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Abstract

We study the conjecture: There exists a square (0,1,-1)-matrix $W = W(w,k)$ of order w satisfying

$$WW^T = kI_w$$

for all $k = 0, 1, \dots, w$ when $w \equiv 0 \pmod{4}$. We prove the conjecture is true for 4, 8, 12, 16, 20, 24, 28, 32, 40 and give partial results for 36, 44, 52, 56.

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ORTHOGONAL (0,1,-1)-MATRICES

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ABSTRACT

We study the conjecture:

There exists a square (0,1,-1)-matrix $W = W(w,k)$ of order w satisfying

$$WW^T = kI_w$$

for all $k = 0, 1, \dots, w$ when $w \equiv 0 \pmod{4}$.

We prove the conjecture is true for 4, 8, 12, 16, 20, 24, 28, 32, 40 and give partial results for 36, 44, 52, 56.

One generalization of Hadamard matrices is to weighing matrices (see Olga Taussky [3]), that is square (0,1,-1)-matrices, W , of order n satisfying

$$WW^T = kI_n, \quad k \leq n, \quad (1)$$

where I_n is the identity matrix of order n , W^T denotes W transposed.

Clearly

$$WW^T = W^T W = kI_n. \quad (2)$$

These matrices have application both in design of weighing experiments (see Raghavarao [2]) and in coding theory.

Write $W(w,k)$ for a weighing matrix satisfying (1).

RELEVANT MATRICES

Clearly $(0,1,-1)$ -matrices satisfying $WW^T = 0$ and $1I$, always exist. Matrices satisfying

$$WW^T = nI_n, \quad n \equiv 0 \pmod{4}$$

are *Hadamard matrices* and if $U = I + W$ is a *skew-Hadamard matrix*

$$WW^T = (n-1)I_n.$$

For up-to-date results about these matrices we refer the reader to [1, 4, 5]. These matrices exist for 2 and all $n \equiv 0 \pmod{4}$, $n \leq 100$.

If $n \equiv 2 \pmod{4}$ a matrix satisfying $U = I + W$ with

$$WW^T = (n-1)I_n, \quad W^T = W$$

is called a *symmetric conference matrix* and these can only exist if

$$n-1 = a^2 + b^2$$

a, b integer (see Raghavarao [2]). These matrices exist for $n \equiv 2 \pmod{4}$, $n-1$ a prime power for $n < 100$.

Write H_n for the Hadamard matrix of order n , J_n for the matrix of order n of all ones, S_n for the matrix of order n with zero diagonal and other elements ± 1 satisfying

$$S_n S_n^T = (n-1)I_n.$$

The symbol \times denotes the Kronecker product and the orders of all

matrices are assumed to be compatible under binary operations.

SOME CONSTRUCTIONS

CONSTRUCTION 1. Provided $AA^T + BB^T + CC^T + DD^T = mI_n$ and for any X , $Y \in \{A, B, C, D\}$, X and Y are $(0,1,-1)$ -matrices and $XY^T = YX^T$, then

$$W = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

satisfies

$$WW^T = mI_{4n}$$

CONSTRUCTION 2. Provided $\sum_{i=1}^8 A_i A_i^T = mI_n$, and each A_i is a $(0,1,-1)$ -matrix and $A_i A_j^T = A_j A_i^T$, then

$$W = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ -A_2 & A_1 & A_4 & -A_3 & A_6 & -A_5 & A_8 & -A_7 \\ -A_3 & -A_4 & A_1 & A_2 & -A_7 & A_8 & A_5 & -A_6 \\ -A_4 & A_3 & -A_2 & A_1 & A_8 & A_7 & -A_6 & -A_5 \\ \hline -A_5 & -A_6 & A_7 & -A_8 & A_1 & A_2 & -A_3 & A_4 \\ -A_6 & A_5 & -A_8 & -A_7 & -A_2 & A_1 & A_4 & A_3 \\ -A_7 & -A_8 & -A_5 & A_6 & A_3 & -A_4 & A_1 & A_2 \\ -A_8 & A_7 & A_6 & A_5 & -A_4 & -A_3 & -A_2 & A_1 \end{bmatrix} \quad (3)$$

satisfies

$$WW^T = mI_{8n}.$$

CONSTRUCTION 6. If there exist two circulant $(0,1,-1)$ -matrices A and B of order n satisfying

$$AA^T + BB^T = kI_n$$

then

$$W = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix} \quad (5)$$

satisfies

$$WW^T = kI_{2n}.$$

CONSTRUCTION 7. If there exist two circulant $(0,1,-1)$ -matrices A and B of order n satisfying

$$AA^T + BB^T = (k+2)I - 2J$$

$$AJ = 0, \quad BJ = J$$

then

$$W = \left[\begin{array}{cc|cc} 0 & 1 & e & e \\ 1 & 0 & e & -e \\ \hline e^T & -e^T & A & B \\ e^T & e^T & -B^T & A^T \end{array} \right] \quad (6)$$

satisfies

$$WW^T = \left[\begin{array}{cc|c} 2n+1 & 0 & 0 \\ 0 & 2n+1 & 0 \\ \hline 0 & 0 & kI_{2n} \end{array} \right] \quad (7)$$

CONSTRUCTION 8. Suppose $S^T = S = W(w, w-1)$

$$\begin{bmatrix} S & S \\ S & -S \end{bmatrix}, \begin{bmatrix} S & S+I & S & S-I \\ -S-I & S & -S+I & S \\ \hline -S & S-I & S & -S-I \\ -S+I & -S & S+I & S \end{bmatrix}, \begin{bmatrix} 0 & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix},$$

$$\begin{bmatrix} I & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix}, \begin{bmatrix} S & S & S & S \\ -S & S & S & -S \\ -S & -S & S & S \\ -S & S & -S & S \end{bmatrix}, \begin{bmatrix} 0 & S & S+I & S-I \\ -S & 0 & -S+I & S+I \\ \hline -S-I & S-I & 0 & -S \\ -S+I & -S-I & S & 0 \end{bmatrix} \text{ are}$$

$W(2w, 2w-2)$, $W(4w, 4w-2)$, $W(4w, 3w-3)$, $W(4w, 3w-2)$,
 $W(4w, 4w-4)$, $W(4w, 3w-1)$ respectively.

CONSTRUCTION 9. Let C be a $(0,1,-1)$ -matrix with zero diagonal satisfying

$$CC^T = cI_d$$

and let B be a $(0,1,-1)$ -matrix satisfying

$$BB^T = aI_c - J_c, \quad BJ = 0.$$

Consider

$$K = I \times J + C \times B,$$

then

$$\begin{aligned} KK^T &= I \times cJ + cI_d \times (aI - J) + \\ &\quad + C^T \times JB^T + C \times BJ^T \\ &= acI_{cd} \end{aligned}$$

and hence is a $W(cd, ac)$.

[Since these conditions are always satisfied when $a + 1 = c + 1 = d$ is the order of a conference matrix or a skew-Hadamard matrix we have a $W(d(d-1), (d-1)^2)$ for these orders.]

SOME RESULTS ON THE CONJECTURE

First we give a theorem and then some results.

THEOREM. *There can only exist $W(2n, k)$ constructed of two circulant matrices A and B of order n , of the form*

$$W = \begin{bmatrix} A & B \\ -B^T & A^T \end{bmatrix},$$

where

$$k = a^2 + b^2.$$

PROOF. Let $T = (t_{ij})$ of order n be given by

$$t_{1j} = \begin{cases} 1 & j = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$t_{ij} = t_{1, j-i+1},$$

then $A = \sum_{i=1}^n a_i T^i$, $B = \sum_{i=1}^n b_i T^i$ where $a_i, b_i = 0, 1, -1$.

Now

$$AA^T + BB^T = kI = \left(\sum_{i=1}^n a_i T^i \right) \left(\sum_{i=1}^n a_i T^{n-i} \right) + \left(\sum_{i=1}^n b_i T^i \right) \left(\sum_{i=1}^n b_i T^{n-i} \right).$$

This is the matrix representation of

$$\left(\sum_{i=1}^n a_i \omega^i \right) \left(\sum_{i=1}^n a_i \omega^{n-i} \right) + \left(\sum_{i=1}^n b_i \omega^i \right) \left(\sum_{i=1}^n b_i \omega^{n-i} \right) = k,$$

where ω is an n th root of unity. This must be true for all n th roots

of unity including 1 so

$$\left(\sum_{i=1}^n a_i\right)^2 + \left(\sum_{i=1}^n b_i\right)^2 = k,$$

and we have the result.

COROLLARY. *There can only exist $W(2n, k)$ constructed of two circulant matrices A and B of order n for*

$$k < n, \text{ and } k = 0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 20, 25, 26, \\ 29, 34, 41, \dots$$

LEMMA 1. (i) *If there exists a $W(w, k)$ then $W(w, k) \oplus W(w, k)$ is a $W(2w, k)$ and $W(w, k) \times H_2$ is a $W(2w, 2k)$.*

(ii) *If there exist $W_1(w_1, k)$ and $W_2(w_2, k)$ then $W_1(w_1, k) \oplus W_2(w_2, k)$ is a $W(w_1 + w_2, k)$.*

(iii) *If there exist $W_1(w_1, k_1)$ and $W_2(w_2, k_2)$ then $W_1(w_1, k_1) \times W_2(w_2, k_2)$ is a $W(w_1 w_2, k_1 k_2)$.*

LEMMA 2. *If the conjecture is true for w then there exist*

$$(i) \quad W(2w, k), \quad 0 \leq k \leq w,$$

$$(ii) \quad W(2w, 2k), \quad 0 \leq k \leq w,$$

$$(iii) \quad W(2w, w + 1).$$

PROOF. Use Lemma 1 for (i) and (ii). For (iii) use the matrix

$$\begin{bmatrix} W(w, w) & I_w \\ I_w & -W^T(w, w) \end{bmatrix}.$$

LEMMA 3. *The conjecture is true for $w = 2, 4, 8, 16$.*

PROOF. (i) For $w = 2$, the required matrices are $0, I_2, H_2$;

(ii) for $w = 4$, the result follows using part (i), Lemma 2 and S_4 ;

(iii) for $w = 8$, by part (ii), Lemma 2 and S_8 we have the conjecture;

(iv) for $w = 16$, by part (iii), Lemma 2 and S_{16} we have that

$W(8,k)$ exists for $k = 0, 1, 2, \dots, 8, 9, 10, 12, 14, 15, 16$.

Now with $S = S_4$, and $H = J_4 - 2I_4$, in the matrices

$$\begin{bmatrix} 0 & S & S+I & S+I \\ -S & 0 & -S-I & S+I \\ -S+I & S-I & 0 & -S \\ -S+I & -S+I & S & 0 \end{bmatrix} \text{ and } \begin{bmatrix} I & H & H & H \\ -H & I & H & -H \\ -H & -H & I & H \\ -H & H & -H & I \end{bmatrix},$$

we have the result for 16.

LEMMA 4. If there exists a $W(w,k) = A$ then

$$\begin{bmatrix} A & A \\ A^T & -A^T \end{bmatrix}, \begin{bmatrix} A & A & I_w \\ A^T & -A^T & I_w \\ I_w & -A^T & -A \\ I_w & -A^T & A \end{bmatrix}, \begin{bmatrix} A & A & A \\ A^T & -A^T & A \\ A^T & -A^T & -A \\ A^T & -A^T & A \end{bmatrix}, \begin{bmatrix} A & A & I & I \\ A^T & -A^T & I & -I \\ I & I & -A^T & -A^T \\ I & -I & -A & A \end{bmatrix}$$

are $W(4w, 2k)$, $W(4w, 2k + 1)$, $W(4w, 3k)$ and $W(4k, 2k + 2)$ respectively.

LEMMA 5. If there exists a $W(w, w - 1) = S$ with $S^T = -S$ then

$$\begin{bmatrix} S & S+I \\ S-I & -S \end{bmatrix},$$

$$\begin{bmatrix} S & S+I & S & S+I \\ S-I & -S & S-I & -S \\ \hline S & S+I & -S & -S-I \\ S-I & -S & -S+I & S \end{bmatrix}, \begin{bmatrix} S & S+I & S+I & S+I \\ S-I & -S & S-I & -S+I \\ \hline S-I & S+I & -S & -S-I \\ S-I & -S-I & -S+I & S \end{bmatrix}$$

are $S_{2w} = W(2w, 2w - 1)$, $W(4w, 4w - 2)$, $S_{4w} = W(4w, 4w - 1)$ respectively, while

$$\begin{bmatrix} 0 & S & S+I & S+I \\ -S & 0 & -S-I & S+I \\ -S+I & S-I & 0 & -S \\ -S+I & -S+I & S & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & S & S & S \\ -S & 0 & S & -S \\ -S & -S & 0 & S \\ -S & S & -S & 0 \end{bmatrix} \text{ and } \begin{bmatrix} S & S & S & S \\ -S & S & S & -S \\ -S & -S & S & S \\ -S & S & -S & S \end{bmatrix}$$

are $W(4w, 3w - 1)$, $W(4w, 3w - 3)$ and $W(4w, 4w - 4)$ respectively.

LEMMA 6. If the conjecture is true for $w \equiv 0 \pmod{4}$ then there exists $\alpha(0,1,-1)$ $W = W(4w,k)$ of order $4w$ satisfying

$$WW^T = kI_{4w}$$

for $k = 0, 1, \dots, 2w+2, 2w+4, 2w+8, \dots, 4w$ and $3, 6, 9, \dots, 3w-3, 3w-1, 3w$.

PROOF. Since $W(w,k)$, $0 \leq k \leq w$ exists so does $W(4w,k) = W(w,k) \oplus W(w,k) \oplus W(w,k) \oplus W(w,k)$. By Lemma 4, for $k = \frac{1}{2}w, \frac{1}{2}w + 1, \dots, w$ we get a $W(4w,l)$ with $l = w+1, w+2, \dots, 2w+1, \frac{3}{2}w, \frac{3}{2}w+3, \dots, 3w$, and $2w+2$.

By Lemma 2 the existence of $W(w,k)$ implies the existence of $W(2w,2k)$ and hence $W(4w,4k)$.

Thus we have the result.

LEMMA 7. The conjecture is true for $w = 32$.

PROOF. By Lemmas 3 and 6 there exists a $(0,1,-1)$ W of order 32 satisfying

$$WW^T = kI_{32}$$

for $k = 0, 1, \dots, 18, 20, 21, 23, 24, 28, 32$.

Since S_8 exists, by Lemma 5, (2) is satisfied for $k = 30, 31$.

By Lemmas 2 and 3 (2) is satisfied for $k = 22, 26$.

$$\text{For } k = 25, \text{ use } A_1 = \dots = A_6 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} = A,$$

$$A_7 = I_4, A_8 = 0.$$

For $k = 29$ use $A_1 = A_2 = \dots = A_7 = A$, $A_8 = I$, in (3) to get $W(32,25)$ and $W(32,29)$.

$$\text{Now let } C = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right] \text{ and } D = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

then $CD^T = DC^T$. Now choosing $A_1 = A_2 = A_3 = A_4 = C$.

$A_5 = D$, $A_6 = I_4$, $A_7 = A_8 = 0$ in (3) we get a $W(32,19)$ and choosing

$A_1 = A_2 = \dots = A_6 = C$, $A_7 = D$, $A_8 = I$ in (3) we get $W(32,27)$.

Thus we have the conjecture for 32.

LEMMA 8. *There exists a $(0,1,-1)$ -matrix W of order 6 satisfying*

$$WW^T = kI_6$$

for $k = 0, 1, 2, 4, 5$ i.e., there exists a $W(6,k)$ for $k \in \{0,1,2,4,5\}$.

PROOF. Clearly the Lemma is true for $k = 0, 1$. The symmetric conference matrix of order 6 gives the result for $k = 5$. The required matrices for 2 and 4 are

$$H_2 \times I_3 \quad \text{and} \quad \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \end{array} \right].$$

LEMMA 9. *The conjecture is true for $w = 12$.*

PROOF. By Lemmas 2 and 8 we have a $W(12,k)$ for $k = 0,1,2,4,5,8,10$.

The existence of an Hadamard and skew-Hadamard matrix of order 12 gives a $W(12,k)$ for $k = 11, 12$. For

- (i) $k = 3$ use $A = B = C = I_3, D = 0$,
- (ii) $k = 5$ use $A = J_3 - I_3, B = J_3 - 2I_3, C = D = 0$,
- (iii) $k = 6$ use $A = J_3 - I_3, B = J_3 - 2I_3, C = I_3, D = 0$
- (iv) $k = 7$ use $A = J_3 - I_3, B = J_3 - 2I_3, C = D = I_3$,
- (v) $k = 9$ use $A = J_3, B = C = D = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

respectively, in construction 1.

LEMMA 10. *The conjecture is true for $w = 24$.*

PROOF. From Lemmas 2 and 9 there exists a $W = W(24,k)$ satisfying (2) for $k = 0, 1, \dots, 14, 16, 18, 20, 22, 24$. Since S_{12} exists, S_{24} exists and (2) is satisfied for $k = 23$.

Write $J = J_3, K = J_3 - 2I, I = I_3$. Then using for

$$k = 15, A_1 = J, A_2 = A_3 = A_4 = K, A_5 = A_6 = A_7 = I, A_8 = 0$$

$$k = 17, A_1 = A_2 = A_3 = J - I, A_4 = A_5 = A_6 = K, A_7 = A_8 = I,$$

$$k = 19, A_1 = J, A_2 = A_3 = A_4 = A_5 = K, A_6 = J - I, A_7 = A_8 = I,$$

in construction 2 and using the following first rows in construction 4

$$10: 0 - 1 1 - -, \quad 0 1 1 1 - 1, \quad 0 0 0 0 0 0, \quad 0 0 0 0 0 0$$

$$12: 0 1 1 0 1 -, \quad 0 1 1 0 1 -, \quad 0 1 1 0 1 -, \quad 0 0 0 0 0 0$$

$$16: 0 1 1 0 1 -, \quad 0 1 1 0 1 -, \quad 0 1 1 0 1 -, \quad 0 1 1 0 1 -$$

$$17: - 1 1 1 1 1, \quad - 1 1 - 1 -, \quad 0 - 1 1 1 -, \quad 0 0 0 0 0 0$$

20: 0 1 - - - 1, 0 - 1 1 - 1, 0 1 - 1 1 1, 0 1 1 1 1 -
0 - 1 1 - 1, 0 - 1 1 - -, 0 1 1 1 - 1, 0 1 1 1 - 1
22: 1 1 - - - 1, 0 - 1 1 - 1, - 1 1 1 1 1, 0 - 1 - 1 1
23: - 1 1 1 1 1, - 1 1 - 1 -, 0 - 1 1 1 -, 0 1 1 0 1 -
24: - 1 1 1 1 1, - - 1 1 1 1, - 1 - 1 1 1, - 1 1 - 1 -

where, as before, - denotes -1, we have the conjecture for $w = 24$.

LEMMA 11. *There exists a $(0,1,-1)$ -matrix $W = W(10,k)$ satisfying (2) for $k = 0, 1, 2, 4, 5, 8, 9$.*

PROOF. The result is clear for $k = 0, 1, 2$ and the symmetric conference matrix of order 10 gives the result for 9. For $k = 4$ use

$$A = \begin{bmatrix} . & 1 & . & . & 1 \\ 1 & . & 1 & . & . \\ . & 1 & . & 1 & . \\ . & . & 1 & . & 1 \\ 1 & . & . & 1 & . \end{bmatrix} \text{ and } B = \begin{bmatrix} . & 1 & . & . & -1 \\ -1 & . & 1 & . & . \\ . & -1 & . & 1 & . \\ . & . & -1 & . & 1 \\ 1 & . & . & -1 & . \end{bmatrix}$$

in

$$\begin{bmatrix} A & B \\ B^T & -A^T \end{bmatrix} . \quad (5)$$

For $k = 5$ use

$$A = \begin{bmatrix} -1 & 1 & . & 1 & . \\ . & -1 & 1 & . & 1 \\ 1 & . & -1 & 1 & . \\ . & 1 & . & -1 & 1 \\ 1 & . & 1 & . & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & . & . & . \\ . & 1 & 1 & . & . \\ . & . & 1 & 1 & . \\ . & . & . & 1 & 1 \\ 1 & . & . & . & 1 \end{bmatrix}$$

in (5).

For $k = 8$ use (for the first rows of) for A and B

$$\begin{matrix} 0 & 1 & -1 & -1 & -1 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 1 & -1 & 1 & 1 \end{matrix}$$

in (5).

We note

$$\begin{matrix} 0 & 1 & -1 & -1 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} -1 & 1 & 1 & 1 & 1 \end{matrix}$$

may also be used to obtain $W(10,9)$.

LEMMA 12. *The conjecture is true for $w = 20$.*

PROOF. By Lemmas 1 and 11 we have a W satisfying (2) for

$k = 0, 1, 2, 4, 5, 8, 9, 10, 16, 18$. There is an Hadamard matrix and a skew-Hadamard matrix of order 20 so we have a W for $k = 19, 20$.

$W(8,6) \oplus W(12,6)$ and $W(8,7) \oplus W(12,7)$ give the result for $k = 6$ and 7.

The following first rows may be used to generate circulant matrices which can then be used in construction 1:

$$\begin{array}{llll} 3: & 1 & 0 & 0 & 0 & 0, & 1 & 0 & 0 & 0 & 0, & 1 & 0 & 0 & 0 & 0, & 0 & 0 & 0 & 0 & 0 \\ 8: & 0 & 1 & - & - & 1, & 0 & 0 & 1 & 1 & 0, & 0 & 1 & 0 & 0 & 1, & 0 & 0 & 0 & 0 & 0 \\ 9: & - & 1 & 1 & 1 & 1, & 0 & 1 & - & - & 1, & 0 & 0 & 0 & 0 & 0, & 0 & 0 & 0 & 0 & 0 \\ 10: & - & 1 & 1 & 1 & 1, & 0 & 1 & - & - & 1, & 1 & 0 & 0 & 0 & 0, & 0 & 0 & 0 & 0 & 0 \\ 11: & - & 1 & 1 & 1 & 1, & - & 0 & 1 & 1 & 0, & - & 1 & 0 & 0 & 1, & 0 & 0 & 0 & 0 & 0 \\ 12: & - & 1 & 1 & 1 & 1, & - & 0 & 1 & 1 & 0, & - & 1 & 0 & 0 & 1, & 1 & 0 & 0 & 0 & 0 \\ 18: & 1 & 1 & - & - & 1, & 0 & 1 & 1 & 1 & 1, & 0 & 1 & - & - & 1, & - & 1 & - & - & 1 \end{array}$$

If we use the following first rows to generate circulant matrices in construction 4:

$$\begin{array}{llll} 13: & 1 & 1 & 0 & 0 & 1, & 0 & 1 & 0 & 0 & 1, & 0 & 1 & - & - & 1, & 0 & 1 & - & - & 1 \\ 14: & 0 & - & 1 & - & 0, & 0 & 1 & 1 & 1 & 0, & 0 & 1 & 1 & 1 & -, & 0 & 1 & - & - & 1 \\ 17: & 0 & - & 1 & - & 1, & 1 & 1 & - & - & 1, & 0 & 1 & 1 & 1 & 1, & 0 & 1 & - & - & 1. \end{array}$$

$W(20,15)$ may be obtained from construction 5. Thus we have the conjecture for $w = 20$.

LEMMA 13. *There exists a $(0,1,-1)$ -matrix W of order 14 satisfying (2) for $k = 0, 1, 2, 4, 5, 8, 9, 10, 13$.*

PROOF. The result for $k = 0, 1, 2, 4, 5$ follows using Lemmas 1, 3 and 8. $W(14,13)$ exists because there is a symmetric conference matrix of order 14.

Use the following first rows to generate circulant matrices in (4) to obtain the remainder of the results

8:	- 1 1 0 1 0 0,	- 1 1 0 1 0 0
or	0 1 1 0 0 - 1,	0 - 1 0 0 1 1
9:	0 1 1 0 1 0 0,	0 1 1 - 1 - -
10:	0 1 1 0 1 0 0,	1 1 1 - 1 - -
or	- 1 1 1 1 0 0,	- 1 1 - 1 0 0
or	0 1 1 - 0 - 0,	0 1 1 - 0 1 1
13:	- 1 1 - 1 0 1,	- 1 1 1 1 1 -

where - denotes -1.

LEMMA 14. *The conjecture is true for $w = 28$.*

PROOF. Since $W(16,k)$ and $W(12,k)$ exist for $0 \leq k \leq 12$ we have $W(28,k)$ for $0 \leq k \leq 12$. By construction 5 we have $W(28,14)$, $W(28,21)$ and $W(28,28)$. A $W(28,27)$ exists since there is a skew-Hadamard matrix of order 28. There is a symmetric conference matrix of order 14 so $S = W(14,13)$ exists and $S \oplus S$ and $S \times H_2$ and $W(28,13)$ and $W(28,26)$ respectively.

We use the following first rows in construction 4

15:	1 0 0 0 0 0 0,	0 1 1 0 1 0 0,	- 1 1 0 1 0 0,	1 1 1 - 1 - -
16:	0 0 0 0 0 0 0,	1 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
	- 1 1 0 1 0 0,	1 - 1 0 1 0 0,	1 1 - 0 1 0 0,	1 1 1 0 - 0 0
17:	1 0 0 0 0 0 0,	1 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
18:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	0 1 1 - 1 - -,	0 1 1 - 1 - -
19:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	0 1 1 - 1 - -
20:	0 1 1 0 1 0 0,	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	1 1 1 - 1 - -
21:	0 1 1 1 - 1 -,	0 - 1 - - - 1,	0 1 1 - - - 1,	0 1 1 - 1 1 1
22:	1 1 1 0 1 0 0,	- 1 1 0 1 0 0,	1 1 1 - 1 - -,	1 1 1 - 1 - -
23:	0 1 1 0 1 0 0,	1 1 1 - 1 - -,	- 1 1 - 1 0 1,	- 1 1 1 1 1 -
24:	0 1 1 1 - 1 -,	0 - 1 - - - 1,	0 1 1 - - - 1,	0 1 1 - 1 1 1
25:	- 1 1 1 1 1 1,	0 1 1 - 1 - -,	0 1 1 - 1 - -,	0 1 1 - 1 - -
26:	- 1 1 1 1 1 1,	1 1 1 - 1 - -,	0 1 1 - 1 - -,	0 1 1 - 1 - -

we have the conjecture for $w = 28$.

LEMMA 15. *The conjecture is true for $w = 40$.*

PROOF. Since the conjecture is true for 20 by Lemma 2 we have the results for $k = 0, 1, 2, \dots, 20, 21, 22, 24, 26, \dots, 38, 40$.

$W(40, 39)$ exists since there is a skew-Hadamard matrix of order 40. By Lemmas 4 and 11, $W(40, k)$ exists for $k = 0, 3, 6, 12, 15, 24, 27$; and by Lemma 11 and construction 8 we have $W(40, k)$ for $k = 38, 27, 28, 36, 29$.

Let B the matrix generated by the first row

0 1 - - 1.

Then using

$$A_1 = J, A_2 = A_3 = A_4 = A_5 = A_6 = B, A_7 = A_8 = 0$$

in construction 2 gives $W(40,25)$, while using

$$A_1 = J, A_2 = A_3 = B + I, A_4 = A_5 = B - I, A_6 = B, A_7 = A_8 = I$$

gives $W(40,31)$.

Let C and D be the matrices generated by the first rows

$$\begin{pmatrix} - & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} - & 0 & 1 & 1 & 0 \end{pmatrix}$$

respectively then using

$$A_1 = J - I, A_2 = A_3 = A_4 = C, A_5 = A_6 = A_7 = D, A_8 = I$$

in construction 2 gives $W(40,23)$.

Let E and F be the matrices generated by the first rows

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

then $A_1 = A_2 = J - 2I, A_3 = B + I, A_4 = B - I, A_5 = A_6 = B, A_7 = E$ and $A_8 = F$ used in construction 2 gives $W(40,33)$, while using

$A_1 = A_2 = J - 2I, A_3 = A_4 = B + I, A_5 = A_6 = B - I, A_7 = E$ and $A_8 = F$ gives $W(40,35)$.

Thus we have the result

OTHER RESULTS

LEMMA 16. *There exists a $(0,1,-1)$ -matrix $W = W(18,k)$ satisfying (2) for $k = 0, 1, 2, 4, 5, 8, 10, 17$.*

PROOF. Since $W(10,k)$ and $W(8,k)$ exist for $k = 0, 1, 2, 4, 5, 8$, we have $W(18,k) = W(10,k) \oplus W(8,k)$ exists for these k values.

If we use the following first rows to generate the circulant matrices to use in (4) we get the remaining results.

8: 0 1 - 0 0 1 0 1 0, 0 1 1 0 0 - 0 1 0;
 or 1 1 0 1 - 0 0 0 0, 1 1 0 - 1 0 0 0 0;
 10: 1 1 1 0 1 - 0 0 0, - 1 1 0 - 1 0 0 0;
 or - 1 0 - 1 0 0 - 0, 1 1 0 1 1 0 0 - 0;
 17: 1 - - 1 1 1 1 - -, 0 - 1 - - - - 1 - .

(We note this leaves $k = 9, 13, 16$ which were not ruled out by the theorem).

LEMMA 17. *There exists a $(0,1,-1)$ -matrix $W = W(36,k)$ satisfying (2) for $k = 0, 1, 2, \dots, 22, 25, 27, 32, 33, 34, 35, 36$.*

PROOF. Since there exists a $W(20,k)$ and $W(16,k)$ for $0 \leq k \leq 16$ there exists a $W(36,k) = W(20,k) \oplus W(16,k)$ for these k . $W(36,20), W(36,34)$ and $W(36,17)$ exist using Lemmas 1 and 15. $W(36,36)$ and $W(36,35)$ exist because there is a skew-Hadamard matrix of order 36.

If we use the following first rows in construction 4, we get the other results of the lemma.

18: 0 1 - 0 0 1 0 1 0, 0 1 1 0 0 - 0 1 0
 1 1 1 0 1 - 0 0 0, - 1 1 0 - 1 0 0 0
 19: 1 - - 1 1 1 1 - -, 0 - 1 - - - - 1 -,
 1 0 0 0 0 0 0 0 0, 1 0 0 0 0 0 0 0 0
 21: 1 - - 1 1 1 1 - -, 0 - 1 - - - - 1 -,
 1 1 0 0 0 0 0 0 0, 1 - 0 0 0 0 0 0 0
 22: 1 1 1 1 1 - 0 0 0, 1 1 1 - 1 - 0 0 0,
 - 1 1 0 - 1 0 0 0, - 1 1 0 - 1 0 0 0

25: 0 1 - 0 0 1 0 1 0, 0 1 1 0 0 - 0 1 0,
1 - - 1 1 1 1 - -, 0 - 1 - - - - 1 -
27: 1 1 1 0 1 - 0 0 0, - 1 1 0 - 1 0 0 0,
1 - - 1 1 1 1 - -, 0 - 1 - - - - 1 -
32: 0 - 1 - - 1 1 - 1, 0 - 1 1 1 1 1 1 -,
0 1 - - - 1 1 1 -, 0 1 - 1 1 1 1 - 1
33: 1 - 1 - - 1 1 - 1, 0 - 1 1 1 1 1 1 -,
0 1 - - - 1 1 1 -, 0 1 - 1 1 1 1 - 1

LEMMA 18. *There exists a $W(22,k)$ for $k \in \{0,1,2,4,5,8,9,10,16,17,20\}$.*

PROOF. There exists a $W(10,\ell)$ and a $W(12,\ell)$ by Lemmas 9 and 11 for $\ell \in \{0, 1, 2, 4, 5, 8, 9\}$ so $W_1 \oplus W_2$ is the required matrix for ℓ . To obtain the other matrices use the following first rows to generate A and B in (5).

10: 0 0 1 1 - 0 - 0 0 0 1, 0 1 0 0 0 1 0 1 1 - 0 (a)
16: - 1 0 1 1 1 0 0 0 1 0, 0 1 - 1 1 1 - - - 1 - (b)
17: - 1 0 1 1 1 0 0 0 1 0, 1 1 - 1 1 1 - - - 1 - (c)
20: 0 1 1 1 - 1 - 1 1 - 1 0 1 - - 1 1 1 1 1 - - - (d)

LEMMA 19. *There exists a $W(44,k)$ for $k \in \{i : 0 \leq i \leq 20, 21, 25 \leq i \leq 28, 30, 32, 33, 34, 36 \leq i \leq 40, 42 \leq i \leq 44\}$.*

PROOF. Since the conjecture is true for 24 and 20, there exists a $W_1(20,m)$ and a $W_2(24,m)$ for $m \in \{i : 0 \leq i \leq 20\}$ and $W_1 \oplus W_2$ is the required matrix for m .

$W(44,44)$ and $W(44,43)$ exist as there is a skew-Hadamard matrix of order 44.

If we use the matrices used to form $W(22,k)$ with the indicated matrices in construction 4 we get $W(44,n)$ for $n \in \{21, 22, 26, 27, 30, 32, 33, 34, 36, 37, 40\}$

- $k = 21$: (d) 0 and I_{11} ;
- $k = 22$: (d) I_{11}, I_{11} ;
- $k = 26$: (a) and (b);
- $k = 27$: (a) and (c);
- $k = 30$: (a) and (d);
- $k = 32$: (b) and (b);
- $k = 33$: (b) and (c);
- $k = 34$: (c) and (c);
- $k = 36$: (b) and (d);
- $k = 37$: (c) and (d);
- $k = 40$: (d) and (d);

Let A be the circulant incidence matrix of an $(11, 6, 3)$ configuration and $B = J - I - 2A$. Then

$$AA^T = 3I + 3J \text{ and } BB^T = 11I - J,$$

and have 6 and 10 non-zero elements respectively.

Further $A - I$ and $B + I$ satisfy

$$(A - I)(A - I)^T = 5I + 2J \text{ and } (B + I)(B + I)^T = 11I - J,$$

and have 7 and 11 non-zero elements respectively.

So we may use the following matrices in construction 4 to get $W(44, k)$:

k = 25:	J - A	B	B	0
k = 26:	J - A	B	B + I	
k = 27:	A - I	B	B	0
k = 28:	A - I	B	B	I
k = 28:	A - I	B + I	B	0
k = 29:	A - I	B + I	B + I	0
k = 30:	A - I	B + I	B + I	I
k = 36:	A	B	B	B
k = 37:	A	B + I	B	B
k = 38:	A	B + I	B + I	B
k = 39:	A	B + I	B + I	B + I

LEMMA 20. *There exists a $W(26,k)$ for $k \in \{0, 1, 2, 4, 5, 8, 9, 10, 25\}$.*

PROOF. This follows from the existence of $W(12,k)$ and $W(14,k)$ for $k \in \{0, 1, 2, 4, 5, 8, 9, 10\}$.

The following first rows generate matrices which can be used in (5) to form a $W(26,25)$:

$$0 \ 1 \ - \ - \ - \ 1 \ - \ - \ 1 \ - \ - \ - \ 1, \quad 1 \ - \ 1 \ 1 \ - \ - \ - \ - \ - \ 1 \ 1 \ - \ .$$

LEMMA 21. *There exists a $W(52,k)$ for*

$k \in \{i : 0 \leq i \leq 24, 32, 34, 39, 48, 51, 52\}$.

PROOF. The existence of a $W(52,k)$ for $k \in \{i : 0 \leq i \leq 24\}$ follows from the existence of Hadamard matrices of orders 24 and 28. The $W(52,52)$ and $W(52,51)$ exist because there is a skew-Hadamard matrix of order 52.

$W(52,48)$ may be obtained by using the following first rows to generate matrices which are then used in (4):

$0\ 1\ -\ -\ -\ 1\ -\ -\ 1\ -\ -\ -\ 1,\ 0\ -\ 1\ 1\ -\ -\ -\ -\ -\ 1\ 1\ -$
 $0\ -\ -\ -\ 1\ -\ 1\ 1\ -\ 1\ -\ -\ -, \ 0\ 1\ -\ 1\ 1\ -\ -\ -\ -\ 1\ 1\ -\ 1\ .$

Write B for the last of these four matrices.

Let Q be the circulant incidence matrix of a (13, 4, 1) configuration then

$$BB^T = 13I - J \quad \text{and} \quad QQ^T = 12I + J.$$

So if we use the following four matrices in (4) we get W(52,k) for $k \in \{32, 34\}$:

k = 32: Q, Q, B, B;

k = 34: Q, Q, B+I, B-I.

We get a W(52,39) by putting A = B = C = 1, D = 0 in the 52 × 52 Hadamard array.

LEMMA 22. *There exists a W(56,k) for*

$k \in \{i: 0 \leq i \leq 30, 32, 33, 34, 36, 37, 38, 40, 42, 44, 46, 48, 50, 52, 54, 55, 56\}.$

PROOF. This follows from Lemmas 2 and 14, the existence of a skew-Hadamard matrix of order 56 and from construction 8, since there exists a W(14,13).

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