Construction of amicable orthogonal designs

Jennifer Seberry

University of Wollongong, jennie@uow.edu.au

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Construction of amicable orthogonal designs

Abstract
Infinite families of amicable orthogonal designs are constructed with
(i) both of type \((1, q)\) in order \(q + 1\) when \(q = 3, (\text{mod } 4)\) is a prime power,
(ii) both of type \((1, q)\) in order \(2(q+1)\) where \(q = 1, (\text{mod } 4)\) is a prime power or \(q + 1\) is the order of a conference matrix,
(iii) both of type \((2, 2q)\) in order \(2(q+1)\) when \(q = 1, (\text{mod } 4)\) is a prime power or \(q + 1\) is the order of a conference matrix.

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Constructions for amicable orthogonal designs

Jennifer Seberry Wallis

Infinite families of amicable orthogonal designs are constructed with

(i) both of type \((1, q)\) in order \(q + 1\) when \(q \equiv 3 \pmod{4}\) is a prime power,

(ii) both of type \((1, q)\) in order \(2(q+1)\) where \(q \equiv 1 \pmod{4}\) is a prime power or \(q + 1\) is the order of a conference matrix,

(iii) both of type \((2, 2q)\) in order \(2(q+1)\) when \(q \equiv 1 \pmod{4}\) is a prime power or \(q + 1\) is the order of a conference matrix.

Introduction

The concept of an orthogonal design was first introduced in [1]. An \(n \times n\) matrix, \(X\), is an orthogonal design of type \(\{u_1, u_2, \ldots, u_s\}\) on the variables \(x_1, x_2, \ldots, x_s\) in order \(n\) if \(X\) has entries from the set \(\{0, \pm x_1, \ldots, \pm x_s\}\) and

\[
XX^T = \left[ u_1 x_1^2 + u_2 x_2^2 + \cdots + u_s x_s^2 \right] I_n,
\]

where \(I_n\) denotes the identity matrix of order \(n\). It was shown in [1] that if there is a pair of orthogonal designs, \(X, Y\), which satisfy the equation \(XY^T = YX^T\), then these designs became a powerful tool in the

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construction of new orthogonal designs (for example, see Construction 22 of [1]).

The existence of such designs has been studied further in [3] and limits are given on the number of variables possible in each design. We define

**DEFINITION.** Two orthogonal designs, \( X \), \( Y \), of the same order, satisfying

\[
XY^T = YX^T, 
\]

will be called amicable orthogonal designs.

In this note we construct infinite families of amicable orthogonal designs.

**The constructions**

Let \( q = p^n \) be a prime power. Then with \( a_0, a_1, \ldots, a_{q-1} \) the elements of \( \mathbb{GF}(q) \) numbered so that

\[
a_0 = 0, \quad a_{q-i} = -a_i, \quad i = 1, \ldots, q-1,
\]

define \( Q = (x_{i,j}) \) by

\[
x_{i,j} = \chi(a_i-a_j),
\]

where \( \chi \) is the character defined on \( \mathbb{GF}(q) \) by

\[
\chi(x) = \begin{cases} 
0, & x = 0, \\
1, & x = y^2 \text{ for some } y \in \mathbb{GF}(q), \\
-1, & \text{otherwise}.
\end{cases}
\]

Then \( Q \) is a type 1 matrix (see [2; p. 285-291]) with the properties that

\[
\begin{align*}
QQ^T &= qI - J, \\
QJ &= JQ = 0, \\
Q^T &= \begin{cases} 
Q & \text{for } q \equiv 1 \pmod{4}, \\
-Q & \text{for } q \equiv 3 \pmod{4},
\end{cases}
\end{align*}
\]

where \( I \) is the identity matrix and \( J \) the matrix of all ones.
Now let $U = aI + dQ$ where $a, d$ are commuting variables. Define $R = \{ r_{ij} \}$ by

$$
\begin{cases}
1, & a_i + a_j = 0, \\
0, & \text{otherwise}.
\end{cases}
$$

Then, as in [2, p. 289] $UR$ is a symmetric type 2 matrix.

We now consider the matrices

$$
A = \begin{bmatrix} a & b & \ldots & b \\ -b & \vdots & \ddots & \vdots \\ \vdots & aI+bQ & \ddots & \vdots \\ -b & \vdots & \ddots & -b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -c & d & \ldots & d \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & d \\ (aI+dQ)R & \vdots & \ddots & d \end{bmatrix}
$$

of order $q + 1$, where $a, b, c, d$ are commuting variables.

We claim that for $q \equiv 3 \pmod{4}$,

(i) $A$ and $B$ are orthogonal designs, and

(ii) $AB^T = BA^T$ (this follows since $aI + bQ$ is type 1 and $(aI + dQ)R$ is type 2).

Hence we have

**Theorem 1.** Let $q \equiv 3 \pmod{4}$ be a prime power. Then there exists a pair of amicable orthogonal designs of order $q + 1$ and both of type $(1, q)$.

Further we note that for $q \equiv 1 \pmod{4}$ choosing

$$
N = \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 0 \end{bmatrix}
$$

gives a $(0, 1, -1)$ matrix $N$ satisfying

$$
N^T = N, \quad NN^T = qI_{q+1}.
$$

Such matrices have been called symmetric conference matrices (see [2, 293, 452]) and we have
THEOREM 2. Let \( n + 1 \equiv 2 \pmod{4} \) be the order of a symmetric conference matrix. Then there exist pairs of amicable orthogonal designs of order \( 2(n+1) \) and both of the pair of type

(i) \((2, 2n)\),

(ii) \((1, n)\).

Proof. Let \( N \) be a symmetric conference matrix and \( a, b, c, d \) be commuting variables. Then for (i) the required designs are

\[
\begin{bmatrix}
  aI+bn & aI-bn \\
  aI-bn & -aI+bn
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  aI+dn & aI-dn \\
  -aI+dn & aI-dn
\end{bmatrix},
\]

while for (ii) they are

\[
\begin{bmatrix}
  aI & bN \\
  bN & -aI
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  aI & dN \\
  -dN & aI
\end{bmatrix}.
\]

COROLLARY. Let \( q \equiv 1 \pmod{4} \) be a prime power. Then there exist pairs of amicable Hadamard designs of order \( 2(q+1) \) where both of the pair are of type \((2, 2q)\) or of type \((1, q)\).

References

