2017

Curvature contraction flows in the sphere

James A. McCoy
University of Wollongong, jamesm@uow.edu.au

Curvature contraction flows in the sphere

Abstract

We show that convex surfaces in an ambient three-sphere contract to round points in finite time under fully nonlinear, degree one homogeneous curvature flows, with no concavity condition on the speed. The result extends to convex axially symmetric hypersurfaces of the (n+1)-dimensional sphere. Using a different pinching function we also obtain the analogous results for contraction by Gauss curvature.

Disciplines

Engineering | Science and Technology Studies

Publication Details

McCoy, J. A. (2018). Curvature contraction flows in the sphere. Proceedings of the American Mathematical Society, 146 (3), 1243-1256.

Curvature contraction flows in the sphere *

James A. McCoy†

May 8, 2017

Abstract

We show that convex surfaces in an ambient three-sphere contract to round points in finite time under fully nonlinear, degree one homogeneous curvature flows, with no concavity condition on the speed. The result extends to convex axially symmetric hypersurfaces of $\mathbb{S}^{n+1}$. Using a different pinching function we also obtain the analogous results for contraction by Gauss curvature.

1 Introduction

The contraction of convex hypersurfaces in Euclidean space by their curvature has been very well studied since Huisken's seminal work on the mean curvature flow [Hu1]. For a review of many fully nonlinear contraction flows in Euclidean space and their action on convex initial hypersurfaces we refer the reader to [AMZ]; contracting axially symmetric hypersurfaces in Euclidean space are treated in [MMW, Section 7]. Flows of smooth convex hypersurfaces (those with pointwise strictly positive definite Weingarten map) in general Riemannian manifolds have been less thoroughly investigated, but there are fundamental contributions by Huisken [Hu2] in the case of mean curvature flow and by Andrews [A2] for a class of fully nonlinear flows. Many other results are available for particular flows in particular ambient spaces. A large amount of work has also been done on curvature expansion flows; we refer the reader to [LWW] for a recent result and a survey of previous work.

If we specialise to evolving hypersurfaces in the sphere, there are some interesting results in the literature. Huisken considered the mean curvature flow in this setting [Hu3], showing that initial hypersurfaces not necessarily convex but satisfying a pointwise curvature condition either contract in finite time to a single point, or evolve for all time, smoothly approaching asymptotically a smooth totally geodesic hypersurface. Again for mean curvature flow, a different curvature condition on the initial hypersurface was given recently in [LXZ]. Andrews discussed in [A4] a corresponding optimal result of this kind, requiring the weakest condition on the curvature of the initial surface in the three-sphere, by optimising the choice of fully nonlinear speed. More recently,

*This research was supported by Discovery Grant DP150100375 of the Australian Research Council.

†MSC: 35K55, 53C44 Keywords: curvature flow, parabolic partial differential equation, hypersurface, axial symmetry, spherical geometry

Guan and Li [GL] have developed a constrained flow by a function of the mean curvature under which star-shaped hypersurfaces in the sphere evolve asymptotically to a sphere in infinite time. Gerhardt [G] has demonstrated a correspondence between contracting and expanding curvature flows of n-dimensional hypersurfaces in the sphere; our results in this article may be considered a companion of the contraction results of that article, since we replace the condition that the normal speed be concave and inverse concave with the requirement that $n = 2$ or the initial hypersurface is axially symmetric about some axis. Some other results related to our setting are as follows. Andrews, Han, Li and Wei [AHLW] have proved non-collapsing results for classes of flow in the ambient sphere (and in hyperbolic space) similar to those considered in this article. Nguyen has obtained convexity and cylindrical estimates for mean curvature flow in the sphere [N]. Wei and Xiong [WX] have used curvature flows to prove Alexandrov-Fenchel type inequalities for convex hypersurfaces in the sphere (and in hyperbolic space), building on earlier work of Makowski and Scheuer [MS]. Bryan, Ivaki and Scheuer have classified ancient solutions [BIS2] and obtained Harnack inequalities for a class of curvature flows of hypersurfaces of the sphere [BIS1], the latter complementing a similar result for the mean curvature flow [BI].

In this article we will use notation similar to that in [Hu3] and [A2]. We consider a smooth immersion $\varphi_0 : \mathbb{S}^n \to \mathbb{S}^{n+1}$. We seek a solution $\varphi : \mathbb{S}^n \times [0,T) \to \mathbb{S}^{n+1}$ to the following system of partial differential equations

$$\frac{\partial}{\partial t}\varphi(x,t) = -F(\mathscr{W}(x,t))\,\nu(x,t), \tag{1}$$

with initial condition $\varphi(x,0) = \varphi_0(x)$. Here $\nu(x,t)$ is a unit normal in $T\mathbb{S}^{n+1}$ to $\varphi_t(\mathbb{S}^n) = \varphi(\mathbb{S}^n,t) = M_t$ at $\varphi(x,t)$, while $\mathscr{W}(x,t)$ is the Weingarten map of M_t at $\varphi_t(x)$ and F is the normal speed function. F satisfies the following conditions

Conditions 1.1

a) *$F(\mathscr{W}) = f(\kappa(\mathscr{W}))$ where $\kappa(\mathscr{W})$ gives the eigenvalues of $\mathscr{W}$ and f is a smooth, symmetric function defined on the positive cone*

$$\Gamma = \{\kappa = (\kappa_1,\ldots,\kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \textit{ for all } i = 1,2,\ldots,n\}.$$

b) *f is strictly increasing in each argument: $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1,\ldots,n$ at every point of Γ.*

c) *f is degree-1 homogeneous: $f(k\kappa) = kf(\kappa)$ for any $k > 0$.*

d) *f is strictly positive on Γ and $f(1,\ldots,1) = 1$.*

In Sections 5 and 6 we also consider speed $F = K = \det(\mathscr{W})$, ie the Gauss curvature flow. To our knowledge Gauss curvature flow of hypersurfaces of the sphere has not been considered before. The most relevant results for Gauss curvature flow in Euclidean space are Chou's proof that convex initial hypersurfaces shrink to points [T], Chow's similar result for flows by positive powers of the Gauss curvature [C], with roundness of the point under rescaling in case of power $\frac{1}{n}$ and Andrews' proof of the

Firey conjecture [A3], that is, roundness of the point under rescaling of Gauss curvature flow of surfaces. Roundness of the final point was also recently shown for flows of centrally symmetric hypersurfaces in Euclidean space by powers larger than 1 of the Gauss curvature [AGN]. Self-similar axially symmetric surfaces contracting in Euclidean space were considered in [I] while axially symmetric surfaces with boundary conditions were considered in [J].

The new results in this article are the following

Theorem 1.2 *Let M_0 be a closed, smooth, strictly convex hypersurface without boundary smoothly embedded in $\mathbb{S}^{n+1}$ of sectional curvature $\overline{\sigma}$. If $n>2$ suppose M_0 is axially symmetric. Let F satisfy Conditions 1.1, or $F=K$. Then there exists a unique family of smooth, strictly convex surfaces $\{M_t=\varphi(M^n)\}_{0\leq t<T}$ satisfying* (1)*, with initial condition $\varphi(\cdot,0)=\varphi_0(\mathbb{S}^n)=M_0$. The solution exists on a finite maximal time interval $[0,T)$, is axially symmetric if M_0 is axially symmetric and the images converge uniformly to a point $p\in\mathbb{S}^n$ as $t\to T$. The rescaled immersions given by $\tilde{\varphi}_\tau(x)=\frac{1}{2\sqrt{T-t}}\varphi(x-p,t)$ converge in C^∞ to the unit sphere with centre at the origin in Euclidean space. The convergence is exponential with respect to the rescaled time parameter $\tau=-\frac{1}{2}\ln\left(1-\frac{t}{T}\right)$.*

The remainder of this article is structured as follows. In Section 2, we set up further notation, state some known facts and the necessary evolution equations and characterise certain terms in these at local extrema. In Section 3, we discuss the case of surfaces contracting in $\mathbb{S}^3$ by F satisfying Conditions 1.1, while in Section 4 we will consider contracting axially symmetric hypersurfaces contracting by these F. In Sections 5 and 6 we use a different curvature pinching quantity to obtain our results for the Gauss curvature flow of surfaces in $\mathbb{S}^3$ and axially symmetric hypersurfaces in $\mathbb{S}^{n+1}$ respectively.

2 Preliminaries

As further notation we will use $g=\{g_{ij}\}$, $A=\{h_{ij}\}$ and $\mathscr{W}=\left\{h^i_{\ j}\right\}$ to denote respectively the metric, second fundamental form and Weingarten map of M_t. The mean curvature of M_t is $H=g^{ij}h_{ij}=h^i_{\ i}$ and the norm of the second fundamental form is $|A|^2=g^{ij}g^{lm}h_{il}h_{jm}=h^j_{\ l}h^{\ l}_j$, where g^{ij} is the (i,j)-entry of the inverse of the matrix (g_{ij}). Throughout this paper we sum over repeated indices unless otherwise indicated. Raised indices indicate contraction with the metric of M_t.

We will denote by $\left(\dot{F}^{kl}\right)$ the matrix of first partial derivatives of F with respect to the components of its argument:

$$\frac{\partial}{\partial s}F(A+sB)\bigg|_{s=0}=\dot{F}^{kl}(A)B_{kl}.$$

Similarly for the second partial derivatives of F we write

$$\frac{\partial^2}{\partial s^2}F(A+sB)\bigg|_{s=0}=\ddot{F}^{kl,rs}(A)B_{kl}B_{rs}.$$

We will also use the notation

$$\dot{f}_i(\kappa) = \frac{\partial f}{\partial \kappa_i}(\kappa) \text{ and } \ddot{f}_{ij}(\kappa) = \frac{\partial^2 f}{\partial \kappa_i \kappa_j}(\kappa).$$

Relationships between derivatives of F and those of f are well-known (see, for example, [A5]). Unless otherwise indicated, throughout this paper we will always evaluate partial derivatives of F at $\mathscr{W}$ and partial derivatives of f at $\kappa(\mathscr{W})$. Two important relationships are the following: in a local orthonormal frame diagonalising $\mathscr{W}$,

$$\dot{F}^{kl}(\mathscr{W}) = \dot{f}^k(\kappa)\,\delta_{kl}$$

and for any symmetric matrix B we have

$$\ddot{F}^{pq,rs}(\mathscr{W})\,B_{pq}B_{rs} = \ddot{f}^{pr}B_{pp}B_{rr} + 2\sum_{p<r}\frac{\dot{f}^p(\kappa) - \dot{f}^r(\kappa)}{\kappa_p - \kappa_r}(B_{pr})^2. \tag{2}$$

Formula (2) makes sense as a limit in the case of any repeated eigenvalues of $\mathscr{W}$.

There are several geometric identities we will need for hypersurfaces of general Riemannian manifolds N^{n+1}. For details of these we refer the reader to [Hu2, A2].

Lemma 2.1 *The following geometric relations hold for hypersurfaces of* $\mathbb{S}^{n+1}$.

(i) *The Codazzi equations* $\nabla_i h_{jk} = \nabla_j h_{ik}$;

(ii) *The Gauss equations* $R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \overline{R}_{ijkl}$;

(iii) *A Simons' type identity*

$$\begin{aligned}\nabla_i\nabla_j h_{kl} = \nabla_k\nabla_l h_{ij} + h_{ij}h_k^{\ p}h_{pl} - h_{kl}h_i^{\ p}h_{pj} + h_{il}h_k^{\ p}h_{pj} - h_{kj}h_i^{\ p}h_{pl} + h_l^{\ p}\overline{R}_{ikjp} \\ - h_j^{\ p}\overline{R}_{kilp} + h_k^{\ p}\overline{R}_{iljp} - h_i^{\ p}\overline{R}_{kjlp} + h_{ij}\overline{R}_{k0l0} - h_{kl}\overline{R}_{i0j0}.\end{aligned}$$

We adopt a coordinate system as in [Hu2] *for example, where index* 0 *to correspond to the normal direction to* M_t*; other indices run from* 1 *to* n*. Above* ∇ *denotes the covariant derivative on* M_t.

We remark that the Codazzi equations have exactly the same form for hypersurfaces of $\mathbb{S}^{n+1}$ as for those of $\mathbb{R}^{n+1}$, that is, the tensor ∇A is totally symmetric. Further, no $\overline{\nabla}\overline{R}$ terms appear in the Simons' identity above (unlike the case for hypersurfaces of general ambient spaces) since $\mathbb{S}^{n+1}$ has constant curvature. In addition to the remark in part (iii) above, when dealing with axially symmetric hypersurfaces we will use a coordinate system where index 1 corresponds to the direction tangent to the generating curve of M_t.

In view of Conditions 1.1, (b), short-time existence of a smooth, convex solution to (1) given smooth, convex initial data $\varphi_0(M^n)$ is known. Short-time existence for the Gauss curvature flow with convex initial data holds similarly. We shall not concern ourselves with minimal regularity requirements here. We refer the interested reader to [A2] where an appropriate graphical parametrisation is developed in detail that applies in $\mathbb{S}^{n+1}$ and in more general ambient spaces. The reader might also wish to consult [Ha] where (1) is converted into an appropriate form and [Lu, Theorem 8.4.1] is applied. Another general treatment appears in [B]. In the case of axially symmetric initial data, once a parametrisation is fixed to remove the degeneracy with respect to tangential diffeomorphisms in (1), it is clear that evolving hypersurface M_t remains axially symmetric.

We will require the following evolution equations corresponding to geometric quantities of M_t evolving under (1). Derivations are as in [A2] (see also [Hu5] and, for the Euclidean case, [A1], for example). Since we will also be using these for the flow by Gauss curvature, we write the equations for speed F homogeneous of degree α in the principal curvatures.

Lemma 2.2 *Under the flow* (1)*,*

(i) *the metric evolves according to*

$$\frac{\partial}{\partial t} g_{ij} = -2F h_{ij};$$

(ii) *the Weingarten map evolves according to*

$$\begin{aligned}\frac{\partial}{\partial t} h^i{}_j = \mathscr{L} h^i{}_j + \ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} \left(h_k{}^p h_{pl} + \overline{R}_{k0l0} \right) h^i{}_j \\ + g^{im} \dot{F}^{nl} \left(h_l{}^p \overline{R}_{mnjp} - h_j{}^p \overline{R}_{nmlp} + h_n{}^p \overline{R}_{mljp} - h_m{}^p \overline{R}_{njlp} \right),\end{aligned}$$

where $\mathscr{L}$ denotes the operator $\dot{F}^{kl} \nabla_k \nabla_l$. It follows from (ii) that

(iii) *any degree-β homogeneous function $G(\mathscr{W}) = g(\kappa(\mathscr{W}))$ evolves according to*

$$\begin{aligned}\frac{\partial}{\partial t} G = \mathscr{L} G + \left(\dot{G}^{ij} \ddot{F}^{kl,rs} - \dot{F}^{ij} \ddot{G}^{kl,rs} \right) \nabla_i h_{kl} \nabla_j h_{rs} + \beta \dot{F}^{kl} \left(h_k{}^p h_{pl} + \overline{R}_{k0l0} \right) G \\ + 2 \dot{F}^{nl} \dot{G}^{jm} \left(h_l{}^p \overline{R}_{mnjp} - h_j{}^p \overline{R}_{nmlp} \right) + (1-\alpha) \dot{G}^{ij} \left(h_i{}^p h_{pj} + \overline{R}_{i0j0} \right) F. \quad (3)\end{aligned}$$

In particular,

(iv) *the speed function F evolves according to*

$$\frac{\partial}{\partial t} F = \mathscr{L} F + \dot{F}^{kl} \left(h_k{}^p h_{pl} + \overline{R}_{k0l0} \right) F.$$

Remarks:

1. From (iv) above we have, in coordinates that diagonalise $\mathscr{W}$ at a minimum of F,

$$\frac{d}{dt} \min_{M_t} F \geq \dot{f}^k \left(\kappa_k^2 + \overline{\sigma} \right) f,$$

from which we observe that the minimum of F does not decrease under the flow.

2. Under (1), the area of M_t changes according to

$$\frac{d}{dt}|M_t| = -2\int_{M_t} HF\,d\mu.$$

Once we have shown convexity is preserved we can see that the area of M_t is decreasing under (1).

In this paper we will work in particular with functions G that are homogeneous of degree zero. We have the following:

Proposition 2.3 *Let F be homogeneous of degree one. In the case $n = 2$, or if $n \geq 2$ and the hypersurface M_t is axially symmetric, then at a local extremum of a degree zero homogeneous function G of the principal curvatures, where $\dot{G}$ is nondegenerate, the terms in the evolution equation of Lemma 2.2, (iii) simplify, in coordinates that diagonalise the Weingarten map, to*

$$\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} = \frac{2f\dot{g}^1}{\kappa_2\left(\kappa_2-\kappa_1\right)}\left[\left(\nabla_1 h_{12}\right)^2 + \left(\nabla_2 h_{12}\right)^2\right] \tag{4}$$

and

$$\dot{F}^{kl}\dot{G}^{ij}\left(h_l^{\;p}\overline{R}_{ikjp} - h_j^{\;p}\overline{R}_{kilp}\right) = \frac{f\dot{g}^1}{\kappa_2}\left(\kappa_2-\kappa_1\right)\overline{R}_{1212}. \tag{5}$$

Proof of Proposition 2.3: First note that in either setting, since f is degree-one homoegeneous and g is degree zero homogeneous we have

$$\dot{f}^1\kappa_1 + (n-1)\dot{f}^2\kappa_2 = f \text{ and } \dot{g}^1\kappa_1 + (n-1)\dot{g}_2\kappa_2 = 0,$$

where the derivatives are evaluated at points of the form $(\kappa_1, \kappa_2, \dots, \kappa_2)$. For (4), the calculation is the same as in [A7, Section 3] using the homogeneity of F and G and expressions for the first and second derivatives of these functions in terms of those of f and g. Factors of $(n-1)$ appear in the axially symmetric case but in the end these cancel out.

For (5), we have in our choice of coordinates

$$\begin{aligned}\dot{F}^{kl}\dot{G}^{ij}\left(h_l^{\;p}\overline{R}_{ikjp} - h_j^{\;p}\overline{R}_{kilp}\right) &= \dot{f}^k\dot{g}^i\left(\kappa_k\overline{R}_{ikik} - \kappa_i\overline{R}_{kiki}\right)\\ &= (n-1)\left(\dot{f}^1\dot{g}^2 - \dot{f}^2\dot{g}^1\right)\left(\kappa_1-\kappa_2\right)\overline{R}_{1212}.\end{aligned}$$

Using now the homogeneity of f and of g, (5) follows. □

3 Contracting convex surfaces

As in [A7], we consider the function $G(\mathscr{W}) = g(\kappa(\mathscr{W}))$ where

$$g(\kappa_1, \kappa_2) = \frac{(\kappa_2-\kappa_1)^2}{(\kappa_1+\kappa_2)^2}.$$

This is a degree $\beta = 0$ homogeneous function that gives a pointwise measure of the difference between the principal curvatures. The evolution equation for G under (1) is given by Lemma 2.2 (iii). Assuming $\kappa_2 \geq \kappa_1$, the ratio of the principal curvatures is related to g via

$$\frac{\kappa_2}{\kappa_1} = \frac{2}{1-\sqrt{g}} - 1. \tag{6}$$

Theorem 3.1 *Under the flow* (1)*, the maximum value of the curvature pinching ratio does not increase.*

Proof: We may assume the maximum of G is nonzero since otherwise $G \equiv 0$ so M_t is totally umbilic and therefore a sphere (see, for example, [S]). We have, by direct calculation, using coordinates that diagonalise the Weingarten map at the maximum,

$$\dot{g}^1 = -\frac{4\kappa_2 (\kappa_2 - \kappa_1)}{(\kappa_1 + \kappa_2)^3}.$$

In view of (4) it follows that at a spatial maximum of G the gradient term in (3) is non-positive. Similarly, in view of (5), the zero order term in the evolution equation is also non-positive. It follows by the maximum principle that $\max_{M_t} G$ does not increase. In view of (6), it follows that the maximum of the pinching ratio does not increase. □

With the pinching ratio of the evolving surface M_t controlled, our evolution equations are uniformly parabolic and we are in a position to obtain regularity estimates for the evolving hypersurfaces and complete the proof of Theorem 1.2. As the argument is relatively standard, we will just provide an outline here. As a first consequence we have

Corollary 3.2 *The maximal existence time T of a solution to* (1) *is finite. Moreover, we have for all $t < T$,*

$$F \geq \frac{F_{\min}(0)}{\sqrt{1 - 2\underline{c} F_{\min}^2(0)\, t}},$$

where $F_{\min}(0) = \min_{M_0} F$ and $\underline{c} > 0$ is a constant.

Proof: As in Remark 1 following Lemma 2.2, we have that under (1) the minimum of F satisfies for almost every t

$$\frac{d}{dt} \min_{M_t} F \geq \dot{f}^k \left(\kappa_k^2 + \overline{\sigma}\right) f \geq \dot{f}^k \kappa_k^2 \geq \underline{c} f^3,$$

where the last step follows for some constant $\underline{c} > 0$ in view of Theorem 3.1. This implies that $\min_{M_t} F \to \infty$ in finite time T. Moreover, the explicit lower bound follows as in [ALM, Lemma 2.5]. □

Completion of the proof of Theorem 1.2: Contraction of solution surfaces to a point p in finite time follows exactly as in [G, Section 6]. For regularity, first note we have a lower speed bound under the evolution from Corollary 3.2, while an upper bound on time intervals $[\delta, t] \subset [0, T)$ follows via an argument related to that of Tso in Euclidean space [T]; the variant needed here appears in [G, Section 6]. Speed bounds imply curvature bounds via monotonicity of F and Theorem 3.1. Evolution equations are thus uniformly parabolic on time intervals $[\delta, t]$. Introducing geodesic polar coordinates, we may infer that the polar graph function for M_t, denoted u, is $C^{2,\alpha}$ without needing convexity or concavity of F, using the result of Andrews [A7]. Higher order regularity follows via standard Schauder estimates (see, eg [Li]). Estimates may be extended to $[0,t]$ using short-time existence.

Similar arguments as in [G, Section 7] show that under rescaling the solution surfaces converge to the sphere. Let $\Theta(t;T)$ denote the radius of a geodesic sphere that shrinks to a point at time T, the precise extinction time for the flow (1) with given initial data φ_0. The rescaled speed $\tilde{F} := \Theta F$ satisfies a Tso-type estimate ([G, Lemma 7.2]); this provides an upper bound on the rescaled principal curvatures $\Theta\kappa_i$. A lower bound on $\tilde{F}$ now follows via the Krylov-Safonov Harnack inequality [KS] thereby providing a lower bound on the rescaled principal curvatures (the ratio of principal curvatures is unchanged under the rescaling). Given uniform parabolicity, arguments as in the previous paragraph but for the rescaled polar graph function $\tilde{u}(\cdot,\tau) = \frac{u}{\Theta}$ provide regularity for the rescaled equation with rescaled time parameter $\tau = -\log\Theta$. To see finally that the limiting surface (which exists via arguments as in [Hu1, Section 1]) is a sphere we may use the strong maximum principle: the pinching ratio of the rescaled principal curvatures must be strictly decreasing unless it is identically constant. In view of the zero order term in (3), this constant must be zero and consequently the rescaled surface is a sphere. This completes the proof of Theorem 1.2. $\square$

4 Contracting convex axially symmetric hypersurfaces

In this section we adapt the previous argument to the case of convex axially symmetric hypersurfaces contracting in $\mathbb{S}^{n+1}$, similarly to the adaptation of the result of [A7] for convex surfaces in $\mathbb{R}^3$ to axially symmetric hypersurfaces in $\mathbb{R}^{n+1}$ in [MMW].

Theorem 4.1 *Under the flow* (1)*, the maximum value of the pinching ratio of the axially symmetric hypersurface M_t does not deteriorate.*

Proof: The proof is similar to that of Theorem 3.1. The function function G is now

$$G(\mathscr{W}) = \frac{n\left|A^0\right|^2}{H^2}$$

corresponding to

$$g(\kappa(\mathscr{W})) = \frac{n\left(\kappa_1^2 + \ldots + \kappa_n^2\right) - (\kappa_1 + \ldots + \kappa_n)^2}{(\kappa_1 + \ldots + \kappa_n)^2}$$

and we will be working at points of the form $(\kappa_1, \kappa_2, \ldots, \kappa_2)$, that is, $(n-1)$ of the principal curvatures are equal. Again G evolves under (1) according to (3) and we may assume its maximum is positive. We have in coordinates that diagonalise the Weingarten map at a maximum of G,

$$\dot{g}^1 = -\frac{2n(n-1)\kappa_2(\kappa_2 - \kappa_1)}{H^3}.$$

It follows in view of equation (4) that the ∇A term in (3) is nonpositive. From (5) we also have

$$2\dot{F}^{kl}\dot{G}^{ij}\left(h_l{}^p\overline{R}_{ikjp}-h_j{}^p\overline{R}_{kilp}\right)=-\frac{4n(n-1)F}{H^3}(\kappa_2-\kappa_1)^2\overline{R}_{1212}$$

which is also nonpositive.

Hence the maximum value of G does not increase under the flow (1) and so the pinching ratio does not deteriorate under the flow. □

The remainder of the proof of Theorem 1.2 in this case is similar to that in the previous section; the case of axially symmetric hypersurfaces in $\mathbb{R}^{n+1}$ appears in [MMW]. To see that the point to which the solution contracts is asymptotically round we again have for the rescaled limit flow by the strong maximum principle that G is identically constant. If this constant is zero, then the rescaled hypersurface is totally umbilic and therefore a sphere. If this constant is positive, then (3) implies that $\nabla_1 h_{12}\equiv 0$ and $\nabla_2 h_{12}\equiv 0$. The Codazzi equations then imply $\nabla_2 h_{11}\equiv 0$ and $\nabla_1 h_{22}\equiv 0$. Using now that $\nabla G\equiv 0$ we see that $\nabla_1 h_{11}\equiv 0$ and $\nabla_2 h_{22}\equiv 0$, so $\nabla A\equiv 0$ and the rescaled hypersurface is again a sphere (and therefore $G\equiv 0$ in any case). □

5 Surfaces contracting by their Gauss curvature

In this section we obtain a pinching estimate using the degree 2 homogeneous function used in the case of surfaces contracting by their Gauss curvature in Euclidean space in [A3]. This estimate has also been previously noticed by Andrews and Chen [AL].

Setting $Q=2\left|A^0\right|^2$ as in [A3] we have using Lemma 2.2, (iii),

Lemma 5.1 *In the case $n=2$, under the flow (1) with $F=K$, the function Q evolves according to*

$$\frac{\partial}{\partial t}Q=\mathscr{L}Q-2\dot{K}^{ij}\nabla_i H\nabla_j H+2H\ddot{K}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs}-2\overline{\sigma}HQ. \tag{7}$$

Proposition 5.2 *Under the flow (1),*

$$\sup_{M_t}|\kappa_1(x,t)-\kappa_2(x,t)|\leq\sup_{y\in M_0}|\kappa_1(y,0)-\kappa_2(y,0)|=:c_p.$$

Proof: We apply the maximum principle to equation (7), similarly as in [A3, Proposition 3]. As remarked earlier, with the ambient space $\mathbb{S}^3$, the Codazzi equations have exactly the same form as in Euclidean space, so they may be used exactly the same way as in [A3]. The only additional term in (7) as compared with the Euclidean case clearly has the right sign for applying the maximum principle. □

From Remark 1 after Lemma 2.2 we know that the minimum of the Gauss curvature does not decrease under the flow, that is $K\geq\underline{K}:=\min_{M_0}K$. We use this together with Proposition 5.2 to obtain a bound on the curvature pinching ratio.

Corollary 5.3 *There exist constants $0<\underline{C}\leq\overline{C}$ such that, under the Gauss curvature flow (1),*

$$\underline{C}\leq\frac{\kappa_1}{\kappa_2}\leq\overline{C}$$

as long as the solution exists.

Proof: Similarly as in [AC, Corollary 4] for example, we write

$$0 \leq \frac{\kappa_1}{\kappa_2} + \frac{\kappa_2}{\kappa_1} - 2 = \frac{(\kappa_1 - \kappa_2)^2}{\kappa_1 \kappa_2} \leq \frac{c_p}{\underline{K}}$$

and the result follows. □

Remark: In view of Corollary 5.3 and the lower bound on K, we have that each of the principal curvatures are uniformly bounded below by positive constants under (1).

That the surfaces M_t shrink to a point in finite time now follows by similar arguments as Section 3 above. For regularity, it is convenient to work directly with the rescaled flow, with $\tilde{K} = \Theta K$, where $\Theta(t;T)$ is now the radius of the geodesic sphere that shrinks to a point at time T under the Gauss curvature flow, where T is again the extinction time of the flow with given initial data φ_0. A uniform upper bound on the rescaled Gauss curvature flow is obtained in a similar way; together with Corollary 5.3, which is invariant under the rescaling, we obtain an upper curvature bound.

Arguments similar to those in [CRS, Section 6] imply a decaying exponential bound on K, similar bounds on H and on the minimum principal curvature follow via Corollary 5.3. It follows that the rescaled evolution equations are uniformly parabolic on any finite time interval, so standard arguments yield regularity of the rescaled solution on such time intervals. That the solution approaches a sphere can now be argued as follows. For the unnormalised flow, the maximum of Q is strictly decreasing unless M_t is a sphere: if Q attained a later maximum then Q would be identically constant, by the strong maximum principle. Calculations as in Section 6 show that in this case the coefficients of the remaining gradient terms in (7) are nonpositive, as is the zero order term, so, in particular, the zero order term must be identically equal to zero. Since H has a positive lower bound for the unnormalised flow (via pinching and the lower bound on K), it must be that $Q \equiv 0$ and M_t would be a shrinking sphere. Hence the maximum of Q is strictly decreasing unless M_t is a shrinking sphere and if a smooth limit for the rescaled flow exists, then it must be the sphere.

To see now that a smooth limit for the rescaled flow exists, we need to overcome the fact that the flow is not uniformly parabolic (our lower bound on the rescaled principal curvatures decays to zero which could cause a degeneracy of operator $\mathscr{L}$). We may overcome this problem by rewriting the evolution equation for the rescaled Gauss curvature $\tilde{K}$ as a porous medium equation and then applying the Hölder estimate of Di Benedetto and Friedman [DF, Theorem 1.3]. The details of application of this estimate including the conditions to check are almost exactly the same as in [CRS, Section 7.2]; we refer the reader to this article for details. The only difference in our situation is the additional term involving the sectional curvature of $\mathbb{S}^3$ in the evolution equation for $\tilde{K}$, however this leads to no additional complications. (The lower order term introduced into evolution equations for our rescaled quantities has a comparable form to the global term in [CRS].) The resulting uniform Hölder estimate allows us to conclude that a smooth limit hypersurface $\tilde{M}_\infty$ exists, at least for a subsequence of times, by the Arzela-Ascoli theorem and we know this limit hypersurface is the sphere. Exponential convergence of the rescaled solutions to the unit sphere may now be obtained using a standard linearisation of the flow and stability argument. □

6 Contracting axially symmetric hypersurfaces in $\mathbb{S}^{n+1}$ by their Gauss curvature

In this section we extend the results of Section 5 to the case of convex, axially symmetric hypersurfaces of $\mathbb{S}^{n+1}$ contracting by their Gauss curvature. We use the same function, $Q = n\left|A^0\right|^2$ to measure the difference between principle curvatures, however, for $n > 2$, the evolution equation for Q is more complicated. We begin with a geometric result for axially symmetric hypersurfaces.

Lemma 6.1 *For axially symmetric hypersurfaces,*

$$HQ - (n-1)\left(nC - H|A|^2\right) = -(n-1)(n-2)\kappa_1(\kappa_1 - \kappa_2)^2.$$

Proof: For axially symmetric hypersurfaces,

$$HQ = [\kappa_1 + (n-1)\kappa_2](n-1)(\kappa_1 - \kappa_2)^2$$

and

$$nC - H|A|^2 = \sum_{i<j}(\kappa_i + \kappa_j)(\kappa_i - \kappa_j)^2 = (n-1)(\kappa_1 + \kappa_2)(\kappa_1 - \kappa_2)^2,$$

where for the first step we used the calculation as in [Hu4, Lemma 1.4, (iii)]. Combining these gives the result. □

Using Lemma 2.2, (iii) we have the following evolution of Q:

Lemma 6.2 *Under the flow* (8)*,*

$$\frac{\partial}{\partial t}Q = \mathscr{L}Q + \left(\dot{Q}^{ij}\ddot{K}^{kl,rs} - \dot{K}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} \\ + 2K\left[HQ - (n-1)\left(nC - H|A|^2\right)\right] - 2KH_{-1}^{-1}Q\overline{\sigma}, \quad (8)$$

where $H_{-1}^{-1} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \ldots + \frac{1}{\kappa_n}$ is the reciprocal of the harmonic mean curvature.

Proposition 6.3 *Under the flow* (1) *of an axially symmetric hypersurface M_t,*

$$\sup_{M_t}|\kappa_1(x,t) - \kappa_2(x,t)| \leq \sup_{M_0}|\kappa_1(x,t) - \kappa_2(x,t)|.$$

Proof: We apply the maximum principle to the evolution equation to (8). First observe that while M_t is convex the $\overline{\sigma}$ term is clearly nonpositive; the other zero order term is also clearly nonpositive by Lemma 6.1. It remains to check the sign of the gradient terms at a maximum of Q. We may assume at such a point $Q>0$ since otherwise $Q\equiv 0$ and M_t is a shrinking sphere. In view of axial symmetry we have

$$
\begin{aligned}
&\left(\dot{Q}^{ij}\ddot{K}^{kl,rs}-\dot{K}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}\\
&=\left(\dot{q}^i\ddot{k}^{11}-\dot{k}^i\ddot{q}^{11}\right)\left(\nabla_i h_{11}\right)^2+2\left(n-1\right)\left(\dot{q}^i\ddot{k}^{12}-\dot{k}^i\ddot{q}^{12}\right)\nabla_i h_{11}\nabla_i h_{22} \qquad (9)\\
&\quad+\left(n-1\right)^2\left(\dot{q}^i\ddot{k}^{22}-\dot{k}^i\ddot{q}^{22}\right)\left(\nabla_i h_{22}\right)^2+\frac{2\left(n-1\right)}{\kappa_1-\kappa_2}\left[\dot{q}^i\left(\dot{k}^1-\dot{k}^2\right)-\dot{k}^i\left(\dot{q}^1-\dot{q}^2\right)\right]\left(\nabla_i h_{12}\right)^2,
\end{aligned}
$$

where $Q\left(\mathscr{W}\right)=q\left(\kappa\right)$ and $K\left(\mathscr{W}\right)=k\left(\kappa\right)=\kappa_1\kappa_2\ldots\kappa_n$. Since

$$\nabla_i Q=\dot{q}^1\nabla_i h_{11}+\left(n-1\right)\dot{q}^2\nabla_i h_{22}$$

we have

$$\nabla_1 h_{11}=\frac{1}{\dot{q}^1}\left[\nabla_1 Q-\left(n-1\right)\dot{q}^2\nabla_1 h_{22}\right] \text{ and } \nabla_2 h_{22}=\frac{1}{\left(n-1\right)\dot{q}^2}\left[\nabla_2 Q-\dot{q}^1\nabla_2 h_{11}\right].$$

Substituting these into (9) we find

$$
\begin{aligned}
&\left(\dot{Q}^{ij}\ddot{K}^{kl,rs}-\dot{K}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}\\
&=\frac{\dot{q}^1\ddot{k}^{11}-\dot{k}^1\ddot{q}^{11}}{\left(\dot{q}^1\right)^2}\left(\nabla_1 Q\right)^2+\frac{2\left(n-1\right)}{\left(\dot{q}^1\right)^2}\left[\dot{q}^1\left(\dot{q}^1\ddot{k}^{12}-\dot{k}^1\ddot{q}^{12}\right)-\dot{q}^2\left(\dot{q}^1\ddot{k}^{11}-\dot{k}^1\ddot{q}^{11}\right)\right]\nabla_1 h_{22}\nabla_1 Q\\
&\quad+\frac{\left(n-1\right)^2}{\left(\dot{q}^1\right)^2}\left[\left(\dot{q}^2\right)^2\left(\dot{q}^1\ddot{k}^{11}-\dot{k}^1\ddot{q}^{11}\right)-2\dot{q}^2\left(\dot{q}^1\ddot{k}^{12}-\dot{k}^1\ddot{q}^{12}\right)+\left(\dot{q}^1\ddot{k}^{22}-\dot{k}^1\ddot{q}^{22}\right)\right]\left(\nabla_1 h_{22}\right)^2\\
&\quad+\frac{\dot{q}^2\ddot{k}^{22}-\dot{k}^2\ddot{q}^{22}}{\left(\dot{q}^2\right)^2}\left(\nabla_2 Q\right)^2+\frac{2}{\left(\dot{q}^2\right)^2}\left[\dot{q}^2\left(\dot{q}^2\ddot{k}^{12}-\dot{k}^2\ddot{q}^{12}\right)-\dot{q}^1\left(\dot{q}^2\ddot{k}^{22}-\dot{k}^2\ddot{q}^{22}\right)\right]\nabla_2 h_{11}\nabla_2 Q\\
&\quad+\frac{1}{\left(\dot{q}^2\right)^2}\left[\left(\dot{q}^1\right)^2\left(\dot{q}^2\ddot{k}^{22}-\dot{k}^2\ddot{q}^{22}\right)-2\dot{q}^1\left(\dot{q}^2\ddot{k}^{12}-\dot{k}^2\ddot{q}^{12}\right)+\left(\dot{q}^2\ddot{k}^{11}-\dot{k}^2\ddot{q}^{11}\right)\right]\left(\nabla_2 h_{11}\right)^2\\
&\quad+\frac{2\left(n-1\right)}{\kappa_1-\kappa_2}\left[\left(\dot{q}^2\dot{k}^1-\dot{q}^1\dot{k}^2\right)\left(\nabla_1 h_{12}\right)^2+\left(n-1\right)\left(\dot{q}^2\dot{k}^1-\dot{q}^1\dot{k}^2\right)\left(\nabla_2 h_{12}\right)^2\right].
\end{aligned}
$$

At a maximum of Q, all the $\nabla_i Q$ terms vanish. Using the Codazzi symmetry, Lemma 2.1, (i), It remains to check the coefficients of $\left(\nabla_1 h_{22}\right)^2$ and $\left(\nabla_2 h_{11}\right)$. Using

$$\dot{q}^i=2n\kappa_i-2H,\ddot{q}^{ij}=2n\delta_{ij}-2,\dot{k}^i=\frac{K}{\kappa_i} \text{ and } \ddot{k}^{ij}=\frac{K}{\kappa_i\kappa_j}-\frac{K}{\kappa_i\kappa_j}\delta_{ij},$$

we find that the coefficient of $\left(\nabla_1 h_{22}\right)^2$ is equal to $-2\left(n-1\right)\left(n^2+1\right)\kappa_2^{n-1}$ and the coefficient of $\left(\nabla_2 h_{11}\right)^2$ is equal to $-2\left(n-1\right)n^2\kappa_1\kappa_2^{n-2}$. Each of these are clearly negative. We may therefore conclude that the maximum of Q does not decrease under the flow, completing the proof. □

The proof of Theorem 1.2 in this case is now completed using similar arguments as in the previous sections. Observe that for the analogue of Corollary 5.2 we would get a factor of κ_2^{n-2} on the right hand side. However, Proposition 6.3 together with an upper speed bound that may be obtained as in [G, Section 6] but for the unrescaled flow, allow us to conclude that the curvatures are bounded above, while M_t encloses a small geodesic ball. Hence we obtain that the curvature pinching ratio is bounded. We obtain regularity of the unrescaled solution up to any particular time prior to the maximal time. The curvature pinching ratio bounds also hold for the rescaled solutions; similar arguments as in Section 5 complete the proof. □

Remark: A similar result to that of Section 6 holds in the case of ambient Euclidean space, generalising Andrews' proof of the Firey conjecture [A3] to the fate of rolling axially symmetric hyperstones. This setting has also been considered by Ben Andrews and Haizhong Li [AL].

7 Acknowledgments

This work was completed while the author was supported by DP150100375 of the Australian Research Council. The author would like to thank Prof Graham Williams for his interest in this work and Prof Ben Andrews, Dr Glen Wheeler and Dr Valentina Wheeler for useful discussions. The author would also like to thank the anonymous referee whose suggestions have led to improvements in the article.

jamesm@uow.edu.au
Institute for Mathematics and its Applications, University of Wollongong,
Northfields Av, Wollongong, NSW 2522, Australia

References

[A1] B Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171.

[A2] ———, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), 407–431.

[A3] ———, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), 151–161.

[A4] ———, *Positively Curved Surfaces in the Three-Sphere*, Proceedings of the International Congress of Mathematicians (Beijing, 2002), Vol. 2, Higher Ed. Press, Beijing, 2002, pp. 221–230.

[A5] ———, *Pinching estimates and motion of hypersurfaces by curvature functions*, J. Reine Angew. Math. **608** (2007), 17–33.

[A6] ———, *Fully nonlinear parabolic equations in two space variables*, available at `http://arxiv.org/abs/math/0402235`.

[A7] ———, *Moving surfaces by non-concave curvature functions*, Calc. Var. Partial Differential Equations **39** (2010), no. 3–4, 649–657.

[AL] B Andrews and H Li, *Axially symmetric hypersurfaces moving by powers of Gauss curvature* (2015), preprint.

[AC] B Andrews and X Chen, *Curvature flow in hyperbolic spaces*, J. Reine Angew. Math., posted on 2015, to appear, DOI 10.1515/crelle-2014-0121.

[AGN] B Andrews, P Guan, and L Ni, *Flow by powers of the Gauss curvature*, Adv. Math. **299** (2016), 174–201.

[AHLW] B Andrews, X Han, H Li, and Y Wei, *Non-collapsing for hypersurface flows in the sphere and hyperbolic space*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), no. 1, 331–338.

[ALM] B Andrews, M Langford, and J A McCoy, *Convexity estimates for surfaces moving by curvature functions*, J. Differential Geom. **99** (2015), 47–75.

[AMZ] B H Andrews, J A McCoy, and Y Zheng, *Contracting convex hypersurfaces by curvature*, Calc. Var. Partial Differential Equations **47** (2013), no. 3–4, 611–665.

[B] C Baker, *The mean curvature flow of submanifolds of high codimension*, PhD Thesis, The Australian National University (2010), `arXiv:1104.4409`.

[BI] P Bryan and M Ivaki, *Harnack estimate for mean curvature flow on the sphere* (2015), `arXiv:1508.02821`.

[BIS1] P Bryan, M Ivaki, and J Scheuer, *Harnack inequalities for evolving hypersurfaces on the sphere* (2015), to appear in Comm. Anal. Geom., `arXiv:1512.03374v1`.

[BIS2] ———, *On the classification of ancient solutions to curvature flows on the sphere* (2016), `arXiv:1604.01694v1`.

[CRS] E Cabezas-Rivas and C Sinestrari, *Volume-preserving flow by powers of the mth mean curvature*, Calc. Var. Partial Differential Equations **38** (2010), no. 3–4, 441–469.

[C] B Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature*, J. Differential Geom. **22** (1985), no. 1, 117–138.

[DF] E DiBenedetto and A Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. 357 **357** (1985), 1–22.

[G] C Gerhardt, *Curvature flows in the sphere*, J. Differential Geom. **100** (2015), no. 2, 301–347.

[GL] P Guan and J Li, *A Mean Curvature Type Flow in Space Forms*, Int. Math. Res. Not. IMRN **2015**, no. 13, 4716-4740.

[Ha] D Hartley, *Stability of geodesic spheres in $\mathbb{S}^{n+1}$ under constrained curvature flows*, J. Evol. Equ., posted on 2016, to appear, DOI 10.1007/s00028-016-0378-7.

[Hu1] G Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geometry **20** (1984), no. 1, 237–266.

[Hu2] ———, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), 463–480.

[Hu3] ———, *Deforming Hypersurfaces of the Sphere by Their Mean Curvature*, Math. Z. **195** (1987), 205–219.

[Hu4] ———, *The volume preserving mean curvature flow*, J. Reine Angew. Math. **382** (1987), 35–48.

[Hu5] ———, *Evolution of hypersurfaces by their curvature in Riemannian manifolds*, International Congress of Mathematicians (Berlin, 1998), Doc. Math., vol. Extra Vol. II, 1998, pp. 349–360.

[I] N Ishimura, *Self-similar solutions for the Gauss curvature evolution of rotationally symmetric surfaces*, Nonlinear Anal. **33** (1998), 97–104.

[J] T D Jeffres, *Gauss curvature flow on surfaces of revolution*, Adv. Geom. **9** (2009), no. 2, 189–197.

[KS] N V Krylov and M V Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk **40** (1980), 161–175; English transl., Math USSR Izv. **16** (1981), 151–164.

[LWW] H Li, X Wang, and Y Wei, *Surfaces expanding by non-concave curvature functions*, available at `arXiv:1609.00570v1`.

[LXZ] Y Li, H Xu, and E Zhao, *Contracting pinched hypersurfaces in the spheres by their mean curvature*, Pure Appl. Math. Q. **11** (2015), no. 2, 329–368.

[Li] G M Lieberman, *Second order parabolic partial differential equations*, World Scientific, Singapore, 1996.

[Lu] A Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, Basel, 1995.

[MS] M Makowski and J Scheuer, *Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere*, Asian J. Math **20** (2016), no. 5, 869–892.

[MMW] J A McCoy, F Y Y Mofarreh, and V-M Wheeler, *Fully nonlinear curvature flow of axially symmetric hypersurfaces*, Nonlinear Differ. Equ. Appl. **22** (2015), 325–343.

[N] H T Nguyen, *Convexity and cylindrical estimates for mean curvature flow in the sphere*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4517–4536.

[S] M Spivak, *A comprehensive introduction to differential geometry*, Vol. 3 and 4, Publish or Perish, Boston, 1979.

[T] K-S Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), 867–882.

[WX] Y Wei and C Xiong, *Inequalities of Alexandrov-Fenchel type for convex hypersurfaces in hyperbolic space and in the sphere*, Pacific J. Math. **277** (2015), no. 1, 219–239.