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A list of balanced incomplete block designs for r < 30

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Abstract

A balanced incomplete block design consists of a set of v elements arranged into b k-element subsets called blocks such that each element occurs r times and each pair of elements appears in lambda distinct blocks. The numbers v,b,r,k,lambda are called the parameters of the design. A necessary condition that a design exist is that the parameters be integers satisfying:

(1) vr = bk

(2) r(k-1) = lambda (v-1)

(32)

A List of (v,b,r,k,λ) Designs for $r \leq 30$

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(1) $$vr = bk$$

(2) $$r(k-1) = \lambda(v-1)$$

An interest in these structures as actual experimental designs prompted the earliest tabulation of known designs in 1943 by Fisher and Yates whose tables showed all designs with $r \leq 10$ known up to 1943. More than ten replications ($r = 10$) was thought to be impracticable since the application of the analysis of variance to the experimental data had to be done by hand and desk calculator. Blanks were left if it was unknown whether a design existed with a given set of parameters satisfying (1) and (2). In the original Fisher-Yates tables 12 blanks were left and of these only two are undecided today. These are $(v,b,r,k,\lambda) = (46, 69, 9, 6, 1)$ and (51, 85, 10, 6, 1). In 1961, C.R. Rao[4] extended the tables to designs with $11 \leq r \leq 15$ and in 1962, D.A. Sprott[5]

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published a list with $16 \leq r \leq 20$. Many block designs come from difference sets. A list of difference sets was compiled by Takeuchi [6] in 1962. A listing of block designs classified by v and k appears in the paper by Collens [1]. Another family listing of designs for v from 21 to 30 is an unpublished work of B. Gardiner to which the authors had access. The most widely known listing of block designs with $r \leq 15$ appears in M. Hall's book [3]. This list is a reworking of the Fisher-Yates and Rao tables with the addition of designs given by finite geometries over finite fields. Hall's table lists 116 parameter sets of which 26 are designated "solution unknown." Since 1967, 8 of these 26 have been constructed.

The present list grew out of the work of Jennifer Seberry Wallis and W.D. Wallis on Hadamard matrices. From a normalized Hadamard matrix of order 4t we may construct a balanced incomplete block design with parameters $v=b=4t-1$, $r=k=2t-1$, $\lambda=t-1$ and conversely, given such a design we can construct a normalized Hadamard matrix. Whenever a balanced incomplete block design (v,v,k,k,λ) exists then the

(i) derived design $(k,\ v-1,\ k-1,\ \lambda,\ \lambda-1)$ exists, as does

(ii) residual design $(v-k,\ v-1,\ k,\ k-\lambda,\ \lambda)$.

A residual design is obtained from a symmetric design (i.e., $v=b$, $r=k$) by deleting all elements in one block. If $\lambda=1$ or 2, a design having the parameters of a residual design is a residual design. This is not true for $\lambda \geq 3$. If a symmetric design with $\lambda=1$ or 2 does not exist, say by the Bruck, Chowla, Ryser Theorem, (see Hall [3] for details) then no design exists with the parameters of the corresponding

residual design.

The list which follows gives the status of balanced incomplete block designs (v,b,r,k,λ) for $r \leq 30$, and for $6 \leq k \leq \frac{1}{2}v$ by reference to a valid construction if a design with the given parameters has been shown to exist. Designs with $k < 6$ are omitted since they are all known to exist except for (15, 21, 7, 5, 2) which in non-existent. If a design exists with (v,b,r,k,λ), multiples, that is, designs with parameters $(v, tb, tr, k, t\lambda)$, have been omitted. However, parameters which are multiples of designs which are unknown or non-existent are listed. A code for interpreting the notation under the heading "comment" precedes the table as does the listing of constructions given by (i) in the table. Other more general references appear at the end of the paper. No attempt has been made to trace the earliest source of any known block design.

The authors wish to thank the many persons, who from attendance at several combinatorics conferences, knew of the existence of the Wallis' listing of block designs and urged its publication.

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Codes to interpret comments in the following lists:

Code:

??	unknown
NE1	non-existent by BCR theorem
NE2	non-existent as residual of a NE1 with $\lambda = 1$ or 2.

Code for existent designs: in every case, the reference is to a correct design

H"number"	"number" from Hall's list.
F"number"	"number" from Fisher and Yates 6th edition.
T"number"	"number" from Takeuchi's list.
R"number"	"number" from Rao's list.
S	from Sprott's list.
R:"number"	residual of "number" in this list which exists.
D:"number"	derived from "number" in this list which exists.
("number")	see "number" in list below
PG	exists as a projective geometry.
WSW	a (4t-1, 4t-1, 2t-1, 2t-1, t-1) design is equivalent to an Hadamard matrix of order 4t which can be found in Appendix A of Wallis, Street and Wallis

Other Codes:

"code"*	although this design either does not exist or is unknown, a multiple is given in Gardiner's List.
??R:"number"	unknown; a residual of "number" of this list.
??D:"number"	unknown; a derived design of "number" of this list.
??M:"number"	unknown; a multiple of a design which is either unknown or does not exist.

The constructions indicated by (i) are given below:

(1) M. Hall Jr., R. Lane, and D. Wales, Designs derived from permutation groups, J. Combinatorial Th. Ser A. 8 (1970), 12-22.

(2) Michael Aschbacher, On collineation groups of symmetric block designs, J. Combinatorial Theory, 3(1971), 272-281.

(3) R.G. Stanton and D.G. Gryte, A family of BIBD's, Combinatorial Structures and their Applications, Gordon and Breach, N.Y. 1970, 411-412.

(4) E. Seiden, A method of contruction of resolvable BIBD, Sankhyā 25A (1963), 393-394.

(5) A. Sillito, An extension property of class of balanced imcomplete block designs, Biometrika 44, 278-279.

(6) W.H. Clatworthy and R.J. Lewyckyj, Comments on Takeuchi's table of difference sets generating balanced incomplete block designs Review of Int. Stat. Inst. 36 (1968), 12-18

(7) B. Gardiner, (v, k) Family listing of BIBD's for v = 21 to 30, unpublished.

(8) S.S. Shrikhande, On a two-parameter family of balanced incomplete block desings, Sankhya 24A (1962), 3[illegible]-40.

(9) D.A. Sprott, A note on balanced incomplete block designs, Caned. J. Math. 6 (1954), 341-346.

(10) K. Takeuchi, A table of difference sets generating balanced incomplete block designs, Review of Int. Stat. Inst. 30 (1962), 361-366.

(11) D.A. Sprott, Some series of balanced incomplete block designs, Sankhya, 17 (1956), 185-192.

(12) M. Hall, Jr. [3], p. 141, Type B.

(13) Richard M. Wilson, Construction of symmetric designs, (to appear)

NO	v	b	r	k	λ	COMMENT
1	31	31	6	6	1	H12
2	16	16	6	6	2	H10
3	36	42	7	6	1	NE2
4	43	43	7	7	1	NE1
5	22	22	7	7	2	NE1*
6	15	15	7	7	3	H16
7	21	28	8	6	2	NE2
8	49	56	8	7	1	H24
9	57	57	8	8	1	H25
10	29	29	8	8	2	NE1
11	46	69	9	6	1	??
12	16	24	9	6	3	H28
13	28	36	9	7	2	H33
14	64	72	9	8	1	H36
15	73	73	9	9	1	H37
16	37	37	9	9	2	H34
17	25	25	9	9	3	F20
18	19	19	9	9	4	H30
19	51	85	10	6	1	??
20	21	30	10	7	3	H39
21	36	45	10	8	2	NE2
22	81	90	10	9	1	H45
23	91	91	10	10	1	H46
24	46	46	10	10	2	NE1
25	31	31	10	10	3	H40
26	12	22	11	6	5	H49
27	45	55	11	9	2	R:30
28	100	110	11	10	1	??R:29
29	111	111	11	11	1	??
30	56	56	11	11	2	(1)
31	23	23	11	11	5	WSW
32	61	122	12	6	1	??
33	21	42	12	6	3	H58
34	13	26	12	6	5	H56
35	22	33	12	8	4	??*
36	33	44	12	9	3	D:41
37	55	66	12	10	2	NE2
38	121	132	12	11	1	H68
39	133	133	12	12	1	H69
40	67	67	12	12	2	NE1
41	45	45	12	12	3	T100A
42	34	34	12	12	4	NE1
43	66	143	13	6	1	??
44	14	26	13	7	6	R:53
45	27	39	13	9	4	H71
46	40	52	13	10	3	??R:51
47	66	78	13	11	2	R:50
48	144	156	13	12	1	??R:49
49	157	157	13	13	1	??
50	79	79	13	13	2	(2)
51	53	53	13	13	3	NE1
52	40	40	13	13	4	H75
53	27	27	13	13	6	H72
54	36	84	14	6	2	H87
55	15	35	14	6	5	H62
56	85	170	14	7	1	??
57	43	86	14	7	2	H88
58	29	58	14	7	3	H86
59	22	44	14	7	4	F64
60	78	91	14	12	2	NE2
61	169	182	14	13	1	R:62
62	183	183	14	14	1	PG
63	92	92	14	14	2	NE1
64	76	190	15	6	1	??
65	26	65	15	6	3	H99
66	16	40	15	6	5	F75
67	91	195	15	7	1	H111
68	16	30	15	8	7	R:81
69	21	35	15	9	6	R:80
70	136	204	15	10	1	??
71	46	69	15	10	3	??
72	28	42	15	10	5	??*
73	56	70	15	12	3	??R:78
74	91	105	15	13	2	NE2
75	196	210	15	14	1	NE2
76	211	211	15	15	1	NE1
77	106	106	15	15	2	NE1
78	71	71	15	15	3	??
79	43	43	15	15	5	NE1
80	36	36	15	15	6	T82A
81	31	31	15	15	7	PG
82	81	216	16	6	1	??
83	21	56	16	6	4	S
84	113	226	16	8	1	??
85	29	58	16	8	4	T65
86	17	34	16	8	7	D:111
87	145	232	16	10	1	??
88	25	40	16	10	6	??*R:100
89	33	48	16	11	5	?? R:99
90	177	236	16	12	1	? ?
91	45	60	16	12	4	??R:98
92	65	80	16	13	3	??R:97
93	105	120	16	14	2	??R:96
94	225	240	16	15	1	??R:95
95	241	241	16	16	1	??
96	121	121	16	16	2	??
97	81	81	16	16	3	??
98	61	61	16	16	4	??
99	49	49	16	16	5	??
100	41	41	16	16	6	??

NO	v	b	r	k	λ	COMMENT
101	18	51	17	6	5	T33
102	35	85	17	7	3	??
103	120	255	17	8	1	(4)
104	18	34	17	9	8	R:111
105	52	68	17	13	4	??R:110
106	120	136	17	15	2	??
107	256	272	17	16	1	R:108
108	273	273	17	17	1	PG
109	137	137	17	17	2	NE1
110	69	69	17	17	4	(13)
111	35	35	17	17	8	WSW
112	91	273	18	6	1	??
113	46	138	18	6	2	??M:11
114	19	57	18	6	5	T37
115	145	290	18	9	1	??
116	49	98	18	9	3	??
117	55	99	18	10	3	??
118	100	150	18	12	2	??
119	34	51	18	12	6	??
120	85	102	18	15	3	??
121	136	153	18	16	2	??
122	289	306	18	17	1	R:123
123	307	307	18	18	1	PG
124	154	154	18	18	2	??
125	103	103	18	18	3	NE1
126	52	52	18	18	6	NE1
127	96	304	19	6	1	??
128	153	323	19	9	1	??
129	20	38	19	10	9	R:138
130	39	57	19	13	6	??
131	96	114	19	16	3	??R:136
132	153	171	19	17	2	NE2
133	324	342	19	18	1	??R:134
134	343	343	19	19	1	??
135	172	172	19	19	2	NE1
136	115	115	19	19	3	??
137	58	58	19	19	6	NE1
138	39	39	19	19	9	WSW
139	51	170	20	6	2	??
140	21	70	20	6	5	??
141	36	90	20	8	4	??M:21
142	181	362	20	10	1	??
143	61	122	20	10	3	??
144	46	92	20	10	4	??M:24
145	37	74	20	10	5	T86
146	21	42	20	10	9	D:189
147	111	185	20	12	2	??
148	45	75	20	12	5	??
149	141	188	20	15	2	??
150	57	76	20	15	5	??
151	36	48	20	15	8	(5)
152	76	95	20	16	4	R:157
153	171	190	20	18	2	??R:156
154	361	380	20	19	1	R:155
155	381	381	20	20	1	PG
156	191	191	20	20	2	??
157	96	96	20	20	4	(6)
158	77	77	20	20	5	NE1
159	106	371	21	6	1	??
160	36	126	21	6	3	??M:3
161	22	77	21	6	5	(7)
162	16	56	21	6	7	(8)
163	127	381	21	7	1	??
164	64	192	21	7	2	??
165	43	129	21	7	3	(9)
166	22	66	21	7	6	??M:5
167	19	57	21	7	7	(11)
168	57	133	21	9	3	??
169	190	399	21	10	1	??
170	22	42	21	11	10	R:189
171	232	406	21	12	1	??
172	274	411	21	14	1	??
173	92	138	21	14	3	??
174	40	60	21	14	7	??
175	295	413	21	15	1	??
176	50	70	21	15	6	??R:187
177	64	84	21	16	5	R:186
178	85	105	21	17	4	??
179	120	140	21	18	3	??R:184
180	190	210	21	19	2	??R:183
181	400	420	21	20	1	??R:182
182	421	421	21	21	1	??
183	211	211	21	21	2	??
184	141	141	21	21	3	??
185	106	106	21	21	4	NE1
186	85	85	21	21	5	PG
187	71	71	21	21	6	??
188	61	61	21	21	7	NE1
189	43	43	21	21	10	WSW
190	111	407	22	6	1	??
191	133	418	22	7	1	??
192	100	220	22	10	2	??M:28
193	221	442	22	11	1	??
194	111	222	22	11	2	??M:29
195	45	90	22	11	5	??
196	287	451	22	14	1	??
197	45	66	22	15	7	??R:206
198	56	77	22	16	6	??R:205
199	133	154	22	19	3	??R:204
200	210	231	22	20	2	NE2

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NO	v	b	r	k	λ	COMMENT
201	441	462	22	21	1	*NE*2
202	463	463	22	22	1	*NE*1
203	232	232	22	22	2	*NE*1
204	155	155	22	22	3	??
205	78	78	22	22	6	??
206	67	67	22	22	7	*NE*1
207	24	92	23	6	5	??*
208	70	230	23	7	2	??
209	24	69	23	8	7	??*
210	70	161	23	10	3	??
211	231	483	23	11	1	??
212	24	46	23	12	11	*R*:217
213	231	253	23	21	2	*NE*2
214	484	506	23	22	1	*NE*2
215	507	507	23	23	1	*NE*1
216	254	254	23	23	2	*NE*2
217	47	47	23	23	11	*WSW*
218	121	484	24	6	1	??
219	61	244	24	6	2	??*M*:32
220	41	164	24	6	3	(9)
221	25	100	24	6	5	(9)
222	169	507	24	8	1	??
223	85	255	24	8	2	??
224	43	129	24	8	4	(9)
225	29	87	24	8	6	??*M*:10
226	25	75	24	8	7	(9)
227	22	66	24	8	8	??*M*:35
228	55	132	24	10	4	??*M*:37
229	25	60	24	10	9	(5)
230	265	530	24	12	1	??
231	89	178	24	12	3	??
232	67	134	24	12	4	??*M*:40
233	34	68	24	12	8	??*M*:42
234	25	50	24	12	11	*D*:277
235	105	180	24	14	3	??
236	85	136	24	15	4	??
237	46	69	24	16	8	??*R*:249
238	69	92	24	18	6	??
239	115	138	24	20	4	??*R*:247
240	161	184	24	21	3	??
241	49	56	24	21	10	(5)
242	253	276	24	22	2	*NE*2
243	529	552	24	23	1	*R*:244
244	553	553	24	24	1	*PG*
245	277	277	24	24	2	*NE*1
246	185	185	24	24	3	*NE*1
247	139	139	24	24	4	??
248	93	93	24	24	6	*NE*1
249	70	70	24	24	8	??
250	126	525	25	6	1	(3)
251	176	550	25	8	1	??
252	226	565	25	10	1	??
253	76	190	25	10	3	??
254	46	115	25	10	5	??
255	26	65	25	10	9	?? *D:294
256	276	575	25	12	1	??
257	26	50	25	13	12	*R*:277
258	351	585	25	15	1	??
259	51	85	25	15	7	??
260	36	60	25	15	10	??
261	51	75	25	17	8	??
262	76	100	25	19	6	R:274
263	476	595	25	20	1	??
264	96	120	25	20	5	??*R*:273
265	126	150	25	21	4	??
266	176	200	25	22	3	??*R*:271
267	276	300	25	23	2	??*R*:270
268	576	600	25	24	1	??*R*:269
269	601	601	25	25	1	??
270	301	301	25	25	2	??
271	201	201	25	25	3	??
272	151	151	25	25	4	*NE*1
273	121	121	25	25	5	(13)
274	101	101	25	25	6	(12)
275	76	76	25	25	8	*NE*1
276	61	61	25	25	10	(13)
277	51	51	25	25	12	*WSW*
278	27	117	26	6	5	??*
279	92	299	26	8	2	??
280	235	611	26	10	1	??
281	40	104	26	10	6	??*M*:46
282	144	312	26	12	2	??*M*:48
283	313	626	26	13	1	??
284	157	314	26	13	2	??*M*:49
285	105	210	26	13	3	??
286	53	106	26	13	6	(9)
287	40	65	26	16	10	??*R*:294
288	105	130	26	21	5	??*R*:293
289	300	325	26	24	2	*NE*2
290	625	650	26	25	1	*R*:291
291	651	651	26	26	1	*PG*
292	326	326	26	26	2	*NE*1
293	131	131	26	26	5	??
294	66	66	26	26	10	??
295	136	612	27	6	1	??
296	46	207	27	6	3	??*M*:11
297	28	126	27	6	5	??*D:254
298	217	651	27	9	1	??
299	109	327	27	9	2	??
300	55	165	27	9	4	??

NO	v	b	r	k	λ	COMMENT
301	28	84	27	9	3	??*D:356
302	55	135	27	11	5	??
303	100	225	27	12	3	??
304	28	63	27	12	11	D:357
305	325	675	27	13	1	??
306	28	54	27	14	13	R:321
307	190	342	27	15	2	??
308	55	99	27	15	7	??
309	460	690	27	18	1	??
310	154	231	27	18	3	??
311	52	78	27	18	9	??R:320
312	91	117	27	21	6	??
313	208	234	27	24	3	??
314	325	351	27	25	2	??R:317
315	676	702	27	26	1	??R:316
316	703	703	27	27	1	??
317	352	352	27	27	2	??
318	235	235	27	27	3	NE1
319	118	118	27	27	6	NE1
320	79	79	27	27	9	(13)
321	55	55	27	27	13	WSW
322	141	658	28	6	1	??
323	21	98	28	6	7	??
324	169	676	28	7	1	??
325	85	340	28	7	2	??M:56
326	57	228	28	7	3	??
327	25	100	28	7	7	(11)
328	50	175	28	8	4	??
329	225	700	28	9	1	??
330	85	238	28	10	3	??
331	309	721	28	12	1	??
332	78	182	28	12	4	??M:60
333	45	105	28	12	7	??
334	365	730	28	14	1	??
335	92	184	28	14	4	??M:63
336	53	106	28	14	7	(9)
337	29	58	28	14	13	D:373
338	36	63	28	16	12	R:357
339	477	742	28	18	1	??
340	57	84	28	19	9	??R:356
341	561	748	28	21	1	??
342	141	188	28	21	4	??
343	81	108	28	21	7	??R:355
344	57	76	28	21	10	??
345	99	126	28	22	6	??R:354
346	162	189	28	24	4	??R:353
347	225	252	28	25	3	??R:352
348	351	378	28	26	2	NE2
349	729	756	28	27	1	R:350
350	757	757	28	28	1	PG
351	379	379	28	28	2	NE1
352	253	253	28	28	3	??
353	190	190	28	28	4	NE1
354	127	127	28	28	6	??
355	109	109	28	28	7	(12)
356	85	85	28	28	9	??
357	64	64	28	28	12	(8)
358	30	145	29	6	5	(7)
359	175	725	29	7	1	??
360	117	377	29	9	2	??
361	30	87	29	10	9	??*
362	117	261	29	13	3	??
363	378	783	29	14	1	??
364	30	58	29	15	14	R:373
365	88	116	29	22	7	??
366	175	203	29	25	4	??R:371
367	378	406	29	27	2	??R:370
368	784	812	29	28	1	??R:369
369	813	813	29	29	1	??
370	407	407	29	29	2	??
371	204	204	29	29	4	??
372	117	117	29	29	7	NE1
373	59	59	29	29	14	WSW
374	151	755	30	6	1	??
375	76	380	30	6	2	??M:64
376	51	255	30	6	3	??M:19
377	36	135	30	8	6	??M:21
378	271	813	30	10	1	??
379	136	408	30	10	2	??M:70
380	55	165	30	10	5	??
381	46	138	30	10	6	??M:24
382	28	84	30	10	10	??M:72
383	166	415	30	12	2	??
384	56	140	30	12	6	??M:73
385	34	85	30	12	10	??
386	91	210	30	13	4	??M:74
387	196	420	30	14	2	??M:75
388	421	842	30	15	1	??
389	211	422	30	15	2	??M:76
390	141	282	30	15	3	??
391	106	212	30	15	4	??M:77
392	85	170	30	15	5	??
393	71	142	30	15	6	??M:78
394	61	122	30	15	7	(9)
395	43	86	30	15	10	??M:79
396	171	285	30	18	3	??
397	286	429	30	20	2	??
398	96	144	30	20	6	??
399	58	87	30	20	10	??
400	301	430	30	21	2	??

NO	v	b	r	k	λ	COMMENT
401	116	145	30	24	6	??
402	145	174	30	25	5	R:409
403	261	290	30	27	3	??R:408
404	406	435	30	28	2	NE2
405	841	870	30	29	1	R:406
406	871	871	30	30	1	PG
407	436	436	30	30	2	NE1
408	291	291	30	30	3	??
409	175	175	30	30	5	(10)
410	146	146	30	30	6	NE1
411	88	88	30	30	10	NE1

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