Explicit stress-strain equations for modelling frictional materials

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Abstract
In this paper, a comprehensive study on simulating the shearing behavior of frictional materials is performed. A set of two explicit equations, describing the relationship among the shear stress ratio and the distortional strain and the volumetric strain, are formulated independently. The equations contain three stress parameters and three strain parameters and another parameter representing the nonuniformity of stress and strain during softening. All the parameters have clear physical significance and can be determined experimentally. It is demonstrated that the proposed equations have the capacity of simulating the complicated shearing behavior of many types of frictional materials including geomaterials. The proposed equations are used to simulate the stress-strain behavior for 27 frictional materials with 98 tests. These materials include soft and stiff clays in both reconstituted and structured states, silicon sands and calcareous sands, silts, compacted fill materials, volcanic soils, decomposed granite soils, cemented soils (both artificially and naturally cemented), partially saturated soils, ballast, rocks, reinforced soils, tire chips, sugar, wheat, and rapeseed. It has been demonstrated that the proposed explicit constitutive equations have the capacity to capture accurately the shearing behavior of frictional materials both qualitatively and quantitatively. A study on model parameters has been performed.

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Explicit stress–strain equations for modelling frictional materials

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ABSTRACT

In this paper, a comprehensive study on simulating the shearing behaviour of frictional materials is carried out. A set of two explicit equations, describing the relationship among the shear stress ratio and the distortional strain and the volumetric strain, are formulated independently. The equations contain three stress parameters and three strain parameters and
another parameter representing the non-uniformity of stress and strain during softening. All the parameters have clear physical significance and can be determined experimentally. It is demonstrated that the proposed equations have the capacity of simulating the complicated shearing behaviour of many types of frictional materials including geo-materials. The proposed equations are employed to simulate the stress-strain behaviour for twenty-seven frictional materials with ninety-eight tests. These materials include soft and stiff clays in both reconstituted and structured states, silicon sands and calcareous sands, silts, compacted fill materials, volcanic soils, decomposed granite soils, cemented soils (both artificially and naturally cemented), partially saturated soils, ballast, rocks, reinforced soils, tyre chips, sugar, wheat, and rapeseed. It has been demonstrated that the proposed explicit constitutive equations have the capacity to capture accurately, the shearing behaviour of frictional materials both qualitatively and quantitatively. A study on model parameters has been performed.

**KEYWORDS:** clay, constitutive models, frictional materials, gravel, rock, sand, shearing

### 1. Introduction

The mechanical properties of materials are independent of the size of the specimens for physical tests. Those properties are applicable for analysing the performance of engineering structures composed of these materials, usually _viz._ numerical analyses. Modelling the stress-strain behaviour of engineering materials is usually complicated. Taking the work in geotechnical engineering as an example, researchers usually divide the highly diverse geo-materials in groups and formulate different models for each group (e.g., Lade and Kim 1995; Liu and Carter 2002; Masín 2007; Pedroso, Sheng, and Zhao 2009). Clays can often be divided into six groups. They are intact soft structured clay, intact stiff structured clay, reconstituted clay,
partially saturated clay, fissured clay, and reinforced clay. This method of constitutive modelling is required simply because of its complexity. Moreover, even for elastoplastic models formulated specially for a particularly group, they are found to less reliable when the stress and strain conditions vary or the stress paths vary (e.g., Muir-Wood, Mackenzie, and Chan 1993; Liu and Carter 2003a). There are some attempts to formulate explicit stress-strain equations (e.g., Wroth and Bassett 1965; Duncan and Chang 1970; Prat and Bazant 1991). The stress-strain relationships may represent well the highly plastic and non-linear behaviour of geomaterials if the values of the material parameters are appropriately selected. Guidance for selecting the values of soil parameters forms an important part of the “stress path method” of geotechnical analysis introduced by Lambe (1973). The accuracy of the performance of the equations will be improved if the stress paths of the tests from which values of the soil parameters are obtained are designed to be similar to those in the field. Because of its simplicity, explicit stress and strain equations are widely used in engineering practice (e.g., Potts and Zdravkovic 1999; Chai and Carter 2011).

In this paper, a set of two explicit equations, describing the relationship between the shear stress ratio and the distortional strain and that between the volumetric strain and the distortional strain, are formulated independently. The features of the constitutive equations are demonstrated, and it is seen that the proposed equations have the capacity to represent a wide range of behavior patterns of geomaterials. The proposed equations are then employed to simulate the stress-strain behaviour of a wide range of frictional materials for about one hundred tests. The constitutive equations are evaluated based on these simulations. An extensive study on model parameters has also been performed.

2. Proposed Explicit Stress and Strain Equations
A set of two equations is proposed, suitable for describing the behaviour of frictional materials under monotonic shearing. The distortional deformation and volumetric deformation are firstly formulated independently. Then the equations are examined according to values of soil parameters identified. Two sets of equations parameters are selected, one set for the strength and the other for the stiffness. This is rational because the two controlling factors in engineering designs are the strength and the deformation of the structure.

The factors, which influence the response of a material, may be divided into two categories: external and internal features. Internal features are those such as mineralogy, grain size, grain shape, and arrangement of the grains. External features are those such as stress history, pressure level and initial packing state including initial density.

The definitions of stress and strain terms are given in the Appendix.

2.1. Conceptual ideas and assumptions

Assumption 1

The shearing response of frictional materials is dependent on shear stress ratio, not the absolute shear stress. Therefore, material properties for shearing behaviour are described in terms of shear stress ratio.

A new distortional modulus, $G$, is suggested, as seen equation (A9) for secant distortional modulus and equation (A10) for tangent distortional modulus. These two moduli are related to the stress ratio directly.

Two types of material behavior
Two types of shearing response of frictional materials are identified. Following Schofield and Wroth (1968), “dry behaviour” is defined as the response of a material to monotonic shearing that reaches a peak strength firstly then drops to a final strength, and “wet behaviour” is defined as the response that approaches monotonically the final strength as the distortional increases (Figure 1).

Assumption 2

The shearing response of a frictional material can be expressed as an exponential function of characteristic factors modified by a linear function of response factors, i.e.,

\[
\eta = f_1(\xi, \varepsilon_d) * e^{f_2(\xi - \varepsilon_d)}
\]  

(1)

where \(\eta\) and \(\varepsilon_d\) are the shear stress ratio and the distortional strain separately (see Appendix), respectively; \(\xi\) and \(\zeta\) are influence factors; and \(f_1\) and \(f_2\) are two linear functions, respectively.

2.2 Modelling distortional deformation

Assumption 3

The final state for the deformation of a frictional material under monotonic shearing is that the material can be continuously distorted with its stress state and voids ratio remaining constant.

The final state is a perfect plastic state, and soil has no resistance to further increase of shear stress ratio. For sands and most clays, the final state is the critical state of deformation. For clay with predominantly platy particles, the final state may be the residual strength if the platy
particles have formed a smooth sliding plane along the failure surface. The residual strength behaviour is not studied in this paper. A comprehensive study of the residual strength behaviour can be found from works such as Skempton (1985).

The relationships between the tangent distortional modulus and the distortional strain, for both a dense soil and a loose soil, are shown in Figure 2(a). The two curves intersect twice on this plot, at points I and C. Point C corresponds to the final failure state, or the critical state of deformation. If the coordinates for point I are denoted as \((\varepsilon_{d,i}(\eta), G_i)\), then the quantities \(\varepsilon_{d,i}(\eta)\) and \(G_i\) may be defined as the characteristic distortional strain and the characteristic distortional modulus.

**Assumption 4**

There exists a common point, the characteristic point, in the \(\varepsilon_d(\eta)\) and \(G_t\) curves for all shearing tests on a frictional material of the same mineralogy.

The coordinates of the characteristic point I, \(\varepsilon_{d,i}(\eta)\) and \(G_i\), are therefore independent of the initial stress and strain states of the material. Point I in Figure 2(a) is therefore a mathematical convergent point of a frictional material in the \(\varepsilon_{d,i}(\eta)\) and \(G_t\) space.

**Assumption 5**

There is no distortional strain if the shear stress ratio is equal to be zero.

\[
\varepsilon_d = 0 \text{ if } \eta = 0 \tag{2}
\]

Assumptions (3) and (4) can be expressed as
\[
\begin{align*}
\eta & \xrightarrow{\text{as } \varepsilon_d \text{ increases}} M_f \\
\left. \frac{\partial \eta}{\partial \varepsilon_d} \right|_{\varepsilon_d = \varepsilon_{d_i}} &= G_i
\end{align*}
\] (3)

It is found that the following equation, a special form of mathematical expression (1), satisfies conditions (2) and (3).

\[
\eta = M_f \left[ 1 + \left( \frac{\varepsilon_d}{\varepsilon_{d_{i,u}}} \right) \left( \frac{\varepsilon_{d_{i,u}}}{\varepsilon_{d_i}} - 1 \right) \right]
\] (4)

\(\varepsilon_{d_{i,u}}\) is a model parameter.

Under standard test procedures, a specimen of frictional materials may be expected to deform with fairly uniform stress and strain distribution for the entire process of “wet behaviour” and for the hardening process of “dry behaviour”. It is normally expected that non-uniformity in stress and strain states will occur during softening. As a result, the deformation of the softening is affected by the factors such as bulge, fracture, and rapture. An effective way to describe the influence of this non-uniformity is to modify constitutive equations through the introduction of a function representing the relative effects of the non-homogeneous behaviour. Equation (4) is modified by a relative non-uniformity function \(D(\varepsilon_d)\) as follows:

\[
\eta = M_f \left[ 1 + D(\varepsilon_d) \left( \frac{\varepsilon_d}{\varepsilon_{d_{i,u}}} - 1 \right) \right] e^{\frac{\varepsilon_d}{\varepsilon_{d_{i,u}}}}
\] (5)

where is \(D(\varepsilon_d)\) expressed as

\[
D(\varepsilon_d) = e^{B \left( \frac{\varepsilon_d}{\varepsilon_{d_{i,u}} + \varepsilon_d} \right)}
\] (6)

in which
\[
B = \begin{cases} 
0 & \text{for } \varepsilon_d \leq \varepsilon_{d,u} + \varepsilon_{d,i} \\
\beta & \text{for } \varepsilon_d > \varepsilon_{d,u} + \varepsilon_{d,i}
\end{cases}
\] (7)

It can be seen that equation (5) also satisfies conditions (2) and (3). Only the second part of the equation (4) is modified by the non-uniformity of stress and strain states because the first part represents the final strength, which is found not influenced by the non-uniformity. The proposed equation describes the distortional strain of frictional materials under monotonic loading as a function of the current shear stress ratio.

It is noted that the description of the non-uniformity in stress and strain states of the specimens in constitutive equations usually is not important. In this case, \( B = 0 \) can be assumed.

2.3 Modelling volumetric deformation

The plastic flow rule is one of the essential parts in modelling plastic deformation. Many flow rules for geomaterials are proposed, e.g., by Roscoe, Schofield, and Wroth (1958), Rowe (1962), Mroz, Norris, and Zienkiewicz (1981), Khalili, Habte, and Valliappan (2005), Horpibulsuk et al. (2010) Liu, Carter, and Airey (2011), and Suebsuk and Horpibulsuk (2010 and 2011). For geomaterials, the plastic flow rule during monotonic shearing is usually assumed to be dependent on the critical state strength and the current stress ratio. For an example, the flow rule in the Cam Clay model (Schofield and Wroth 1968) is described by

\[
\frac{d\varepsilon^p_v}{d\varepsilon^p_d} = M - \eta
\] (8)

where \( \varepsilon^p_d \) is plastic distortional strain, \( \varepsilon^p_v \) is plastic volumetric strain, and \( M \) is the critical state stress ratio.
The volumetric deformation of frictional materials is rather completed and varies with a particular material. After examining a large body of experimental data, the following equation is proposed to represent the relationship between the distortional strain and volumetric strain increments, and there is no imposed dependency on material strength or the current shear stress ratio in the proposed equation.

\[
\frac{d\varepsilon_v}{d\varepsilon_d} = M^\gamma - \eta^\gamma
\]  

(9)

where

\[
\eta^\gamma = M^\gamma \left[ 1 + \left( \frac{\varepsilon_d}{\varepsilon_{d,\text{in}}} - 1 \right) e^{-\frac{\varepsilon_d}{\varepsilon_{d,\text{in}}}} \right]
\]  

(10)

\(M^\gamma, \eta^\gamma, \varepsilon^\gamma_{d,\text{in}},\) and \(\varepsilon^\gamma_{d,\text{in}}\) are parameters to describe the volumetric deformation of frictional materials. Meanwhile the value of \(\eta^\gamma\) can be calculated from \(M^\gamma, \varepsilon_d, \varepsilon^\gamma_{d,\text{in}},\) and \(\varepsilon^\gamma_{d,\text{in}}.\) Hence, there are only three parameters for the flow rule. As shown in in Figure 2(b), \(\varepsilon^\gamma_{d,\text{in}}\) is the value of the distortional strain at the convergent point J.

Consequently, the total volumetric strain can be computed by integration:

\[
\varepsilon_v = \int_0^{\varepsilon_v} \left( M^\gamma - \eta^\gamma \right) d\varepsilon_d
\]  

(11)

Substituting Equation (10) into (11), the following explicit expression for the total volumetric strain is obtained through integration:

\[
\varepsilon_v = e^\gamma_{d,\text{in}} M^\gamma \left[ 1 - \frac{\varepsilon_d}{\varepsilon_{d,\text{in}}} \right] + \frac{\varepsilon^\gamma_{d,\text{in}}}{\varepsilon_{d,\text{in}}} \left( \frac{\varepsilon_d}{\varepsilon_{d,\text{in}}} - 1 \right) e^{-\frac{\varepsilon_d}{\varepsilon_{d,\text{in}}}} \right]
\]  

(12)
3. Features of the Proposed Constitutive Equations

3.1 Model parameters

Seven parameters are needed to define the two constitutive equations represented by Eqns (5) and (12). Four of the parameters are needed for the distortional strain and shear stress ratio equation. They are $M_f$, $\varepsilon_{d,u}$, $\varepsilon_{d,i}$, and $B$ in Eqn (7). The other three parameters are required for the volumetric strain and the distortional strain equation, and they are $M^*$, $\varepsilon^{v}_{d,u}$, and $\varepsilon^{v}_{d,i}$.

$M_f$ is the value of the final shear strength of a frictional material, which is a material constant. Parameter $\varepsilon_{d,i}$ is the distortional strain at the characteristic point I in the $\varepsilon_d(\eta)\sim G_t$ coordinates (Figure 2a). The characteristic point I is assumed to be independent of the stress and strain state of the material, and is a convergent point of geo-material behaviour in the $\varepsilon_d(\eta)\sim G_t$ space.

Parameter $\varepsilon_{d,u}$ is the distortional strain at $\eta = M$ before the peak strength is reached, and consequently it represents the relative shear stiffness for a given material (Figure 1). With a decrease in the value of $\varepsilon_{d,u}$, the material reaches the peak strength at a lower shear strain. The value of $\varepsilon_{d,u}$ can be measured directly from a shear stress-strain $\varepsilon_d(\eta)\sim \eta$ curve.

Parameter $B$ (Eqn 7) describes the softening of a material. The influence if this parameter is illustrated in section 3.5.

According to Equation (12), $M^*$ is equal to the value of dilatancy at $\eta = 0$. Theoretically speaking, the dilatancy at $\eta = 0$ should be zero for an isotropic material. Then the flow rules such as the proposed one and those in Cam Clay Model (Schofield and Wroth 1968) and Rowe’s dilatancy (1962) are not correct. However, it has been observed that those flow rules describe
satisfactorily experimental data on shearing behaviour of soils (See Muir-Wood 1990). Obviously such equations are not valid for isotropic compression. The effect of the problem should be insignificant for a model concerning the effect of monotonic shearing only. For the identification, $M^v$ can be measured directly from a volumetric and distortional curve (Figure 1b) because a non-zero value for the initial dilatancy is clearly found in most shearing test data.

As shown in Figure 2, $\varepsilon_{v,i}^d$ is the distortional strain at the convergent point for curvature rate, and it is a material constant. $\varepsilon_{v,u}^d$ is the value of distortional strain at the peak volumetric strain (Figure 1b).

It is noticed that difficulty may arise in determining parameters $\varepsilon_{d,u}$ and $\varepsilon_{d,u}^v$ from “wet behaviour”. The distortional strain at the $\eta = M_f$ before peak state and that at peak volumetric strain state usually cannot clearly defined. For “wet behaviour”, a practical method for determining those parameters is by the means of best fitting.

3.2 Peak strength of a frictional material

The proposed stress-strain equations predict a peak strength for frictional materials, irrespective of initial stress and strain states. It can be shown mathematically that the peak stress ratio $\eta_p$ and the corresponding distortional strain at the peak $\varepsilon_{d,p}$ are given by:

$$\eta_p = M_f \left[ 1 + e^{\frac{1+\varepsilon_{d,u}^v}{\varepsilon_{d,i}^v}} \right]$$  \hspace{1cm} (13)

$$\varepsilon_{d,p} = \varepsilon_{d,i}^v + \varepsilon_{d,u}$$  \hspace{1cm} (14)

Furthermore, from Equation (14) it can be shown that
\[
\begin{cases}
\eta_p < 1.027M_f \text{ when } \frac{\varepsilon_{d,i}}{\varepsilon_{d,u}} < 0.25 \\
\eta_p < 1.0002M_f \text{ when } \frac{\varepsilon_{d,i}}{\varepsilon_{d,u}} < 0.1
\end{cases}
\]

(15)

For frictional material with “wet behaviour”, values of \(\varepsilon_{d,i}/\varepsilon_{d,u}\) less than 0.25 are normally expected. Therefore, the difference between the peak strength and the final strength is practically negligible.

### 3.3 Features of the shear stress ratio and distortional strain curve

In order to demonstrate the capability of the proposed general equation (5), nine cases of simulation have been made. The values of the material parameters for these cases are listed in Table 1. Non-uniformity of stress and strain state is not considered in this calculation, therefore, \(\beta\) is assigned to be zero as per Eqn (7). The variation of shear stress ratio \(\eta\) with the distortional strain \(\varepsilon_d\) is indicated in Figure 3. The numbers in Figure 3 indicate the cases simulated with the values of model parameters explained in Table 1. The values of material parameters are selected to be in consistence with the engineering materials that it has the capacity to represent. For curves (1) and (2), the values selected for \(M\) are different to all other cases, because the materials represented are very stiff and they have much higher final strength than most geo-materials.

As is expected from the mathematical format of equation (5), the following general features of the curves are observed.

1. The absolute magnitude of the shear stress is controlled by the value of the final strength \(M_f\).
(2) The absolute magnitude of the distortional strain is controlled by the value of the characteristic strain $\varepsilon_{d,i}$.

(3) The shape and curvature of the stress-strain relationship is controlled by the ratio $\varepsilon_{d,i}$ over $\varepsilon_{d,u}$.

It is seen from the simulations that the proposed simple model has the capability to represent a wide range of behaviour of engineering materials. For the convenience of discussion, three categories of materials are divided: hard materials, geo-materials, and soft materials, depending on the value of characteristic strain $\varepsilon_{d,i}$.

The geo-materials considered have two typical patterns of behaviour, viz. the “wet behaviour” and the “dry behaviour”. The “wet behaviour” can exhibit a wide range of different stiffness.

The so-called “hard” material has a basically linear shear stress-strain relationship prior to the peak, and reaches the peak strength at very small strain. If $\alpha \leq 0.25$, there is virtually no softening, and the material behaves as would a perfectly plastic material after the strength is reached. If $\alpha \geq 0.25$, softening is observed.

If $\varepsilon_{d,i} \to 0$ the shear stress-strain equation (9) is simplified as

$$\eta = \langle /\tau \lambda > M_f $$ (16)

However, if $\varepsilon_{d,i} \leq 0.001$ and $\varepsilon_{d,i}/\varepsilon_{d,i} \leq 0.1$, the distortional strain virtually varies linearly with the shear stress ratio before the shear stress ratio reaches $M_f$. When the shear stress ratio reaches $M_f$, the material has reached plastic perfect state. Mathematically the proposed stress-strain equation (4) is simplified as
\[
\begin{cases}
\eta = \left( \frac{\varepsilon_d}{\varepsilon_{i,d} + \varepsilon_{d,\alpha}} \right) M_f \quad \text{when } \eta < M_f \\
\frac{d\varepsilon_d}{d\eta} = 0 \quad \text{when } \eta = M_f
\end{cases}
\]

(17)

In this case, the material is extremely hard and behaves in the same manner as the classical elastic perfectly-plastic material. Equation (17) has been widely used in metal plasticity (e.g. Calladine 1985).

The so-called “soft” material has almost a linear relationship between the shear stress ratio and the shear strain. If \( \varepsilon_{d,i} \rightarrow \infty \) the shear stress-strain equation is simplified to

\[
\eta = \left( \frac{\varepsilon_d}{\varepsilon_{d,\alpha}} \right) M_f
\]

(18)

In this case, the material is extremely soft, and the distortional strain increases linearly with shear stress ratio.

### 3.4 Features of the volumetric strain and distortional strain curve

In order to demonstrate features of the proposed general equation (12), eight cases of simulation have been made. The values of the material parameters are listed in Table 2. Non-uniformity of stress and strain state is not considered in this calculation, therefore, \( \beta = 0 \) is assumed. The variation of volumetric strain \( \varepsilon_v \) with distortional strain \( \varepsilon_d \) is indicated in Figure 4.

The following features of the strain curve are observed, which is useful for the identification of model parameters.
When $M^v > 0$, the initial volumetric deformation is compressive; and when $M^v < 0$, the initial volumetric deformation is expansive. There is no volumetric deformation if $M^v = 0$.

(2) The total volumetric deformation at the failure is zero if $\varepsilon^v_{d,i} / \varepsilon^v_{d,u} = 1$.

(3) If $\varepsilon^v_{d,i} / \varepsilon^v_{d,u} < 1$, the volumetric deformation is monotonic. If $\varepsilon^v_{d,i} / \varepsilon^v_{d,u} > 1$, the volumetric deformation changes signs. For $M^v > 0$, the volumetric deformation changes from initial compression to expansion at failure. For $M^v < 0$, the volumetric deformation changes from initial expansion to compression at failure. All these types of volumetric deformation are observed experimentally (e.g., Airey, Carter, and Liu 2011).

### 3.5 Effect of $\beta$ on shearing deformation

The effect of non-uniformity of stress and strain state on softening behaviour is demonstrated. In this simulation, it is assumed that $M^v = M_f$, $\varepsilon^v_{d,i} = \varepsilon_{d,i}$, and $\varepsilon^v_{d,u} = \varepsilon_{d,u}$. The values of the material parameters are listed in Table 3. The influence of $\beta$ on strain state during softening is shown in Figure 5a and 5b.

The influence of parameter $\beta$ on the softening behaviour is clearly demonstrated. Non-uniformity usually occurs during softening. A detailed study of its influence on the behaviour of frictional materials is reported by Read and Hegemier (1984), and the behaviour pattern is similar to that shown in Figure 5.

### 3.6 The final state of frictional material during shearing
It can be seen from equations (5) and (12) that when a material reaches the final strength, \( \eta = M_f \), the distortional strain is predicted to be infinite and the volumetric strain is constant. Therefore, the final failure state of deformation predicted is that the material can be distorted continuously at constant voids ratio and with the stress state remaining unchanged.

The characteristics of soil deformation at the critical state satisfy the condition of deformation for the final state. Therefore, the critical state of deformation can be described by the proposed constitutive equations. The applicability of the concept of the critical state of deformation has been observed for a wide range of geo-materials such as clays, sands; cemented soils, soft rocks, and hard rocks (e.g., Carter and Airey 1994; Novello and Johnston 1995; Liu and Carter 2003b).

The volumetric strain at the final state \( \varepsilon_{v,f} \) is found to be

\[
\varepsilon_{v,f} = \left(1 - \frac{\varepsilon_{d,i}^v}{\varepsilon_{d,u}^v}\right) \varepsilon_{d,i}^v M^v
\]  

(19)

The feature of the volumetric strain at final state is listed in Table 4.

In addition, the following features of the proposed model should be notice.

(1) Although the proposed explicit constitutive equations are expressed in terms of general stress and strain parameters, as seen in the Appendix of the paper, some further work are needed to apply the equations for general stress and strain conditions such as multi-axial deformation or strain tensor states. All the evaluation of the proposed model is made based on conventional triaxial tests, and thus is for two dimensional (2D) stress and strain conditions. As demonstrated in details in the work by Khalili and Liu (2008), a study of the failure surface of the material in the \( \pi \) plane.
is the minimum requirement for the generalization of a 2D constitutive model into 3D.

(2) Usually geomaterials exhibit some elastic deformation, and elastic deformation may be the major part of deformation for materials with high peak strength. No separation of elastic and plastic deformation is made in this study, similar to Duncan-Chang’s model (1970). As a result, the model is suitable for monotonic loadings only.

(3) When the mean effective is vanishingly small, the shear stress ratio will become indefinitely large. This is mathematically a singular point, where the material fails. Consequently, it is indicated that the proposed model is only suitable for representing pure frictional materials without cohesion.

4. Validation of the Proposed Equations

The explicit constitutive equations are employed to simulate the mechanical of frictional materials. A wide selection of frictional materials has been selected. There are twenty-seven different types of frictional materials with ninety-eight tests in total. All the experimental data are obtained from previous publications. Equation parameters are firstly determined according to the methods introduced in the previous section. Then, equations (5) and (12) are employed to simulate the observed stress and strain relationship. The detailed simulations for ten frictional materials and comparisons with experimental data are presented in this paper. The values of soil parameters identified for all tests are used in the parametric study in the next section. A summary of the materials and tests is listed in Table 5. All the tests are carried by means of conventional triaxial apparatuses.

The simulations of the shearing behaviour of ten frictional materials are presented. They are Fuji sand, Cambria sand, reconstituted Corinth marl, intact Corinth marl, compacted Grand-
Maison filter, cemented volcanic ashes, granular rock, oolitic limestone, sugar, and rape. The values of soil parameters used to generate the theoretical curves are listed in Table 6. Comparisons of the theoretical simulations and experimental data are shown in Figures 6–15, where both the distortional strain and shear stress ratio relationships and the distortional strain and the volumetric strain relationships are shown. In all the simulations, B = 0 is assumed, and thus the influence of the non-uniformity of the deformation of the specimens is not considered in this study.

The shearing behaviour of frictional materials is very complicated, and enormous variation of material behaviour in both magnitudes and patterns is seen in the experimental data. Despite of decades of research on and with hundreds, if not thousands, of constitutive models developed for geomaterials, reliable representing geomaterial behaviour, either qualitatively or quantitatively, is highly challenging. Most exiting models, with explicit or inexplicit stress and strain equations, may only capture two types of material behaviour, that is, wet behaviour and dry behaviour as defined by Schofield and Wroth (1968). As seen in the simulations presented in this section, the overall agreement between the test results and the simulations is highly satisfactory. It has been demonstrated that the proposed explicit constitutive equations have the capacity to capture accurately the mechanical behaviour of frictional materials both qualitatively and quantitatively.

5. Study of the Model Parameters

In section 4 model parameters for twenty-seven types of frictional materials are identified. Based on these data, a study on the model parameters is made here. The correlation between parameter $M_f$ and $M'$ is shown in Figure 16. The range of final failure stress ratio is in a
band of 0.5 to 2.5. This strength corresponds to critical state strength in critical state soil mechanics. The average value of strength is 1.5. The value of the initial volumetric slope $M_v$ is between 0 to 2.2. On average, the following relationship is obtained

$$M_v = 0.6M_f \quad (20)$$

The correlation between parameter $\varepsilon_{d,i}$ and $e_{d,i}'$ is shown in Figure 17. The range of $\varepsilon_{d,i}$ is from 0 to 0.08, and the range for $e_{d,i}'$ is about in a band of 0 to 0.3. On the average,

$$e_{d,i}' = 3\varepsilon_{d,i} \quad (21)$$

The correlation between parameter $\varepsilon_{d,u}$ and $e_{d,u}'$ is shown in Figure 18. The range of $\varepsilon_{d,i}$ is from 0 to 0.45, and the range for $e_{d,u}'$ is in a band of 0 to 0.5. On the average,

$$e_{d,u}' = 1.25\varepsilon_{d,u} \quad (22)$$

The following the constitutive equation is suggested if there is not reliable data to identify the volumetric strain parameters, i.e., $M_v$, $\varepsilon_{d,u}$, and $e_{d,i}'$.

$$\varepsilon_i = 1.8\varepsilon_{d,i} M_f \left[ 1 - \frac{2.4\varepsilon_{d,i}'}{\varepsilon_{d,u}} + \frac{3\varepsilon_{d,i} + \varepsilon_d}{\varepsilon_{d,u}} - 1 \right] \left( \frac{\varepsilon_d}{\varepsilon_{d,i}} \right) \quad (23)$$

Consequently, Eqns (4) and (23) are the simplified constitutive equations for frictional materials and there are only three parameters, i.e., $M_f$, $\varepsilon_{d,u}$, and $\varepsilon_{d,i}$.

6. Conclusions

Based on the assumption that there is a characteristic distortional strain at which the tangent distortional modulus is independent of the initial stress and strain states of the material, a
A general explicit shear stress-strain equation for frictional materials has been formulated. Assuming a relaxed form of the dilatancy law proposed in the original Cam Clay model and applying it for the total strain, an explicit equation for the volumetric strain is also obtained. Physical meanings of the model parameters are seen from the introduction of them in equations derivation, and the values of these parameters are measurable from physical tests.

The proposed constitutive equations have been used to simulate the shearing behaviour of twenty-seven frictional materials with ninety-eight tests. Despite the complexity in the behaviour of frictional materials and the variations in test conditions, the overall agreement between the test results and the simulations is very good. It has been demonstrated that the proposed explicit constitutive equations have the capability to simulate accurately the mechanical behaviour of frictional materials both qualitatively and quantitatively. The study on the model parameters provides some useful guideline on the understanding and estimation of the parameters. In the simplified form, there are only three parameters in the explicit constitutive equations, i.e., $M_f$, $\varepsilon_{d,u}$, and $\varepsilon_{d,i}$.

Acknowledgements

The authors would like to express their thanks to Professors Poulos H. G. and Carter J. P. for the useful discussions and helps in preparing this paper. The last author is grateful to the financial support from Suranaree University of Technology and the Thailand Research Fund under the TRF Senior Research Scholar program Grant No. RTA5980005.

References


Calladine, C. R. 1985. Plasticity for engineers, Chichester: Ellis Horwood Ltd.


### Table 1. Values of material parameters.

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<th>Soft material</th>
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<td>$M_f$</td>
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<td>$\varepsilon_{d,i}$</td>
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<td>0.005</td>
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<td>$\varepsilon_{d,i}/\varepsilon_{d,u}$</td>
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### Table 2. Values of material parameters.

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<td>1</td>
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<td>$-1$</td>
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<td>$\varepsilon_v \equiv 0$</td>
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<td>0.03</td>
<td>0.03</td>
<td>0.15</td>
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<td>2</td>
<td>6</td>
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<td>3</td>
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<td>$\beta$</td>
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### Table 4 Features of the final volumetric strain

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<th>$\varepsilon_{d,3}'/\varepsilon_{d,2}'$</th>
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<tr>
<td>$&gt;0$</td>
<td>$&lt;1$</td>
<td>Compressive</td>
</tr>
<tr>
<td>$&lt;0$</td>
<td>$&lt;1$</td>
<td>Expansive</td>
</tr>
<tr>
<td>$&lt;0$</td>
<td>$&gt;1$</td>
<td>Compressive</td>
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Table 5. Lists of frictional materials and no of tests and references.

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<th>Materials</th>
<th>Reference</th>
<th>No. of tests</th>
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<td>Fuji</td>
<td>Tatsuoka 1972</td>
<td>3</td>
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<tr>
<td></td>
<td>Cambrai</td>
<td>Yamamuro and Lade 1996</td>
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<td></td>
<td>Ham River</td>
<td>Daramola 1980</td>
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<tr>
<td></td>
<td>Soma</td>
<td>Ladd et al. 1977</td>
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</tr>
<tr>
<td>Clay</td>
<td>Ancona (Soft)</td>
<td>Canestrari and Scarpelli 1993</td>
<td>2</td>
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<tr>
<td></td>
<td>Corinth marl (Stiff both intact and reconstituted)</td>
<td>Burland et al. 1996</td>
<td>6</td>
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<tr>
<td></td>
<td>Intact Weald</td>
<td>Henkel 1956</td>
<td>2</td>
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<tr>
<td></td>
<td>Intact Nanticoke</td>
<td>Lo 1972</td>
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<td>Calcareous sand</td>
<td>NR sand</td>
<td>Kaggwa 1988</td>
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<td></td>
<td>Bass Strait</td>
<td>Poulos, Uesugi, and Young 1982</td>
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<tr>
<td></td>
<td>Ballyconneely</td>
<td>Golightly and Hyde 1988</td>
<td>5</td>
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<tr>
<td></td>
<td>Dogs Bay</td>
<td>Golightly and Hyde</td>
<td>6</td>
</tr>
<tr>
<td>Material Type</td>
<td>Description</td>
<td>Authors</td>
<td>Year</td>
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<td>-----------------------------</td>
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<tr>
<td>Volcanic ashes</td>
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<td>Decomposed granite soil</td>
<td>Decomposed granite soil</td>
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<td>Ballast</td>
<td>Alva-Hurtado, McMahon, and Steward 1981</td>
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<td>Clay-sand mixture</td>
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<td>Marachi et al. 1969</td>
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<td>Cemented soil</td>
<td>Artificially, carbonate sand</td>
<td>Huang 1994</td>
<td>12</td>
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<td>Compacted material</td>
<td>Grand-Maison filter</td>
<td>Dendani, Flavigny, and Fry 1988</td>
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<td>Cazzuffi et al. 1994</td>
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<td>Limestone</td>
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<td>Granular rock</td>
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<td>Sugar</td>
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<td></td>
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<td></td>
<td>1990</td>
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<td>Tire chips</td>
<td>Wu, Christopher, and</td>
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<td>Robert 1997</td>
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Table 6. Values of soil parameters for Fuji sand.

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<tr>
<th>Soils and Figs</th>
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<th>$M_f$</th>
<th>$e_{d,i}$</th>
<th>$e_{d,u}$</th>
<th>$M^v$</th>
<th>$\varepsilon_{v,i}$</th>
<th>$\varepsilon_{v,u}$</th>
<th>$\beta$</th>
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<td>Fuji sand</td>
<td>$e_i = 0.52$</td>
<td>1.5</td>
<td>0.037</td>
<td>0.01</td>
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<td>0.037</td>
<td>0.01</td>
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<tr>
<td></td>
<td>$e_i = 0.78$</td>
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<td>0.037</td>
<td>0.04</td>
<td>0.7</td>
<td>0.06</td>
<td>0.04</td>
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<tr>
<td></td>
<td>$e_i = 0.85$</td>
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<td>0.037</td>
<td>0.2</td>
<td>0.94</td>
<td>0.037</td>
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<td>Cambria sand</td>
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<td>1</td>
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<td>0.4</td>
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<tr>
<td></td>
<td>MPa</td>
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<td>$\sigma_{3i} = 5.8$ MPa</td>
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<td>0.25</td>
<td>0.8</td>
<td>0.2</td>
<td>0.5</td>
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<td>$\sigma_{3i} = 4$ MPa</td>
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<td>0.505</td>
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<td>$\sigma_{3i} = 2.1$ MPa</td>
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<td>0.04</td>
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<tr>
<td>Corinth marl (reconstituted)</td>
<td>OCR = 7</td>
<td>1.6</td>
<td>0.015</td>
<td>0.015</td>
<td>−0.368</td>
<td>0.02</td>
<td>0.008</td>
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<td></td>
<td>OCR = 2.4</td>
<td>1.6</td>
<td>0.025</td>
<td>0.035</td>
<td>0.5</td>
<td>0.04</td>
<td>0.037</td>
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<td>OCR = 1.4</td>
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<td>0.027</td>
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<td>0.848</td>
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<td>0.04</td>
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<td>Corinth marl (intact)</td>
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<tr>
<td>Grand-Maison</td>
<td>500 kPa</td>
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<td>Grand-Maison</td>
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<td>0.03</td>
<td>0.045</td>
<td>0.61</td>
<td>0.069</td>
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<td>800 kPa</td>
<td>1.78</td>
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<td>0.01</td>
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<td>$\sigma'_{3i}$</td>
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<td>0.02</td>
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<td>0.09</td>
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<tr>
<td><strong>limestone</strong></td>
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<td>0.05</td>
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<td><strong>Sugar</strong></td>
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<td><strong>Figure 14</strong></td>
<td>$\sigma'_{3i} = 200$ kPa</td>
<td>1.36</td>
<td>0.0183</td>
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<td>0.8</td>
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<td>$\sigma'_{3i} = 200$ kPa</td>
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<td>1.3</td>
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<td>0.14</td>
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Figure 1. Stress-strain relationship of frictional materials during shearing.
Figure 2. Characteristic points of shearing behaviour of frictional materials.
Figure 3. Features of the proposed distortional strain and stress ratio equation.
Figure 4. Features of the proposed distortional strain and volumetric strain equation.
Figure 5. Effect of $b$ on shear behaviour of frictional materials.
Figure 6. Shearing behaviour of Fuji sand (data after Tatsuoka 1972).
**Figure 7.** Shearing behaviour of dense Cambrai sand (data after Yamamuro and Lade 1996).
Figure 8. Shearing behaviour of reconstituted Corinth marl (data after Burland et al, 1996).
Figure 9. Shearing behaviour of intact Corinth marl (data after Burland et al., 1996).

(a) Distortional behaviour of intact Corinth marl

(b) Volumetric behaviour of intact Corinth marl
Figure 10. Shearing behaviour of Grand-Maison filter material (data after Dendani, Flavigny, and Fry 1988).
Figure 11. Shearing behaviour of cemented volcanic soil (data after O’Rourke and Crespo 1988).
Figure 12. Shearing behaviour of a granular rock (data after Michelis et al., 1981).
Figure 13. Shearing behaviour of oolitic limestone (data after Elliott 1983).
Figure 14. Shearing behaviour of sugar (data after Kolymbos et al., 1990).
Figure 15. Shearing behaviour of rapeseed (data after Kolymbos et al., 1990).
Figure 16. Correlation between parameter $\varepsilon_f$ and $\varepsilon_v$. 
Figure 17. Correlation between parameter $\varepsilon_{d,i}$ and $\varepsilon'_{d,i}$. 
Figure 18. Correlation between parameters $\epsilon_{d,u}$ and $\tilde{\epsilon}_{d,u}$. 

\[ \epsilon_{d,u} = 1.25 \epsilon_{du} \]
APPENDIX

Some of the terms and symbols used in the paper are defined by the following equations.

\( \sigma \): vector of 3D Cartesian stress components

\[
\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T \quad (A1)
\]

\( \varepsilon \): vector of 3D Cartesian strain components

\[
\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})^T \quad (A2)
\]

\( p' \): mean effective stress

\[
p' = \frac{1}{3}(\sigma'_{11} + \sigma'_{22} + \sigma'_{33}) \quad (A3)
\]

\( q \): distortional stress

\[
q = \frac{1}{\sqrt{2}} \sqrt{\left(\sigma_{11}' + \sigma_{22}'\right)^2 + \left(\sigma_{22}' + \sigma_{33}'\right)^2 + \left(\sigma_{33}' + \sigma_{11}'\right)^2 + 6\left(\sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{31}'^2\right)} \quad (A4)
\]

\( \eta \): stress ratio = \( q/p' \) \quad (A5)

\( \varepsilon_v \): volumetric strain = \( \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \) \quad (A6)

\( \varepsilon_d \): distortional strain = \( \frac{\sqrt{2}}{3} \sqrt{\left(\varepsilon_{11} - \varepsilon_{22}\right)^2 + \left(\varepsilon_{22} - \varepsilon_{33}\right)^2 + \left(\varepsilon_{33} - \varepsilon_{11}\right)^2 + 6\left(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2\right)} \) \quad (A7)

M: the critical state stress ratio

\( \eta_p \): the peak stress ratio

\( e_v \): voids ratio
\( \Phi \): state parameter (Figure 2) = \( T_{cs} - \lambda \ln(p) - e \gamma \) (A8)

\( \varepsilon_{d,u} \): distortional strain at \( \eta = M \) before the peak strength is reached (Figure 1)

\( \varepsilon_{d,p} \): distortional strain at peak strength (Figure 1)

\( \varepsilon_{d,i} \): characteristic distortional strain (Figure 1) = \( \varepsilon_{d,p} - \varepsilon_{d,u} \)

\( G_s \): secant distortional modulus = \( \frac{\eta}{\varepsilon_d} \) (A9)

\( G_t \): tangent distortional modulus = \( \frac{d\eta}{d\varepsilon_d} \) (A10)

\( G_o \): initial tangent distortional modulus (Figure 1)

\( G_{s,u} \): secant distortional modulus at \( \varepsilon_d = \varepsilon_{d,u} \) (Figure 1)

\( G_i \): characteristic distortional modulus, i.e. the tangent distortional modulus at \( \varepsilon_d = \varepsilon_{d,i} \) (Figure 1)

\( \alpha \): a basic material parameter = \( \frac{\varepsilon_{d,i}}{\varepsilon_{d,u}} \) (A11)

\( e \): natural exponent and \( e = 2.7182818 \).