	3							1781	3			1981	4
				1283	12			1483	3	1583	8			1783	7	1883	6		
						1385	3											1985	3
1087	18	1187	6			1387	3	1487	12					1787	4			1987	16
										1589	3	1689	3			1889	4		
				1291	3			1491	3										
		1193	4					1493	4			1693	7	1793	4	1893	6	1993	15
														1795	5				
						1397	4												
		1199	3	1299	3			1499	16			1699	3					1999	3

If an odd number q does not appear in the list it means there is an Hadamard matrix of order 4q. When q appears in the list the number t next to q indicates there is an Hadamard matrix of order $2^t q$ and no smaller power of 2.

This result can be used to prove

THEOREM. *Let* q *be any odd natural number. Then there exist an integer* t *such that there exists a symmetric Hadamard matrix with constant diagonal of order* $2^{2s}q^2$ *for every natural number* $s \geq t$.

THE LIST.

As mentioned previously Paley [10] constructed a list of orders ≤ 200 for which Hadamard matrices were known in 1933. Various other lists appeared (see, for example, Florek [5], Raghavarao [12], Wallis [23]).

The current listing, which is available for $q < 10,000$ although we only give $q < 2,000$ in this note, has no entry after q if an Hadamard matrix of order 4q is *known*. If no Hadamard matrix of order 4q is known then the smallest t such that an Hadamard matrix of order $2^t q$ is known is given. We note that Wallis' theorem gives an upper bound on t and often Hadamard matrices are known for smaller powers of 2 (but greater than 2) than that theorem indicates.

The computer tape on which these results are stored in order to be easily updated also has an indication of whether a skew-Hadamard matrix is known.

ACKNOWLEDGEMENTS.

I wish to thank Emma Lehmer and D.H. Lehmer who suggested to me the idea of computerizing this list. The largest part of the programming was done by Dr. Ian S. Williams while a student in the Research School of Physical Sciences at A.N.U. on the School's DEC-10 system. The updating and printing programmes were written by Mr. N. Wormald.

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