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Abstract

Recent advances in the construction of Hadamard matrices have depended on the existence of Baumert-Hall arrays and Williamson-type matrices. These latter are four $(1,-1)$ matrices A,B,C,D , of order m , which pairwise satisfy

$$(i) \quad MN^T = NM^T, \quad M,N \in \{A,B,C,D\},$$

$$\text{and (ii) } AA^T + BB^T + CC^T + DD^T = 4mI_m, \text{ where } I \text{ is the identity matrix.}$$

Currently Williamson matrices are known to exist for all orders less than 100 except: 35,39,47,53,59,65,67,70,71,73,76,77,83,89,94.

This paper gives two constructions for Williamson matrices of even order, $2n$. This is most significant when no Williamson matrices of order n are known. In particular we give matrices for the new orders 2.39,2.203,2.303,2.333,2.689,2.915, 2.1603.

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WILLIAMSON MATRICES OF EVEN ORDER

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ABSTRACT

Recent advances in the construction of Hadamard matrices have depended on the existence of Baumert-Hall arrays and Williamson-type matrices. These latter are four $(1,-1)$ matrices A, B, C, D , of order m , which pairwise satisfy

$$(i) \quad MN^T = NM^T, \quad M, N \in \{A, B, C, D\},$$

$$\text{and (ii) } AA^T + BB^T + CC^T + DD^T = 4mI_m, \quad \text{where } I \text{ is the identity matrix.}$$

Currently Williamson matrices are known to exist for all orders less than 100 except: 35, 39, 47, 53, 59, 65, 67, 70, 71, 73, 76, 77, 83, 89, 94.

This paper gives two constructions for Williamson matrices of even order, $2n$. This is most significant when no Williamson matrices of order n are known. In particular we give matrices for the new orders 2.39, 2.203, 2.303, 2.333, 2.689, 2.915, 2.1603.

1. INTRODUCTION AND BASIC DEFINITIONS

A matrix with every entry $+1$ or -1 is called a $(1,-1)$ -matrix. An Hadamard matrix $H = (h_{ij})$ is a square $(1,-1)$ matrix of order n which satisfies the equation

$$HH^T = H^TH = nI_n.$$

We use J for the matrix of all 1's and I for the identity matrix. The Kronecker product is written \times .

A Baumert-Hall array of order t is a $4t \times 4t$ array with entries $A, -A, B, -B, C, -C, D, -D$ and the properties that:

- (i) in any row there are exactly t entries $\pm A$, t entries $\pm B$, t entries $\pm C$, and t entries $\pm D$; and similarly for columns;
- (ii) the rows are formally orthogonal, in the sense that if $\pm A, \pm B, \pm C, \pm D$ are realised as elements of any commutative ring then the distinct rows of the array are pairwise orthogonal; and similarly for columns.

The Baumert-Hall arrays are a generalisation of the following array of Williamson:

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix},$$

which gives, when A, B, C, D are replaced by matrices of Williamson-type - that is, $(1, -1)$ matrices of order m which pairwise satisfy

$$(i) \quad MN^T = NM^T,$$

$$\text{and } (ii) \quad AA^T + BB^T + CC^T + DD^T = 4mI_m,$$

- an Hadamard matrix of order $4m$.

The status of knowledge about Williamson matrices and Baumert-Hall arrays is summarised below; these, together with the following theorem, give many infinite families of Hadamard matrices.

THEOREM 1. (Baumert and Hall) If there exists a Baumert-Hall array of order t and a Williamson matrix of order m then there exists an Hadamard matrix of order $4mt$.

STATEMENT 1. There exist Baumert-Hall arrays of order

- (i) $\{3, 5, 7, \dots, 59\} = B$,
- (ii) $\{1+2^a \cdot 10^b \cdot 26^c : a, b, c \text{ natural numbers}\} = A$,
- (iii) $5b, b \in A \cup B$.

STATEMENT 2. There exist Williamson-type matrices of order

- (i) $\{1, 3, 5, 7, \dots, 29, 37, 43\}$,
- (ii) $\frac{1}{2}(p+1)$, $p \equiv 1 \pmod{4}$ a prime power,
- (iii) 9^d , d a natural number,
- (iv) $\frac{1}{2}p(p+1)$, $p \equiv 1 \pmod{4}$ a prime power,
- (v) $s(4s+3), s(4s-1)$, $s \in \{1, 3, 5, \dots, 25\}$,
- (vi) 93.

This leaves the following orders less than 100 for which Williamson-type matrices are not yet known: 35, 39, 47, 53, 59, 65, 67, 70, 71, 73, 76, 77, 83, 89, 94.

Four $(1, -1)$ matrices A, B, C, D of order m with the properties

- (i) $MN^T = NM^T$ for $M, N \in \{A, B, C, D\}$,
 - (ii) $(A-I)^T = -(A-I)$, $B^T = B$, $C^T = C$, $D^T = D$,
 - (iii) $AA^T + BB^T + CC^T + DD^T = 4mI_m$,
- (1)

will be called good matrices. These are used in [2], [7], [12] to form skew-Hadamard matrices and exist for odd $m \leq 25$.

Let S_1, S_2, \dots, S_n be subsets of V , an additive abelian group of order v , containing k_1, k_2, \dots, k_n elements respectively. Write T_i for the totality of all differences between elements of S_i (with repetitions), and T for the totality of elements of all the T_i . If T contains each non-zero element a fixed number of times, λ say, then the sets S_1, S_2, \dots, S_n will be called n - $\{v; k_1, k_2, \dots, k_n; \lambda\}$ supplementary difference sets. This will be abbreviated to sds. If $n = 1$ we have a (v, k, λ) difference set which is cyclic or abelian according as V is cyclic or abelian. Henceforth we assume V is always an additive abelian group of order v with elements g_1, g_2, \dots, g_v .

The type 1 $(1, -1)$ incidence matrix $M = (m_{ij})$ of order v of a subset X of V is defined by

$$m_{ij} = \begin{cases} +1 & g_j - g_i \in X, \\ -1 & \text{otherwise;} \end{cases}$$

while the type 2 $(1, -1)$ incidence matrix $N = (n_{ij})$ of order v of a subset Y of V is defined by

$$n_{ij} = \begin{cases} +1 & g_j + g_i \in Y, \\ -1 & \text{otherwise.} \end{cases}$$

It is shown in [12] that if M is a type 1 $(1, -1)$ incidence matrix and N is a type 2 $(1, -1)$ incidence matrix of $2-(v; k_1, k_2; \lambda)$ supplementary difference sets then

$$MN^T = NM^T.$$

Also in [12], it is shown that $R = (r_{ij})$ of order v , defined on V by

$$r_{i,j} = \begin{cases} 1 & \text{if } g_i + g_j = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

then if M is type 1, MR is type 2.

Hence if M and N are type 1 of order v , $MN = NM$ and $M(NR)^T = (NR)M^T$.

In general the $(1, -1)$ incidence matrices A_1, \dots, A_n of $n-(v; k_1, k_2, \dots, k_n; \lambda)$ supplementary difference sets satisfy

$$\sum_{i=1}^n A_i A_i^T = 4 \left(\sum_{i=1}^n k_i - \lambda \right) I + [nv - 4 \left(\sum_{i=1}^n k_i - \lambda \right)] J.$$

Let $v = ef + 1 = p^a$ (p a prime). Let x be a primitive element of $GF(v) = F$ and write $G = \{z_1, \dots, z_{v-1}\}$ for the cyclic group of order $v-1$ generated by x .

Define the cyclotomic classes, C_i , of G (see Storer [4] for more details) by

$$C_i = \{x^{ej+i} : 0 \leq j \leq f-1\}, \quad 0 \leq i \leq e-1.$$

For any results implied, but unproved in this paper, on forming supplementary difference sets from cyclotomic classes, the reader is referred to [9] or [12].

2. USING GOOD MATRICES

THEOREM 2. Let A, B, C, D be four good matrices of order s . Suppose there exist four $(1, -1)$ matrices of order p, X, Y, P, Q which satisfy

$$(i) \quad \underline{SR^T = RS^T, \quad \text{for } R, S \in \{X, Y, P, Q\},}$$

$$(ii) \quad PP^T + QQ^T = 2aI + (2p-2a)J,$$

$$(iii) \quad XX^T + YY^T = 2(p+1)I - 2J.$$

Then there exist Williamson-type matrices of order $2sp$, when $4s = p-a+1$.

PROOF. Let

$$M = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix}, \quad N = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix}.$$

Then

$$MN^T = NM^T,$$

$$MM^T = I_2 \times (2aI + (2p-2a)J),$$

$$NN^T = I_2 \times (2(p+1)I - 2J).$$

Now consider

$$A_1 = I \times M + (A-I) \times N$$

$$A_2 = B \times N$$

$$A_3 = C \times N$$

$$A_4 = D \times N.$$

Clearly

$$A_i A_j^T = A_j A_i^T, \quad i, j = 1, 2, 3, 4,$$

and

$$\begin{aligned} \sum_{i=1}^4 A_i A_i^T &= I \times MM^T + (4s-1)I \times NN^T \\ &= I_s \times I_2 \times [(2a+2(4s-1)(p+1))I + (2p-2a-8s+2)J] \\ &= 8spI_{2sp}, \quad \text{when } s = (p-a+1)/4. \end{aligned}$$

Hence we have the result.

COROLLARY 1. Suppose there exist $2-(p; k_1, k_2; k_1+k_2-\frac{1}{2}(p+1)+2s)$ sds, X_1, X_2 , and $2-(p; \frac{1}{2}(p-1); \frac{1}{2}(p-3))$ sds, X_3, X_4 , with the property that

$$\underline{x \in X_i \Rightarrow -x \in X_i}, \quad \underline{i = 2, 3, 4},$$

where s is the order of a good matrix. Then there exist Williamson-type matrices of order $2sp$.

PROOF. Let Q, X, Y be the type 1 $(1, -1)$ incidence matrices of X_2, X_3, X_4 . Then Q, X, Y are symmetric.

Let P be the type 2 $(1, -1)$ incidence matrix of X_1 . Then (i) of the theorem is satisfied. Further

$$PP^T + QQ^T = (2p+2-8s)I + (8s-2)J,$$

$$XX^T + YY^T = (2p+2)I - 2J.$$

Hence there exist Williamson-type matrices of order $2sp$.

COROLLARY 2. If $p = 4f+1$, f odd, is a prime power of the form $9+4t^2$ or $25+4t^2$ and there exist good matrices of order $(f+1)/8$, then there exist Williamson-type matrices of order $(f+1)(4f+1)/4$.

PROOF. We note that for $9+4t^2$ and $25+4t^2$, C_0, C_0+C_2 and C_1, C_0+C_2 respectively are $2-\{4f+1; 2f, f; (5f-3)/4\}$ sds. Then using C_0+C_2 and C_1+C_3 for the other sds in the previous corollary we have the result.

COROLLARY 3. If $p = 4f+1$, f odd, is a prime power of the form $1+4t^2$ or $49+4t^2$ and there exist good matrices of order $(f-1)/8$, then there exist Williamson-type matrices of order $(f-1)(4f+1)/4$.

PROOF. We note that for $1+4t^2$ and $49+4t^2$, $\{0\}+C_0, C_0+C_2$ and $\{0\}+C_1, C_0+C_2$ respectively are $2-\{4f+1; 2f, f+1; (5f-1)/4\}$ sds. Then using C_0+C_2 and C_1+C_3 for the other sds in corollary 1 we have the result.

For $f = 25$ and 57 we get Williamson matrices for the following orders $2n$ where no Williamson matrix of order n is known: 2.303, 2.1603.

COROLLARY 4. If $p = 4f+1$ is a prime power and $(p-1)/4$ is the order of a good matrix, then there exist Williamson type matrices of order $\frac{1}{2}p(p-1)$.

PROOF. Use the (p, p, p) and $(p, p-1, p-2)$ difference sets to form the $2-\{p; p, p-1; 2p-2\}$ sds for the corollary.

For $p = 13$ we find there is a good matrix of order $3 = (p-1)/4$ and hence Williamson-type matrices of order 2.39 even though Williamson-type matrices of order 39 are not yet known. This corollary also gives us Williamson-type matrices for the following orders $2n$ where no Williamson matrix of order n is known: 2.203, 2.333,

2.689, 2.915.

COROLLARY 5. Let $p \equiv 1 \pmod{4}$ be a prime power. Further suppose there exists a (p, k, λ) difference set; then if there exist good matrices of order

$$(i) \ s = [2(\lambda - k) + p + 1]/4; \quad (ii) \ s = [2(\lambda - k) + p - 1]/4$$

respectively, there exist Williamson-type matrices of order $2sp$.

PROOF. Use $Q = J$, $Q = J - 2I$ respectively in the theorem and form X and Y from the type 1 incidence matrices of $C_0 + C_2$ and $C_1 + C_3$ respectively. For P use the type 2 incidence matrix of the difference set.

3. USING SOME OTHER WILLIAMSON MATRICES

THEOREM 3. Let $I+R, I-R, S, S$ be four Williamson matrices of order s . Suppose there exist four $(1, -1)$ matrices of order p, X, Y, P, Q which satisfy

$$(i) \ ZW^T = WZ^T, \text{ for } Z, W \in \{X, Y, P, Q\},$$

$$(ii) \ PP^T + QQ^T = 2aI + (2p - 2a)J,$$

$$(iii) \ XX^T + YY^T = 2(p+1)I - 2J.$$

Then there exist Williamson matrices of order $2ps$, where $s = \frac{1}{2}(p-a+1)$.

PROOF. Let

$$M = \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix}, \quad N = \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix}.$$

Then, as before,

$$MN^T = NM^T,$$

$$MM^T = I_2 \times (2aI + (2p - 2a)J),$$

$$NN^T = I_2 \times (2(p+1)I - 2J).$$

Now consider

$$A_1 = I \times M + R \times N$$

$$A_2 = S \times N$$

$$A_3 = I \times -M + R \times N$$

$$A_4 = S \times N.$$

Clearly

$$A_i A_j^T = A_j A_i^T, \quad i, j = 1, 2, 3, 4,$$

and

$$\begin{aligned} \sum_{i=1}^4 A_i A_i^T &= 2I \times MM^T + 2(RR^T + SS^T) \times NN^T \\ &= I \times I_2 \times [4aI + 2(2p-2a)J] + I \times I_2 \times [4(2s-1)(p+1)I - 4(2s-1)J] \\ &= 8spI_{2sp}, \quad \text{when } s = \frac{1}{2}(p-a+1); \end{aligned}$$

which gives the result.

COROLLARY 1. Suppose there exist Williamson-type matrices, $I+R, I-R, S, S$, of order s . Suppose there exist $2-\{p; k_1, k_2; k_1+k_2+s-\frac{1}{2}(p+1)\}$ sds with incidence matrices P and Q , and $2-\{p; \frac{1}{2}(p-1); \frac{1}{2}(p-3)\}$ sds with incidence matrices X and Y which satisfy

$$ZW^T = WZ^T \quad \text{for } Z, W \in \{P, Q, X, Y\}.$$

Then there exist Williamson matrices of order $2ps$.

COROLLARY 2. Suppose there exist Williamson-type matrices $I+R, I-R, S, S$, of order $p = \frac{1}{2}(s-1)$. Suppose there exists a symmetric Hadamard matrix of order $s+1 \equiv 0 \pmod{4}$. Then there exist Williamson matrices of order $s(s-1)$.

PROOF. Normalize the Hadamard matrix to the form

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & E & \\ 1 & & & \end{bmatrix}, \quad E^T = E,$$

and use $P = J$, $Q = J-2I$, $X = Y = E$ in the theorem.

COROLLARY 3. Let $p \equiv 1 \pmod{4}$ be a prime power. Suppose there exists a symmetric Hadamard matrix of order $p+3$. Then there exist Williamson matrices of order $(p+2)(p+1)$.

PROOF. There exist Williamson matrices of order $\frac{1}{2}(p+1)$ of the required form.

This gives Williamson matrices of the following orders $2n$ where none are known for n : 2.105, 2.171, 2.903.

COROLLARY 4. Let $p \equiv 1 \pmod{4}$ be a prime power. Suppose there exists a (v, k, λ) difference set, where v is a prime power and $\lambda = k + \frac{1}{2}(p-v)$. Then there exist Williamson matrices of order $v(p+1)$.

PROOF. Let $P = J, Q$ be the type 2 $(1, -1)$ incidence matrix of the (v, k, λ) difference set; let $X = Y$ be the type 1 $(1, -1)$ incidence matrix of the $(v, (v-1)/2, (v-3)/4)$ difference set for $v \equiv 3 \pmod{4}$ and X, Y be the type 1 $(1, -1)$ incidence matrices of $2-\{v; (v-1)/2; (v-3)/2\}$ sds for $v \equiv 1 \pmod{4}$.

COROLLARY 6. Let $p \equiv 1 \pmod{4}$ be a prime power. Suppose there exists a (v, k, λ) difference set, where v is a prime power and $\lambda = k + (p-v)/4$. Then there exist Williamson matrices of order $v(p+1)$.

PROOF. For $P = Q$ use the type 2 incidence matrix of the difference set. Form X and Y as in the previous corollary.

Neither of the last two corollaries give interesting matrices for small orders.

COROLLARY 7. Let $p = 4f+1$ (f odd) be a prime power of the form $9+4t^2$ or $25+4t^2$. Suppose $(f-1)/2 \equiv 1 \pmod{4}$ is a prime power. Then there exist Williamson matrices of order $\frac{1}{2}(f+1)(4f+1)$.

PROOF. For P and Q in the theorem use the type 2 and type 1 incidence matrices respectively of C_0 and C_0+C_2 or C_1 and C_0+C_2 which are $2-\{4f+1; 2f, f; (5f-3)/4\}$ sds for the prime powers of the theorem. For X and Y use the type 1 incidence matrices of C_0+C_2 and C_1+C_3 which are $2-\{4f+1; 2f; 2f-1\}$ sds.

COROLLARY 8. Let $p = 4f+1$ (f odd) be a prime power of the form $1+4t^2$ or $49+4t^2$. Suppose $(f-3)/2 \equiv 1 \pmod{4}$ is a prime power. Then there exist Williamson matrices of order $\frac{1}{2}(f-1)(4f+1)$.

PROOF. Proceed as in the previous corollary but use $\{0\}+C_0$ or $\{0\}+C_1$ to form P .

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