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THE CONCEPTUAL INTERFACE BETWEEN SECONDARY AND UNIVERSITY MATHEMATICS: A SCHEMA-BASED ANALYSIS

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Abstract

In this paper I provide a theoretical analysis of difficulties experienced by students in having to transfer concepts they have learnt in high school to university mathematics. This analysis is based on the assumption that mathematical knowledge can be represented in the form of organised structures called schemas. The organisational quality of students' mathematical schemas is a function of spread of network and strength of links between pieces of information. Well-developed schemas are argued here to facilitate assimilation of university mathematical concepts and the use of this new information in the solution of novel problems.

INTRODUCTION

A major task for university educators today is to identify difficulties faced by students enrolled in university courses. Within the university curriculum the learning difficulties experienced by students who are new to the university academic culture have received some attention in recent times (Tall, 1992; Harel and Trgalova, 1996). The problem is particularly acute in the case of students who are enrolled in mathematics courses in North American, British and Australian universities. For example, in a survey of students enrolled in university mathematics courses, both school leavers and mature-age students found university mathematics relatively difficult (Leder and Forgasz, 1998). The complex nature of the subject-matter of mathematics, the learning of mathematics, individual differences and teaching style of university mathematics lecturers make the task of helping students a formidable one. However, the task would become more manageable and worth our efforts if we were to focus our attention on an area of mathematical learning that would contribute to our understanding of impediments to the progress that we expect our students to make. In this regard, an analysis of students' learning problems that are associated with the transition from secondary to university mathematics constitutes an important exercise as it has the potential to generate useful strategies for bridging the gap between secondary and university mathematics.

Despite its importance we have limited information about how students deal with problems they encounter in the first year of their university mathematics courses. This state of affairs could be due to the fact that the relationship between development of mathematical understanding at the secondary school level and the transfer of this understanding to university mathematics has received little attention from researchers and mathematics educators. The examination of the nexus between secondary and university level mathematical concept development, however, should be based on a clear understanding of what we mean by concepts and concept-related processes. Such an understanding will provide us with a solid theoretical base from which to analyse problems experienced by students in assimilating new mathematical contents and procedures that are taught in first-year university mathematics courses.

The purpose of this paper is to provide this theoretical base and recommend procedures for the analysis of difficulties in conceptual transition experienced by students undertaking university mathematics courses. The analysis will focus on the elucidation of organisational aspects of mathematical knowledge development and its effect on how students come to understand and solve problems. Accordingly, the accessing of previously-learnt knowledge and its role in the acquisition of new concepts and problem solving skills is discussed within two key theoretical frameworks: schema development and problem representation. A number of examples from senior school calculus and algebra will be used in order to illustrate the importance of extending schemas in understanding and remediating student difficulties with university mathematics.

INTERFACE BETWEEN SECONDARY AND UNIVERSITY MATHEMATICS

Notwithstanding other transitional difficulties, students are expected to make major adaptations in coming to terms with new mathematical concepts and principles in the university courses. Students' abilities at integrating new ideas are put under further stress by the fact in most of the courses they have to cope with large amounts of new material in a short period of time. A number of studies have examined mathematical learning difficulties encountered by students in the university.

Tall and Vinner (1981) provided an influential theoretical model to analyse conceptual transition from school to university mathematics. This model consists of two major constructs: *concept image and concept definition.* The authors used the term concept image to describe 'the total cognitive structure that is associated with the concept, which includes mental picture and associated processes' (p.152). Concept definition, on the other hand, referred to words and symbols that were employed to specify that concept. Both these constructs constitute a major advancement in our understanding of development of mathematical understandings at the secondary/university interface. The strength of model is that it supports the prevailing view that individual students construct different images or understandings about a particular concept. However, there is a lack of specificity about the nature of relations that one could build not only between concept definition and image, but between images and definitions. Given that definitions consists of symbols and words, and some of these could be used elsewhere in association with others, students need to differentiate and integrate the different meanings of mathematical symbols and words. Likewise, the model fails to inform about links between images about a concept. For example, what are the relations between the symbolic and graphical images of functions?

Other studies on the transition to university mathematics tended to focus on specific concepts and their misconceptions. In a study about limits, Davis and Vinner (1986) argued that students experienced problems in extending their understanding of rate of change to the notion of limit. Likewise, Dreyfus and Vinner (1989) identified inconsistencies in the way university students understood the notion of functions. Results of their study showed that students tend to develop a 'formula' view of a function, $y=f(x)$, where for each value of x, one can generate a value of y. There was a general lack of understanding of function as a one-to-one relationship between dependent and independent variables. These studies are useful because they highlight the existence of specific conceptual gaps among university students. The proliferation of bridging courses in departments of mathematics at universities is a further acknowledgment that there is a problem in level of prerequisite knowledge and skills of some secondary students aiming to study university courses. While the above investigations about the state of students' understanding of particular concepts are valuable, they are not grounded in a solid theoretical base about mathematical cognition.

In sum, there is a general admission that the transition to university mathematics is a difficult one. However, both theoretical and empirical research on the issue do not provide sufficient information about the nature of the relationships between the various concepts taught in school and university mathematics, and the mechanisms involved in transfer of learning between the two systems.

SCHEMAS AND MATHEMATICAL KNOWLEDGE DEVELOPMENT

Recent research from cognitive psychologists and mathematics educators has advanced several theoretical frameworks about concepts and how they are represented in memory (Rumelhart and Ortony, 1977). In this paper, I adopt the network perspective in making judgments about mathematical knowledge development (Anderson, 1977; Marshall, 1995). According to this view conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes. As students' experiences with a concept or a set of concepts increase, they come to form organised meaningful wholes called *schemas*. Schemas can be visualised as knowledge structures or networks having one or more core concepts which are connected to other concepts by relational statements. The relations that are found between concepts that form a schema could denote a number of features including information about (a) similarities and dissimilarities between those concepts, (b) procedures for using the concepts for solving problems and (c) affective factors about those concepts. Chinnappan (1998), for example, provided data that showed that schemas in the domain of geometry could be organised around axioms or theorems about Euclidean geometry.

According to Anderson (1995), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. A qualitatively superior schema can be characterised as having a large number of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes to the accessing and use of the schema in problem-solving and other situations. A high quality schema can also benefit students by helping them assimilate new mathematical ideas because such a schema has many conceptual points to link with. As a theoretical construct schemas provide a useful way to interpret the growth of advanced mathematical knowledge by identifying pedagogically important relations.

ROLE OF SCHEMAS IN PROBLEM REPRESENTATION

It is assumed that performance in mathematical tasks is to a large measure dependent on accessing and using prior knowledge that is organised in the form of schemas. A major advantage of having knowledge stored in clusters or chunks is that they facilitate retrieval of the required knowledge from the long-term memory into the working memory during information processing. In problem-solving contexts schemas play an influential role during the construction of a representation for the problem. Cognitive psychologists argue that the solution of mathematical problems can be greatly enhanced if students are taught to construct useful representations of problems (Frederikson, 1984).

Building a problem representation can be a deliberate process in which students attempt to establish *meaningful* links between bits of information in the problem statement and knowledge embedded in their schemas that can be related to the problem. Students' repertoires of problem-related schemas could include, but not are restricted to, (a) knowledge of procedures and strategies associated with tackling a group of problems that are similar to the problem in question, (b) mathematical concepts and (c) knowledge about previous experiences with similar problems. Hence, building a representation of the problem involves, among other things, making decisions about what to select from the above range of schemas. This point was made by Hayes and Simon (1977) who have suggested that 'the representation of the problem must include the initial conditions of the problem, its goal, and the operators for reaching the goal from the initial state' (p.21).

The information gap between available schemas and problem goal might be filled if the student could decompose the problem into subproblems, a strategy that has long been argued to facilitate problem solving (Newell and Simon, 1972). In a more recent investigation, Catrombone (1998) showed that transfer of learning during the solution of novel problems could be facilitated if students understand the structure of subproblems and subgoals.

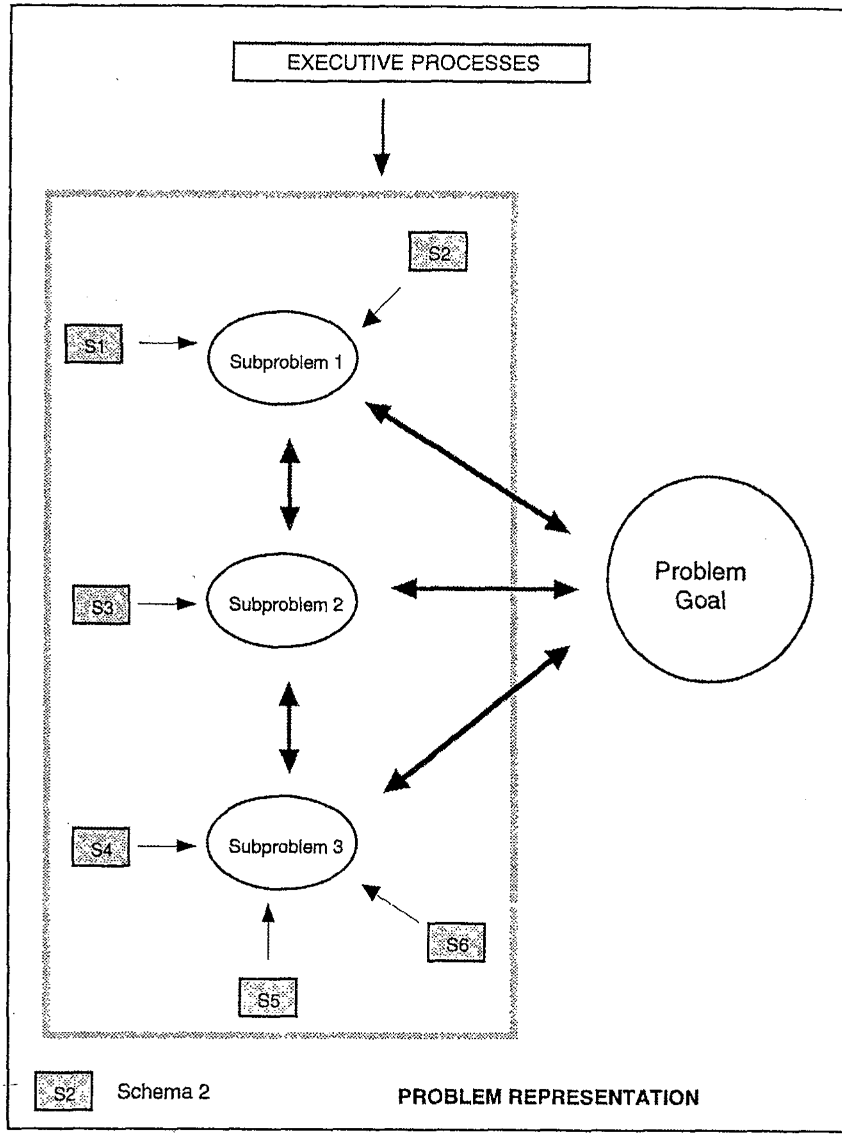

Figure 1: Schemas in problem representation

It appears that better organised schemas could help students break up the problem into subproblems. The nature and number of the subproblems depend on their understanding of the multiple relations between the subproblems and problem goals (Catrombone, 1998). This phase of the students' thinking process which involves generating relevant connections between the subproblems and the problem goal, lies at the heart of problem representation. The construction of representations is a cyclic event where students continue to refine one representation or change to a different one until the correct match is found between schemas that have been accessed and the goal. The goal could be unknown value that has to be determined or a mathematical result that has to proved via a chain of reasoning.

Hence, a complex representation of the problem can be expected to accompany many levels of schema generation and establishment of connections. These relationships between schemas, subproblems and attainment of the problem goal are illustrated in Figure 1. For simplicity, I have shown three subproblems and six schemas that could be related to the subproblems. The dark arrows emphasise the two-way actions between problem goal and subproblem decomposition. The executive processes in Figure 1 refer to control actions such as planning, monitoring and regulating subproblem decomposition, schema retrieval and progress to the problem goal. The influence of executive processes on the activation of schemas is important as it highlights the interaction between domain and strategic knowledge (Alexander and Judy, 1998). The figure does not show the cycles of refinement that can be expected in the course of most problem-solving attempts.

The above model suggests that instructional methods that would help students decompose problems into subproblems would benefit them in three ways. Firstly, students might be expected to access previously-acquired schemas from their memory by examining what is given in the problem. Secondly, the accessed schemas could be deployed in solution of subproblems. Thirdly, students could relate the subproblems in ways that would help them reach the problem goal. The net effect of teaching for problem representation is that students are encouraged to access and use a greater proportion of their previously-learnt knowledge.

SCHEMAS AND UNIVERSITY MATHEMATICS

Students enrolled in first-year university mathematics courses may face a range of learning-related and other problems. Among these, the conceptual transition to advanced mathematics is a significant one (Tall, 1992). Collectively, such problems could be the result of students having an incomplete understanding of mathematical concepts, methods, techniques and assumptions. While the nature of individual understandings of mathematics may vary, schemas provide a useful vehicle to analyse them. A useful way to begin a schema-based analysis would be to consider a focus concept (node) in the schema and examine the number and quality of relations (links) that students have constructed around that focus concept. According this analysis, poor understandings are reflected in the development of fewer and mathematically weaker links with the focus concept.

Let us now look at some examples of students' conceptual difficulties and examine them within the framework of schemas. These examples provided here are based on North American and Australian situations. In the high school mathematics curriculum, the teaching of functions is an important area. The instructional methods employed by teachers, in general, tend to focus on helping students learn the equation of a linear function, $y = mx+c$. Teachers use a range of routine exercise problems completed either in the classroom and at home to reinforce the meaning of the symbols embedded in the equation and their geometric significance (Leinhardt, Zaslavsky, and Stein; 1990). Students are given limited exposure, if any, to a) the exploration of relationship among variables in a function and b) alternative representations of functions (Kaput, 1987). As a result the function schemas that students have developed at the end of their high school life tend to be limited to two major concepts: the symbols in the equation and the representation of the equation within a rectangular coordinate system.

The above schema is deficient in a number of ways. Firstly, there is a lack of information about range and domain of a function. Secondly, the schema does not include information about alternative ways of visualising a function as a correspondence or relationship between two or more variables. We can extend this analysis to other deficiencies that are manifested in students' inability to understand new but related concepts that are taught in the university courses. Suffice to say that a student who comes into a first year calculus course whose understanding of functions is based on the above schema is likely to experience difficulty in making sense of notions such as the range and domain of functions. This is so because the schemas they can access from their Long Term Memory do not have information about these dual characteristics of functions. Students' difficulties could be seen most dramatically in situations which require them to find the limit of a given function or solve problems involving linear programming where the notions of range and domain play a crucial role in the interpretation of corner or critical points.

The limitations of a poorly-developed function schema such as the one described above could also influence the quality of students' problem-solving attempts. For instance, if a particular problem requires that students analyse and interpret it for variables, and rate of change of one or more variables, then the accessing of a function schema which has no link to the idea of variables will not be helpful in representing the problem correctly or the representation constructed by students may be inappropriate as it is unlikely to have a subproblem that draws out the variables in question (see Figure 1).

An impoverished schema of the type mentioned above is not conducive to making sense of advanced mathematical concepts because it does not help students extend their prior knowledge to new conceptual territories. Such schemas can be characterised as being less powerful and lack the sophistication that is required to acquire and use new concepts appropriately.

The above example illustrates the value of schemas in visualising cognitive structures that students develop as they learn mathematics at secondary and university levels. The difference between an average and a low-achieving student could be that the average-achiever has built up schemas that are more complex and better organised than his low-achieving peers. This analysis suggests that one effective way to help low-achieving students would be to analyse their schemas with a view to identifying knowledge gaps and determining the organisational quality.

In order for students to develop a sophisticated schema, say, about functions, they need to increase the number and quality of connections between the definition of functions, families of functions and use of functions among others. As the schema expands one might expect information about related concepts such as derivatives and optimisation become more easily incorporated. In other words students' understanding of derivatives and optimisation is supported by an existing schema which has the relevant prerequisite knowledge. In this way schemas can be argued to provide a measure of the depth of understanding students develop about mathematical ideas. More critically, schemas provide a useful tool for the analysis of conceptual links between university and secondary mathematics.

UNIVERSITY TEACHING AND THE DEVELOPMENT OF SCHEMAS

There is sufficient evidence that secondary students experience difficulties in learning and applying advanced mathematical ideas they encounter in university mathematics courses. At the conceptual level such difficulties could be interpreted as students not being able to 'see' the connection between what they have acquired at the school and what is presented at lectures (ICMI, 1998). The question is what can be done to help students acquire schemas that are more conducive to learning the type of mathematics encountered at the university level. On the basis of schema analysis, I suggest that university teaching adopt the following two strategies:

1. Examine the nature of mathematical schemas that students bring to university courses and identify the strengths and weaknesses of these schemas

For example, in algebra, students often have developed some understanding of methods used in the solution of linear equations with two variables. That is, they have a schema for solving two linear equations. While such a schema may provide information about algorithmic procedures for solving similar equations it may be deficient in terms of having a basic understanding of the meaning of a) *variables*, b) *solution* of the equations and c) a *system* of equations. The latter components of the schema are necessary for students' understanding of and making progress with core concepts in their first-year course on the applications of linear algebra in solving commerce-related problems.

Hence, the first step here is to provide students with experiences in which we can observe deficiencies in their schemas. However, this has to be done in a non-threatening, non-evaluative environment as students in general are rather reticent to expose their weaknesses. I suggest that free-recall and cued-recall procedures be used to elicit the necessary information. These techniques are widely employed by researchers in their attempts to characterise domain knowledge (Lawson and Chinnappan, 1994). In the *free-recall* format, student would be met individually. During the interview each student would be asked to recall as much as they can about the topic or concept in question. In the second interview, we could adopt the cued-recall format in which the student would be given prompts to expand on ideas from the first interview. For instance, if a student fails to mention about variables in the free-recall, we could provide a hint about x and y in the cued-recall format. In their study on geometry learning, Lawson and Chinnappan (1994) have shown that this procedure generates important qualitative data about schemas built by students.

2. Draw on the above information for the development of powerful schemas

Once we have some information about the gaps and weaknesses in the students' schemas, we are in a better position to devise strategies to help students develop appropriate schemas or modify existing schemas. Using the example I have given above about functions, we could adopt the following three strategies in order to develop schemas and facilitate the transition to more advanced schemas involving a system of three or more linear functions.

Firstly, tutorial-type classes can be effective in 'reteaching' the target ideas such as systems of equations, variables, solution of equations and geometric interpretation of solution of two equations. The nature of the target ideas will naturally depend on the prerequisite knowledge that lecturers consider as necessary for the next level course.

Secondly, students could be encouraged to work in groups on a series of activities that are developed in response to improving the above schema. For instance, we could provide a practical problem that requires the generation and solution of a system of two linear equations. Student could attempt to solve this as a group, after which they could brain storm the problem and their solution in terms of the three concepts above. This activity has the potential to facilitate the construction of new links that were non-existent in the schema of the students in the first instance. Teachers could act as critical friends during this exercise.

Computers could also be used as an evaluative tool to check the quality of students' prior knowledge schemas. A useful strategy here would be to ask students to compare and contrast concepts and procedures that are found in the solutions produced by computers with that of their own. For instance, we could ask students to find the limit of a rational function, $f(x)$ with and without the use of computers. Students' own solution attempts would reveal attributes of their schema in this area. Students could then be required to find the limit of the same function with the aid of computer programs such as MAPLE. These programs have in-built facilities that help them visualise the function as well generate a table of values that demonstrate the link between values of x and the limit of $f(x)$. That is, computers provide relatively easy and rapid access to multiple representation of the problem and associated concepts. This technique of using technology to help university students build and refine schemas that are rich in conceptual information about calculus was supported by Palmiter (1991).

The comparison of students' answers and that produced by the computer could thus be used as an important learning and diagnostic activity. The more enriched interpretation of the problem provided by the computer solution has significant pedagogical value in that it would help students not only understand the limitations of their schema, but more importantly, demonstrate in a dynamic manner the relationship between the x, $f(x)$, and the limit of $f(x)$. We can go a step further by asking students to justify their solutions and computer-generated solution to peers, and explain any apparent contradictions. This activity would further enlarge their schema for the concept of limit.

CONCLUSION

There is consensus that success in mathematical learning at the university level is based on a sound understanding of a number of basic concepts and techniques. The acquisition and application of this knowledge could be argued to play a vital role for progress and further participation in more advanced courses in university mathematics. However, there is an urgent need to evaluate and understand the mathematical knowledge base of students aspiring to successfully complete our courses. The schema analysis provided here constitutes a new way of thinking about understanding the character of knowledge that students bring to university mathematics and its effect on learning more advanced mathematics.

While schema-based instruction seems to be a worthwhile activity, further research is needed in this area in order to generate insight into how such an approach could benefit students. In particular, any information about changes in the quality of students' schema as they shift from secondary to university mathematics could prove to be useful in the design of instruction at both the levels.

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