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# Nonlinear $\sigma$ model for odd-frequency triplet superconductivity in superconductor/ferromagnet structures

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We consider some properties of odd-frequency triplet superconducting condensates. In order to describe fluctuations we construct a supermatrix  $\sigma$  model for the superconductor/ferromagnet or superconductor/normal-metal structures. We show that an odd frequency triplet superconductor, when in isolation or coupled to a normal metal, generally displays behavior comparable to a superconductor with the usual singlet pairings. However, for spin dependent systems such as the superconductor/ferromagnet the two types of superconductor have quite different behavior. We discuss this difference by considering transformations under which the  $\sigma$  model is invariant. Finally, we calculate the low energy density of states in a ferromagnet coupled to a singlet superconductor. If odd frequency triplet components are induced in the ferromagnet the density of states will decrease relative to the usual bulk solution but will not vanish.

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## I. INTRODUCTION

The Pauli principle imposes important restrictions on the symmetry of the superconducting condensate in superconductors. The most common condensate is a singlet where the Cooper pairs have antiparallel spins (*s*- or *d*-wave). In this case, the wave function describing Cooper pairs is assumed to be invariant under the exchange of electron coordinates. Another possibility is a triplet pairing with the total spin of the pair equal to unity. In this case the wave function of the pair is assumed to change sign if the electrons exchange coordinates. The most famous example of the triplet pairing (*p*-wave) is superfluid He<sup>3</sup> (Ref. 1) but triplet superconductivity has been recently discovered.<sup>2,3</sup>

However, a characterization of the superconductor in terms of space symmetries of the wave function of Cooper pairs is somewhat oversimplified. The full information about the superconducting condensate is in fact given by an anomalous Green's function (Gorkov function)  $F(\epsilon)$ . This function depends not only on the coordinates of the Cooper pair but also on the frequency  $\epsilon$ . The previous discussion about the properties of the wave function of the Cooper pairs corresponds to the case when the condensate function  $F(\epsilon)$  is an even function of the frequency  $\epsilon$  although nothing forbids the function  $F(\epsilon)$  from being an odd function of  $\epsilon$ . If this alternative possibility were realized one would have a situation where the condensate function  $F(\epsilon)$  is invariant under the permutation of electrons with triplet pairing but would change sign in the singlet case. So, odd condensate functions of frequencies allow, at least theoretically, *p*-wave singlets and *s*- and *d*-wave triplets.

In this paper we shall discuss some aspects of triplet Cooper pairings which are odd in frequency and even in momentum. A superconductor with an odd frequency triplet condensate was introduced by Berezinskii<sup>4</sup> as a possible candidate for a phase of He<sup>3</sup>,<sup>3</sup> though this was later found to not be the

case. One may also consider other symmetry variations. For example, in Ref. 5 an odd singlet superconductor (one which is odd in both frequency and momentum) was discussed. Unfortunately, the authors of Refs. 4 and 5 did not find a microscopic model that would lead to the odd frequency condensate.

Recently, it was found that the odd triplet condensate can be induced in a superconductor/ferromagnet structure provided the magnetization in the ferromagnet is inhomogeneous.<sup>6</sup> In this situation one does not need a special kind of an electron-electron interaction. It is sufficient that the ferromagnet is coupled to a standard singlet superconductor. This shows that, independent of whether the odd superconductivity can be obtained as the ground state of a microscopic model or not, a detailed study of its properties based just on the symmetry of the condensate may be of interest because it can be realized at least as a proximity effect.

In this paper we compare properties of the odd triplet superconductivity with those of the conventional singlet. We first consider a superconductor with odd frequency triplet pairings ( $S_t$ ). We construct the Gorkov Green's functions and write them in terms of an integral over supervectors, which allows us to obtain a supermatrix  $\sigma$  model. It turns out that the form of the Green's functions closely resembles those of a standard singlet superconductor ( $S_s$ ). In fact, one can show that in many cases an  $S_t$  will have very similar properties to an  $S_s$ . Differences appear when one considers spin dependent structures such as a superconductor coupled to a ferromagnet ( $S_s/F$  or  $S_t/F$ ). These two types of superconductors have different symmetries of the order parameter which leads to differences in the Josephson effect. A qualitative discussion about the proximity effect in  $S_t/F$  structures may be made from considering transformations under which the  $\sigma$  model is invariant. From these transformations one can determine which types of Cooper pairs are induced in the ferromagnet and whether the penetration is long-ranged or short-ranged.

Generally, it is simpler to just solve the saddle point equation, but if the ferromagnet has a complicated inhomogeneous structure consideration of the transformational invariances may be useful.

It is well known that the density of states of an  $S_s$  in isolation (and also an  $S_t$ ) has an energy gap equal to the value of the order parameter. A normal metal has no energy gap. However, fluctuations in the density of states due to the proximity effect in various hybrid structures of superconductors, metals and ferromagnets have been studied using quasiclassical methods such as the Eilenberger equation,<sup>9,10</sup> the Usadel equation<sup>11</sup> and the Bugolubov-de Gennes equation.<sup>12,13</sup> These structures often exhibit an oscillating density of states which gradually decreases as the energy decreases. By consideration of the fluctuations about the supersymmetric saddle point in an  $S_s/N$  structure it has been shown that the density of states in the normal metal decreases quadratically at low energies and vanishes completely at zero energy.<sup>7,8</sup> This vanishing density of states is called a “micro-gap” and is a consequence of long-ranged Cooper pairs being induced in the normal metal. For most  $S_s/F$  structures no long-ranged Cooper pairs are induced in the ferromagnet and so there is no micro-gap comparable to the micro-gap found in  $S_s/N$  structures. However, as a result of some inhomogeneities in the ferromagnets (domain walls can be an example), a long-ranged odd triplet condensate may be induced.<sup>6</sup> We consider such a  $S_s/F$  structure and calculate the low energy C-mode fluctuations about the saddle point solution. From this we can calculate the density of states. We find that the density of states in the ferromagnet does not have a micro-gap but it is lower than what would be obtained in a bulk ferromagnet. As concerns an  $S_t/F$  structure, the low energy behavior is the same as for an  $S_s/N$  and will always exhibit a micro-gap.

## II. GORKOV GREENS FUNCTIONS

In this section we construct the Green’s functions for an odd frequency triplet condensate and compare it to the Green’s functions of an even frequency singlet. The construction of the single Green’s function is well known and details can be found in, for example, Ref. 14. The construction of the odd triplet Green’s function is similar to that of the singlet but with some significant differences which will be discussed here. We begin with the general form of the superconductor Hamiltonian,

$$H = \int dr \left[ \psi_\alpha^\dagger(r) \mathcal{H}(r) \psi_\alpha(r) + \psi_\alpha^\dagger(r) V_{\alpha\beta}(r) \psi_\beta(r) + \frac{1}{2} \int dr' \psi_\delta^\dagger(r) \psi_\gamma^\dagger(r') U_{\delta\gamma\alpha\beta}(r, r') \psi_\alpha(r') \psi_\beta(r) \right], \quad (1)$$

where  $\mathcal{H}$  is the one-particle Hamiltonian,  $V_{\alpha\beta}$  is the exchange field which may have some spatial dependence,  $U_{\delta\gamma\alpha\beta}$  is the two-particle potential and  $\psi_\xi$  and  $\psi_\xi^\dagger$  are fermionic destruction and annihilation operators. This form of the Hamiltonian is completely general with regards to spin, time and position symmetries. Since  $H$  must be Hermitian  $V=V^\dagger$  and  $U=U^\dagger$ . In coupled systems such as  $S/F$  and  $S/N$  the

superconductor is defined to lie along the negative  $x$ -axis and the ferromagnet or normal-metal lies along the positive  $x$ -axis. The two-particle potential  $U$  is just defined in the superconductor so vanishes in  $F$  and  $N$ . The exchange field vanishes in  $S$  and  $N$ .

From the above Hamiltonian and using the conventional mean field approximation we can construct the Green’s functions that have both normal and anomalous components represented by  $G$  and  $F$ , respectively. For more details see Ref. 14. Usually one complements the Green’s function equations with the self-consistency equation

$$\Delta_{\xi\alpha}(X, X') = U_{\xi\alpha\gamma\beta}(X, X') \langle \Psi_\gamma(X') \Psi_\beta(X) \rangle, \quad (2)$$

where  $\Psi$  and  $\Psi^\dagger$  are the operators in the Heisenberg representation and  $X=(r, t)$ . By definition the anomalous Green’s function  $F$  and the order parameter  $\Delta$  must have the same symmetry and due to the Pauli principle they must be anti-symmetric under simultaneous position-time and spin exchange. In the singlet state  $S_s$  the order parameter is even in time and position exchange. In the triplet state  $S_t$  considered here the order parameter is odd in time exchange but even in position exchange. However, we emphasize that we cannot and do not try to present a microscopic model that would determine the odd triplet superconducting order parameter  $\Delta$  but write it purely phenomenologically. Note that the odd triplet condensate function  $F$  can exist due to the proximity effect in  $S/F$  structures.<sup>6</sup> In the  $S_s$  case it is usual to simplify the order parameter by setting  $\Delta(X, X') \propto \delta(X-X')$ . However, this is not possible for an  $S_t$  as it will destroy the odd symmetry in the time component. An acceptable form of the order parameter for both the  $S_s$  and the  $S_t$  is  $\Delta(X, X') = \Delta(r, t, t') \delta(r-r')$  where the symmetry in  $r$  and  $r'$  implies we are considering an  $s$ -wave. For the conventional singlet superconductivity the function  $\Delta(r, t, t')$  is invariant under the exchange of  $t$  and  $t'$  whereas in the triplet case considered here it changes sign.

After taking a Fourier transform the advanced and retarded Gorkov Green’s functions represented in particle-hole space are

$$\begin{pmatrix} \epsilon \pm i\delta/2 - \mathcal{H} - V & \Delta(x, \epsilon) \\ (-1)^{S+1} \Delta^*(x, -\epsilon) & -\epsilon \mp i\delta/2 - H - V^* \end{pmatrix} \mathcal{G}^{R,A}(x, x', \epsilon) = \delta(x-x'), \quad (3)$$

$$\mathcal{G} = \begin{pmatrix} G & F \\ F^\dagger & G^\dagger \end{pmatrix},$$

where  $S$  is the total spin of the Cooper pair and  $\delta$  is a small positive real number, the sign in front of which determines the advanced or retarded nature of the Green’s function. We see that the difference between the equations for the conventional singlet and odd triplet superconductivities is minimal. Note that the spin dependence is hidden inside  $G$ ,  $F$ ,  $\Delta$ , and  $V$ .

If the spin is represented by the Pauli matrices  $\sigma$  we can expand the order parameter as  $\Delta = \sum_{i=0}^3 \Delta_i \sigma_i$  and we may write each component in terms of a phase,  $\Delta_i = |\Delta_i| e^{i\theta_i}$ . We

represent the triplet components of  $\Delta$  by  $\sigma_0$ ,  $\sigma_1$ , and  $\sigma_3$  and the singlet ones by  $\sigma_2$ . With this choice we satisfy the symmetry relations  $\Delta = -\Delta^T$  for the conventional singlet superconductivity and  $\Delta = \Delta^T$  for the odd triplet. For conventional even frequency superconductors the order parameter is often assumed to be energetically independent, however, in the case of an  $S_t$  the order parameter must be odd in energy so we choose the simplest possibility  $\Delta(x, \epsilon) = \text{sgn}(\epsilon)\Delta(x)$ .

In order to study mesoscopic fluctuations we use the supersymmetry method.<sup>15</sup> Within this technique one can write the solution of Eq. (3) in terms of a functional integral over supervectors  $\psi$ ,<sup>8,15-17</sup>

$$\mathcal{G}^{R,A}(x, x', \epsilon) = i \int \psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x') \exp[-\mathcal{L}_{s,t}] \mathcal{D}\psi$$

$$\mathcal{L}_s = i \int \bar{\psi}(y) \begin{pmatrix} \epsilon - i\delta\Lambda/2 - \mathcal{H} - V & \Delta(y) \\ -\Delta^*(y) & -\epsilon + i\delta\Lambda/2 - \mathcal{H} - V^* \end{pmatrix} \times \psi(y) dy,$$

$$\mathcal{L}_t = i \int \bar{\psi}(y) \begin{pmatrix} \epsilon - i\delta\Lambda/2 - \mathcal{H} - V & \text{sgn}(\epsilon)\Delta(y) \\ -\text{sgn}(\epsilon)\Delta^*(y) & -\epsilon + i\delta\Lambda/2 - \mathcal{H} - V^* \end{pmatrix} \times \psi(y) dy, \quad (4)$$

where  $\mathcal{L}_{s,t}$  is the action for the singlet and the odd triplet superconductivity, respectively, and all other terms have the standard definitions. If we perform the gauge transformation  $\psi \rightarrow \psi e^{i(\pi/4)[\text{sgn}(\epsilon)-1]\tau_3}$  and  $\bar{\psi} \rightarrow \bar{\psi} e^{-i(\pi/4)[\text{sgn}(\epsilon)-1]\tau_3}$  where  $\tau$  represents Pauli matrices of the particle-hole space we find that, if we ignore the spin dependence, the triplet action is no different from the singlet action but the coefficient of the exponential becomes  $[\psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x')]_{mn} \rightarrow [\psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x')]_{mn} [\text{sgn}(\epsilon)]^{m-n}$  where  $m$  and  $n$  represent components of the particle-hole space. Thus, if spin is not important the normal odd triplet Green's functions  $G$  are identical to the normal singlet Green's functions but the anomalous triplet Green's functions  $F$  differ from that of the singlet by a factor of  $\text{sgn}(\epsilon)$ , i.e., the singlet's anomalous Green's functions are even in  $\epsilon$  but the triplet's are odd, as expected from the initial symmetry requirements. As the normal Green's functions determine the density of states the bulk singlet and the bulk triplet have the same density of states. Also, a  $S_t/N$  structure should be similar to a  $S_s/N$  structure since in these cases spin is not important.

### III. TRANSFORMATIONAL INVARIANCES OF THE $\sigma$ MODEL

From Eq. (4) the construction of a  $\sigma$  model is fairly straightforward. Using the standard method of derivation developed for the singlet superconductor the  $\sigma$  model action may be shown to be<sup>8,16</sup>

TABLE I. Transformational invariances of the  $\sigma$  model action. The matrix  $A$  is invariant under the transform  $C$  if  $A = CA^T C^T$ . The singlet superconductor action is invariant under the  $C_0$ ,  $C_1$ ,  $C_2$  and  $C_3$  transforms whereas the triplet superconductor action is only invariant under the  $C_4$  transform. In addition both types of superconductor actions must have charge conjugation and convergence symmetry.

Transform	Invariance
$C_0 = i\tau_1$	$\sigma_2, \tau_1\sigma_2, \tau_2\sigma_2, \tau_3\sigma_{0,1,3}$
$C_1 = \tau_2\sigma_1$	$\sigma_3, \tau_1\sigma_{0,1,2}, \tau_2\sigma_{0,1,2}, \tau_3\sigma_{0,1,2}$
$C_2 = \tau_1\sigma_2$	$\sigma_{1,2,3}, \tau_1\sigma_{1,2,3}, \tau_2\sigma_{1,2,3}, \tau_3$
$C_3 = \tau_2\sigma_3$	$\sigma_1, \tau_1\sigma_{0,2,3}, \tau_2\sigma_{0,2,3}, \tau_3\sigma_{0,2,3}$
$C_4 = \tau_2$	$\sigma_2, \tau_1\sigma_{0,1,3}, \tau_2\sigma_{0,1,3}, \tau_3\sigma_{0,1,3}$

$$S = \frac{\pi\nu}{16} \text{str} \int [D(\partial Q)^2 + 4iQ(\epsilon\tau_3 - i\delta\Lambda\tau_3/2 - \tilde{\Delta} - \text{Re } V - i\tau_3\rho_3 \text{Im } V)], \quad (5)$$

where  $\rho_3$  is the third Pauli matrix in the time-reversal space,  $Q$  is a  $32 \times 32$  supermatrix,  $\nu$  is the bulk normal-metal density of states per spin and

$$\begin{aligned} \tilde{\Delta} = & i\tau_2\rho_3[\sigma_0|\Delta_0|\exp(-i\theta_0\tau_3\rho_3) + \sigma_1|\Delta_1|\exp(-i\theta_1\tau_3\rho_3) \\ & + \sigma_3|\Delta_3|\exp(-i\theta_3\tau_3\rho_3)] - \sigma_2\tau_1\rho_3|\Delta_2|\exp(-i\theta_2\tau_3\rho_3). \end{aligned} \quad (6)$$

The  $Q$ -matrices in Eq. (5) must satisfy as usual the charge conjugation symmetry and integrals with the action  $S$  must converge. In addition, one can find several transformations under which  $Q$  is invariant in the bulk superconductor (when  $V=0$ ).<sup>7</sup> We define  $A$  to be invariant under the transformation  $C$  if  $A = CA^T C^T$ . Table I defines five transformations and the terms with which they are invariant. All the terms in the action of a triplet superconductor are invariant under the  $C_4$  transform while the singlet superconductor action is invariant under the other four transforms. This appears to disagree with what was found in Ref. 7 where it was claimed that the singlet was invariant under the  $C_4$  transform. The difference is due to the spin dependence of our  $\sigma$  model. In general the ferromagnetic exchange field is of the form  $V = h_0\sigma_0 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$  (all the  $h_i$  must be real since  $V = V^\dagger$ ). In the ferromagnet  $Q$  is not required to be invariant under any of the transforms in Table I but they can help in determining the form of  $Q$  in the ferromagnet.

To illustrate how the transformational invariances may be used we discuss a simple example. Consider an  $S_t/F$  structure with different exchange fields. The saddle-point equation of a superconductor  $\sigma$  model is also known as the Usadel equation. The quasiclassical Green's function which satisfies the Usadel equation is the saddle point solution of the  $\sigma$  model and is represented by

$$g_0 = \begin{pmatrix} g & f \\ f^\dagger & g^\dagger \end{pmatrix} \quad (7)$$

in the particle-hole space with the constraint  $g_0^2=1$ . If we assume that the temperature is just below the superconducting transition temperature or the tunnelling resistivity is very large, the Green's function in the ferromagnet is  $g^{A,R} = -g^{A,R\dagger} \sim \mp 1$ . In this case the Usadel equation may be linearized and the retarded anomalous triplet Green's function can be shown to satisfy

$$iD\partial_x^2 f - 2\epsilon f + Vf - fV^* = 0 \quad (8)$$

in the ferromagnet (having dropped the superscript). This is the same as the linearized equation in the ferromagnetic region of an  $S_s/F$  structure but, due to the boundary conditions, the spin structure of  $f$  must be different.<sup>6,18</sup> The boundary conditions at the interface are

$$\begin{aligned} \partial_x f(0^+) &= [\rho(+)/R_b]f(0^-), & T \ll 1, \\ f(0^+) &= f(0^-), & T \sim 1, \end{aligned} \quad (9)$$

where “-” is the superconducting side of the interface and “+” is the ferromagnetic side,  $T$  is the transparency of the interface,  $\rho(\pm)$  is the resistivity and  $R_b$  is the tunnelling resistivity. As  $x \rightarrow -\infty$  the Green's function must approach the bulk superconductor solution and as  $x \rightarrow \infty$  it must approach the bulk ferromagnet solution. Assuming that the proximity effect on the superconductor is small the well known bulk solution may be taken in the entire superconducting region  $x < 0$  where  $V=0$  so  $f(x < 0) = \text{sgn}(\epsilon)\Delta/\sqrt{\epsilon^2 - |\Delta|^2}$ . This is the same solution as for a bulk  $S_s$  but with the extra term  $\text{sgn}(\epsilon)$  which gives the required odd energy dependence and  $\Delta$  has a different spin dependence. In the ferromagnetic region  $x > 0$  the anomalous Green's function is of the form  $f(x > 0) = \sum_{i=0}^3 f_i(x)\sigma_i$  (assuming we have both triplet and singlet components). The boundary condition at  $x \rightarrow \infty$  is that all the  $f_i$  must vanish.

If the magnetization is of the form  $V=h\sigma_j$ ,  $j=1,2,3$  then the solution of the linearized Usadel equation is that each  $f_i$  will exponentially decay. Two components will decay at a rate independent of the exchange field,  $\kappa_\epsilon$  and the other two will decay at the rate  $\kappa = \sqrt{\kappa_\epsilon^2 \pm \kappa_h^2}$  where  $\kappa_\epsilon^2 = -2i\epsilon/D$  and  $\kappa_h^2 = -2ih/D$ . For example, if  $V=h\sigma_3$  the  $\sigma_3$  and  $\sigma_0$  components of the anomalous Green's function decay at the rate  $\kappa_\epsilon$  while the  $\sigma_1$  and  $\sigma_2$  components decay at the rate  $\kappa$ . When  $h$  is large, as it generally is in such structures, the  $\sigma_{0,3}$  components are long-ranged while the other two are short-ranged. The boundary conditions at the interface require that the  $\sigma_2$  component vanishes at the interface. Inducing long-ranged triplet components  $\sigma_{0,3}$  in the ferromagnet of a  $S_s/F$  structure with exchange field  $h\sigma_3$  should not be surprising. However, if  $V=h\sigma_2$  we find that the  $\sigma_0$  and  $\sigma_2$  components decay rapidly at the rate  $\kappa$  and the  $\sigma_3$  and  $\sigma_1$  components decay slowly at the rate  $\kappa_\epsilon$ . The boundary conditions at the interface will make the  $\sigma_2$  component vanish at the interface. A comparison of the results obtained with  $V=h\sigma_3$  and  $V=h\sigma_2$  show that we are not merely rotating the structure. In contrast to an  $S_s/F$ , boundary conditions in an  $S_s/F$  structure with a homo-

geneous ferromagnet potential only allow the  $\sigma_2$  anomalous component in the ferromagnet which always decays rapidly at the rate  $\kappa$ .

The invariant transforms are a useful tool because one can show that the anomalous components which are invariant under the same transform as the ferromagnet part of the  $\sigma$  model will decay at the short-ranged  $h$  dependent rate in the ferromagnet. In other words, the transform which is invariant with the ferromagnet part of the  $\sigma$  model is also invariant with those components which couple to the ferromagnet, thereby providing a simple way to determine which components are affected by the ferromagnet. If we have  $S_s/F$  with  $V=h\sigma_3$  the transforms under which the ferromagnet part of the associated  $\sigma$  model is invariant are  $C_1$  and  $C_2$ . These invariances are shared by  $\tau_{1,2}\sigma_{1,2}$  so one may conclude that the  $\sigma_1$  and  $\sigma_2$  components of the anomalous Green's functions decay quickly at the  $h$  dependent rate  $\kappa$ . The other two anomalous components,  $\sigma_0$  and  $\sigma_3$ , are not invariant under the  $C_1$  and  $C_2$  transformations so decay at the rate  $\kappa_\epsilon$  which is independent of  $h$ . Similarly, if  $V=h\sigma_2$  the action is invariant under the  $C_1$  and  $C_3$  transforms, as are the terms  $\tau_{1,2}\sigma_{0,2}$ . Therefore the  $\sigma_{0,2}$  components of the anomalous Green's functions are short-ranged, decaying at the rate  $\kappa$ , while the other two components  $\sigma_{1,3}$  are long-ranged, decaying at the rate  $\kappa_\epsilon$ . This is a trivial example of how the invariant transforms may be used. In a more complicated problem, such as that discussed in the following section we can use the invariant transforms to immediately reject certain components, thus making the calculations much simpler.

One case of particular interest is when a superconductor is coupled to an inhomogeneous ferromagnet. It has been shown that at an  $S_s/F$  interface it is possible to induce both a singlet and an odd frequency triplet component in the ferromagnet if, for example,  $V=h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$ .<sup>6</sup> Here  $\alpha = Ax$  for some constant  $A$  when  $0 < x < w$  and  $\alpha = Aw$  when  $x > w$  where  $w$  is some positive constant. We shall briefly describe how the anomalous components induced in the ferromagnet may be determined from the transformational invariances of the action. At the interface the ferromagnet potential introduces the term  $\tau_0\sigma_3$  into the action so at this point the action is invariant under the  $C_1$  and  $C_2$  transforms. As  $x$  increases a  $\tau_3\sigma_2$  component appears in the action. Now the action is invariant only under the  $C_1$  transform. Invariance under the  $C_1$  and  $C_2$  transforms at the interface implies short-ranged (decay is  $h$  dependent) anomalous components  $\sigma_{1,2}$  and long-ranged (decay is  $h$  independent) components  $\sigma_{0,3}$ . However, as  $x$  increases we lose the invariance under the  $C_2$  transform. When  $C_1$  is the only transformational invariance the short-ranged components are  $\sigma_{0,1,2}$  and only  $\sigma_3$  is long-ranged. However, the boundary conditions cause the coefficient of the  $\sigma_3$  component to vanish. We may conclude that, if the total rotation  $Aw$  is small the solution within the domain wall will be approximately similar to the solution at the  $S_s/F$  interface. Thus we would expect the  $\sigma_0$  component to be long-ranged. If the rotation is increased the loss of invariance under the  $C_2$  transform has a more significant effect on the range of the  $\sigma_0$  component and it vanishes more rapidly. This result is shown in Fig. 2 of Ref. 6 in which the Usadel equation for this  $S_s/F$  structure was solved, however, due to a spin rotation of  $\sigma_1$  the authors find the  $\sigma_1$  component to be long-ranged.

#### IV. LOW ENERGY DENSITY OF STATES

The full solution of the  $\sigma$  model is obtained by considering fluctuations about the saddle point solution. There are several different types of fluctuations which are relevant to different cases. The low energy C-mode fluctuations about the Usadel saddle-point solution are defined as being diagonal in the advanced-retarded space and are therefore quantum corrections to the Usadel solution. They have the further property that they are independent of the order parameter and any magnetic field. The C-modes dominate at energies below the Thouless energy  $D/L^2$  where  $L$  is the length of the ferromagnet and therefore, to study the low energy properties only the C-modes need be considered.<sup>8</sup> We shall find the C-mode fluctuations for an  $S_s/F$  structure with  $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$ . We then derive the low energy density of states. We are interested in seeing how the triplet component induced in the ferromagnet affects the density of states. Our method closely follows that of Ref. 8 where an  $S_s/N$  structure was considered.

If the solution of the Usadel equation is  $Q_U$  and we represent the C-mode corrections by the matrix  $T$  then the full solution of the supermatrix is  $Q = TQ_U T^{-1}$ . One can show<sup>8</sup> that at very low energies the dominant C-mode is spatially constant, the so-called zero-mode. In addition,  $Q_U$  has a very slow spatial variation. The matrix  $Q$  must satisfy the convergence symmetry and the charge conjugation symmetry. The convergence symmetry is

$$Q = \eta Q^\dagger \eta^{-1}, \quad \eta = E_{11} \tau_3 \Lambda + E_{22}, \quad (10)$$

and the charge conjugation symmetry is

$$Q = \tau Q^T \tau^{-1}, \quad \tau = E_{22} i \rho_2 + E_{11} \rho_1. \quad (11)$$

We have defined

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\text{bf}}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\text{bf}}, \quad (12)$$

where the subscript ‘‘bf’’ indicates boson-fermion space. Since  $Q_U$  must also satisfy the above symmetries we may define the fluctuations as  $T = e^W$  where  $W$  must satisfy

$$W^\dagger = -\eta W \eta^{-1}, \quad W^T = -\tau W \tau^{-1}. \quad (13)$$

The C-mode fluctuations must be insensitive to the superconducting order parameter and magnetic fields so we require

$$[W, \sigma_2 \tau_1 \rho_3], [W, \sigma_2 \tau_2] = 0,$$

the order parameter commutes through;

$$[W, \tau_3 \rho_3] = 0, \quad \text{the magnetic field commutes through.} \quad (14)$$

For a solution of  $W$  we may use the zero-mode derived in Ref. 8 but we must include some spin dependence:

$$\begin{aligned} T &= v u a_1 a_2 a_3, \\ a_1 &= \exp(i \frac{1}{2} \theta_1 E_{22} \tau_1 \rho_1 \sigma_1), \\ a_2 &= \exp(i \frac{1}{2} \theta_2 E_{22} \tau_2 \rho_1 \sigma_2), \end{aligned}$$

$$a_3 = \exp(i \frac{1}{2} \theta_3 E_{22} \tau_1 \rho_2 \sigma_3),$$

$$u = \exp(i y E_{22} \rho_3),$$

$$v = \exp \begin{pmatrix} 0 & \lambda - \mu \rho_3 \\ \mu + \lambda \rho_3 & 0 \end{pmatrix}_{\text{bf}}, \quad (15)$$

where  $y$  is some complex variable and  $\lambda$  and  $\mu$  are Grassmann variables. The above solution is sufficiently general for our choice of  $V$ . Terms which satisfy the symmetry requirements and are not included in  $T$  are superfluous to our density of states calculation. We could have chosen, for example, spin dependent fluctuations with the matrix structures  $E_{22} \tau_1 \rho_2 \sigma_1$ ,  $E_{22} \tau_2 \rho_2 \sigma_2$ , and  $E_{22} \tau_1 \rho_1 \sigma_3$  as they also satisfy the symmetry requirements. However, they would add nothing extra to the final solution. The extra terms will either vanish or make a contribution identical to the one already obtained from  $a_{1,2,3}$ . One should note that the invariant transform of the action of the ferromagnet part of the  $\sigma$  model with  $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$  is  $C_1$  and that  $T$  is also invariant under the  $C_1$  transform so we have chosen  $T$  so that it couples to the ferromagnet. If we chose a different exchange field, for example  $V = h(\sigma_3 \cos \alpha + \sigma_1 \sin \alpha)$ , we should choose a different form of  $T$ . The above choice of  $a_3$  will not contribute to the action and should be replaced with  $\exp(i \frac{1}{2} \theta_3 E_{22} \tau_1 \rho_1 \sigma_3)$ . In this case the invariant transform of both the action and the fluctuations is  $C_2$ . Deriving a suitable form of  $T$  can be quite tedious and the task is considerably shortened if one chooses  $T$  to have the same invariance transform as the action under consideration. As stated above, this will not give the most general form of  $T$ , but gives those parts which contribute uniquely to the density of states.

The solution of the Usadel saddle point equation is  $Q_U = g_0$ . One can show that the part diagonal in particle-hole space which describes the normal Green's function is  $g \tau_3$ , i.e.,  $g = -g^\dagger$ . The off-diagonals in particle hole space  $f$  and  $f^\dagger$  describe the anomalous Green function and may in general contain the terms  $\tau_1$  and  $\tau_2 \rho_3$  multiplied by the spin components  $\sigma_{0,1,3}$  and  $\sigma_2 \rho_3$ . The spin components which actually appear in the solution of  $Q_U$  will of course be dependent on the spin structure of the exchange field  $V$ . On substituting the general solution of  $Q_U$  with the fluctuations  $T$  into the action given in Eq. (5) with  $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$ , one finds that all the anomalous components vanish. The singlet components vanish because they are proportional to the order parameter which commutes with  $T$  while the triplet components give zero supertrace. One can show this is true even with the most general form of  $T$ , which is why it is unnecessary to find the most general form. One finds the zero-mode action to be

$$\begin{aligned} S &= -2i\bar{s}(\cos \theta_1 \cos \theta_2 \cos \theta_3 - 1) \\ &\quad + 2ih_1 \sin \theta_1 \sin \theta_2 \cos \theta_3 - 2ih_2 \sin \theta_1 \sin \theta_3 \cos \theta_2, \end{aligned} \quad (16)$$

where

$$\begin{aligned}\tilde{s} &= \pi\epsilon\nu \int g dx, \\ h_1 &= \pi h\nu \int g \cos \alpha dx, \\ h_2 &= \pi h\nu \int g \sin \alpha dx.\end{aligned}\quad (17)$$

Since the C-modes are diagonal in the advance-retarded space we need only consider the retarded part so  $Q$  has been reduced to a  $16 \times 16$  supermatrix and we may set  $\delta=0$ . The density of states with C-type zero-mode fluctuations is

$$\begin{aligned}\rho &= \frac{\nu}{8} \text{Re} \langle \text{str } Q \sigma_3^{\text{bf}} \tau_3 \rangle \\ &= 2\nu \text{Re } g \left[ 1 - \frac{1}{2} \langle \mu \lambda (1 - \cos \theta_1 \cos \theta_2 \cos \theta_3) \rangle \right],\end{aligned}\quad (18)$$

where the averaging is weighted by the action in Eq. (16) and we must perform a path integration over the fluctuating part of  $Q$  (which means an integral over the three  $\theta$ 's,  $\lambda$ ,  $\mu$ , and  $y$ ). This form of the density of states and action is quite general and one would obtain something similar for any exchange field of the form  $V=h(\sigma_i \cos \alpha + \sigma_j \sin \alpha)$ .

The Jacobian for the integral over the fluctuating parts may be found from evaluating  $\text{str}(dQ)^2$ .<sup>15</sup> This Jacobian is extremely complex if one wishes to include all three  $\theta$ 's. If we assume that the total rotation  $Aw$  is very small  $\theta_3$  will not play a significant role and we may set it to zero. Note that  $a_1$  and  $a_2$  are both invariant with  $C_1$  and  $C_2$  whereas  $a_3$  is not invariant with  $C_2$  and, as discussed in the previous section, the  $C_2$  transform becomes relevant when the total rotation is small. Setting  $\theta_3=0$  and integrating over all but  $\theta_1$  and  $\theta_2$  gives

$$\begin{aligned}\rho &= 2\nu \text{Re } g \left[ 1 - \frac{1}{2} \int_0^\pi d\theta_1 d\theta_2 (1 - \cos 2\theta_1 \cos 2\theta_2)^{1/2} \right. \\ &\quad \left. \times \exp(2ih_1 \sin \theta_1 \sin \theta_2 - 2i\tilde{s}(1 - \cos \theta_1 \cos \theta_2)) \right].\end{aligned}\quad (19)$$

We shall discuss the behavior of this density of states in the homogeneous part of the ferromagnet.

There is no exact solution of  $g$  for an inhomogeneous ferromagnet. In Ref. 6 the Usadel solution of the anomalous Green's function  $f$  was studied in the limit of small  $w$  and a large tunnelling resistivity  $R_b$  between the superconductor and the ferromagnet (this study is also valid for a general tunnelling resistivity when near the phase transition but this requires  $\Delta \ll \epsilon$  which does not satisfy our small energy requirement). In this approximation  $f$  is very small and  $g \sim 1$  inside the ferromagnet but inside the superconductor we may take the bulk solution  $g = \epsilon / \sqrt{\epsilon^2 - |\Delta|^2}$  which is vanishingly small. In the ferromagnet  $g$  may be found from  $f$  by using  $g_0=1$  so  $g = \sqrt{1 - f^\dagger f} \sim 1 - \frac{1}{2} f^\dagger f$ . In Ref. 6 it is shown that in the ferromagnet  $f$  contains both a singlet component and a triplet component. If the ferromagnetic exchange field is

large compared to the energy then the singlet part is much smaller than the triplet in the region  $w < x < L$  so may be neglected. The coefficient of the triplet component is derived in Ref. 6 although some care must be taken as one must perform two rotations to make it compatible with the matrix structures used here. The result is, when taking just the triplet component,  $f^\dagger f \sim -C^2$  where

$$\begin{aligned}C^{R,A} &= \mp iAB(0) \sinh[\kappa_\epsilon(L-x)] [\kappa_\epsilon \cosh \Theta_\epsilon \cosh \Theta_3 \\ &\quad + \kappa_3 \sinh \Theta_\epsilon \sinh \Theta_3]^{-1},\end{aligned}\quad (20)$$

for  $w < x < L$  and where  $B(0) = (\rho \xi_h / 2R_b) f_s$ ,  $f_s = \Delta / \sqrt{\epsilon^2 - \Delta^2}$ ,  $\Theta_\epsilon = \kappa_\epsilon L$ ,  $\Theta_3 = \kappa_3 L$ ,  $\kappa_3 = \sqrt{A^2 + \kappa_\epsilon^2}$ .

To evaluate equation (19) we require

$$\tilde{s} = \pi\epsilon\nu \left( \int_{-\infty}^0 g dx + \int_0^w g dx + \int_w^L g dx \right).\quad (21)$$

In the small energy limit  $g$  is very small in the superconductor so we will neglect the integral over negative  $x$ . Since we assume that  $w$  is small we may also neglect the second integral. So now  $\tilde{s}$  just depends on the value of  $g$  in the homogeneous part of the ferromagnet which we have found to be

$$\begin{aligned}g &\sim 1 - \frac{1}{2} A^2 B(0)^2 \sinh^2[\kappa_\epsilon(L-x)] \\ &\quad \times (\kappa_\epsilon \cosh L\kappa_\epsilon \cosh w\sqrt{A^2 + \kappa_\epsilon^2} \\ &\quad + \sqrt{A^2 + \kappa_\epsilon^2} \sinh L\kappa_\epsilon \sinh w\sqrt{A^2 + \kappa_\epsilon^2})^{-2},\end{aligned}\quad (22)$$

and therefore

$$\begin{aligned}\tilde{s} &= \pi\epsilon\nu(L-w) \left[ 1 + \frac{1}{4} A^2 B(0)^2 \left( 1 - \frac{1}{2} (L-w)^{-1} \kappa_\epsilon^{-1} \right. \right. \\ &\quad \left. \left. \times \sinh[2\kappa_\epsilon(L-w)] \right) (\kappa_\epsilon \cosh L\kappa_\epsilon \cosh w\sqrt{A^2 + \kappa_\epsilon^2} \right. \\ &\quad \left. + \sqrt{A^2 + \kappa_\epsilon^2} \sinh L\kappa_\epsilon \sinh w\sqrt{A^2 + \kappa_\epsilon^2})^{-2} \right],\end{aligned}\quad (23)$$

which, in the small energy limit gives  $\tilde{s} = D\epsilon$  for constant  $D$ . Similarly we find that  $h_1 = Dh \cos(Aw)$ . Substituting these solutions for  $\tilde{s}$ ,  $h_1$ , and  $g$  into Eq. (19) gives the low energy density of states within the homogeneous part of the ferromagnet ( $x > w$ ) in the limit of a large tunnelling resistivity, large  $h$  and small  $Aw$ . However, it is true in general that  $\tilde{s} \propto \epsilon$  and  $h_1 \propto h$ .

To analyze the energy dependence of the density of states we first consider Eq. (19) as  $h$  becomes vanishingly small. In this case  $\theta_2$  becomes irrelevant and may be set to zero. The integral over  $\theta_1$  is easy to solve and

$$\rho = 2\nu \text{Re } g \left[ 1 - \frac{\sin 4\tilde{s}}{4\tilde{s}} - \frac{1 - \cos 4\tilde{s}}{4i\tilde{s}} \right],\quad (24)$$

which is, as expected, equivalent to the low energy density of states derived in Ref. 8 for  $S_s/N$ . This density of states is quadratic in  $\epsilon$  and vanishes when  $\epsilon=0$ .<sup>7,8</sup> If  $h$  is not large and Eq. (19) is expanded with respect to  $\tilde{s}$ , we find that the low energy density of states is linear in  $\tilde{s}$ , and therefore linear in  $\epsilon$ . Also, this density of states does not in general vanish when  $\epsilon=0$  so there is no micro-gap. If  $h \gg 1$ , as we have assumed previously and is usually the case in practice, there is only a slight reduction in the density of states from the bulk solu-

tion  $\rho=2\nu$ . This reduction is not due to fluctuations about the Usadel solution since the integral in Eq. (19) approaches zero at large  $h$  but is due to the reduction in  $g$  from the bulk solution  $g=1$ , as in the high resistivity case given in Eq. (22). These conclusions are true even if we did not neglect  $\theta_3$  as they are a consequence of the form of the action rather than the form of the Jacobian. It should be stressed that the choice of the matrices  $a_1$ ,  $a_2$ , and  $a_3$  are very important. The wrong choice may lead to an action which has no  $h$  dependence and this would result in a quadratically increasing density of states with a micro-gap.

In an  $S_s/F$  structure with  $V=h\sigma_3$  we would also obtain an equation of the form (19) so we may also claim that there is a linear reduction in the density of states with respect to energy if  $h$  is not too large. However, for large  $h$  the density of states, when measured some distance from the  $S/F$  interface, will retain the bulk solution because there are no long-ranged anomalous components and therefore  $g=1$ . An equivalent calculation for an  $S_t/F$  structure is much simpler. The C-mode fluctuations are defined to commute with the order parameter so in the case of a triplet superconductor these fluctuations must be independent of spin. Therefore an  $S_t/F$  is similar to an  $S/N$  and one can show that Eq. (24), which is exact for an  $S/N$  but only true for an  $S_s/F$  if  $h$  is extremely small, is exact for an  $S_t/F$ . In the ferromagnetic part of an  $S_t/F$  the form of the low energy density of states is the same as in the normal metal of an  $S/N$  structure, displaying a micro-gap as the energy vanishes.

## V. CONCLUSION

We have considered an unusual type of triplet Cooper pairing which is defined by an order parameter which is even in the momentum (or position) and odd in the frequency (or time). It was found that, for the most part, a superconductor with odd triplet Cooper pairs is much like the standard singlet superconductor (even in position and time). In the bulk these superconductors would appear to be much the same,

and also when coupled to a normal metal. The main difference between the two superconductors is their spin structure. Another difference is the energy dependence of the order parameter though, in many cases, this is not important.

If we consider a situation where the spin is unimportant we may obtain equations for  $S_t$  from equations for  $S_s$  by simply replacing the order parameter  $\Delta$  with  $\text{sgn}(\epsilon)\Delta$ . However, in density of states calculations, for example, this change of sign is irrelevant. Where we do observe a difference between the  $S_t$  and the  $S_s$  is in cases where the spin is important. When an  $S_s$  is coupled to an inhomogeneous ferromagnet it is possible to induce a long-ranged triplet anomalous Green's function component as well as a short-ranged singlet component in the ferromagnet. However, when an  $S_t$  is coupled to any type of ferromagnet a long-ranged triplet component always exists in the ferromagnet. One can determine which anomalous components will dominate the ferromagnet by considering the transformational invariances of the  $\sigma$  model. We considered the low-energy fluctuations about the Usadel solution of an  $S_s/F$  structure with a nonhomogeneous exchange field in order to see if the long-range triplet has a significant effect. We found that an  $S_s/F$  structure which induces a long-range anomalous component in  $F$  will have a smaller density of states compared to the bulk solution. However, in general the fluctuations are not responsible for this reduction. Instead, the reduction is due to a reduction in the Usadel solution of the normal Green's function from the bulk solution of unity. The fluctuations only provide a significant reduction in the density of states if the exchange field  $h$  is small. In such a case the low energy density of states is linear in energy but does not vanish at zero energy. The density of states will only vanish if both  $h$  and  $\epsilon$  approach zero.

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