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Abstract

Given an n×n integer matrix A whose eigenvalues are strictly greater than 1 in absolute value, let σA be the transformation of the n-torus Tn = Rn/Zn defined by σA(e2πix) = e2πiAx for x ∊ Rn. We study the associated crossed-product C∗-algebra, which is defined using a certain transfer operator for σA, proving it to be simple and purely infinite and computing its K-theory groups.

Keywords

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Purely Infinite Simple C^*-Algebras Associated to Integer Dilation Matrices

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ABSTRACT. Given an $n \times n$ integer matrix A whose eigenvalues are strictly greater than 1 in absolute value, let σ_A be the transformation of the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ defined by $\sigma_A(e^{2\pi i x}) = e^{2\pi i A x}$ for $x \in \mathbb{R}^n$. We study the associated crossed-product C^*-algebra, which is defined using a certain transfer operator for σ_A, proving it to be simple and purely infinite and computing its K-theory groups.

1. INTRODUCTION

Exel has recently introduced a new kind of crossed product for an endomorphism α of a C^*-algebra B [6]. The crucial ingredient in his construction is a *transfer operator*, which is a positive linear map $L : B \to B$ satisfying $L(\alpha(a)b) = aL(b)$. In the motivating example, $B = C(X)$, X is a compact Hausdorff space, α is the endomorphism $\alpha : f \mapsto f \circ \sigma$ associated to a covering map $\sigma : X \to X$, and L is defined by

$$L(f)(x) = \frac{1}{|\sigma^{-1}(\{x\})|} \sum_{\sigma(y)=x} f(y) \tag{1.1}$$

(see [3,7,8], for example; transfer operators themselves have been used in dynamics for many years [26]). Exel's crossed product $B \rtimes_{\alpha,L} \mathbb{N}$ can be constructed in several ways, but here we view it as the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$ of a right-Hilbert B-bimodule M_L constructed from L, as discussed in [2] (see also Section **2.2** below).

We became interested in this circle of ideas when we noticed that the bimodule M_L associated to the covering map $\sigma : z \mapsto z^N$ of the unit circle $\mathbb{T}$ plays a key

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role in work of Packer and Rieffel on projective multi-resolution analyses [18]–[21]. The module elements $m \in M_L$ such that $\langle m, m\rangle$ is the identity of $C(X)$ are precisely the quadrature mirror filters arising in signal processing and wavelet theory, and orthonormal bases for M_L are what engineers call "filter banks with perfect reconstruction" (as observed and exploited in [16] and [10], for example.) We then noticed further, using results from [8], that the associated crossed product $C(\mathbb{T}) \rtimes_{\alpha_N, L} \mathbb{N}$, where α_N is the endomorphism of $C(\mathbb{T})$ given by σ, is simple, and accordingly computed its K-theory, finding that $K_0 = \mathbb{Z} \oplus (\mathbb{Z}/(N-1)\mathbb{Z})$ and $K_1 = \mathbb{Z}$. But then we saw this K-theory occurring elsewhere, and we gradually realised that the C^*-algebra $C(\mathbb{T}) \rtimes_{\alpha_N, L} \mathbb{N}$ had already been studied by many authors under other guises. (An almost certainly incomplete list includes [1, Theorem 2.12; 5, Example 3; 12, Example 4.1; 14, Appendix A; 29, Theorem 2.1]

Multiplication by N, however, is just one of many dilations of interest in wavelet theory (see, for example, [27]). Here we consider the covering maps σ_A of $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ induced by integer matrices A whose eigenvalues λ satisfy $|\lambda| > 1$, and the crossed products of the associated systems $(C(\mathbb{T}^d), \alpha_A, L)$, where α_A is the endomorphism of $C(\mathbb{T}^d)$ given by σ_A.

We will show, using results from [8] and [14], that the crossed products $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ are simple and purely infinite, and hence by the Kirchberg-Phillips theorem are classified by their K-theory. The computation of the K-theory groups of $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ therefore has a special significance, and one of the main goals of this paper is to carry out this computation.

Since $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ is a Cuntz-Pimsner algebra, one should in principle be able to compute its K-theory using the exact sequence of [23, Theorem 4.8], but in practice we were not able to compute some of the homomorphisms in that sequence. So we have argued directly from the six-term exact sequence associated to the Toeplitz algebra of the bimodule M_L, and we hope that our computation will be of independent interest.

Our computation is based on a six-term exact sequence which is valid for any system (B, α, L) for which the bimodule M_L is free as a right Hilbert B-module. Using an orthonormal basis for M_L, we build a homomorphism $\Omega : B \to M_N(B)$ which has the property that $\Omega \circ \alpha(a)$ is the diagonal matrix $a1_N$ with N copies of a down the diagonal, and which we view as a K-theoretic left inverse for α. When the bimodule is obtained from an integral matrix A, as above, this map is closely associated to the classical adjoint of A.

We then show that there is an exact sequence

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\mathrm{id} - \Omega_*} & K_0(B) & \xrightarrow{j_{B*}} & K_0(\mathcal{O}(M_L)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}(M_L)) & \xleftarrow{j_{B*}} & K_1(B) & \xleftarrow{\mathrm{id} - \Omega_*} & K_1(B) \end{array}$$

in which j_B is the canonical embedding of B in the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$. When $(B, \alpha, L) = (C(\mathbb{T}^d), \alpha_A, L)$, we know from [20] that $C(\mathbb{T}^d)_L$ is free, so this exact sequence applies; since we also know from [11] that $K_*(C(\mathbb{T}^d)) = K^*(\mathbb{T}^d)$ is isomorphic to the exterior ring generated by a copy of $\mathbb{Z}^d$ in $K^1(\mathbb{T}^d)$, we can in this case compute Ω_*, and derive explicit formulas for $K_i(\mathcal{O}(M_L))$.

2. Crossed Products by Endomorphisms

2.1. Cuntz-Pimsner algebras. A *right-Hilbert bimodule* over a C^*-algebra B, also known as a *correspondence*, is a right Hilbert B-module M with a left action of B implemented by a homomorphism φ of B into the C^*-algebra $\mathcal{L}(M)$ of adjointable operators on M. In this paper B is always unital, the bimodule M is always *essential* in the sense that $1 \cdot m = m$ for $m \in M$, and the bimodule has a *finite Parseval frame* or *quasi-basis*: a finite subset $\{m_j : 0 \le j \le N-1\}$ for which we have the *reconstruction formula*

$$m = \sum_{j=0}^{N-1} m_j \cdot \langle m_j, m\rangle \quad \text{for every } m \in M. \tag{2.1}$$

The reconstruction formula can be reformulated in terms of the rank-one operators $\Theta_{m,n} : x \mapsto m \cdot \langle n, x\rangle$ as

$$\varphi(a) = \sum_{j=0}^{N-1} \Theta_{a\cdot m_j, m_j} \quad \text{for every } a \in B \tag{2.2}$$

and hence implies that the homomorphism φ takes values in the algebra $\mathcal{K}(M)$ of compact operators on M.

The obvious examples of Parseval frames are orthonormal bases:

Lemma 2.1. *Suppose that $\{m_j : 0 \le j \le N-1\}$ are vectors in a right-Hilbert bimodule M over a unital C^*-algebra B. If the m_j generate M as a Hilbert B-module and satisfy $\langle m_j, m_k\rangle = \delta_{j,k} 1_B$, then $\{m_j : 0 \le j \le N-1\}$ is a finite Parseval frame for M, and $m \mapsto (\langle m_j, m\rangle)_j$ is an isomorphism of M onto B^N.*

Proof. A quick calculation gives the reconstruction formula for m of the form $m_k \cdot b$, and then linearity and continuity give it for arbitrary m. For the last assertion, check that $(b_0, \ldots, b_{N-1}) \mapsto \sum_j m_j \cdot b_j$ is an inverse. □

Remark 2.2. If $P \in \mathcal{L}(M)$ is a projection and $\{n_j\}$ is an orthonormal basis for M, then $\{Pn_j\}$ is a Parseval frame for $P(M)$, and Frank and Larson have shown that every Parseval frame $\{m_j\}$ has this form because $m \mapsto (\langle m_j, m\rangle)_j$ is an isomorphism of M onto a complemented submodule of B^N [9, Theorem 5.8]. However, many interesting bimodules have Parseval frames but are not obviously presented as direct summands of free modules. For example, for a bimodule of the form $C(X)_L$, one can construct a Parseval frame directly using a partition of unity (see, for example, [8, Proposition 8.2]).

A *Toeplitz representation* of a right-Hilbert bimodule M in a C^*-algebra C consists of a linear map $\psi : M \to C$ and a homomorphism $\pi : B \to C$ satisfying $\psi(m)^*\psi(n) = \pi(\langle m,n\rangle)$ and $\psi(\varphi(a)m) = \pi(a)\psi(m)$; we then also have $\psi(m \cdot a) = \psi(m)\pi(a)$. The *Toeplitz algebra* $\mathcal{T}(M)$ is generated by a universal Toeplitz representation (i_M, i_B) of M (either by theorem [23] or by definition [9]).

The following lemma is implicit in the proof of [2, Corollary 3.3].

Lemma 2.3. *Suppose M is an essential right-Hilbert bimodule over a unital C^*-algebra B and (ψ,π) is a Toeplitz representation of M on a Hilbert space $\mathcal{H}$. Then the subspace $\pi(1)\mathcal{H}$ is reducing for (ψ,π), and*

$$(\psi,\pi) = (\psi_{\pi(1)\mathcal{H}} \oplus 0, \pi_{\pi(1)\mathcal{H}} \oplus 0).$$

Proof. It is standard that $\pi = \pi_{\pi(1)\mathcal{H}} \oplus 0$, and each $\psi(m) = \psi(1 \cdot m) = \pi(1)\psi(m)$ has range in $\pi(1)\mathcal{H}$, so it suffices to show that $h \perp \pi(1)\mathcal{H}$ implies $\psi(m)h = 0$. Suppose $h \perp \pi(1)\mathcal{H}$. Then $\pi(\langle m,m\rangle)h \in \pi(1)\mathcal{H}$, so that

$$\begin{aligned}\|\psi(m)h\|^2 &= (\psi(m)h \mid \psi(m)h) = (\psi(m)^*\psi(m)h \mid h) \\ &= (\pi(\langle m \mid m\rangle)h \mid h) = 0.\end{aligned}$$

□

Remark 2.4. Lemma 2.3 implies that the Toeplitz algebra $\mathcal{T}(M)$ is universal for Toeplitz representations (ψ,π) in which π is unital, and we shall assume from now on that in all Toeplitz representations (ψ,π), π is unital.

For every Toeplitz representation (ψ,π) of M, there is a unique representation $(\psi,\pi)^{(1)}$ of the algebra $\mathcal{K}(M)$ of compact operators on M such that

$$(\psi,\pi)^{(1)}(\Theta_{m,n}) = \psi(m)\psi(n)^* \quad \text{for } m,n \in M$$

(see, for example, [9, Proposition 1.6]). When[1] $\varphi : B \to \mathcal{L}(M)$ has range in $\mathcal{K}(M)$, we say that (ψ,π) is *Cuntz-Pimsner covariant* if $\pi = (\psi,\pi)^{(1)} \circ \varphi$, and the *Cuntz-Pimsner algebra* $\mathcal{O}(M)$ is the quotient of $\mathcal{T}(M)$ which is universal for Cuntz-Pimsner covariant representations. The algebra $\mathcal{O}(M)$ is generated by a canonical Cuntz-Pimsner covariant representation (j_M, j_B).

Now we investigate what this all means when M has an orthonormal basis. Compare with [7, Section 8] and [8, Proposition 7.1], which use quasi-bases.

Lemma 2.5. *Suppose that M is an essential right-Hilbert bimodule over a unital C^*-algebra B, and that $\{m_j : 0 \le j \le N-1\}$ is an orthonormal basis for M. Let (ψ,π) be a Toeplitz representation of M. Then*

(1) *$\{\psi(m_j) : 0 \le j \le N-1\}$ is a Toeplitz-Cuntz family of isometries such that $\sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*$ commutes with every $\pi(a)$; and*
(2) *(ψ,π) is Cuntz-Pimsner covariant if and only if $\{\psi(m_j) : 0 \le j \le N-1\}$ is a Cuntz family.*

---

[1] As is always the case here; when the left action on the bimodule M contains non-compact operators, there are several competing definitions of $\mathcal{O}(M)$.

Proof. (1) The relations $\psi(m_j)^*\psi(m_j) = \pi(\langle m_j, m_j\rangle) = \pi(1)$ and our convention that $\pi(1) = 1$ (see Remark 2.4) imply that the $\psi(m_j)$ are isometries. Next, we fix $a \in B$, let $q := \sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*$, and compute the following using the reconstruction formula (2.1):

$$
\begin{aligned}
(2.3) \qquad q\pi(a)q &= \sum_{j,k=0}^{N-1} \psi(m_j)\psi(m_j)^*\pi(a)\psi(m_k)\psi(m_k)^* \\
&= \sum_{j,k=0}^{N-1} \psi(m_j)\pi(\langle m_j, a\cdot m_k\rangle)\psi(m_k)^* \\
&= \sum_{k=0}^{N-1}\Big(\sum_{j=0}^{N-1} \psi(m_j\cdot\langle m_j, a\cdot m_k\rangle)\Big)\psi(m_k)^* \\
&= \sum_{k=0}^{N-1} \psi(a\cdot m_k)\psi(m_k)^* = \pi(a)q.
\end{aligned}
$$

Taking $a = 1$ in (2.3) shows that $q^2 = q$, and since q is self-adjoint, it is a projection. Since each $\psi(m_j)$ is an isometry, each $\psi(m_j)\psi(m_j)^*$ is a projection, and since their sum is a projection, their ranges must be mutually orthogonal. Thus $\{\psi(m_j)\}$ is a Toeplitz-Cuntz family. Next we use (2.3) again to see that $q\pi(a) = (\pi(a^*)q)^* = (q\pi(a^*)q)^* = q\pi(a)q = \pi(a)q$, and we have proved (1).

(2) Suppose that (ψ, π) is Cuntz-Pimsner covariant. Plugging the formula (2.2) for $a = 1$ into $(\psi,\pi)^{(1)}(\varphi(1)) = \pi(1) = 1$ shows that $\sum_j \psi(m_j)\psi(m_j)^* = 1$, so $\{\psi(m_j)\}$ is a Cuntz family. On the other hand, if $\{\psi(m_j)\}$ is a Cuntz family, then we can deduce from (2.2) that

$$
(\psi,\pi)^{(1)}(\varphi(a)) = \sum_{j=0}^{N-1} \psi(m_j)\psi(a^*\cdot m_j)^* = \sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*\pi(a) = \pi(a),
$$

and (ψ, π) is Cuntz-Pimsner covariant. □

2.2. Exel systems and crossed products. Let α be an endomorphism of a unital C^*-algebra B. A *transfer operator* L for (B, α) is a positive linear map $L : B \to B$ such that $L(\alpha(a)b) = aL(b)$ for all a, $b \in B$. We call the triple (B, α, L) an *Exel system.*

Given an Exel system (B, α, L), we construct a right-Hilbert B-module M_L over B as in [6] and [2]. Let B_L be a copy of the underlying vector space of B. Define a right action of $a \in B$ on $m \in B_L$ by $m \cdot a = m\alpha(a)$, and a B-valued pairing on B_L by

$$\langle m, n\rangle = L(m^*n) \quad \text{for } m, n \in B_L.$$

Modding out by $\{m : \langle m, m\rangle = 0\}$ and completing yields a right Hilbert B-module M_L. The action of B by left multiplication on B_L extends to an action of B by adjointable operators on M_L which is implemented by a unital homomorphism $\varphi : B \to \mathcal{L}(M_L)$, and thus makes M_L into a right-Hilbert bimodule over B.

Exel's crossed product is constructed in two stages. First he forms a Toeplitz algebra $\mathcal{T}(B, \alpha, L)$, which is isomorphic to $\mathcal{T}(M_L)$ (see [2, Corollary 3.2]). Then the crossed product $B \rtimes_{\alpha,L} \mathbb{N}$ is the quotient of $\mathcal{T}(M_L)$ by the ideal generated by the elements

$$i_B(a) - (i_{M_L}, i_B)^{(1)}(\varphi(a)) \quad \text{for } a \in K_\alpha := \varphi^{-1}(\mathcal{K}(M_L)) \cap \overline{B\alpha(B)B}$$

(see [2, Lemma 3.7]). When M_L has a finite Parseval frame and the projection $\alpha(1)$ is full, we have $\varphi^{-1}(\mathcal{K}(M_L)) = B = K_\alpha$, and $B \rtimes_{\alpha,L} \mathbb{N}$ is the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$.

For us, the main examples of Exel systems come from surjective endomorphisms σ of a compact group K with finite kernel: the corresponding Exel system $(C(K), \alpha, L)$ has $\alpha(f) = f \circ \sigma$ and L defined by averaging over the fibres of σ, as in (1.1). The next lemma is a mild generalisation of [20, Proposition 1].

Lemma 2.6. *Suppose that $\sigma : K \to K$ is a surjective endomorphism of a compact abelian group K with $N := |\ker \sigma| < \infty$, and $(C(K), \alpha, L)$ is the corresponding Exel system. Then the norm on $C(K)_L$ defined by the inner product is equivalent to the usual sup-norm, and $C(K)_L$ is complete. It has an orthonormal basis $\{m_j : 0 \le j \le N-1\}$.*

Proof. The assertions about the norm and the completeness are proved in [16, Lemma 3.3], for example. Since $\gamma \mapsto \gamma|_{\ker\sigma}$ is surjective and $|(\ker\sigma)^\wedge| = |\ker\sigma| = N$, we can find a subset $\{\gamma_i : 0 \le i \le N-1\}$ of $\hat{K}$ such that $\{\gamma_i|_{\ker\sigma}\}$ is all of $(\ker\sigma)^\wedge$. Then

$$\begin{aligned}
\langle \gamma_i, \gamma_j\rangle_L(k) &= \frac{1}{N} \sum_{\sigma(\ell)=k} \overline{\gamma_i(\ell)}\gamma_j(\ell) \\
&= \frac{1}{N} \sum_{\zeta\in\ker\sigma} \overline{\gamma_i(\zeta\ell_0)}\gamma_j(\zeta\ell_0) \quad \text{for any fixed } \ell_0 \text{ such that } \sigma(\ell_0) = k \\
&= \frac{1}{N}\overline{\gamma_i(\ell_0)}\gamma_j(\ell_0) \sum_{\zeta\in\ker\sigma} (\overline{\gamma_i}\gamma_j)(\zeta).
\end{aligned}$$

If $i \ne j$, then $(\gamma_i^{-1}\gamma_j)|_{\ker\sigma}$ is a nontrivial character of $\ker\sigma$, and its range is a nontrivial subgroup of $\mathbb{T}$, so the sum vanishes. If $i = j$, then the sum is N. So $\{\gamma_j\}$ is orthonormal.

We still need to see that $\{\gamma_j\}$ generates $C(K)_L$ as a Hilbert module. The Stone-Weierstrass theorem implies that the characters of K span a dense $*$-subalgebra of $C(K)$, and hence by the equivalence of the norms, they also span a dense subspace of $C(K)_L$. So it suffices to show that each $\gamma \in \hat{K}$ is in the submodule generated by $\{\gamma_j\}$. Since $(\ker\sigma)^\wedge = \{\gamma_j\}$, there exists j such that $\gamma|_{\ker\sigma} = \gamma_j$. Then $\gamma_j^{-1}\gamma$ vanishes on $\ker\sigma$, and there is a character χ such that $\gamma_j^{-1}\gamma = \chi\circ\sigma$. This equation unravels as $\gamma = \gamma_j(\chi\circ\sigma) = \gamma_j\alpha(\chi) = \gamma_j\cdot\chi$, so it implies that γ belongs to the submodule generated by $\{\gamma_j\}$. □

Example 2.7. Suppose that $A \in M_d(\mathbb{Z})$ is an integer matrix with $|\det A| > 1$, and σ_A is the endomorphism of $\mathbb{T}^d$ given by $\sigma_A(e^{2\pi ix}) = e^{2\pi iAx}$ for $x \in \mathbb{R}^d$. Then σ_A is surjective (because $A : \mathbb{R}^d \to \mathbb{R}^d$ is), and $\ker\sigma_A$ has $N := |\det A|$ elements. A function $m \in C(\mathbb{T}^d)_L$ such that $\langle m, m\rangle(z) = 1$ for all z is called a *quadrature mirror filter* for dilation by A, and an orthonormal basis for $C(\mathbb{T}^d)_L$ is a *filter bank*. Lemma 2.6 says that for every A, filter banks exist.

Remark 2.8. Although the dilation matrices A are greatly relevant to wavelets, the filter banks we constructed in the proof of Lemma 2.6 are not the kind which are useful for the construction of wavelets. There one wants the first filter m_0 to be low-pass, which means roughly that $m_0(1) = N^{1/2}$, m_0 is smooth near 1, and m_0 does not vanish on a sufficiently large neighbourhood of 1; for the basis in Lemma 2.6, we have $|m_0(z)| = 1$ for all z, and m_0 is *all-pass*. The *matrix completion problem* considered in [20] asks whether, given a low-pass filter m_0, one can find a filter bank $\{m_j\}$ which includes the given m_0. This amounts to asking that the submodule $m_0^\perp := \{m \in C(\mathbb{T}^d)_L : \langle m, m_0\rangle = 0\}$ is free. In [20, Section 4], Packer and Rieffel show by example that it need not be free if $|\det A| > 2$ and $d > 4$. Of course, since $m_0^\perp$ is a direct summand of a free module, it always has a Parseval frame.

When α is the endomorphism of $C(K)$ coming from a surjective endomorphism σ of K, we know from Lemma 2.6 that $M_L = C(K)_L$ admits an orthonormal basis, and the associated endomorphism $\alpha : f \mapsto f\circ\sigma$ is unital, so $\alpha(1) = 1$ is certainly full. Thus for the systems of interest to us, Exel's crossed product $B \rtimes_{\alpha,L} \mathbb{N}$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$. We will use this identification without comment.

3. The Six-Term Exact Sequence

We assume throughout this section that (B, α, L) is an Exel system and that $\{m_j : 0 \le j \le N-1\}$ is a Parseval frame for M_L. We write Q for the quotient map from $\mathcal{T}(M_L) \to \mathcal{O}(M_L)$, and (ψ, π) for the universal Toeplitz covariant representation of M_L in $\mathcal{T}(M_L)$.

To construct our exact sequence for $K_*(\mathcal{O}(M))$, we analyse the six-term exact sequence

$$(3.1)\qquad \begin{array}{ccccc} K_0(\ker Q) & \xrightarrow{\iota_*} & K_0(\mathcal{T}(M_L)) & \xrightarrow{Q_*} & K_0(\mathcal{O}(M_L)) \\ \uparrow{\scriptstyle\delta_1} & & & & \downarrow{\scriptstyle\delta_0} \\ K_1(\mathcal{O}(M_L)) & \xleftarrow{Q_*} & K_1(\mathcal{T}(M_L)) & \xleftarrow{\iota_*} & K_1(\ker Q). \end{array}$$

We begin by recalling from [23, Theorem 4.4] that the homomorphism $\pi : B \to \mathcal{T}(M_L)$ induces an isomorphism of $K_i(B)$ onto $K_i(\mathcal{T}(M_L))$, so we can replace $K_i(\mathcal{T}(M_L))$ with $K_i(B)$ provided we can identify the maps. Next we introduce our "K-theoretic left inverse" for α, and then we will work towards showing that B is a full corner in $\ker Q$, so that we can replace $K_i(\ker Q)$ with $K_i(B)$.

Part (3) of the next result will not be used in this section; it is included here because it shows how Ω relates to α, and gives a hint of why we view it as a "K-theoretic left inverse" for α. (Rieffel used a similar construction in the proof of [24, Proposition 2.1], but the technicalities seem different.)

Lemma 3.1. *Define $\Omega : B \to M_N(B)$ by $\Omega(a) = (\langle m_j, a \cdot m_k \rangle)_{j,k}$. Then*

(1) *Ω is a homomorphism of C^*-algebras;*
(2) *Ω is unital if and only if $\{m_j : 0 \le j \le N-1\}$ is an orthonormal basis;*
(3) *if B is commutative and $\{m_j : 0 \le j \le N-1\}$ is orthonormal, then $\Omega(\alpha(a))$ is the diagonal matrix $a1_N$ with diagonal entries a.*

Proof. For (1), we let a, $b \in B$ and compute first

$$\begin{aligned}(\Omega(a)\Omega(b))_{j,k} &= \sum_{\ell=0}^{N-1} \langle m_j, a \cdot m_\ell \rangle \langle m_\ell, b \cdot m_k \rangle \\ &= \Big\langle m_j, a \cdot \Big(\sum_{\ell=0}^{N-1} m_\ell \cdot \langle m_\ell, b \cdot m_k \rangle \Big) \Big\rangle \\ &= \langle m_j, a \cdot (b \cdot m_k) \rangle = \Omega(ab)_{j,k},\end{aligned}$$

and then

$$\Omega(a^*) = (\langle m_j, a^* \cdot m_k \rangle)_{j,k} = (\langle a \cdot m_j, m_k \rangle)_{j,k} = (\langle m_k, a \cdot m_j \rangle^*)_{j,k} = \Omega(a)^*.$$

Part (2) is easy. For (3), we let $q_L : B_L \to M_L$ be the quotient map, and consider $m = q(b) \in q(B_L)$. Then commutativity of B gives

$$m \cdot a = q(b \cdot a) = q(b\alpha(a)) = q(\alpha(a)b) = \alpha(a) \cdot q(b) = \alpha(a) \cdot m,$$

and this formula extends to $m \in M_L$ by continuity. Thus

$$\Omega(\alpha(a))_{j,k} = \langle m_j, \alpha(a) \cdot m_k \rangle = \langle m_j, m_k \cdot a \rangle = \langle m_j, m_k \rangle a = \delta_{j,k} a,$$

as required. □

To describe $\ker Q$, we need some standard notation. We write $M_L^{\otimes i}$ for the i-fold internal tensor product $M_L \otimes_B \cdots \otimes_B M_L$, which is itself a right-Hilbert bimodule over B. There is a Toeplitz representation $(\psi^{\otimes i}, \pi)$ of $M_L^{\otimes k}$ in $\mathcal{T}(M_L)$ such that $\psi^{\otimes k}(\xi) = \prod_{i=1}^{i} \psi(\xi_i)$ for elementary tensors $\xi = \xi_1 \otimes \cdots \otimes \xi_k$ in

$M_L^{\otimes k}$ (see [9, Proposition 1.8], for example). By convention, we set $M_L^{\otimes 0} := B$ and $\psi^{\otimes 0} := \pi$. Then from [9, Lemma 2.4] we have

$$(3.2)\qquad \mathcal{T}(M_L) = \overline{\operatorname{span}}\left\{\psi^{\otimes k}(\xi)\psi^{\otimes \ell}(\eta)^* : k, \ell \geq 0,\ \xi \in M_L^{\otimes k},\ \eta \in M_L^{\otimes \ell}\right\}.$$

We also recall from Lemma 2.5 (1) that the element $q := \sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*$ of $\mathcal{T}(M_L)$ is a projection which commutes with every $\pi(a)$.

Lemma 3.2. *With the preceding notation, we have*

(1) $1 - q = 1 - \sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*$ *is a full projection in* $\ker Q$;

(2) $(1-q)\psi^{\otimes k}(\xi) = 0$ *for all* $\xi \in M_L^{\otimes k}$ *with* $k \geq 1$; *and*

(3) $\ker Q = \overline{\operatorname{span}}\{\psi^{\otimes k}(\xi)(1-q)\psi^{\otimes \ell}(\eta)^* : k, \ell \geq 0,\ \xi \in M_L^{\otimes k},\ \eta \in M_L^{\otimes \ell}\}$.

Proof. (1) The reconstruction formula implies that $\varphi(a) = \sum_{j=0}^{N-1} \Theta_{a\cdot m_j, m_j}$, and so

$$(3.3)\qquad (\psi, \pi)^{(1)}(\varphi(a)) = \sum_{j=1}^{N-1} \psi(a \cdot m_j)\psi(m_j)^* = \pi(a)q.$$

This implies in particular that

$$Q(1-q) = Q(\pi(1) - \pi(1)q) = Q(\pi(1) - (\psi, \pi)^{(1)}(\varphi(1))) = 0,$$

so $1-q$ belongs to $\ker Q$. Since $\ker Q$ is by definition the ideal in $\mathcal{T}(M_L)$ generated by the elements $\pi(a) - (\psi, \pi)^{(1)}(\varphi(a))$ for $a \in B$, (3.3) also implies that $\ker Q$ is generated by the elements $\pi(a)(1-q)$, and hence by the single element $1-q$. This says precisely that the projection $1-q$ is full.

(2) First we consider $m \in M_L^{\otimes 1} = M_L$. The reconstruction formula gives

$$q\psi(m) = \sum_{j=0}^{N-1} \psi(m_j)\psi(m_j)^*\psi(m) = \sum_{j=0}^{N-1} \psi(m_j \cdot \langle m_j, m\rangle) = \psi(m),$$

so $(1-q)\psi(m) = 0$. Now for $k > 1$ and an elementary tensor $\xi = \xi_1 \otimes \cdots \otimes \xi_k$, we have $(1-q)\psi^{\otimes k}(\xi) = (1-q)(\prod_{i=1}^{i} \psi(\xi_i)) = 0$, and the result extends to arbitrary $\xi \in M^{\otimes k}$ by linearity and continuity.

(3) In consideration of part (2), we are able to deduce from (3.2) that $\ker Q = \mathcal{T}(M_L)(1-q)\mathcal{T}(M_L)$ is spanned by the elements of the form

$$\psi^{\otimes k}(\xi)\pi(a)^*(1-q)\pi(b)\psi^{\otimes \ell}(\eta)^* = \psi^{\otimes k}(\xi \cdot a)(1-q)\psi^{\otimes l}(\eta \cdot b^*)^*$$

for $\xi \in M^{\otimes k}$, $\eta \in M^{\otimes \ell}$ and $a, b \in B$, which gives (3). $\square$

Lemma 3.3. *There is a homomorphism* $\rho : B \to \ker Q$ *such that* $\rho(a) = \pi(a)(1-q)$*, and* ρ *is an isomomorphism of* B *onto* $(1-q)\ker Q(1-q)$.

Proof. Lemma 3.2 says that $\pi(a)(1-q)$ belongs to $\ker Q$, and Lemma 3.1 says that q commutes with every $\pi(a)$, so there is a homomorphism $\rho : B \to (1-q)\ker Q(1-q) \subset \ker Q$ such that $\rho(a) = \pi(a)(1-q)$. From parts (2) and (3) of Lemma 3.2 we get

$$\begin{aligned}(1-q)\ker Q(1-q) &= \overline{\operatorname{span}}\{(1-q)\psi^{\otimes k}(\xi)(1-q)\psi^{\otimes l}(\eta)^*(1-q) : k,\ell \geq 0\}\\ &= \overline{\operatorname{span}}\{(1-q)\pi(a)(1-q)\pi(b)(1-q) : a,b \in B\}\\ &= \overline{\operatorname{span}}\{(1-q)\pi(ab) : a,b \in B\},\end{aligned}$$

which is precisely the range of ρ. So ρ is surjective.

To see that ρ is injective we choose a faithful representation $\pi_0 : B \to B(\mathcal{H})$ and consider the Fock representation (ψ_F, π_F) of M_L induced from π_0, as described in [9, Example 1.4]. The underlying space of this Fock representation is $F(M_L) \otimes_B \mathcal{H} := \bigoplus_{k\geq 0}(M_L^{\otimes k} \otimes_B \mathcal{H})$; B acts diagonally on the left, and M_L acts by creation operators. The crucial point for us is that each $\psi_F(m)^*$ is an annihilation operator which vanishes on the subspace $B \otimes_B \mathcal{H} = M_L^{\otimes 0} \otimes_B \mathcal{H}$ of $F(M_L) \otimes_B \mathcal{H}$.

Now suppose that $a \in B$. Then

$$\begin{aligned}0 = \psi_F \times \pi_F(\rho(a)) &= \psi_F \times \pi_F(\pi(a)(1-q))\\ &= \pi_F(a)\Big(1 - \sum_{j=0}^{N-1} \psi_F(m_j)\psi_F(m_j)^*\Big).\end{aligned}$$

Since $\psi(m_j)^*$ vanishes on $B \otimes_B \mathcal{H}$, we have

$$\begin{aligned}\rho(a) = 0 &\implies \pi_F(a)\Big(1 - \sum_{j=0}^{N-1} \psi_F(m_j)\psi_F(m_j)^*\Big)(1 \otimes_B h) = 0 \quad \text{for all } h \in \mathcal{H}\\ &\implies \pi_F(a)(1 \otimes_B h) = 0 \quad \text{for all } h \in \mathcal{H}\\ &\implies a \otimes_B h = 0 \quad \text{for all } h \in \mathcal{H}\\ &\implies \pi_0(a)h = 0 \quad \text{for all } h \in \mathcal{H},\end{aligned}$$

which implies that $a = 0$ because π_0 is faithful. □

Lemma 3.3 implies that we can replace $K_i(\ker Q)$ in (3.1) by $K_i(B)$, as claimed. Now we need to check what this replacement does to the map ι_*.

Proposition 3.4. *The following diagram commutes for $i = 0$:*

$$\begin{array}{ccc} K_i(B) & \xrightarrow{\ \mathrm{id} - \Omega_*\ } & K_i(B) \\ \rho_* \downarrow & & \downarrow \pi_* \\ K_i(\ker Q) & \xrightarrow[\ \iota_*\ ]{} & K_i(\mathcal{T}(M_L)). \end{array} \tag{3.4}$$

If $\{m_j : 0 \le j \le N-1\}$ is orthonormal, then the diagram also commutes for $i = 1$.

Since $\Omega : B \to M_N(B)$, the Ω_* in the diagram is really the composition of $\Omega_* : K_i(B) \to K_i(M_N(B))$ with the isomorphism $K_i(M_N(B)) \to K_i(B)$; the latter is induced by the map which views an element in $M_r(M_N(B))$ as an element of $M_{rN}(B)$.

The proof needs two standard lemmas. The first says, loosely, that if we rewrite an $r \times r$ matrix of $N \times N$ blocks as an $N \times N$ matrix of $r \times r$ blocks, then the resulting $rN \times rN$ matrices are unitarily equivalent. We agree that this cannot be a surprise to anyone, and we apologise for failing to come up with more elegant notation.

Lemma 3.5. *Suppose that B is a C^*-algebra, $r \ge 1$ and $N \ge 2$ are integers, and*

$$\{b_{j,s;k,t} : 0 \le j,k \le N-1 \text{ and } 0 \le s,t < r\}$$

is a subset of B. For m, n satisfying $0 \le m,n \le rN-1$, we define

$$c_{m,n} = b_{j,s;k,t} \quad \text{where } m = sN + j \text{ and } n = tN + k,$$

and

$$d_{m,n} = b_{j,s;k,t} \quad \text{where } m = jr + s \text{ and } n = kr + t.$$

Then there is a scalar unitary permutation matrix U such that the matrices $C := (c_{m,n})$ and $D := (d_{m,n})$ are related by $C = UDU^*$.

Proof. For $0 \le p,q \le rN-1$, we define

$$u_{p,q} = \begin{cases} 1 & \text{if there exist } k,\, t \text{ such that } p = tN + k \text{ and } q = kr + t, \\ 0 & \text{otherwise.} \end{cases}$$

Each row and column contains exactly one 1, so $U := (u_{p,q})$ is a scalar permutation matrix, and we can verify that both $(CU)_{m,q}$ and $(UD)_{m,q}$ are equal to $b_{j,s;k,t}$ where $m = sN + j$ and $q = kr + t$, so $CU = UD$. □

Lemma 3.6. *Suppose that S is an isometry in a unital C^*-algebra B. Then*

$$U := \begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix}$$

is a unitary element of $M_2(B)$ and its class in $K_1(B)$ is the identity.

Proof. A straightforward calculation shows that U is unitary.

Let $\mathcal{T} = C^*(v)$ be the Toeplitz algebra. By Coburn's Theorem [4] there is a homomorphism $\pi_S : \mathcal{T} \to B$ such that $\pi_S(v) = S$. Since $K_1(\mathcal{T}) = 0$ (see, for example, [28, Remark 11.2.2]),

$$\left[\begin{pmatrix} v & 1 - vv^* \\ 0 & v^* \end{pmatrix}\right] = [1] \quad \text{in } K_1(\mathcal{T}),$$

and hence

$$\begin{aligned}\left[\begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix}\right] &= (\pi_S)_*\left(\left[\begin{pmatrix} v & 1 - vv^* \\ 0 & v^* \end{pmatrix}\right]\right) \\ &= (\pi_S)_*([1]) = [1] \quad \text{in } K_1(B).\end{aligned}$$ □

Proof of Proposition 3.4. We start with $i = 0$. Let $a = (a_{s,t})$ be a projection in $M_r(B)$. For $\pi : A \to B$, we write π_r for the induced homomorphism of $M_r(A)$ into $M_r(B)$. Then we have

$$\begin{aligned}\rho_*([a]) = [(\rho(a_{r,s})] &= [(\pi(a_{s,t})(1-q))] = [(\pi(a_{s,t}))(1-q)1_r)] \\ &= [\pi_r(a)((1-q)1_r)] = [\pi_r(a)] - [(\pi_r(a)(q1_r))],\end{aligned}$$

and

$$\pi_* \circ (\mathrm{id} - \Omega_*)([a]) = [\pi_r(a)] - \pi_* \circ \Omega_*([a]),$$

so it suffices to show that $[\pi_r(a)(q1_r)] = \pi_* \circ \Omega_*([a])$ in $K_0(\mathcal{T}(M_L))$. The class $\pi_* \circ \Omega_*([a])$ appears as the class of the $r \times r$ block matrix $\pi_{rN}(\Omega_r(a))$ whose (s,t) entry is the $N \times N$ block $(\pi(\langle m_j, a_{s,t} \cdot m_k \rangle))_{j,k}$. In other words, with $b_{j,s;k,t} = \pi(\langle m_j, a_{s,t} \cdot m_k \rangle)$, the matrix $\pi_{rN}(\Omega_r(a))$ is the matrix $C = (c_{m,n})$ in Lemma 3.5.

We now consider the matrix T in $M_N(M_r(\mathcal{T}(M_L)))$ defined by

$$T = \begin{pmatrix} \psi(m_0)1_r & \cdots & \psi(m_{N-1})1_r \\ 0_r & \cdots & 0_r \\ \vdots & \cdots & \vdots \end{pmatrix}. \tag{3.5}$$

Computations show that $TT^* = (q1_r) \oplus 0_{r(N-1)}$, and since $\pi_r(a)$ is a projection which commutes with $q1_r$, we deduce that $(\pi_r(a) \oplus 0_{r(N-1)})T$ is a partial isometry which implements a Murray-von Neumann equivalence between $T^*(\pi_r(a) \oplus 0_{r(N-1)})T$ and $(\pi_r(a) \oplus 0_{r(N-1)})TT^* = (\pi_r(a)(q1_r)) \oplus 0_{r(N-1)}$. Thus we have

$$\begin{aligned}[\pi_r(a)(q1_r)] &= [\pi_r(a)(q1_r) \oplus 0_{r(N-1)}] \\ &= [T^*(\pi_r(a) \oplus 0_{r(N-1)})T].\end{aligned}$$

Another computation shows that the (j,k) entry of $T^*(\pi_r(a)\oplus 0_{r(N-1)})T$ is the $r\times r$ matrix $(\pi_r(\langle m_j, a_{s,t}\cdot m_k\rangle 1_r))_{s,t}$. Thus with the same choice of $b_{j,s;k,t}=\pi(\langle m_j, a_{s,t}\cdot m_k\rangle)$, $T^*(\pi_r(a)\oplus 0_{r(N-1)})T$ is the matrix $D=(d_{m,n})$ in Lemma 3.5. Since unitarily equivalent projections have the same class in K_0, we can therefore deduce from Lemma 3.5 that

$$\begin{aligned}(3.6)\qquad [\pi_r(a)(q1_r)] &= [T^*(\pi_r(a)\oplus 0_{r(N-1)})T]\\ &= [\pi_{rN}(\Omega_r(a))] = \pi_*\circ\Omega_*([a]).\end{aligned}$$

Thus Diagram 3.4 commutes when $i=0$.

Now consider $i=1$, where we assume in addition that $\{m_j\}$ is orthonormal. Let u be a unitary in $M_r(B)$. To compute $\rho_*: K_1(B)\to K_1(\ker Q)$, we observe that ρ is the composition of a unital isomorphism of B onto $(1-q)\ker Q(1-q)$, which takes $[u]$ to

$$[\rho_r(u)] = [\pi_r(u)((1-q)1_r)],$$

with the inclusion of $(1-q)\ker Q(1-q)$ as a full corner in the non-unital algebra $\ker Q$, which takes $[\pi_r(u)((1-q)1_r)]$ to $[\pi_r(u)((1-q)1_r)+q1_r]\in K_1((\ker Q)^+)=K_1(\ker Q)$. On the other hand,

$$\pi_*\circ(\mathrm{id}-\Omega_*)([u]) = [\pi_r(u)] - [\pi_{rN}\circ\Omega_r(u)].$$

So we need to show that

$$\begin{aligned}(3.7)\qquad &[(\pi_r(u)((1-q)1_r)+q1_r)\oplus 1_{r(N-1)}]\\ &\quad= [\pi_r(u)\oplus 1_{r(N-1)}] - [\pi_{rN}\circ\Omega_r(u)]\end{aligned}$$

in $K_1(\mathcal{T}(M_L))$. To this end, we note that the left-hand side of (3.7) is unchanged by pre- or post-multiplying by any invertible matrix $C\in M_{2rN}(\mathcal{T}(M_L))$ whose K_1 class is 1. In particular, we can do this when C is

- a unitary of the form

$$C=\begin{pmatrix} S & 1-SS^* \\ 0 & S^*\end{pmatrix},$$

  where $S\in M_{rN}(\mathcal{T}(M_L))$ is an isometry (see Lemma 3.6);
- an upper- or lower-triangular matrix of the form

$$C=\begin{pmatrix}1 & A\\ 0 & 1\end{pmatrix}\quad\text{or}\quad C=\begin{pmatrix}1 & 0\\ A & 1\end{pmatrix}$$

  (which are connected to 1_{2rN} via $t\mapsto\left(\begin{smallmatrix}1 & tA\\ 0 & 1\end{smallmatrix}\right)$ and its transpose);
- any constant invertible matrix C in $M_{2rN}(\mathbb{C})$ (because $GL_{2rN}(\mathbb{C})$ is connected); this implies that we can perform row and column operations without changing the class in K_1.

Since $\{m_j\}$ is an orthonormal basis, the matrix T defined at (3.5) is an isometry in $M_{rN}(\mathcal{T}(M_L))$. Thus

$$\begin{aligned}
&\big[(\pi_r(u)((1-q)1_r)+q1_r)\oplus 1_{r(N-1)}\big]\\
&\quad=\left[\begin{pmatrix}(\pi_r(u)((1-q)1_r)+q1_r)\oplus 1_{r(N-1)} & 0_{rN}\\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix}T & 1_{rN}-TT^*\\ 0_{rN} & T^*\end{pmatrix}\right]\\
&\quad=\left[\begin{pmatrix}(\pi_r(u)((1-q)1_r)+q1_r)\oplus 1_{r(N-1)} & 0_{rN}\\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix}T & (1-q)1_r\oplus 1_{r(N-1)}\\ 0_{rN} & T^*\end{pmatrix}\right]\\
&\quad=\left[\begin{pmatrix}((\pi_r(u)((1-q)1_r)+q1_r)\oplus 1_{r(N-1)})T & \pi_r(u)((1-q)1_r)\oplus 1_{r(N-1)}\\ 0_{rN} & T^*\end{pmatrix}\right],
\end{aligned}$$

which, since $(1-q)\psi(m_i)=0$ by Lemma 3.2 (2), is

$$\begin{aligned}
&=\left[\begin{pmatrix}T & \pi_r(u)((1-q)1_r)\oplus 1_{r(N-1)}\\ 0_{rN} & T^*\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}T & \pi_r(u)((1-q)1_r)\oplus 1_{r(N-1)}\\ 0_{rN} & T^*\end{pmatrix}\right]\left[\begin{pmatrix}1_{rN} & T^*(\pi_r(u)\oplus 1_{r(N-1)})\\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}T & \pi_r(u)\oplus 1_{r(N-1)}\\ 0_{rN} & T^*\end{pmatrix}\right]
\end{aligned}$$

since $TT^*=q1_r\oplus 0_{r(N-1)}$ and $(q1_r)\pi_r(u)=\pi_r(u)(q1_r)$. By an elementary row operation this is

$$\begin{aligned}
&=\left[\begin{pmatrix}\pi_r(u)\oplus 1_{r(N-1)} & T\\ T^* & 0_{rN}\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}\pi_r(u)\oplus 1_{r(N-1)} & T\\ T^* & 0_{rN}\end{pmatrix}\right]\left[\begin{pmatrix}1_{rN} & -(\pi_r(u^{-1})\oplus 1_{r(N-1)})T\\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}\pi_r(u)\oplus 1_{r(N-1)} & 0_{rN}\\ T^* & -T^*(\pi_r(u^{-1})\oplus 1_{r(N-1)})T\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}1_{rN} & 0_{rN}\\ -T^*(\pi_r(u^{-1})\oplus 1_{r(N-1)}) & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix}\pi_r(u)\oplus 1 & 0_{rN}\\ T^* & -T^*(\pi_r(u^{-1})\oplus 1)T\end{pmatrix}\right]\\
&=\left[\begin{pmatrix}\pi_r(u)\oplus 1_{r(N-1)} & 0_{rN}\\ 0_{rN} & -T^*(\pi_r(u^{-1})\oplus 1_{r(N-1)})T\end{pmatrix}\right]\left[\begin{pmatrix}1_{rN} & 0_{rN}\\ 0_{rN} & -1_{rN}\end{pmatrix}\right]\\
&=\big[\pi_r(u)\oplus 1_{r(N-1)}\big]+\big[T^*(\pi_r(u^{-1})\oplus 1_{r(N-1)})T\big].
\end{aligned}$$

Now we recall from the argument in the second paragraph (see (3.6)) that

$$[T^*(\pi_r(u^{-1})\oplus 1_{r(N-1)})T]=[\pi_{rN}(\Omega_r(u^{-1}))]=-[\pi_{rN}\circ\Omega_r(u)],$$

and we see that we have proved what we wanted. □

Theorem 3.7. *Let (B,α,L) be an Exel system with B unital and separable, and suppose that M_L has an orthonormal basis $\{m_j\}_{j=0}^{N-1}$. Let (j_{M_L},j_B) be the canonical*

Cuntz-Pimsner covariant representation of M_L in $\mathcal{O}(M_L)$. Then there is an exact sequence

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\mathrm{id}-\Omega_*} & K_0(B) & \xrightarrow{j_{B*}} & K_0(\mathcal{O}(M_L)) \\ \uparrow{\scriptstyle \rho_*^{-1}\circ\delta_1} & & & & \downarrow{\scriptstyle \rho_*^{-1}\circ\delta_0} \\ K_1(\mathcal{O}(M_L)) & \xleftarrow{j_{B*}} & K_1(B) & \xleftarrow{\mathrm{id}-\Omega_*} & K_1(B). \end{array} \tag{3.8}$$

Proof. The canonical representation (j_{M_L}, j_B) is the composition of the universal Toeplitz representation (ψ, π) of M_L in $\mathcal{T}(M_L)$ with the quotient map Q, and in particular $j_B = Q \circ \pi$. Since B is separable, [18, Theorem 4.4] says that the homomorphism $\pi : B \to \mathcal{T}(M_L)$ induces an isomorphism $\pi_* : K_i(B) \to K_i(\mathcal{T}(M_L))$, and since $\rho : B \to \ker Q$ is an isomorphism onto a full corner, ρ_* is an isomorphism. So splicing the commutative diagram of Proposition 3.4 into (3.1) gives the result. □

4. Endomorphisms Arising from Dilation Matrices

Throughout this section, d is an integer ≥ 2 and $A \in M_d(\mathbb{Z})$ is an integer dilation matrix, by which we mean that all the complex eigenvalues λ of A satisfy $|\lambda| > 1$. We consider the surjective endomorphism σ_A of $\mathbb{T}^d$ defined by $\sigma_A(e^{2\pi i x}) = e^{2\pi i A x}$ for $x \in \mathbb{R}^d$, which has $|\ker \sigma_A| = |\det A|$, and the associated Exel system $(C(\mathbb{T}^d), \alpha_A, L)$, where α_A is the endomorphism of $C(\mathbb{T}^d)$ given by σ_A.

We start by showing that $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N} = \mathcal{O}(M_L)$ is simple and purely infinite. We deduce simplicity from results of Exel and Vershik [8] on crossed products by endomorphisms, and pure infiniteness from results of Katsura [14] on the C^*-algebras of topological graphs. So we need to note that the map $f \mapsto N^{1/2} f$ is an isomorphism of the bimodule $M_L = C(K)_L$ onto the bimodule of the topological graph E with $E^0 = \mathbb{T}^d$, $E^1 = \mathbb{T}^d$, $r = \mathrm{id}$ and $s = \sigma_A$, and hence the crossed product $C(\mathbb{T}^d) \rtimes_{\alpha, L} \mathbb{N} = \mathcal{O}(M_L)$ can also be viewed as the C^*-algebra $C^*(E)$ studied in [13, 14].

We need the following lemma on the operator norms of A^n acting on $\mathbb{R}^d$; it must be well known, but we lack a reference.

Lemma 4.1. *We have $\|A^{-n}\| \to 0$ as $n \to \infty$.*

Proof. For each n in $\mathbb{N}$, let $a_n = \|A^{-n}\|^{1/n}$. By the spectral radius formula we know that the limit of the a_n, as n tends to infinity, coincides with the spectral radius $r(A^{-1})$. By hypothesis we know that $r(A^{-1}) < 1$, so we may choose c with $r(A^{-1}) < c < 1$. Therefore, for all sufficiently large n, we have that $a_n < c$, and hence

$$\lim_{n\to\infty} \|A^{-n}\| = \lim_{n\to\infty} a_n^n \leq \lim_{n\to\infty} c^n = 0. \qquad \square$$

Proposition 4.2. *The Cuntz-Pimsner algebra $\mathcal{O}(M_L)$ is simple and purely infinite.*

Proof. We show that $\mathcal{O}(M_L)$ is simple using [8, Theorem 11.2], which says that $C(\mathbb{T}^d) \rtimes_{\alpha,L} \mathbb{N}$ is simple if and only if σ_A is irreducible. We recall from [8, Section 11] that $x, y \in \mathbb{T}^d$ are *trajectory-equivalent*, written $x \sim y$, if there are $n, m \in \mathbb{N}$ such that $\sigma_A^n(x) = \sigma_A^m(y)$, and a subset $Y \subseteq \mathbb{T}^d$ is *invariant* if $x \sim y \in Y$ implies that $x \in Y$; σ_A is *irreducible* if the only closed invariant sets are $\emptyset$ and $\mathbb{T}^d$.

Let Y be a non-empty closed invariant subset of $\mathbb{T}^d$, and pick a point $e^{2\pi i y} \in Y$. We need to show that $Y = \mathbb{T}^d$. Fix $e^{2\pi i z} \in \mathbb{T}^d$. Since the unit cube in $\mathbb{R}^d$ has diameter $\sqrt{d}$, for every $n \in \mathbb{N}$ we can find $k_n \in \mathbb{Z}^d$ such that $|A^n z - (y + k_n)| \le \sqrt{d}$. Then $x_n := A^{-n}(y + k_n)$ has $\sigma_A^n(e^{2\pi i x_n}) = e^{2\pi i A^n x_n} = e^{2\pi i y} \in Y$, and invariance implies that $e^{2\pi i x_n} \in Y$ also. Lemma 4.1 implies that

$$|z - x_n| \le \|A^{-n}\|\,|A^n z - (y + k_n)| \le \|A^{-n}\|\sqrt{d} \to 0 \quad \text{as } n \to \infty,$$

so $x_n \to z$ in $\mathbb{R}^d$ and $e^{2\pi i x_n} \to e^{2\pi i z}$. Since Y is closed, this implies that $e^{2\pi i z} \in Y$, as required. Thus σ_A is irreducible, and $\mathcal{O}(M_L)$ is simple.

To show that $\mathcal{O}(M_L)$ is purely infinite, we realise $\mathcal{O}(M_L) = C(\mathbb{T}^d) \rtimes_{\alpha,L} \mathbb{N}$ as $C^*(E)$ with $E = (\mathbb{T}^d, \mathbb{T}^d, \mathrm{id}, \sigma_A)$. Since $C^*(E) = \mathcal{O}(M_L)$ is simple, E is minimal by [14, Proposition 1.11]. So by [14, Theorem A] it suffices to prove that E is contracting at some vertex $v_0 \in E^0$ in the sense of Definition 2.3 of [14]; we will show that E is contracting at $v = (1, 1, \dots, 1)$. First, we need to see that the positive orbit $\{z : \sigma_A^n(z) = v\}$ of v is dense in $E^0 = \mathbb{T}^d$. The positive orbit of v contains all points of the form $e^{2\pi i A^{-n}k}$ for $n \in N$ and $k \in \mathbb{Z}^d$, and it follows from our proof of the irreducibility of σ_A above (with $y = 0$) that this positive orbit is dense in E^0.

Second, we fix a neighbourhood V of v; we need to show that V contains a contracting open set W (see [14, Definition 2.3]). For this, it suffices to find an open neighbourhood W of v such that $W \subset V$ and $\overline{W} \subsetneq \sigma_A^k(W)$ for some $k \ge 1$. By Lemma 4.1 we can choose k such that $\|A^{-k}\| < 1$. Then for every $\varepsilon > 0$ and every x in the closed unit ball $\overline{B(0,\varepsilon)}$ in $\mathbb{R}^d$, we have $|A^{-k}x| < \varepsilon$, so $x = A^k(A^{-k}x)$ belongs to $A^k(B(0,\varepsilon))$. Thus $\overline{B(0,\varepsilon)} \subset A^k(B(0,\varepsilon))$. The inequality $\|A^k A^{-k}\| \le \|A^k\|\,\|A^{-k}\|$ implies that $\|A^k\| > 1$, so for every $\varepsilon > 0$ there exists $y \in B(0,\varepsilon)$ such that $|A^k y| > \varepsilon$, and $\overline{B(0,\varepsilon)} \subsetneq A^k(B(0,\varepsilon))$. If ε is small enough to ensure that $x \mapsto e^{2\pi i x}$ is one-to-one on $A^k(B(0,\varepsilon))$, then $W := \{e^{2\pi i x} : x \in B(0,\varepsilon)\}$ satisfies $\overline{W} \subsetneq \sigma_A^k(W)$, and by taking ε smaller still we can ensure that $W \subset V$. Thus E is contracting, and the result follows from [14, Theorem A]. □

We now want to calculate the K-theory of $C(\mathbb{T}^d) \rtimes_{\alpha_A,L} \mathbb{N} = \mathcal{O}(M_L)$, and we aim to use Theorem 3.7. To do this, we need descriptions of $K_*(C(\mathbb{T}^d))$ and the map Ω_*.

Lemma 4.3. *Suppose that (B, α, L) is an Exel system with B commutative, that M_L admits an orthonormal basis $\{m_j : 0 \leq j \leq N-1\}$, and that $\Omega : B \to M_N(B)$ is the homomorphism described in Lemma 3.1. Then $\Omega_* \circ \alpha_*$ is multiplication by N on both $K_0(B)$ and $K_1(B)$.*

Proof. We know from Lemma 3.1 (3) that $\Omega \circ \alpha(a) = a1_N$. Now take $b = (b_{ij})$ in $M_r(B)$. Then $(\Omega \circ \alpha)_r(b)$ is the $r \times r$ block matrix with (i, j) block equal to $b_{ij}1_N$. If we view $(\Omega \circ \alpha)_r(b)$ as an element of $M_N(M_r(B))$, as in Lemma 3.5, it becomes $b \oplus b \oplus \cdots \oplus b$. Whether b is a projection or a unitary, $[b \oplus \cdots \oplus b] = N[b]$. Thus by Lemma 3.5, we have

$$\Omega_* \circ \alpha_*([b]) = (\Omega \circ \alpha)_*([b]) = [(\Omega \circ \alpha)_r(b)] = [b \oplus \cdots \oplus b] = N[b]. \qquad \square$$

Ji proved in [11] that the Chern character is a $\mathbb{Z}/2$-graded ring isomorphism of $K_*(C(\mathbb{T}^d)) = K^*(\mathbb{T}^d)$ onto the integral cohomology ring

$$H^*(\mathbb{T}^d, \mathbb{Z}) := \bigoplus_{k \in \mathbb{Z}}^{\infty} H^k(\mathbb{T}^d, \mathbb{Z}) = \bigoplus_{k=0}^{d} H^k(\mathbb{T}^d, \mathbb{Z}),$$

which in turn is isomorphic as a $\mathbb{Z}$-graded ring to the exterior algebra $\bigwedge^* \mathbb{Z}^d$. Thus the ring $H^*(\mathbb{T}^d, \mathbb{Z})$ is generated by $H^1(\mathbb{T}^d, \mathbb{Z})$, which is isomorphic to the set of homotopy classes of continuous functions from $\mathbb{T}^d$ to $\mathbb{T}$, and is the free abelian group generated by the coordinate functions $u_k : z = (z_1, \ldots, z_n) \mapsto z_k$. Since the homomorphism α_* is induced by a continuous map $\sigma_A : \mathbb{T}^d \to \mathbb{T}^d$, the corresponding ring homomorphism on $H^*(\mathbb{T}^d, \mathbb{Z})$ is the map σ_A^*, which respects the $\mathbb{Z}$-grading. Thus we can compute α_* on $\bigwedge^* \mathbb{Z}^d$ by working out what σ_A^* does on $H^1(\mathbb{T}^d, \mathbb{Z})$ using the basis $\{e_k := [u_k] : 1 \leq k \leq d\}$, and then taking exterior powers. Once we know what α_* is, we can use the formula for $\Omega_* \circ \alpha_*$ in Lemma 4.3 to work out what Ω_* is.

Lemma 4.4. *With respect to the basis $\{[u_k]\}$, $\alpha_* : \mathrm{span}\{[u_k]\} \to \mathrm{span}\{[u_k]\}$ is multiplication by the transpose A^T of A.*

Proof. We have $\alpha_*([u_k]) = [\alpha(u_k)] = [u_k \circ \sigma_A]$. Since

$$\begin{aligned} u_k \circ \sigma_A(e^{2\pi i x}) &= u_k(e^{2\pi i A x}) = e^{2\pi i \sum_j a_{k,j} x_j} \\ &= \prod_j e^{2\pi i a_{k,j} x_j} = \prod_j (e^{2\pi i x_j})^{a_{k,j}} = \prod_j u_j(e^{2\pi i x})^{a_{k,j}}, \end{aligned}$$

we have $u_k \circ \sigma_A = \prod_j u_j^{a_{k,j}}$. Hence $[u_k \circ \sigma_A] = \sum_j a_{k,j}[u_j]$. $\square$

Since the 0-graded component is isomorphic to $H^0(\mathbb{T}^d, \mathbb{Z})$, the free abelian group generated by the connected components, the action of α_* on the 0-component $\bigwedge^0(\mathbb{Z}) = \mathbb{Z}$ is the identity map. For $n = 1$, Lemma 4.4 implies that $\alpha_* = A^T$.

For $n > 1$, we use the basis

$$\mathcal{E}_n = \Big\{ e_J = e_{j_1} \wedge \cdots \wedge e_{j_n} : J \subset \{1, \ldots, d\},\ |J| = n,\ J = \{j_1 < \cdots < j_n\} \Big\}$$

for $\bigwedge^n \mathbb{Z}^d$. For $e_K \in \mathcal{E}_n$, we write $K' = \{1, \ldots, d\} \setminus K$. With K and K' listed in increasing order as $K = \{k_1 < \cdots < k_n\}$ and $K' = \{k_{n+1} < \cdots < k_d\}$, we let τ_K be the permutation $i \mapsto k_i$ for $1 \le k \le d$. For subsets K, J of the same size, we write $A_{K,J}$ for the submatrix of A whose entries belong to the rows in K and the columns in J. The following lemma is essentially Lemma 1 of Chapter 5 in [17]; we have included a short proof because the conventions of [17] are different (matrices act on the right of vector spaces, for example).

Lemma 4.5. *Let $1 \le n \le d$. The matrix C_n of $\alpha_*| : \bigwedge^n \mathbb{Z}^d \to \bigwedge^n \mathbb{Z}^d$ with respect to the basis $\mathcal{E}_n$ has (J, K) entry* $\det A_{K,J}$.

Proof. Fix $e_K \in \mathcal{E}_n$ with $K = \{k_1 < \cdots < k_n\}$. Then

$$\begin{aligned}
\Big(\bigwedge^n A^T\Big)(e_K) = \Big(\bigwedge^n A^T\Big)(e_{k_1} \wedge \cdots \wedge e_{k_n}) &= A^T e_{k_1} \wedge \cdots \wedge A^T e_{k_n} \\
&= \sum_{m_1=1,\ldots,m_n=1}^{d} a_{k_1,m_1} \ldots a_{k_n,m_n} (e_{m_1} \wedge \cdots \wedge e_{m_n}) \\
&= \sum_{e_J \in \mathcal{E}_n} \sum_{\{m_1,\ldots,m_n\}=J} a_{k_1,m_1} \ldots a_{k_n,m_n} (e_{m_1} \wedge \cdots \wedge e_{m_n}) \\
&= \sum_{e_J \in \mathcal{E}_n} \sum_{\sigma \in S_n} a_{k_1,\sigma(j_1)} \ldots a_{k_n,\sigma(j_n)} (e_{\sigma(j_1)} \wedge \cdots \wedge e_{\sigma(j_n)}) \\
&= \sum_{e_J \in \mathcal{E}_n} \sum_{\sigma \in S_n} (-1)^{\deg \sigma} a_{k_1,\sigma(j_1)} \ldots a_{k_n,\sigma(j_n)} (e_{j_1} \wedge \cdots \wedge e_{j_n}) \\
&= \sum_{e_J \in \mathcal{E}_n} (\det A_{K,J}) e_J. \qquad \square
\end{aligned}$$

We are now ready to compute the matrix B_n of Ω_* on $\bigwedge^n \mathbb{Z}^d$ with respect to the same basis $\mathcal{E}_n$. The answer must, of course, be an integer matrix. But Lemma 4.3 implies that C_n is invertible as a real matrix, and hence if we can find matrices B_n such that $B_n C_n = N1_n$, then uniqueness of the real inverse tells us that B_n is the matrix of Ω_*.

Proposition 4.6. *Let $B_0 = |\det A|$, $B_d = \operatorname{sign}(\det A)$, and*

$$B_n = \begin{cases} \Big((-1)^{\deg(\tau_K \tau_L)} \det(A_{K',L'})\Big)_{K,L} & \text{if } \det A > 1, \\ -\Big((-1)^{\deg(\tau_K \tau_L)} \det(A_{K',L'})\Big)_{K,L} & \text{if } \det A < -1. \end{cases}$$

(1) *Then $B_n C_n = |\det A| 1$, where 1 is the $\binom{d}{n} \times \binom{d}{n}$ identity matrix.*

(2) *We have* $1 - B_0 = 1 - |\det A| < 0$, $\det(1 - B_n) \neq 0$ *for* $1 \leq n < d$, *and*

$$1 - B_d = \begin{cases} 0 & \text{if } \det A > 1, \\ 2 & \text{if } \det A < -1. \end{cases}$$

For the proof of Proposition 4.6 we need the following lemma; its first part appears as equation (5.3.7) in [17], for example.

Lemma 4.7. *Fix* n *satisfying* $1 \leq n \leq d - 1$.

(1) *If* $e_J \in \mathcal{E}_n$, *then*

$$\det A = \sum_{e_K \in \mathcal{E}_n} (-1)^{\deg(\tau_K \tau_J)} \det(A_{K,J}) \det(A_{K',J'}).$$

(2) *If* $e_J, e_L \in \mathcal{E}_n$ *and* $L \neq J$, *then*

$$\sum_{e_K \in \mathcal{E}_n} (-1)^{\deg(\tau_K \tau_J)} \det(A_{K,J}) \det(A_{K',L'}) = 0.$$

Proof. (1) Fix $e_J \in \mathcal{E}_n$. We have

$$\begin{aligned} \det A &= \sum_{\sigma \in S_d} (-1)^{\deg \sigma} a_{\sigma(1),1} \ldots a_{\sigma(d),d} \\ &= (-1)^{\deg \tau_J} \sum_{\sigma \in S_d} (-1)^{\deg \sigma} a_{\sigma(1),j_1} \ldots a_{\sigma(d),j_d}, \end{aligned}$$

which, by reordering the sum according to the image of $I_n := \{1, \ldots, n\}$ under σ, is

$$= (-1)^{\deg \tau_J} \sum_{e_K \in \mathcal{E}_n} \sum_{\{\sigma : \sigma(I_n) = K\}} (-1)^{\deg \sigma} a_{\sigma(1),j_1} \ldots a_{\sigma(d),j_d}. \tag{4.1}$$

Note that for fixed $\sigma \in S_n$ such that $\sigma(I_n) = K$, we have

$$\sigma = (\sigma_K \times \sigma_{K'}) \circ \tau_K,$$

where $\sigma_K(k_i) := \sigma(i)$ and $\sigma_{K'}(k_\ell) := \sigma(\ell)$. So

$$\begin{aligned} (4.1) &= (-1)^{\deg \tau_J} \sum_{e_K \in \mathcal{E}_n} \sum_{\{\sigma : \sigma(I_n) = K\}} (-1)^{\deg \tau_K} (-1)^{\deg(\sigma_K \times \sigma_{K'})} a_{\sigma_K(k_1),j_1} \ldots a_{\sigma_K(k_n),j_n} \\ &\qquad \cdot a_{\sigma_{K'}(k_{n+1}),j_{n+1}} \ldots a_{\sigma_{K'}(k_d),j_d} \\ &= (-1)^{\deg \tau_J} \sum_{e_K \in \mathcal{E}_n} (-1)^{\deg \tau_K} \sum_{\alpha \in S_K, \beta \in S_{K'}} (-1)^{\deg \alpha} a_{\alpha(k_1),j_1} \ldots a_{\alpha(k_n),j_n} \\ &\qquad \cdot (-1)^{\deg \beta} a_{\beta(k_{n+1}),j_{n+1}} \ldots a_{\beta(k_d),j_d} \\ &= \sum_{e_K \in \mathcal{E}_n} (-1)^{\deg(\tau_K \tau_J)} \det(A_{K,J}) \det(A_{K',J'}). \end{aligned}$$

(2) If $L \neq J$, then $L' \neq J'$ and $L' \cap J \neq \emptyset$. Consider the matrix D whose entries are those of A except that the $L \setminus J$ columns of D have been replaced by copies of the $J \setminus L$ columns of A. Thus $\det D = 0$. Note that $A_{K,J}$ and $D_{K,L}$ have the same columns up to permutation, so $\det(A_{K,J}) = \pm\det(D_{K,L})$. For every K we have $D_{K',L'} = A_{K',L'}$, so using (1) we get

$$\sum_{e_K \in \mathcal{E}_n} (-1)^{\deg(\tau_K \tau_J)} \det(A_{K,J}) \det(A_{K',L'})$$
$$= \pm \sum_{e_K \in \mathcal{E}_n} (-1)^{\deg(\tau_K \tau_J)} \det(D_{K,L}) \det(D_{K',L'}) = \det D = 0. \qquad \square$$

Remark 4.8. In [17, page 92], it is observed that the coefficient $(-1)^{\deg(\tau_K\tau_J)}$ can be realised as the product $\prod_{i=1}^n (-1)^{j_i+k_i}$. To see this, first observe that $(-1)^{\deg(\tau_J)} = \prod_{i=1}^n (-1)^{j_i - i}$ (because $j_n - n$, for example, is the number of transpositions required to move j_n to its correct place in J' without changing the ordering of J'), and then $(-1)^{\deg(\tau_K\tau_J)} = \prod_{i=1}^n (-1)^{(j_i-i)+(k_i-i)}$.

Proof of Proposition 4.6. Say $\det A > 1$. Then the (J, L) entry of $C_n B_n$ is

$$\sum_{e_K \in \mathcal{E}_n} \det(A_{K,J})(-1)^{\deg(\tau_K\tau_L)} \det(A_{K',L'}),$$

which, by Lemma 4.7, equals $\delta_{J,L}(\det A)1$. If $\det A < -1$, the same calculation gives $-\delta_{J,L}(\det A)1 = \delta_{J,L}|\det A|1$. Thus $C_n B_n = |\det A|1 = B_n C_n$. This gives (1).

(2) The statements in (2) about B_0 and B_d are immediate, so we suppose $1 \leq n \leq d-1$. To compute $\det(I - B_n)$ we work over $\mathbb{C}$, and choose a basis for $\mathbb{C}^d$ such that A is upper-triangular. We claim that if

$$J = \{j_1 < \cdots < j_n\} > K = \{k_1 < \cdots < k_n\}$$

in the lexicographical order, then $\det(A_{J,K}) = 0$. If $J > K$, then there exists m such that $j_i = k_i$ for $i < m$ and $j_m > k_m$. Since A is upper-triangular, $j_m > k_m$ implies $a_{j_m,k_m} = 0$. Moreover, $j_{n-m+1} > \cdots > j_{m+1} > j_m > k_m$, so $A_{J,K}$ has the form

$$A_{J,K} = \begin{pmatrix} U & * \\ 0 & V \end{pmatrix},$$

where U is an $(m-1) \times (m-1)$ upper-triangular matrix and V is a square matrix with the first column consisting of zeros. Thus $\det(A_{J,K}) = 0$, as claimed. So if we order $\mathcal{E}_n$ with the lexicographic order, then the matrix $(\det(A_{K,J}))_{J,K}$ of $\alpha_*| = \bigwedge^n A^T$ is lower-triangular. Hence its inverse $(\det A)^{-1} B_n$ is also lower-triangular, and so is B_n. The diagonal entries of B_n are $\det(A_{K',K'}) = \prod_{k \in K'} a_{k,k}$; since each $a_{k,k}$ is an eigenvalue of A, we have $|a_{k,k}| > 1$, and each diagonal entry of B_n has absolute value greater than 1. Since B_n is lower-triangular, it follows that $\det(1 - B_n) \neq 0$. $\square$

Theorem 4.9. *Let A be a dilation matrix in $GL_d(\mathbb{Z})$ with $d \geq 1$, and define B_n as in Proposition 4.6. Let M_L be the bimodule for the Exel system $(C(\mathbb{T}^d), \alpha_A, L)$ and for which $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N} = \mathcal{O}(M_L)$.*

(1) *If* $\det A > 1$, *then*

$$K_0(\mathcal{O}(M_L)) = \Big(\bigoplus_{n \text{ even},\, n<d} \operatorname{coker}(1 - B_n) \Big) \oplus \mathbb{Z},$$

$$K_1(\mathcal{O}(M_L)) = \Big(\bigoplus_{n \text{ odd},\, n<d} \operatorname{coker}(1 - B_n) \Big) \oplus \mathbb{Z}.$$

(2) *If* $\det A < -1$, *then*

$$K_0(\mathcal{O}(M_L)) = \bigoplus_{n \text{ even},\, n\leq d} \operatorname{coker}(1 - B_n),$$

$$K_1(\mathcal{O}(M_L)) = \bigoplus_{n \text{ odd},\, n\leq d} \operatorname{coker}(1 - B_n).$$

Proof. We identify

$$K_1(C(\mathbb{T}^d)) \cong \bigoplus_{n \text{ odd},\, n\leq d} \bigwedge^n \mathbb{Z}^d \quad \text{and} \quad K_0(C(\mathbb{T}^d)) \cong \bigoplus_{n \text{ even},\, n\leq d} \bigwedge^n \mathbb{Z}^d.$$

Suppose that $\det A > 1$. By Lemma 4.3, $(\Omega \circ \alpha)_*$ is multiplication by $|\det A|$, and by Proposition 4.6 (1) the matrix C_n of $\alpha_*|$ has inverse $|\det A|^{-1} B_n$; it follows that the map $\mathrm{id} - \Omega_*$ appearing in Diagram 3.8 is

$$\bigoplus_{\text{even } n\leq d} (1 - B_n) \quad \text{and} \quad \bigoplus_{\text{odd } n\leq d} (1 - B_n)$$

on $K_0(C(\mathbb{T}^d))$ and $K_1(C(\mathbb{T}^d))$, respectively. By Proposition 4.6 (2), each $1 - B_n$ with $n < d$ is injective, and $1 - B_d = 0$.

Suppose that d is odd. Then $\bigoplus_{\text{even } n\leq d}(1 - B_n)$ is injective and

$$\ker\Big(\bigoplus_{\text{odd } n\leq d} (1 - B_n) \Big) = \ker(1 - B_d) = \mathbb{Z}.$$

Thus Diagram 3.8 gives

$$K_1(\mathcal{O}(M_L)) \cong \bigoplus_{n \text{ odd},\, n\leq d} \operatorname{coker}(1 - B_n)$$

and an exact sequence

$$0 \longrightarrow \bigoplus_{n \text{ even},\, n<d} \operatorname{coker}(1 - B_n) \longrightarrow K_0(\mathcal{O}(M_L)) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z}$ is free this sequence splits, and the formula for K_0 follows.

The proof for even d is similar, though this time the zero summand $1 - B_d$ appears in the map $\mathrm{id} - \Omega_*$ on $K_1(C(\mathbb{T}^d))$ rather than in the map on $K_0(C(\mathbb{T}^d))$.

For part (2), we just note that Proposition 4.6 (2) implies that

$$\bigoplus_{n \text{ even},\, n \le d} (1 - B_n) \quad \text{and} \quad \bigoplus_{n \text{ odd},\, n \le d} (1 - B_n)$$

are injective, and the result follows. □

For small d, we can identify the B_n in more familiar terms. Both B_0 and B_d are just numbers (or rather, multiplication by those numbers on $\mathbb{Z}$). Next we have the following result:

Proposition 4.10. *For every d, we have $B_1 = |\det A|(A^T)^{-1}$. If we list the basis for $\bigwedge^{d-1}\mathbb{Z}^d$ as $f_k := e_{\{1,\dots,d\}\setminus\{k\}}$, then B_{d-1} is the matrix with (k,ℓ) entry $(-1)^{k+\ell}a_{k\ell}$ (if $\det A > 0$) or $(-1)^{k+\ell+1}a_{k,\ell}$ (if $\det A < 0$).*

Proof. For each singleton set $\{k\}$, the permutation $\tau_{\{k\}}$ is the cycle which pulls k to the front and moves the elements $1, \dots, k-1$ to the right, which has degree k. The complements $\{k\}'$ are the sets $\hat{k} := \{1, \dots, k\} \setminus \{k\}$, and the number

$$(-1)^{\deg(\tau_K\tau_L)} \det A_{\{k\}',\{\ell\}'} = (-1)^{\deg\tau_K + \deg\tau_L} \det A_{\hat{k},\hat{\ell}} = (-1)^{k+\ell} \det A_{\hat{k},\hat{\ell}}$$

is the (ℓ,k) entry in $(\det A)A^{-1}$, and the (k,ℓ) entry in $(\det A)(A^T)^{-1}$. The extra minus sign in the formula for B_1 when $\det A < 0$ shows that B_1 is $(|\det A|)(A^T)^{-1}$.

The (k,ℓ) entry in the matrix of B_{d-1} with respect to the basis $\{f_k\}$ is the $(\hat{k},\hat{\ell})$ entry in the matrix with respect to the basis $\mathcal{E}_{d-1}$. For $K = \hat{k}$, τ_K is the cycle which moves k to the back and the last $d-k$ terms one forward, which has degree $d-k+1$. Since $A_{(\hat{k})',(\hat{\ell})'}$ is the 1×1 matrix with entry $a_{k,\ell}$, we have

$$\begin{aligned}(-1)^{\deg(\tau_K\tau_L)} \det A_{(\hat{k})',(\hat{\ell})'} &= (-1)^{(d-k+1)+(d-\ell+1)} a_{k,\ell} \\ &= (-1)^{2(d+1)-(k+\ell)} a_{k,\ell} = (-1)^{k+\ell} a_{k,\ell}.\end{aligned}$$

This immediately gives the result for $\det A > 0$, and for $\det A < 0$, the extra minus sign in the formula for B_{d-1} means we need to replace $(1)^{k+\ell}$ by $(-1)^{k+\ell+1}$. □

We can now sum up our results for small d: The first statement of Corollary 4.11 is well-known, as we observed in the introduction, but the second statement and Corollary 4.12 were a bit of a surprise.

Corollary 4.11. *Suppose N is a non-zero integer, and consider the Exel system $(C(\mathbb{T}), \alpha_N, L)$ associated to the covering map $z \mapsto z^N$.*

(1) *If $N > 1$, then*

$$K_0(C(\mathbb{T}) \rtimes_{\alpha_N,L} \mathbb{N}) = (\mathbb{Z}/(N-1)\mathbb{Z}) \oplus \mathbb{Z}$$

and

$$K_1(C(\mathbb{T}) \rtimes_{\alpha_N,L} \mathbb{N}) = \mathbb{Z}.$$

(2) *If $N < -1$, then*

$$K_0(C(\mathbb{T}) \rtimes_{\alpha_N,L} \mathbb{N}) = \mathbb{Z}/(|N|-1)\mathbb{Z}$$

and

$$K_1(C(\mathbb{T}) \rtimes_{\alpha_N,L} \mathbb{N}) = \mathbb{Z}/2\mathbb{Z}.$$

Corollary 4.12. *Suppose that $A = (a_{ij}) \in M_2(\mathbb{Z})$ is a dilation matrix. Then*

$$K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \begin{cases} \mathbb{Z}/(|\det A|-1)\mathbb{Z} \oplus \mathbb{Z} & \text{if } \det A > 1, \\ (\mathbb{Z}/(|\det A|-1)\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) & \text{if } \det A < -1, \end{cases}$$

and

$$K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \begin{cases} \mathbb{Z} \oplus \operatorname{coker}\begin{pmatrix} 1-a_{11} & a_{12} \\ a_{21} & 1-a_{22} \end{pmatrix} & \text{if } \det A > 1, \\ \operatorname{coker}\begin{pmatrix} 1+a_{11} & -a_{12} \\ -a_{21} & 1+a_{22} \end{pmatrix} & \text{if } \det A < -1. \end{cases}$$

Proof. The statement about K_0 follows immediately from Theorem 4.9. For K_1, we use the description of $B_1 = B_{2-1}$ in Proposition 4.10. (If we had used the description of B_1 as $|\det A|(A^T)^{-1}$, we would have gotten a different matrix, because we would then be calculating it with respect to the basis $\{e_1, e_2\}$ rather than $\{f_1, f_2\} = \{e_2, e_1\}$. However, the two matrices are conjugate in $M_2(\mathbb{Z})$, and hence have isomorphic cokernels.) □

We now look at the implications of these results for some concrete examples of dilation matrices. The first two were used in [19] to provide examples of projective multi-resolution analyses.

Examples 4.13.

(1) For $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ we have $\det A = -2 < -1$. So $K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N})$ is the cokernel of $\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$; since this matrix has determinant -1, it is invertible over $\mathbb{Z}$, and we have

$$K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = 0.$$

These K-groups are the same as those of $\mathcal{O}_3$, but the class of the identity is different. To see the last statement, note that the class $[1]$ of the identity in $K_0(C(\mathbb{T}^2))$

is the image of $1 \in \mathbb{Z} = \bigwedge^0 \mathbb{Z}^2$, and when $|\det A| = 2$, $1 - B_0$ is invertible, so $[1]$ belongs to the range of $\mathrm{id} - \Omega_*$. Thus the class of the identity $1_{C(\mathbb{T}^2)\rtimes\mathbb{N}} = j_{C(\mathbb{T})}(1)$ in $K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N})$ is 0. For $\mathcal{O}_3$, on the other hand, $[1]$ is the generator of $K_0(\mathcal{O}_3)$.

In $K_0(M_2(\mathcal{O}_3))$, however, the class of the identity is twice the generating class $[1_{\mathcal{O}_3} e_{11}]$, which is also zero in $K_0(\mathcal{O}_3) = \mathbb{Z}/2\mathbb{Z}$. Thus the classification theorem of Kirchberg and Phillips [15, 22, 25] implies that $C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}$ is isomorphic to $M_2(\mathcal{O}_3)$. (We thank the referee for pointing this out.)

(2) The matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ has $\det A = 2 > 1$. So Corollary 4.12 implies that

$$K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z} \quad \text{and} \quad K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z}.$$

(3) The vanishing of K_1 in Example (1) is something of an accident. Consider the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ k & 0 \end{smallmatrix}\right)$ for $k \in \mathbb{Z}$ with $k > 2$. We have $\det A = -k < 0$, the map $(m, n) \mapsto m + n \pmod{(k-1)}$ induces an isomorphism of $\operatorname{coker}(1 - B_1) = \operatorname{coker}\left(\begin{smallmatrix} 1 & -1 \\ -k & 1 \end{smallmatrix}\right)$ onto $\mathbb{Z}/(k-1)\mathbb{Z}$, and we have $K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z}/(k-1)\mathbb{Z}$.

(4) The matrix $A = \left(\begin{smallmatrix} 2 & 1 \\ -1 & 2 \end{smallmatrix}\right)$ has $\det A = 5 > 1$. Thus

$$K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = (\mathbb{Z}/4\mathbb{Z}) \oplus \mathbb{Z} \quad \text{and} \quad K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

(5) The matrix $A = \left(\begin{smallmatrix} 2 & -1 \\ 1 & -3 \end{smallmatrix}\right)$ has determinant -5, and

$$K_0(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad K_1(C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}) = \mathbb{Z}/5\mathbb{Z}.$$

It is interesting to compare this with the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix}\right)$ in Example (3), which also has $\det A = -5$, but has $K_1 = \mathbb{Z}/4\mathbb{Z}$. So the K-theory of $C(\mathbb{T}^2) \rtimes_{\alpha_A,L} \mathbb{N}$ is not completely determined by $\det A$.

Remark 4.14. The referee asked us whether we could describe the range of the K-invariant on our class of algebras. We know from Proposition 4.6 that the matrices $1 - B_n$ appearing in the formulas in Theorem 4.9 are all invertible over $\mathbb{Q}$, and hence all the cokernels appearing there are finite abelian groups. In the examples above, the cokernels happen to be cyclic, but in general they will not be, as is easily seen by considering diagonal matrices A. For large d, therefore, the torsion parts of K_0 and K_1 could be large direct sums of finite cyclic groups. The rank of K_0 and K_1, on the other hand, is completely determined by Theorem 4.9: it is 1 if $\det A > 1$, and 0 if $\det A < -1$.

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