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Abstract

In this paper, a completely new integral equation for the price of an American put option as well as its optimal exercise price is successfully derived. Compared to existing integral equations for pricing American options, the new integral formulation has two distinguishable advantages; i) it is in a form of one-dimensional integral, and ii) it is in a form that is free from any discontinuity and singularities associated with the optimal exercise boundary at the expiry time. These rather unique features have led to a significant enhancement of the computational accuracy and efficiency as shown in the examples.

AMS(MOS) subject classification.

Keywords. Integral equation, American put options, computational accuracy and efficiency.

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1 Introduction

Financial derivatives are becoming increasingly popular means of investment, speculation and risk management; market demands on faster and more accurate valuation for these contracts have prompted researchers to continue seeking alternative solution approaches for pricing various derivative contracts. Options, as one kind of the most well-known and useful derivatives, have received a lot of attention ever since Black-Scholes [1] derived a simple and elegant pricing formula for European options with the underlying price following a geometric Brownian motion. However, it is widely acknowledged that pricing American options is a much more intriguing problem [11, 16]; the main reason is the inherent characteristic that an American option can be exercised at any time before the expiry time. This additional right for the holder of an American option over that of its European counterpart has cast the American option pricing problem into a free boundary problem, and the so-called “optimal exercise price” (hereafter referred to as “optimal exercise boundary”) at which the option contract should be early exercised needs to be determined together with the option price itself. Mathematically, the existence of the optimal exercise boundary has made the problem of pricing American options highly nonlinear since the domain of such a problem is not only unknown in advance but also “moving” with time, and a closed-form analytic solution is not attainable unless in some special cases, such as perpetual American options and the series solutions for American puts in Zhu [22]. Thus, much of the research in pricing American options involves the development of accurate and efficient valuation methods.

Among various numerical approaches proposed in the literature, one of the most common methods is to numerically solve the partial differential equation (PDE) governing the price of an American option (referred to as the PDE approach hereafter), and this particular approach can be further divided into several sub-categories. On one hand, with the optimal exercise boundary being implicitly located, the pricing problem can be transformed into a linear complementarity problem [10], which can be solved with various numerical algorithms [15, 20]. On the other hand, the optimal exercise boundary can be explicitly tracked and simultaneously found together with the price function. A well-known approach in the latter sub-category is the finite difference method (FDM), based on which many different algorithms have been developed [19, 21]. Other kinds of numerical approaches often used in solving the option pricing problem are the Monte Carlo simulation technique and the tree approach. Typical examples are a least square Monte Carlo method proposed by Longstaff & Schwartz [16] and a modified binomial tree method for pricing American options mentioned in [26].

A main disadvantage of purely numerical approaches is that errors are introduced at very early stage of computation. One way to overcome such a disadvantage is to develop semi-analytical approaches, in which analytical analysis is performed until a point beyond which numerical calculations must be resorted to. There are several well-known papers in this category. For example, Geske & Johnson [8] proposed the compound-option approximation method such that an American option is decomposed into a finite number of European options, while Carr [2] presented a semi-explicit approximation with a randomization technique. Zhu [23] developed an analytic approximation method for American options with a pseudo-steady-state approximation of the moving boundary. In order to seek a good balance between maximizing analytical tractability and minimizing the computational time, integral equation approaches are a good compromise between the two[1]. The essence of this particular method is to cast the differential equation into an integral equation, so that the analytical tractability is preserved in the form of an integral equation and yet the eventual numerical calculations, should numerical values need to be computed, can be completed with a relatively efficient algorithm. Of course, a crucial measure of the performance of this approach is the specific form of the integral equation analytically derived as various forms have been proposed in the past.

[1] Ideally, a closed-form solution in terms of elementary functions like the Black-Scholes pricing formula for European options is ultimately preferred, in terms of rendering both analytical tractability as well as computational efficiency. Unfortunately, such kind of solution has not been found yet.

McKean [17] seems to be the first to derive an integral equation for American option prices using the technique of incomplete Fourier transform. The advantage of this particular integral equation is that it only involves two one-dimensional integrals. However, there are two main drawbacks for this representation, which may cause problems when conducting numerical experiments; one is that the presence of the derivative of the optimal exercise boundary can create numerical difficulties because of the infinite slope of the optimal exercise boundary at maturity [6], and another is that the value of the integral equation at the optimal exercise boundary only equals to half of its original value due to the fact that the inverted Fourier transform of a discontinuous function will converge to the midpoint of the discontinuity [7]. Jamshidian [12] derived a different integral equation for American option prices by transforming the homogeneous Black-Scholes equation into an inhomogeneous one. Although this formulation does not involve the derivative of the optimal exercise boundary, the integral equation contains a two-dimensional integral, which is much more computational intensive than those involving one-dimensional integrals only. One of the most famous integral equations for American option prices was derived by Kim [13] through taking the limit of compound option prices. A very useful feature of Kim's formulation is its quantification of the value of an American option in two parts; a base value that corresponds to its European option and an early exercise premium that is associated exclusively with the early exercise right of an American option. On the other hand, one of its main drawbacks is still the relatively excessive computational time needed for the computation of the two-dimensional integrals involved in finding the unknown optimal exercise boundary.

In this paper, we present a new integral equation (IE) formulation for American put option prices under the Black-Scholes model. Our derivation procedure involves several steps. Firstly, we cast the original problem into a new free boundary problem for the option Theta (the first-order derivative with respect to the time to maturity as one of the important Greeks in option pricing). Taking the advantage of the free boundary being a monotonic decreasing function of the time to expiry, we adopt a novel approach in which the optimal exercise price itself is taken as an independent variable first, replacing an original independent variable, the time to expiry. Mathematically, we transform a problem governed by a linear PDE defined on a domain bounded by an unknown free boundary into one governed by a nonlinear PDE with a fixed boundary. After applying a Fourier transform to this particular nonlinear PDE, an analytical solution in the Fourier space is successfully derived. However, our approach should not be regarded as successful if we could not obtain the analytical inversion of the solution since the numerical inversion of a function in the Fourier space is not desirable and should be avoided whenever possible. Fortunately, we have finally managed to derive a simple and elegant integral equation after analytically performing the inverse Fourier transform. Once this integral equation is solved, the optimal exercise price as a function of the time to expiry can be retrieved.

It should be pointed out here that this newly derived integral equation possesses two distinguishable advantages over all the existing IE formulations. The first one is that the integral equation only involves a one-dimensional integral. The advantage associated with this is clearly its numerical realization being far less computational intensive than that involving two-dimensional integrals. The second unique feature of the newly derived integral equation is that it does not suffer from any discontinuity problem, and singularities associated with the optimal exercise boundary at the expiry time are totally avoided as a result of taking the moving boundary itself as an independent variable in the newly formulated nonlinear PDE system; the computational accuracy and efficiency can thus be further enhanced.

It should also be remarked here that our method can be extended to the valuation problem of American option prices under other models, such as stochastic volatility models and jump-diffusion models. For any stochastic volatility model such as the well-known Heston model [9], although volatility becomes another state variable (see [24]), the free boundary is essentially only “moving” in the direction of the underlying price. Thus, the same technique presented here would still apply, transforming a two-dimensional free boundary surface, instead of a one-dimensional free boundary curve like in the Black-Scholes' case, into a two-dimensional fixed boundary first and then a "retrieving" process similar to what has been presented here is used to restore the needed two-dimensional free boundary surface. On the other hand, when jump-diffusion models are to be dealt with, there is an added integral component in the PDE, so that a partial integro-differential equation needs to be solved. Our proposed technique can still be adopted, with a difference that the ordinary differential equation (ODE) in the Fourier space presented in Theorem 1 of this paper becomes an ODE with a modified coefficient. A challenge then is to find the analytical solution for this modified ODE and to perform the Fourier inversion analytically as we did in this paper. Therefore, specifically dealing with these issues will be left in future research with results shown in a forthcoming paper.

The rest of the paper is organized as follows. In Section 2, a new PDE system governing the Theta of American puts is presented. This new free boundary problem is further transformed into a fixed boundary problem through a novel approach, and a new integral equation containing only a one-dimensional integral is derived. In Section 3, numerical experiments and related discussions are presented, followed by some concluding remarks given in the last section.

2 A new integral equation

In this section, we use the Black-Scholes PDE system for American put option prices to illustrate our approach. The Black-Scholes PDE system is first transformed into another system with two different free boundary conditions by simply differentiating the PDE with respect to the time to expiry. This new free boundary problem is then formulated into a fixed boundary problem by using the free boundary as the new variable to replace the time to expiry. The PDE of this fixed boundary problem is actually nonlinear, from which we obtain a new integral equation with the aid of the Fourier transform.

2.1 A new PDE system

If the underlying price S follows a geometric Brownian motion under the risk-neutral measure as

$$\frac{dS}{S} = rdt + \sigma dW_t, \tag{2.1}$$

with r and σ representing the risk-free interest rate and the volatility respectively, and W_t being the standard Brownian motion, the PDE system[2] governing the American put price $P(S,t)$ can be derived according to the Feynman-Kac formula

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + rS\dfrac{\partial P}{\partial S} - rP = 0, \ S > S_f(t), \\ P(S,t)|_{S=S_f(t)} = K - S_f(t), \\ \dfrac{\partial P}{\partial S}|_{S=S_f(t)} = -1, \\ \lim\limits_{S\to+\infty} P(S,t) = 0, \\ P(S,t)|_{t=T} = \max(K-S,0), \end{cases} \tag{2.2}$$

in which T is the expiry time, K denotes the strike price, and $S_f(t)$ is the optimal exercise boundary with K being its terminal value, i.e., $S_f(t)|_{t=T} = K$. It should be noted that there are altogether three boundary conditions for the reason that the existence of the unknown free boundary has added one degree of freedom and thus a second-order PDE system needs to be supplied with an additional boundary condition to properly close the system. Details on the existence and uniqueness of the solution to System (2.2) can be found in [4].

To efficiently solve System (2.2), all variables are firstly non-dimensionalized with the following transform

$$x = \ln\frac{S}{K}, p = \frac{P}{K}, \tau = \frac{\sigma^2}{2}(T-t), \tag{2.3}$$

[2]The problem of pricing American options is a non-linear problem, if one looks from the whole PDE system point of view, even though the involved PDE itself is a linear one. The main reason is due to the existence of the unknown free boundary, which needs to be determined as part of the solution.

from which we can obtain a dimensionless PDE system

$$\begin{cases} \dfrac{\partial p}{\partial \tau} = \dfrac{\partial^2 p}{\partial x^2} + (k-1)\dfrac{\partial p}{\partial x} - kp, \ x > b(\tau) \\ p(x,\tau)|_{x=b(\tau)} = 1 - e^{b(\tau)}, \\ \dfrac{\partial p}{\partial x}|_{x=b(\tau)} = -e^{b(\tau)}, \\ \lim\limits_{x \to +\infty} p(x,\tau) = 0, \\ p(x,\tau)|_{\tau=0} = \max(1-e^x, 0), \end{cases} \tag{2.4}$$

where $k = \dfrac{2r}{\sigma^2}$ and $b(\tau) = \ln \dfrac{S_f(\tau)}{K}$ with the initial condition

$$b(\tau)|_{\tau=0} = 0. \tag{2.5}$$

From System (2.4), it is not very difficult to obtain the following two identities[3]

$$\frac{\partial p}{\partial \tau}|_{x=b(\tau)} = 0, \tag{2.6}$$

$$\frac{\partial^2 p}{\partial x \partial \tau}|_{x=b(\tau)} = -k\frac{db}{d\tau}. \tag{2.7}$$

Clearly, these two identities are the conditions defined on the free boundary for the Theta of American puts, that is, $\theta(x,\tau) \triangleq \dfrac{\partial p}{\partial \tau}$.

To form a complete PDE system for θ, we still need an initial condition and a boundary condition. In particular, if $\tau = 0$ is substituted into the PDE in System (2.4), the following equation can be derived

$$\begin{aligned} \frac{\partial p}{\partial \tau}|_{\tau=0} &= \frac{\partial^2 p}{\partial x^2}|_{\tau=0} + (k-1)\frac{\partial p}{\partial x}|_{\tau=0} - kp|_{\tau=0} \\ &= \frac{\partial^2 p_0(x)}{\partial x^2} + (k-1)\frac{\partial p_0(x)}{\partial x} - kp_0(x), x \geq 0 \end{aligned} \tag{2.8}$$

[3]Although these two properties have already been presented in [5], their derivation is still included the in the Appendix, for the easiness of reference and completeness of this paper.

where $p_0(x) = \max(1 - e^x, 0)$. This further leads to

$$\theta(x,\tau)|_{\tau=0} = \frac{\partial p}{\partial \tau}|_{\tau=0} = \delta(x), x \geq 0, \tag{2.9}$$

where $\delta(x)$ is the Dirac delta function.

To close the PDE system, we need another boundary condition of $\theta(x,\tau)$ along the x direction as $x \to +\infty$. Since the price of an American option can be expressed as the sum of the corresponding European option price and an early exercise premium in an integral form [3, 13], such an expression can be utilized to derive the needed boundary condition through taking the first-order derivative of the expression with respect to the time to expiry and then taking the limit with $x \to +\infty$, which lead to

$$\lim_{x\to+\infty} \theta(x,\tau) = \lim_{x\to+\infty} \frac{\partial p}{\partial \tau} = 0, \tag{2.10}$$

Therefore, differentiating the PDE in System (2.4) with respect to the time to expiry τ, and collecting boundary conditions (2.6), (2.7), (2.10) and the initial condition (2.9), we arrive at a new PDE system for $\theta(x,\tau)$ as

$$\begin{cases} \dfrac{\partial \theta}{\partial \tau} = \dfrac{\partial^2 \theta}{\partial x^2} + (k-1)\dfrac{\partial \theta}{\partial x} - k\theta, \;\; x > b(\tau), \\ \theta(x,\tau)|_{x=b(\tau)} = 0, \\ \dfrac{\partial \theta}{\partial x}|_{x=b(\tau)} = -k\dfrac{db}{d\tau}, \\ \lim\limits_{x\to+\infty} \theta(x,\tau) = 0, \\ \theta(x,\tau)|_{\tau=0} = \delta(x), x \geq 0. \end{cases} \tag{2.11}$$

It should be remarked that the PDE system governing $\theta(x,\tau)$ does not possess any essential change on the fundamental characteristics as it is still a free boundary problem with a linear PDE. However, it has facilitated the conversion of a moving boundary problem to a fixed boundary problem as a result of the second boundary on $x = b(\tau)$ being inserted into the

PDE as demonstrated in the next subsection.

2.2 Analytical pricing formula for American put

In this subsection, the free boundary problem contained in System (2.11) is transformed into a fixed boundary problem with a non-linear PDE defined on a fixed domain. The solution of the new PDE system in the Fourier space is then obtained by applying the Fourier transform, the analytical inversion of which yields an integral equation.

In the literature, there are many ways in which a free boundary problem can be converted into a fixed boundary problem. Specifically, in the context of transforming the American option pricing problem into a fixed boundary problem, Zhu [22] adopted the well-known Landau transform [21], and obtained an exact and explicit pricing formula for American puts in an infinite series form. However, this technique is not suitable for the integral equation approach, as the adoption of Landau transform would introduce the derivative of the optimal exercise boundary into the non-linear PDE and consequently in the final expression of the integral equation as well. To overcome this problem, we take the advantage of a well-known property of the optimal exercise boundary, namely, the monotonicity of the optimal exercise boundary, and introduce an alternative method with one of the independent variables in the original PDE system being replaced by a new independent variable, the free boundary itself being treated as "parameter".

More specifically, from the monotonicity of $b(\tau)$, we can take b as a new independent variable and view $\theta(x,\tau)$ as $\theta(x,b(\tau))$. Then, using the chain rule on the left hand side of the first equation in (2.11), we obtain

$$\frac{\partial \theta}{\partial b}\frac{db}{d\tau} = \frac{\partial^2 \theta}{\partial x^2} + (k-1)\frac{\partial \theta}{\partial x} - k\theta. \tag{2.12}$$

On the other hand, the second boundary condition in (2.11) can be rewritten as

$$\frac{\partial \theta}{\partial x}|_{x=b} = -k\frac{db}{d\tau}, \tag{2.13}$$

which is now used to facilitate the elimination of $\frac{db}{d\tau}$ in (2.12) and complete the change of one of the independent variables as far as the new PDE is concerned[4]. The remaining two boundary conditions in (2.11) simply become the boundary conditions of the new system at $x = b$, which is no longer viewed as a function of τ for the time being. Taking (2.5) into account, the PDE system for $\theta(x, b)$ can be summarized as

$$\begin{cases} -\frac{1}{k}\frac{\partial \theta}{\partial x}|_{x=b}\frac{\partial \theta}{\partial b} = \frac{\partial^2 \theta}{\partial x^2} + (k-1)\frac{\partial \theta}{\partial x} - k\theta, \ \ x > b, \\ \theta(x, b)|_{x=b} = 0, \\ \lim\limits_{x \to +\infty} \theta(x, b) = 0, \\ \theta(x, b)|_{b=0} = \delta(x), x \geq 0. \end{cases} \tag{2.14}$$

It should be pointed out that the PDE system (2.14) is obtained after introducing a new independent variable to the original PDE system (2.11). Thus, the existence and uniqueness of the solution to the new system are preserved as a result of the one-to-one explicit relationship between the new independent variable b and the old independent variable τ. To obtain the solution, a Fourier transform is performed as presented in the following theorem. It should be remarked here that although the PDE in (2.14) is a nonlinear one due to the presence of a product of the unknown function and its first order derivative, one can still establish an integral equation by performing a Fourier transform, which would normally be a powerful tool only for solving linear PDEs. The key of the success hinges on a careful observation that the PDE in System (2.14) can be treated as a linear PDE with constant coefficients as far as a Fourier transform with respect to x is concerned, because the source of the nonlinearity, $\frac{\partial \theta}{\partial x}|_{x=b}$, is a function b only, albeit unknown. Once the solution

[4] Note: the subtle difference on the left hand side of (2.13) from that of the second boundary condition in (2.11) is a result of taking b as a new independent variable.

for $\theta(x,b)$ is obtained for every given x and b, $b(\tau)$ can be then easily retrieved from using essentially the third equation in (2.11) and $p(x,\tau)$ can be recovered from integrating $\theta(x,b)$ with respect to τ.

Theorem 1 *If $\theta(x,b)$ is the solution to the PDE system (2.14), then an integral equation for $\theta(x,b)$ can be derived as*

$$\begin{aligned}\theta(x,b) &= \frac{e^{-k^2 m(b)-\frac{1}{2}m(b)[k-1+\frac{x}{km(b)}]^2}}{2\sqrt{\pi k m(b)}} \\ &+ \int_0^b \frac{e^{-k^2[m(b)-m(y)]-\frac{1}{2}[m(b)-m(y)]\{k-1+\frac{x-y}{k[m(b)-m(y)]}\}^2}}{2\sqrt{\pi k[m(b)-m(y)]}}dy, \end{aligned} \tag{2.15}$$

where $m(b)$ is defined as

$$m(b) = -\int_0^b \frac{1}{\frac{\partial\theta(x,s)}{\partial x}|_{x=s}}ds. \tag{2.16}$$

Proof.

We begin with treating the nonlinear PDE in System (2.14) as a linear one with constant coefficients in the process of performing an incomplete Fourier transform, after denoting $\frac{\partial\theta}{\partial x}|_{x=b}$ as $f(b)$[5]. With such a notation deliberately emphasizing that all the coefficients of the PDE in System (2.14) are a function of b only, we can perform an incomplete Fourier transform on the PDE in System (2.14) with respect to x, the operator of which, $F(\cdot)$, is defined as

$$F[g(x)] = \int_b^{+\infty} e^{-i\phi x} g(x)dx, \tag{2.17}$$

where i denotes the imaginary unit. It should be pointed out here that the incomplete Fourier transform can be viewed as an ordinary one if we assume that the function $g(x)$ is also defined on $(-\infty,b)$ where $g(x)=0$.

[5]It should be remarked that this elegant treatment does not mean that the Fourier transform technique can be extended to solve nonlinear equations in general; we have merely utilized the fact that when the source of the nonlinearity becomes a known function of one variable only, as in this case, the power of the Fourier transform can still be "displayed" as the PDE would appear to be a pseudo "linear" as long as the Fourier transform is performed against another variable (or variables in a more general case).

After applying the incomplete Fourier transform to the unknown function $\theta(x,b)$ and its derivatives with respect to x, we have

$$\begin{aligned}
F(\frac{\partial\theta}{\partial x}) &= i\phi\bar{\theta}, \\
F(\frac{\partial^2\theta}{\partial x^2}) &= -e^{-i\phi b}f(b) - \phi^2\bar{\theta}, \\
F(\frac{\partial\theta}{\partial b}) &= \frac{d\bar{\theta}}{db},
\end{aligned}$$

where $\bar{\theta}(\phi,b) = F[\theta(x,b)]$. Consequently, the PDE in System (2.14) becomes an ODE

$$\frac{d\bar{\theta}}{db} - \frac{k[\phi^2-(k-1)i\phi+k]}{f(b)}\bar{\theta} = ke^{-i\phi b}, \tag{2.18}$$

with the initial condition

$$\bar{\theta}(\phi,b)|_{b=0} = F[\delta(x)H(x)]. \tag{2.19}$$

where $H(x)$ is a Heaviside function. Such a first-order ODE with variable coefficients can be solved analytically with the solution

$$\bar{\theta} = e^{-\int_0^b \frac{k[\phi^2-(k-1)i\phi+k]}{-f(s)}ds}\{\int_0^b ke^{-i\phi y}e^{\int_0^y \frac{k[\phi^2-(k-1)i\phi+k]}{-f(s)}ds}dy + F[\delta(x)H(x)]\}. \tag{2.20}$$

If we further let

$$m(b) = -\int_0^b \frac{1}{f(s)}ds, \tag{2.21}$$

Equation (2.20) can then be simplified as

$$\bar{\theta} = \int_0^b ke^{-i\phi y}e^{-k[\phi^2-(k-1)i\phi+k][m(b)-m(y)]}dy + e^{-k[\phi^2-(k-1)i\phi+k]m(b)}F[\delta(x)H(x)]. \tag{2.22}$$

This means that we have successfully obtained the analytical solution in the Fourier space and the remaining work is to apply the inverse Fourier transform.

Denoting $F^{-1}[\cdot]$ as the inverse Fourier transform operator, our target solution can then be expressed as

$$\begin{aligned}
\theta &= F^{-1}[\int_0^b ke^{-i\phi y}e^{-k[\phi^2-(k-1)i\phi+k][m(b)-m(y)]}dy] + F^{-1}\{e^{-k[\phi^2-(k-1)i\phi+k]m(b)}F[\delta(x)H(x)]\} \\
&\triangleq U_1 + U_2. \qquad (2.23)
\end{aligned}$$

The first part U_1 can be evaluated as

$$\begin{aligned}
U_1 &= \frac{1}{2\pi}\int_0^b \int_{-\infty}^{+\infty} ke^{i\phi(x-y)}e^{-k[\phi^2-(k-1)i\phi+k][m(b)-m(y)]}d\phi dy \\
&= \frac{k}{2\pi}\int_0^b e^{-k^2[m(b)-m(y)]}\int_{-\infty}^{+\infty} e^{-k[m(b)-m(y)]\{\phi^2-(k-1)i\phi-\frac{x-y}{k[m(b)-m(y)]}\}}d\phi dy \\
&= \frac{k}{2\pi}\int_0^b e^{-k^2[m(b)-m(y)]-\frac{1}{4}k[m(b)-m(y)]\{k-1+\frac{x-y}{k[m(b)-m(y)]}\}^2} \\
&\times \int_{-\infty}^{+\infty} e^{-k[m(b)-m(y)]\{\phi-\frac{1}{2}i[k-1+\frac{x-y}{k[m(b)-m(y)]}]\}^2}d\phi dy. \qquad (2.24)
\end{aligned}$$

In order to work out the integration with respect to ϕ in Equation (2.24), we need to firstly study the property of the function $m(b)$. The definition of $f(b) = -k\dfrac{db}{d\tau}$ implies that $f(b)$ is always positive since b is a monotonically decreasing function of the time to expiry. Moreover, considering the fact that $m(b)$ defined in Equation (2.16) is non-negative as b is non-positive, we can reach the conclusion that a decrease in b will result in an increase in $m(b)$. Therefore, we have $m(b) > m(y)$ when $y > b$, and thus U_1 in Equation (2.24) can be further simplified as

$$U_1 = \int_0^b \frac{e^{-k^2[m(b)-m(y)]-\frac{1}{4}k[m(b)-m(y)]\{k-1+\frac{x-y}{k[m(b)-m(y)]}\}^2}}{2\sqrt{k\pi[m(b)-m(y)]}}dy. \qquad (2.25)$$

Similar to the above derivation process, it is quite straightforward to obtain

$$F^{-1}[e^{-k[\phi^2-(k-1)i\phi+k]m(b)}] = \frac{e^{-k^2m(b)-\frac{1}{4}km(b)[k-1+\frac{x}{km(b)}]^2}}{2\sqrt{k\pi m(b)}}. \qquad (2.26)$$

Using the convolution theorem for inverse Fourier transform, we obtain

$$U_2 = F^{-1}[e^{-k[\phi^2-(k-1)i\phi+k]m(b)}] * F^{-1}\{F[\delta(x)H(x)]\}, \tag{2.27}$$

with $*$ denoting the convolution operator. This further gives

$$\begin{aligned} U_2 &= \int_{-\infty}^{+\infty} \frac{e^{-k^2m(b)-\frac{1}{4}km(b)[k-1+\frac{x-u}{km(b)}]^2}}{2\sqrt{k\pi m(b)}}\delta(u)H(u)du \\ &= \frac{e^{-k^2m(b)-\frac{1}{4}km(b)[k-1+\frac{x}{km(b)}]^2}}{2\sqrt{k\pi m(b)}}. \end{aligned} \tag{2.28}$$

Therefore, Equations (2.25) and (2.28) can lead to the final solution of $\theta(x, b)$. This completes the proof.

Obviously, $\theta(x, b)$ can finally be calculated through Equation (2.15) once the function $m(b)$ is solved. Therefore, in order to obtain $m(b)$, we make use of $\theta(x, b)|_{x=b} = 0$, which yields

$$\frac{e^{-k^2m(b)-\frac{1}{2}m(b)[k-1+\frac{b}{km(b)}]^2}}{2\sqrt{\pi km(b)}} + \int_0^b \frac{e^{-k^2[m(b)-m(y)]-\frac{1}{2}[m(b)-m(y)]\{k-1+\frac{b-y}{k[m(b)-m(y)]}\}^2}}{2\sqrt{\pi k[m(b)-m(y)]}}dy = 0. \tag{2.29}$$

It should be noted that there is no discontinuity in this equation because the value of $\theta(x, b)$ at the boundary condition $x = b$ is zero, which is the same as that outside the continuously holding region. It should also be remarked here that while there always exists a singularity at $\tau = 0$ in all the existing integral equations for optimal exercise boundary of the American option in the literature due to the presence of the negative infinite slope of the optimal exercise boundary, i.e.,

$$\frac{db}{d\tau}|_{\tau=0} = -\infty, \tag{2.30}$$

we have successfully avoided directly dealing with this singularity when eventually solving the integral equation numerically. This is achieved as a direct result of taking the free boundary b itself as an independent variable, rather than an unknown function in the original system. In other words, one can easily verify that there are no singularities in (2.29) when b approaches zero, i.e., the slope of $m(b)$ is actually

$$\frac{dm(b)}{db}|_{b=0} = -\frac{1}{\frac{\partial\theta(x,b)}{\partial x}|_{x=b}}\Bigg|_{b=0} = \frac{1}{k\frac{db}{d\tau}}\Bigg|_{\tau=0} = 0. \tag{2.31}$$

Thus, a notoriously difficult problem of numerically dealing with the resolution near $\tau = 0$ in almost all numerical solution approaches proposed to price American options is avoided here as a result of $b = 0(\tau = 0)$ being dealt with "exactly" in our newly derived integral equation. Furthermore, the singular behavior of $b(\tau)$ at $\tau = 0$ is recovered analytically through Equation (2.32), so we can claim that there is no loss of accuracy in our newly derived integral equation as far as dealing with the well-known singularity at $\tau = 0$ is concerned. In addition, Equation (2.29) only involves a one-dimensional integral and thus its computation should be very efficient.

Once $m(b)$ is found Equation (2.29), the time to expiry that corresponds to each b can be calculated straightforwardly via

$$\tau = km(b), \tag{2.32}$$

which is the result of a direct integration of the ODE

$$\frac{db}{d\tau} = -\frac{1}{k}f(b), \tag{2.33}$$

with the utilization of Equation (2.21) and the initial condition (2.5).

Consequently $\theta(x,\tau)$ can be computed through (2.15) making use of equation (2.32). Finally, the dimensionless option price, $p(x,\tau)$, can be obtained from

$$p(x,\tau) = p(x,0) + \int_0^\tau \theta(x,s)ds = \max(1-e^x) + \int_0^\tau \theta(x,s)ds, \tag{2.34}$$

and the corresponding dimensional option price posted in the original problem can be calculated from

$$P(x,\tau) = K[\max(1-e^x) + \int_0^\tau \theta(x,s)ds], \tag{2.35}$$

where $x = \ln(\frac{S}{K})$ and $\tau = \frac{1}{2}\sigma^2(T-t)$.

In summary, to work out the American option price and the optimal exercise boundary with respect to the time to expiry, the integral equation (2.29) should be numerically solved to obtain $m(b)$ values for each discrete b value step by step[6]. Then, by utilizing the obtained $m(b)$, corresponding τ and $\theta(x,b)$ values can be successfully obtained with Equation (2.32) and (2.15) respectively. Finally, the American put prices can be figured out from Equation (2.35). In the next section, this solution procedure is numerically realized, and the accuracy and efficiency of the newly derived integral equation are demonstrated.

3 Numerical examples and discussions

In this section, the accuracy and efficiency of our integral equation approach will be numerically verified. In the following calculations, the risk-free interest rate r is set to be 0.1, the strike price K is 100, and other model parameters are $\sigma = 0.3$ and $T = 1$.

Unlike the solution procedure for American option prices with other integral equation approaches, where the time to expiry is discretized first and the optimal exercise boundary is obtained by solving the corresponding integral equation step by step, the free boundary b is discretized in our approach before the integral equation (2.29) is solved numerically to obtain $m(b)$. Once $m(b)$ is found, τ, the time to expiry that corresponds to each b is computed by using Equation (2.32) to obtain discrete values of the optimal exercise boundary $b(\tau)$.

[6]To a certain extent, this first part of the solution procedure in the proposed new approach is similar to the concept of inverse finite element method proposed by Zhu & Chen [25] for solving the American option pricing problem, in which a set of unknown function values are "prescribed" first and then the corresponding values of the original independent variable are found through an efficient nonlinear iteration scheme.

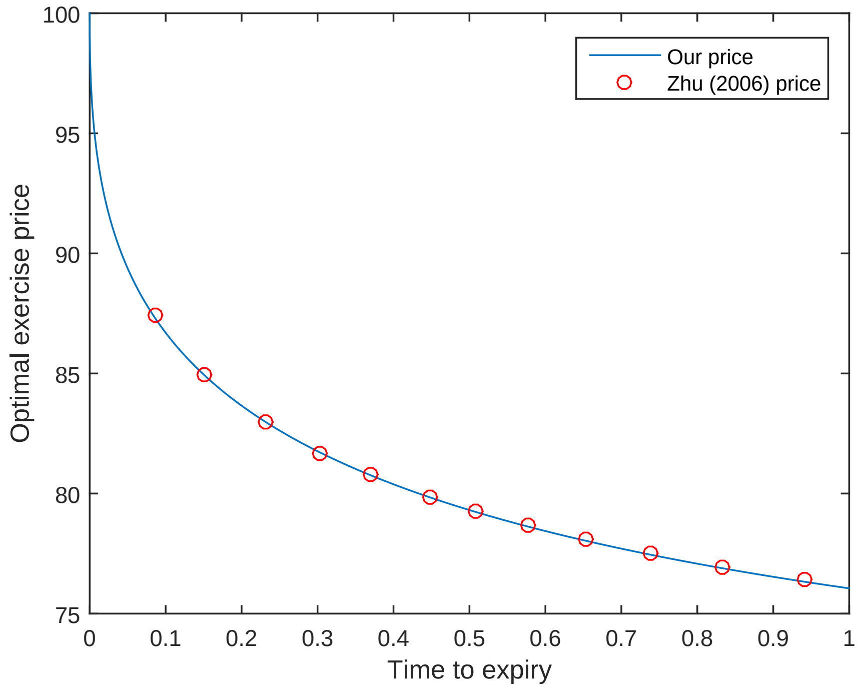

Figure 1: Comparison of optimal exercise prices with two different approaches.

Depicted in Figure 1 is the comparison of optimal exercise prices calculated with our integral equation with those obtained from Zhu's formula. What should be noticed first is that the optimal exercise price of an American put option is a monotonic decreasing function of the time to expiry, which can partially verify our formula. Moreover, it is obvious that our results agree very well with those obtained from Zhu's formula. To further demonstrate this issue, the two sets of optimal exercise prices are listed in Table 1, and clearly the maximum relative error between the two results is less than 0.2%, which can certainly show the accuracy of our integral equation. On the other hand, the speed with which the optimal exercise boundary is calculated by solving our newly formulated integral equation is much, much faster than that through Zhu's formula by summing up an infinite series. As Medvedev and Scaillet [18] pointed out in their paper, Zhu [22] did not focus on computational efficiency at all; in fact, the two approaches are not even comparable in terms of computational efficiency, as it would take hours to compute the optimal exercise boundary on a same resolution if we were trying to compute the optimal exercise boundary with Zhu's approach, whereas it only takes few seconds to do the same job with the current approach.

Table 1: Comparison of optimal exercise prices with our and Zhu's approach.

Time to expiry	Our results	Zhu's results	Relative error
0.0868	87.4347	87.2748	0.18%
0.1515	84.9193	84.9158	0.004%
0.2321	82.9560	82.9710	0.02%
0.3039	81.6967	81.7036	0.008%
0.3697	80.7728	80.7625	0.01%
0.4480	79.8654	79.8408	0.03 %
0.5083	79.2696	79.2349	0.04%
0.5761	78.6813	78.6336	0.06%
0.6521	78.1008	78.0411	0.08%
0.7376	77.5284	77.4571	0.09%
0.8335	76.9635	76.8856	0.10%
0.9413	76.4007	76.3263	0.09%

Table 2: Comparison of the CPU time

	CPU Time
Our equation	6.0s
Kim's equation	11.5s

As far as the computational efficiency is concerned, we make comparison of the CPU times consumed by computing our new integral equation and Kim's integral equation to obtain a single set of the optimal exercise boundary. As can be seen in Table 2, it is clear that to obtain a similar accuracy with a relative difference being in the order of 1%, it only takes 6.0 seconds to compute for one particular set of the optimal exercise prices with the time to expiry $T = 1$. This means that our time savings is about 50% over Kim's approach, which is as expected since solving integral equations requires iteration, and Kim's integral equation actually involves a two-dimensional integral while our integral equation only contains a one-dimensional integral. On the other hand, this significant enhancement of numerical efficiency further justifies the usefulness of the new approach as an alternative to the traditional integral equations derived in the past.

As far as computing the option prices is concerned, there is no need to present some computational results and further discuss the associated computational accuracy and efficiency after an example is provided above for the computation of the optimal exercise boundary. This is because it has been well documented in the literature [11, 14, 22] that the much harder part in pricing American options is to determine the optimal exercise boundary; once the optimal exercise boundary is found, the problem of finding option prices becomes a linear one, which can be straightforwardly solved without much computational effort at all.

4 Conclusion

In this paper, we first present a PDE system for the first-order derivative of American put prices with respect to the time to expiry, and then this new free boundary problem is further transformed into a fixed boundary problem with a novel approach by making the unknown free boundary as a new variable replacing the time to expiry. This new fixed boundary problem actually contains a nonlinear PDE with one initial condition and two fixed boundary conditions, which leads to a new integral equation involving only a one-dimensional integral as one of its main advantages. Another great advantage is that the discontinuity of the integral equation and singularities associated with the optimal exercise boundary at expiry are avoided so that the accuracy and efficiency can be further enhanced. It should be noted that the option Theta, which is one of the important Greeks in option pricing, is computed directly in our formulation.

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Appendix

In order to derive the two identities, i.e., (2.6) and (2.7), we make use of the two free boundary conditions in System (2.4), which are specified as

$$p(x,\tau)|_{x=b(\tau)} = 1 - e^{b(\tau)}, \tag{A-1}$$

and

$$\frac{\partial p}{\partial x}|_{x=b(\tau)} = -e^{b(\tau)}. \tag{A-2}$$

Specifically, if we differentiate Equation (A-1) with respect to τ, we can obtain

$$\frac{\partial p}{\partial x}|_{x=b} \cdot \frac{db}{d\tau} + \frac{\partial p}{\partial \tau}|_{x=b} = -e^{b}\frac{db}{d\tau}.$$

Substituting Equation (A-2) into above equation can yield Equation (2.6). On the other hand, if we make $x = b$ in the PDE of System (2.4), it is very straightforward to obtain

$$\frac{\partial p}{\partial \tau}|_{x=b} = \frac{\partial^2 p}{\partial x^2}|_{x=b} + (k-1)\frac{\partial p}{\partial x}|_{x=b} - kp|_{x=b} = 0,$$

which can be further simplified as

$$\frac{\partial^2 p}{\partial x^2}|_{x=b} = k - e^{b}. \tag{A-3}$$

In addition, differentiating Equation (A-2) with respect to τ can result in

$$\frac{\partial^2 p}{\partial x^2}|_{x=b} \cdot \frac{db}{d\tau} + \frac{\partial^2 p}{\partial x \partial \tau}|_{x=b} = -e^{b}\frac{db}{d\tau}. \tag{A-4}$$

We can then arrive at Equation (2.7) if we combine Equation (A-3) and (A-4) together. This completes the proof.