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A note on orthogonal designs in order eighty

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Abstract

This is a short note showing the existence of all twovariable designs in order 80 except possibly (13, 64) and (15, 62) which have not yet been construced. The designs are constructed using designs in order 8, 16, 20, and 40 and applying lemmas and theorems concerning orthogonal designs. Three-variable designs (a, b, n-a-b), which are useful in constructing Hadamard matrices, are also considered for n = 40 and 80.

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A NOTE ON ORTHOGONAL DESIGNS IN ORDER EIGHTY

Joan Cooper and Jennifer Seberry Wallis

ABSTRACT: This is a short note showing the existence of all two-variable designs in order 80 except possibly (13, 64) and (15, 62) which have not yet been construced. The designs are constructed using designs in order 8, 16, 20, and 40 and applying lemmas and theorems concerning orthogonal designs. Three-variable designs (a, b, n-a-b), which are useful in constructing Hadamard matrices, are also considered for n = 40 and 80.

Introduction.

An *orthogonal design* of order n and type $(u_1, u_2, \ldots, u_s)$ on the commuting variables $x_1, x_2, \ldots, x_s$ is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n .$$

Alternatively, the rows (and hence the columns) of A are formally orthogonal and every row (column) contains u_i entries of the type $\pm x_i$.

The following standard lemmas and corollaries are used in the proofs of this paper.

LEMMA 1.

If there is an orthogonal design of type (a, b) *in order* n, *there are orthogonal designs of types* (a, a, b, b) *in order* 2n *and* (a, a, 2a, b, b, 2b) *in order* 4n.

COROLLARY 2.

If there are orthogonal designs of type (1, k) *in order* n *for* $1 \le k \le \ell$, *then there are designs of type* (1, m) *in order* 2n *for* $1 \le m \le 2\ell + 1$. *In particular, if there are orthogonal designs of the type* (1, k) *in order* n *for* $1 \le k \le n - 1$, *then there are orthogonal designs of the type* (1, m), $1 \le m \le 2^t n - 1$, *in order* $2^t n$. *(*t *a positive integer).*

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LEMMA 3.

If X *is an orthogonal design of order* n *and type* $(u_1, u_2, \ldots, u_s)$ *on the variables* $x_1, \ldots, x_s$, *then there is an orthogonal design of order* n *and type* $(u_1, \ldots, u_i+u_j, \ldots, u_s)$ *on the* x − 1 *variables* $x_1, \ldots, \bar{x}_j, \ldots, x_s$.

COROLLARY 4.

If X *is an orthogonal design of order* n *and type* $(u_1, u_2, \ldots, u_s)$ *on the variables* $x_1, \ldots, x_s$, *then there is an orthogonal design of order* n *and type* $(u_1 \ldots, u_{i-1}, u_{i+1}, \ldots, u_s)$ *on the* s − 1 *variables* $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s$.

LEMMA 5.

If there exists an orthogonal design of order n *and type* $(u_1, \ldots, u_s)$, *then there exist orthogonal designs of the types*

(i) $(\ell_1 u_1, \ell_1 u_2, \ldots, \ell_s u_s)$ where $\ell_i = 1$ or 2

(ii) $(u_1, u_1, fu_2, \ldots, fu_s)$ where $f = 1$ or 2

in order 2n.

LEMMA 6. (Geramita and Verner)

If there exists an orthogonal design of type $(u_1, \ldots, u_s)$ *in order* $n \equiv 0 (\mathrm{mod}\ 4)$ *and*

$$\sum_{i=1}^{s} u = n - 1,$$ *then there exists an orthogonal design of type* $(1, u_1, u_2, \ldots, u_s)$ *in order* n.

It is known that all (1, k) designs exist in order 80, see [2]. In [7] the following theorem was proved.

THEOREM 7.

If all orthogonal designs (a, b, n-a-b) *exist in order* n, *then all orthogonal designs* (x, y, m-x-y) *exist in orders* $m = 2^t n$, t *a positive integer.*

We have therefore considered the existence of orthogonal designs (a, b, n-a-b) in orders 40 and 80 in this note. We acknowledge gratefully the help of Frank Allaire in working through 80.

Results.

LEMMA 8.

The following full orthogonal designs exist in order 40:

(1) (1,2,2,2,16,17); (2) (1,10,14,15); (3) (2,2,14,22);

(4) (1,2,3,34); (5) (1,8,12,19); (6) (5,8,12,15);

(7) (1,2,10,27); (8) (2,4,13,21); (9) (2,2,5,31);

(10) (1,i,j), $i + j = 39$, $i, j \notin \{6,7,8,9\}$;

(11) (1,2,2,11,24); (12) (1,2,12,25); (13) (1,10,10,19);

(14) (1,2,14,23).

Proof.

The designs (10), (11), (12), (13), (14) are given in [6]. The remaining designs are constructed by using the matrices whose first rows are given in Appendix 1 in the (1,1,1,1,1,1,1,1) design in order 8; we use the back-circulant matrix constructed from X and the circulant matrices constructed from the other X_i ($i = 2, 3, \ldots, 8$). Note that (4) is constructed using Lemma 5(iv) of [6].

COROLLARY 9.

*All 3-tuples (*a, b, 40-a-b*) are the types of orthogonal designs in order 40 except possibly*

(1, 7, 32)	(1, 9, 30)	(3, 5, 32)	(3, 6, 31)	(3, 8, 29)
(3, 9, 28)	(4, 7, 29)	(5, 6, 29)	(5, 13, 22)	(6, 7, 27)
(7, 8, 25)	(7, 9, 24)	(7, 11, 22)	(7, 14, 19)	(8, 9, 23)
(8, 11, 21)	(9, 14, 17)			

which are undecided.

Proof.

We use the designs of the theorem and the designs

(1, 1, 9, 9) (1, 5, 5, 9) (2, 5, 5, 8) (5, 5, 5, 5)
(1, 2, 8, 9) (2, 3, 6, 9) (2, 2, 16) (4, 8, 8)

in order 20.

COROLLARY 10.

All 2-tuples are the types of orthogonal designs in order 40 except possible

(7, 32) (9, 30)

which are undecided.

LEMMA 11.

Let M = {(4, 11), (1, 19), (1, 1, 10), (1, 4, 9), (1, 17), (1, 14), (5, 15)} *and* N = {(8, 12), (10, 10), (2, 13), (2, 16), (2, 18), (9, 9), (15), (13), (1, 11), (1, 19)}. *Then there exist orthogonal designs of type* (a, b, 3c, 3d) *in order 80 when* (a, b) $\in$ M *and* (c, d) $\in$ N *or* (a, b) $\in$ N *and* (c, d) $\in$ M $\cup$ N. *In particular, there exist orthogonal designs of types* (1, 6, 14, 48), (1, 6, 14, 54), (2, 3, 16, 51), (2, 3, 16, 57), (2, 15, 18, 45), *and* (3, 10, 10, 57) *in order 80.*

Proof.

Each of the designs in N can be constructed using four circulant symmetric matrices of order 5. Each design in M can be constructed using one back-circulant and three circulant symmetric matrices of order 5. These matrices are used in the (1, 1, 1, 1, 3, 3, 3, 3) design in order 16 found in [5] to obtain the design of the enunciation.

LEMMA 12.

The following orthogonal designs exist in order 80:
(1) (1, 4, 6, 19, 50); (2) (1, 4, 6, 34, 35); (3) (2, 3, 16, 24, 35); (4) (2, 3, 4, 35, 36); (5) (2, 3, 20, 20, 35); (6) (2, 3, 4, 18, 53); (7) (2, 3, 12, 28, 35); (8) (1, 6, 28, 43); (9) (1, 2, 18, 59); (10) (3, 10, 10, 57); (11) (1, 14, 65); (12) (1, 6, 15, 58);

(13) (1, 2, 28, 45); (14) (1, 6, 29, 44); (15) (1, 1, 6, 16, 43);
(16) (2, 3, 14, 26, 35); (17) (2, 2, 3, 6, 32, 35); (18) (4,6,11,54);
(19) (4, 11, 24, 36); (20) (4, 11, 30, 30).

Proof.

We use the designs cited in appendix 2 in designs in order 16 as follows: for (1), ..., (5), use the (1, 1, 2, 2, 2, 2, 3, 3) design found in [5]; for (6), (7), ..., (20), use the (1, 1, 1, 1, 1, 2, 3, 3, 3) design found in [5]. In all cases the first row of the circulant matrix is given; if the first row does not give a circulant symmetric matrix, the back-circulant matrix should be used.

COROLLARY 13.

All 3-tuples (a, b, 80-a-b) *are the types of orthogonal designs in order 80 except possibly*

(3, 13, 64)	(3, 15, 62)	(3, 21, 56)	(5, 11, 64)
(5, 17, 58)	(5, 31, 44)	(6, 9, 65)	(6, 31, 43)
(7, 8, 65)	(7, 9, 64)	(7, 12, 61)	(7, 16, 57)
(7, 22, 51)	(7, 25, 48)	(7, 28, 45)	(8, 29, 43)
(9, 14, 57)	(9, 15, 56)	(9, 16, 55)	(9, 28, 43)
(10, 29, 41)	(11, 14, 55)	(11, 16, 53)	(11, 25, 44)
(12, 27, 41)	(12, 29, 39)	(13, 23, 44)	(14, 19, 47)
(14, 27, 39)	(16, 21, 43)	(16, 23, 41)	(16, 25, 39)
(18, 23, 39)	(19, 28, 33)	(21, 22, 37).	

which are undecided.

Proof.

We use the designs of the theorem together with the designs obtained by using lemmas 1, 3 and 5 on the known designs in 40 and 80 to obtain these results.

COROLLARY 14.

All 2-tuples (a, b) *are the types of orthogonal designs in order 80 except possibly*

(13, 64) (15, 62)

which are undecided.

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Appendix 1

Design	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
(1, 2, 2, 2, 16, 17)	f a a-a-a	a-a a a-a	e b-b-b b	-e b-b-b b	d a-b-b a	-d a-b-b a	c b a a b	-c b a a b
(1, 10, 14, 15)	d b b-b-b	-b b-b-b b	b a b b a	a a-b-b a	b-a a a-a	c a-a-a a	-a c-c-c c	-c c c c c
(2, 2, 14, 22)	c b-b-b b	-c b-b-b b	d a-a-a a	-d a-a-a a	-a a a a a	b a b b a	a a-b-b a	b-a a a-a
(1, 2, 3, 34)	c b-b b-b	a b b-b-b	a b b-b-b	a b b-b-b	b-b b b-b	b-b b b-b	d b b b b	-d b b b b
(1, 8, 12, 19)	a b b-b-b	b-b b b-b	-c d c c d	d-d c c-d	d c-c-c c	-d d d d d	-b b b b b	-b b b b b
(5, 8, 12, 15)	a b a-a b	b-b a a-b	-c d c c d	d-d c c-d	d c-c-c c	-d d d d d	-a a a a a	a a-a-a a
(1, 2, 10, 27)	d b b-b-b	a b-b-b b	-a b-b-b b	b c-c-c c	-c b-b-b b	-c c c c c	b-b b b-b	b b b b b
(2, 4, 13, 21)	-a a b b-a	b-a b b-a	-c b a a b	c a b b a	c b-b-b b	c-a a a-a	d b-b-b b	-d b-b-b b
(2, 2, 5, 31)	a a b b-a	b a-b-b a	b-b b b-b	c b-b-b b	-c b-b-b b	e b-b-b b	-e b-b-b b	b b b b b

Design	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
1, 4, 6, 19, 50)	f e e-e-e	e-e e e-e	-e e e e e	d b-b-b b	-d b-b-b b	b b b b b	a b-b-b b	-a b-b-b b	
1, 4, 6, 24, 35)	e c c-c-c	c-c c c-c	-d c-c-c c	d c-c-c c	-b c c c c	c b b b b	a b-b-b b	-a b-b-b b	
2, 3, 16, 24, 35)	e b b b b	-e b b b b	-c d c c d	c-c d d-c	c d-d-d d	-c c c c c	a b b-b-b	b-b b b-b	
2, 3, 4, 35, 36)	e b b b b	-e b b b b	c d-d-d d	-c d-d-d d	-d d d d d	-d d d d d	a b b-b-b	b-b b b-b	
2, 3, 20, 20, 35)	e b b b b	-e b b b b	c d-d-d d	-d c-c-c c	-c c c c c	-d d d d d	a b b-b-b	b-b b b-b	
2, 3, 4, 18, 53)	a b-b-b b	-a b-b-b b	-b b b b b	d c-c-c c	-d c-c-c c	c c c c c	e c c-c-c	c-c c c-c	
2, 3, 12, 28, 35)	e b b b b	-e b b b b	-c c c c c	-d c-c-c c	-c d-d-d d	c d-d-d d	-d d d d d	a b b-b-b	b-b b b-b
1, 6, 28, 43)	c a a-a-a	0 a-a-a a	0 b-b-b b	a-a a a-a	b b b b b	-b b b b b	-a a a a a	d b-b-b b	-d b-b-b b
1, 2, 18, 59)	c d d-d-d	-a b-b-b b	a b-b-b b	-b b b b b	-b b b b b	d d-d-d d	d-d d d-d	-d d d d d	-d d d d d
3, 10, 10, 57)	a b-b-b b	-b a-a-a a	-b b b b b	-a a a a a	c d d-d-d	c d d-d-d	d-d d d-d	-d d d d d	-d d d d d
1, 14, 65)	c b b-b-b	-a a a a a	b-a a a-a	b-b b b-b	-b b b b b	a-b b b-b	a b-b-b b	-b b b b b	-b b b b b
1, 6, 15, 58)	c b b-b-b	-b b-b-b b	b a b b a	a a-b-b a	b-a a a-a	-a a a a a	d a-a-a a	d-a a a-a	-a a a a a
1, 2, 28, 45)	c a a-a-a	0 a-a-a a	a-a a a-a	d b-b-b b	-d b-b-b b	-b b b b b	0 b-b-b b	-a a a a a	-b b b b b
1, 6, 29, 44)	c a a-a-a	-b a-a-a a	-a b-b-b b	a-a a a-a	b b b b b	-b b b b b	-a a a a a	d b-b-b b	-d b-b-b b
1, 1, 6, 16, 43)	c 0 0 0 0	d 0 0 0 0	0 b-b-b b	0 a a a a	b b b b b	-b b b b b	0 a-a-a a	e b-b-b b	-e b-b-b b
2, 3, 14, 26, 35)	e b b b b	-e b b b b	-d c-c-c c	-c c c c c	d d d d d	-c d-d-d d	c d-d-d d	a b b-b-b	b-b b b-b
2, 2, 3, 6, 32, 35)	e b b b b	-c b b b b	-c d-d-d d	f d d d d	-f d d d d	-c d-d-d d	c d-d-d d	a b b-b-b	b-b b b-b
4, 6, 11, 54)	0 0 a a 0	b-b a-a-b	b 0 b b 0	b b-b-b b	c d-d-d d	c d-d-d d	-c d-d-d d	-d d d d d	-d d d d d
4, 11, 24, 36)	0 0 a a 0	b-b a-a-b	b 0 b b 0	b b-b-b b	-c d c c d	-c d c c d	c-c d d-c	c d-d-d d	-c c c c c
4, 11, 30, 30)	0 0 a a 0	b-b a-a-b	b 0 b b 0	b b-b-b b	c d-d-d d	c d-d-d d	-c d-d-d d	-c c c c c	-d d d d d