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Closed form expressions for two harmonic continued fractions

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Closed form expressions for two harmonic continued fractions

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Introduction.

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots}}}$$

The expression can continue for ever, in which case it is called an *infinite* continued fraction, or it can stop after some term, when we call it a *finite* continued fraction. For irrational numbers, a continued fraction expansion often reveals beautiful number patterns which remain obscured in their decimal expansion. The interested reader is referred to [1] for a collection of many interesting continued fractions for famous mathematical constants.

When we truncate a continued fraction after some number of terms, we get what is called a *convergent*. In particular, the following finite continued fraction

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{\ddots + \cfrac{b_{n-1}}{a_n}}}$$

is called the n^{th} convergent of the above mentioned continued fraction. If the limit, as n approaches ∞, of the n^{th} convergent exists, we say that the infinite continued fraction converges and the limit is its value.

If in the above continued fraction all numerators b_i are 1, we denote it by $[a_0, a_1, a_2, a_3, \ldots]$, and its n^{th} convergent by $[a_0, a_1, a_2, \ldots, a_n]$. If in addition the coefficients a_n are positive integers, it is not hard to prove that the infinite continued fraction always converges. Continued fractions of this type have applications in cryptography [2, 3]. When the coefficients a_n are positive real numbers, there is a classical theorem due to Seidel and Stern [4, 5], dating back to the 1840s:

The Seidel-Stern Theorem. [6, 7] *If $a_n > 0$ then $[a_0, a_1, a_2, a_3, \ldots]$ converges if, and only if, $\sum a_n$ diverges.*

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the Seidel-Stern Theorem, the infinite continued fraction $[\frac{t}{1}, \frac{t}{2}, \frac{t}{3}, \ldots]$ converges for any positive real number t. Let us call these continued fractions the *harmonic continued fractions* and denote them by

$$HCF(t) = \frac{t}{1} + \cfrac{1}{\frac{t}{2} + \cfrac{1}{\frac{t}{3} + \cfrac{1}{\ddots}}}.$$

When $t = 1$ and $t = 2$, the values of the harmonic continued fractions are known.

Theorem 1

$$HCF(1) = \frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\frac{1}{4} + \cfrac{1}{\ddots}}}} = \frac{2}{\pi - 2}$$

Theorem 2

$$HCF(2) = \frac{2}{1} + \cfrac{1}{\frac{2}{2} + \cfrac{1}{\frac{2}{3} + \cfrac{1}{\frac{2}{4} + \cfrac{1}{\ddots}}}} = \frac{1}{2\ln 2 - 1}$$

The two harmonic continued fractions in Theorem 1 and Theorem 2 are derived from the following continued fractions of Euler [8]:

$$1 + \cfrac{1}{1 + \cfrac{1 \times 2}{1 + \cfrac{2 \times 3}{1 + \cfrac{3 \times 4}{\ddots}}}} = \frac{\pi}{2}, \qquad 2 + \cfrac{1 \times 2}{2 + \cfrac{2 \times 3}{2 + \cfrac{3 \times 4}{2 + \cfrac{4 \times 5}{\ddots}}}} = \frac{1}{2\ln 2 - 1}.$$

Euler's proof involved the Wallis product and differential equations. In this paper, we offer an elementary and direct proof of Theorem 1 and Theorem 2. Our proof uses the Euler-Wallis recurrence formula to establish a *closed form formula* for the convergents of the continued fractions $HCF(1)$ and $HCF(2)$.

Theorem 3 *For any natural number $n \geq 1$,*

$$\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\ddots + \cfrac{1}{\frac{1}{2n-1}}}}} = \cfrac{1}{\frac{1}{2n} \frac{2^2 4^2 \ldots (2n)^2}{1^2 3^2 \ldots (2n-1)^2} - 1},$$

$$\frac{1}{1} + \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{\ddots + \cfrac{1}{\frac{1}{2n}}}}} = \cfrac{1}{\frac{1}{2n+1} \frac{2^2 4^2 \ldots (2n)^2}{1^2 3^2 \ldots (2n-1)^2} - 1}.$$

Theorem 4 *For any natural number $n \geq 1$,*

$$\cfrac{2}{1} + \cfrac{1}{\cfrac{2}{2} + \cfrac{1}{\cfrac{2}{3} + \cfrac{1}{\ddots + \cfrac{1}{\cfrac{2}{n}}}}} = \frac{1}{\sum_{i=0}^{n-1} \frac{(-1)^i}{(i+1)(i+2)}}.$$

Euler-Wallis recurrence formulas.

The following theorem due to William Brouncker [1620-1684], the first President of the Royal Society – is called *the fundamental theorem of continued fractions.* It gives us recursive formulas to calculate the numerator p_n and the denominator q_n of the convergents. John Wallis [1616-1703] and Leonhard Euler [1707-1783] in [8] made extensive use of these formulas which are now called the Euler-Wallis formulas.

Theorem 5 *For any $n \geq 0$, the n^{th} convergent can be determined as*

$$[a_0, a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}$$

where the sequences $\{p_n\}_{n\geq -2}$ and $\{q_n\}_{n\geq -2}$ are specified as follows

$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \forall n \geq 0,$$

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_n = a_n q_{n-1} + q_{n-2}, \forall n \geq 0.$$

The theorem can be proved easily by induction, since

$$\begin{aligned}
[a_0, a_1, a_2, \ldots, a_n, a_{n+1}] &= [a_0, a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}] \\
&= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\
&= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\
&= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}.
\end{aligned}$$

Sometimes we want to investigate the sub-sequences $\{p_{2n}\}$, $\{p_{2n-1}\}$, $\{q_{2n}\}$, $\{q_{2n-1}\}$ and the following theorem is useful in those scenarios.

Theorem 6 *The convergents $[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}$ satisfy the following:*

$$\begin{aligned}
p_n &= \left(a_n a_{n-1} + \frac{a_n}{a_{n-2}} + 1\right) p_{n-2} - \frac{a_n}{a_{n-2}} p_{n-4}, \quad \forall n \geq 2, \\
q_n &= \left(a_n a_{n-1} + \frac{a_n}{a_{n-2}} + 1\right) q_{n-2} - \frac{a_n}{a_{n-2}} q_{n-4}, \quad \forall n \geq 2.
\end{aligned}$$

Proof. By the Euler-Wallis formula,

$$
\begin{aligned}
p_{n-2} &= a_{n-2}\ p_{n-3} + p_{n-4},\ \ \forall n \geq 2,\\
p_{n-3} &= \frac{1}{a_{n-2}}\ p_{n-2} - \frac{1}{a_{n-2}}\ p_{n-4},\\
p_{n-1} &= a_{n-1}\ p_{n-2} + p_{n-3}\\
&= a_{n-1}\ p_{n-2} + \frac{1}{a_{n-2}} p_{n-2} - \frac{1}{a_{n-2}} p_{n-4},\\
p_n &= a_n\ p_{n-1} + p_{n-2}\\
&= a_n a_{n-1}\ p_{n-2} + \frac{a_n}{a_{n-2}}\ p_{n-2} - \frac{a_n}{a_{n-2}}\ p_{n-4} + p_{n-2},\\
p_n &= \left(a_n a_{n-1} + \frac{a_n}{a_{n-2}} + 1\right)\ p_{n-2} - \frac{a_n}{a_{n-2}}\ p_{n-4}.
\end{aligned}
$$

The relation for q_n is proved similarly. ■

Finding HCF(1)

The following theorem establishes closed form formulas for the numerator p_n and the denominator q_n of the convergents of $HCF(1)$. Our Theorem 3 follows from this theorem.

Theorem 7 *The numerator p_n and the denominator q_n of the convergents of $HCF(1)$ are*

$$
\begin{aligned}
p_{2n} &= \prod_{i=1}^{n} \frac{2i+1}{2i},\quad \forall n \geq 0,\\
p_{2n+1} &= \prod_{i=1}^{n+1} \frac{2i+1}{2i},\quad \forall n \geq 0,\\
q_{2n} &= \left(\frac{1}{2n+2}\frac{2^2 4^2 \ldots (2n+2)^2}{1^2 3^2 \ldots (2n+1)^2} - 1\right) \prod_{i=1}^{n} \frac{2i+1}{2i},\quad \forall n \geq 0,\\
q_{2n+1} &= \left(\frac{1}{2n+3}\frac{2^2 4^2 \ldots (2n+2)^2}{1^2 3^2 \ldots (2n+1)^2} - 1\right) \prod_{i=1}^{n+1} \frac{2i+1}{2i},\quad \forall n \geq 0.
\end{aligned}
$$

Proof. By Theorem 6,

$$
\begin{aligned}
p_n &= \left(\frac{1}{n(n+1)} + \frac{n-1}{n+1} + 1\right)\ p_{n-2} - \frac{n-1}{n+1}\ p_{n-4}\\
&= \frac{2n^2+1}{n(n+1)}\ p_{n-2} - \frac{n-1}{n+1}\ p_{n-4},\quad \forall n \geq 2
\end{aligned}
$$

So

$$
\begin{aligned}
p_{2n} &= \frac{8n^2+1}{2n(2n+1)}\ p_{2n-2} - \frac{2n-1}{2n+1}\ p_{2n-4},\quad \forall n \geq 1,\\
p_{2n+1} &= \frac{8n^2+8n+3}{(2n+1)(2n+2)}\ p_{2n-1} - \frac{2n}{2n+2}\ p_{2n-3},\quad \forall n \geq 1.
\end{aligned}
$$

For each $n \geq -1$, let

$$
p_{2n} = p'_{2n} \prod_{i=1}^{n} \frac{2i+1}{2i} \quad \text{and} \quad p_{2n+1} = p'_{2n+1} \prod_{i=1}^{n+1} \frac{2i+1}{2i},
$$

then

$$\frac{2n+1}{2n}\frac{2n-1}{2n-2}\ p'_{2n} = \frac{8n^2+1}{2n(2n+1)}\frac{2n-1}{2n-2}\ p'_{2n-2} - \frac{2n-1}{2n+1}\ p'_{2n-4}, \quad \forall n \geq 2,$$
$$\frac{2n+3}{2n+2}\frac{2n+1}{2n}\ p'_{2n+1} = \frac{8n^2+8n+3}{(2n+1)(2n+2)}\frac{2n+1}{2n}\ p'_{2n-1} - \frac{2n}{2n+2}\ p'_{2n-3}, \quad \forall n \geq 1.$$

This simplifies to

$$(2n+1)^2\ p'_{2n} = (8n^2+1)\ p'_{2n-2} - 2n(2n-2)\ p'_{2n-4}, \quad \forall n \geq 2,$$
$$(2n+3)(2n+1)\ p'_{2n+1} = (8n^2+8n+3)\ p'_{2n-1} - 4n^2\ p'_{2n-3}, \quad \forall n \geq 1,$$

and so,

$$(2n+1)^2\ (p'_{2n} - p'_{2n-2}) = 2n(2n-2)\ (p'_{2n-2} - p'_{2n-4}), \quad \forall n \geq 2$$
$$(2n+3)(2n+1)\ (p'_{2n+1} - p'_{2n-1}) = 4n^2\ (p'_{2n-1} - p'_{2n-3}), \quad \forall n \geq 1$$

Therefore,

$$p'_{2n} - p'_{2n-2} = \frac{2n(2n-2)}{(2n+1)^2}\ (p'_{2n-2} - p'_{2n-4}), \quad \forall n \geq 2, \tag{1}$$
$$p'_{2n+1} - p'_{2n-1} = \frac{4n^2}{(2n+3)(2n+1)}\ (p'_{2n-1} - p'_{2n-3}), \quad \forall n \geq 1. \tag{2}$$

We have $p_{-1} = 1$, $p_0 = 1$, $p_1 = \frac{3}{2}$, $p_2 = \frac{3}{2}$, so

$$p'_{-1} = p'_0 = p'_1 = p'_2 = 1.$$

It follows that

$$p'_n = 1, \quad \forall n \geq -1.$$

This gives us the desired closed form formula for p_n:

$$p_{2n} = \prod_{i=1}^{n} \frac{2i+1}{2i} \quad \text{and} \quad p_{2n+1} = \prod_{i=1}^{n+1} \frac{2i+1}{2i}, \quad \forall n \geq 0.$$

Defining a sequence $\{q'_n\}$ for $\{q_n\}$ similar to $\{p'_n\}$ for $\{p_n\}$, we seek for $\{q'_n\}$ equations corresponding to (1) and (2). With $q_{-1} = 0$, $q_0 = 1$, $q_1 = \frac{1}{2}$, $q_2 = \frac{7}{6}$, we have

$$q'_{-1} = 0, \quad q'_0 = 1, \quad q'_1 = \frac{1}{3}, \quad q'_2 = \frac{7}{9}.$$

This gives us

$$
\begin{aligned}
q'_{2n} - q'_{2n-2} &= \frac{2n(2n-2)}{(2n+1)^2} \dots \frac{(4)(2)}{5^2}(q'_2 - q'_0) \\
&= -\frac{2n(2n-2)}{(2n+1)^2} \dots \frac{(4)(2)}{5^2}\frac{2}{9} \\
&= -\frac{1}{2n}\frac{2^2 4^2 \dots (2n)^2}{1^2 3^2 \dots (2n+1)^2}, \quad \forall n \geq 1, \\
q'_{2n+1} - q'_{2n-1} &= \frac{(2n)^2}{(2n+3)(2n+1)} \dots \frac{2^2}{(5)(3)}(q'_1 - q'_{-1}) \\
&= \frac{(2n)^2}{(2n+3)(2n+1)} \dots \frac{2^2}{(5)(3)}\frac{1}{3} \\
&= \frac{1}{2n+3}\frac{2^2 4^2 \dots (2n)^2}{1^2 3^2 \dots (2n+1)^2}, \quad \forall n \geq 1.
\end{aligned}
$$

Simple algebraic manipulation gives us

$$
\begin{aligned}
q'_{2n} - q'_{2n-2} &= \frac{1}{2n+2}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - \frac{1}{2n}\frac{2^2 4^2 \dots (2n)^2}{1^2 3^2 \dots (2n-1)^2}, \quad \forall n \geq 1, \\
q'_{2n+1} - q'_{2n-1} &= \frac{1}{2n+3}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - \frac{1}{2n+1}\frac{2^2 4^2 \dots (2n)^2}{1^2 3^2 \dots (2n-1)^2}, \quad \forall n \geq 1.
\end{aligned}
$$

Replacing n by i and summing over i, from 1 to n, we find that

$$
\begin{aligned}
q'_{2n} - q'_0 &= \frac{1}{2n+2}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - \frac{1}{2}\frac{2^2}{1^2}, \quad \forall n \geq 0, \\
q'_{2n+1} - q'_1 &= \frac{1}{2n+3}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - \frac{1}{3}\frac{2^2}{1^2}, \quad \forall n \geq 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
q'_{2n} &= \frac{1}{2n+2}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - 1, \quad \forall n \geq 0, \\
q'_{2n+1} &= \frac{1}{2n+3}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - 1, \quad \forall n \geq 0.
\end{aligned}
$$

From here we obtain the desired closed form formula for q_n. ■

Proof of Theorem 1. By Theorem 7,

$$
\begin{aligned}
\frac{q_{2n}}{p_{2n}} &= \frac{1}{2n+2}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - 1, \\
\frac{q_{2n+1}}{p_{2n+1}} &= \frac{1}{2n+3}\frac{2^2 4^2 \dots (2n+2)^2}{1^2 3^2 \dots (2n+1)^2} - 1.
\end{aligned}
$$

We have,

$$
\begin{aligned}
\Gamma\left(n+\frac{3}{2}\right) &= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\dots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{2n+1}{2}\frac{2n-1}{2}\dots\frac{1}{2}\sqrt{\pi}.
\end{aligned}
$$

It follows that

$$\frac{q_{2n}}{p_{2n}} = \frac{\pi}{2}\frac{(n+1)\Gamma(n+1)^2}{\Gamma(n+\frac{3}{2})^2} - 1 \to \frac{\pi}{2} - 1.$$

The above limit is due to the fact [9, 10, 11] that, for any complex number a,

$$\lim_{n\to\infty} \frac{n^a\ \Gamma(n)}{\Gamma(n+a)} = 1,$$

we have used $a = \frac{1}{2}$.

Further

$$\lim_{n\to\infty} \frac{q_{2n+1}}{p_{2n+1}} = \lim_{n\to\infty} \frac{q_{2n}}{p_{2n}} = \frac{\pi}{2} - 1.$$

The reciprocal of this limit shows that $HCF(1) = \frac{2}{\pi-2} \approx 1.75$. ■

Finding HCF(2)

The following theorem establishes closed form formulas for the numerator p_n and the denominator q_n of the convergents of $HCF(2)$. Our Theorem 4 follows from this theorem.

Theorem 8 *For any $n \geq 0$, the numerator p_n and the denominator q_n of the convergents of $HCF(2)$ are*

$$p_n = n+2,$$
$$q_n = (n+2)\sum_{i=0}^{n} \frac{(-1)^i}{(i+1)(i+2)}.$$

Proof. By the Euler-Wallis formula, the convergents $\frac{p_n}{q_n}$ are determined by the following recurrence relation

$$p_{-2} = 0,\ p_{-1} = 1,\ p_n = \frac{2}{n+1}\ p_{n-1} + p_{n-2}, \quad \forall n \geq 0,$$
$$q_{-2} = 1,\ q_{-1} = 0,\ q_n = \frac{2}{n+1}\ q_{n-1} + q_{n-2}, \quad \forall n \geq 0.$$

We have

$$(n+1)p_n = 2p_{n-1} + (n+1)p_{n-2}, \quad \forall n \geq 0,$$
$$(n+1)p_n + np_{n-1} = (n+2)p_{n-1} + (n+1)p_{n-2}, \quad \forall n \geq 0,$$
$$(-1)^n(n+1)p_n - (-1)^{n-1}np_{n-1} = (-1)^n(n+2)p_{n-1} - (-1)^{n-1}(n+1)p_{n-2}, \quad \forall n \geq 0.$$

Taking summation, we have

$$(-1)^n(n+1)p_n = (-1)^n(n+2)p_{n-1} - (-1)^{-1}1p_{-2}, \quad \forall n \geq 0,$$
$$(n+1)p_n = (n+2)p_{n-1} + (-1)^n p_{-2}, \quad \forall n \geq 0.$$

Since $p_{-2} = 0$, it follows that

$$(n+1)p_n = (n+2)p_{n-1}, \quad \forall n \geq 0,$$
$$\frac{p_n}{n+2} = \frac{p_{n-1}}{n+1} = \cdots = \frac{p_{-1}}{1} = 1, \quad \forall n \geq 0,$$
$$p_n = n+2, \quad \forall n \geq 0.$$

Similarly,

$$(n+1)q_n = (n+2)q_{n-1} + (-1)^n q_{-2}, \quad \forall n \geq 0.$$

Since $q_{-2} = 1$, it follows that

$$(n+1)q_n = (n+2)q_{n-1} + (-1)^n, \quad \forall n \geq 0,$$
$$\frac{q_n}{n+2} = \frac{q_{n-1}}{n+1} + \frac{(-1)^n}{(n+1)(n+2)}, \quad \forall n \geq 0.$$

Replacing n by i and summing from $i = 0$ to $i = n$, we obtain

$$\frac{q_n}{n+2} = \frac{q_{-1}}{1} + \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)(i+2)} = \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)(i+2)}, \quad \forall n \geq 0,$$

and this gives the desired closed form formula for q_n. ■

Proof of Theorem 2. By Theorem 8,

$$\frac{q_n}{p_n} = \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)(i+2)}$$

So

$$\begin{aligned}
\lim_{n\to\infty} \frac{q_n}{p_n} &= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)(i+2)} = \lim_{n\to\infty} \sum_{i=0}^{n} (-1)^i \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{(-1)^i}{i+1} + \frac{(-1)^{i+1}}{i+2} \right) \\
&= -1 + 2 \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i+1} = 2\ln 2 - 1.
\end{aligned}$$

The reciprocal of this limit gives $HCF(2) = \frac{1}{2\ln 2 - 1} \approx 2.59$. ■

Conclusion.

We have established explicit formulas for the convergents of the first two harmonic continued fractions $HCF(1)$ and $HCF(2)$ in Theorem 3 and Theorem 4, respectively. From these formulas, we derive the values of $HCF(1)$ and $HCF(2)$ as in Theorem 1 and Theorem 2.

We can see that the result for $HCF(2)$ is relatively easy to establish compared to that for $HCF(1)$, as in Theorem 4 we need only one formula for any length of convergent but in Theorem 3 we have one formula for the odd-length convergents and another one for the even-length convergents of $HCF(1)$.

We have not seen any result on $HCF(t)$ for a general t, but we conjecture that when t is an odd integer, we may need two formulas for odd-length and even-length convergents, and that the value of $HCF(t)$ may involve π when t is odd and involve $\ln 2$ when t is even. Obviously, $HCF(0) = 1$ and $HCF(t) \approx t + \frac{2}{t}$ as $t \to \infty$.

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