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We review the recent construction of semifinite spectral triples for graph C^*-algebras. These examples have inspired many other developments and we review some of these such as the relation between the semifinite index and the Kasparov product, examples of noncommutative manifolds, and an index theorem in twisted cyclic theory using a KMS state.

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SEMIFINITE SPECTRAL TRIPLES ASSOCIATED WITH GRAPH C^*-ALGEBRAS

Abstract. We review the recent construction of semifinite spectral triples for graph C^*-algebras. These examples have inspired many other developments and we review some of these such as the relation between the semifinite index and the Kasparov product, examples of noncommutative manifolds, and an index theorem in twisted cyclic theory using a KMS state.

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Contents

1. Introduction

The extension of the Connes-Moscovici local index theorem in noncommutative geometry to the case of semifinite spectral triples (in which the bounded operators on a Hilbert space are replaced by a general semifinite von Neumann algebra) in [7] and [8] leads naturally to the question of finding interesting novel examples. In this article we will direct our attention to some recent investigations of new noncommutative semifinite spectral triples arising from graph C^*-algebras. Not all graph C^*-algebras lead to spectral triples and we describe in some detail by the example of the Cuntz algebra how badly things fail. The results we describe for the Cuntz algebra are by way of an announcement of a more extensive program in which we study KMS-Dixmier functionals and twisted cyclic theory to shed light on situations where there is no faithful trace on the algebra.

We attempt to give enough detail on notations and definitions to make the account reasonably self contained. We do not try to give proofs referring to the original articles or preprints instead, [2, 5, 6, 10]. Thus the second half of the Introduction will be devoted to the definition of a semifinite spectral triple and some of the analytic subtleties of which one needs to take account in this more general situation. We then describe the semifinite version of the Connes-Moscovici local index theorem in Section 2. A brief review of graph C^*-algebras occupies Section 3 and their Kasparov modules then follow in Section 4. The construction of semifinite spectral triples for graph algebras and the index pairing via spectral flow is described in Section 5 (these Sections follow [26]).

We then describe some results which are the subject of several papers in preparation. The first of these follows [19] where we explain the relationship of semifinite spectral triples to KK theory (Section 6). The explanation of the verification of the axioms of [12] for a noncommutative spin manifold for the case of certain graph C^*-algebras along the lines described in [28] is next in Section 7. For algebras that do not posses a faithful trace a completely different picture emerges. We choose to illustrate this using the Cuntz algebra in Section 8. The first new ingredient is that, imitating the procedure in [26], using the Kasparov module leads to a non-summable spectral triple. By modifying the trace to make it a weight we restore summability but at the expense of having to study twisted cyclic cocycles. Moreover, we find in Section 9 that, there is a natural replacement for K_1 of the Cuntz algebra which we term 'modular K_1'. Modular K_1 pairs, via the semifinite local index theorem, with these twisted cocycles.

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1.1. **Semifinite spectral triples.** We begin with some semifinite versions of standard definitions and results. Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 1.1. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$

2) $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbf{R}$

3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

Definition 1.2. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.*

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{N}$ which need not be (weakly) dense in $\mathcal{N}$. More precisely, $\text{dom}\ \delta = \{T \in \mathcal{N} : \delta(T) \text{ is bounded}\}$. One of the difficulties we face in the semifinite case is that zero may be in the continuous spectrum of $\mathcal{D}$ and for that reason we have to work with $(1 + \mathcal{D}^2)^{1/2}$ instead. Then if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1 + \mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{N}$. In fact, writing $|\mathcal{D}|_1 = (1 + \mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\text{dom}\ \delta^n = \text{dom}\ \delta_1^n \quad \text{for all } n.$$

We also observe that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1 + \mathcal{D}^2)^{1/2}, T]$. The proofs of these statements can be found in [7].

Definition 1.3. *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Thus saying that $\mathcal{A}$ is *smooth* means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrizable and complete.) We will sometimes speak of $\overline{\mathcal{A}} = A$, particularly when $\mathcal{A}$ is represented on Hilbert space and the norm closure $\overline{\mathcal{A}}$ is unambiguous. At other times we regard $i : \mathcal{A} \hookrightarrow A$ as an embedding of $\mathcal{A}$ in a C^*-algebra. We will use both points of view.

It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [16, 33]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. Moreover a smooth algebra has a sensible spectral theory which agrees with that defined using the C^*-closure, and the group of invertibles is open.

Lemma 1.4 ([30]). *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{n,i}(a) = \| \delta^n d^i(a) \|, \quad n \geq 0, \ i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

We call the topology on $\mathcal{A}$ determined by the seminorms q_{ni} of Lemma 1.4 the δ-topology. Thus whenever we have a QC^∞ spectral triple (semifinite or not) it extends to a smooth algebra. This is a necessary ingredient in order to define topological cyclic homology.

2. Summability and the Semifinite Local Index Theorem

In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [14] that if $S \in \mathcal{N}$, the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| \ : \ E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N}, \tau)$ consists of those operators $T \in \mathcal{N}$ such that $\| T \|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will denote the norm on $\mathcal{L}^1(\mathcal{N}, \tau)$ by $\| \cdot \|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1(\mathcal{N}, \tau)$ if $\|T\|_1 := \int_0^\infty \mu_t(T) dt < \infty$.

When $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1(\mathcal{N}, \tau)$ need not be complete in this norm but it is complete in the norm $||.||_1 + ||.||_\infty$. (where $||.||_\infty$ is the uniform norm). Another important ideal for us is the domain of Dixmier traces:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau) = \left\{ T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds < \infty \right\}.$$

The reader should note that $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [11] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t \to \infty} \mu_t(T) = 0.$$

Definition 2.1. *A semifinite spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *relative to* $(\mathcal{N}, \tau)$ *with* $\mathcal{A}$ *unital is* $(1, \infty)$*-summable if*

$$(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau) \quad \textit{for all} \quad \lambda \in \mathbf{C} \setminus \mathbf{R}.$$

We need to briefly discuss the notion of Dixmier trace. For more information on semifinite Dixmier traces, see [9]. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$, $T \geq 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$$

is bounded. For certain $\omega \in L^\infty(\mathbf{R}_*^+)^*$, [9, 11], we obtain a trace on $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ by setting

$$\tau_\omega(T) = \omega(F_T), \quad T \geq 0$$

and extending to $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ by linearity. We will not go into the properties that ω must satisfy to give a trace leaving that to the well established literature see [9]. However, given such an ω associated to the semifinite normal trace τ, we denote by τ_ω the corresponding Dixmier trace.

The Dixmier trace τ_ω vanishes on the ideal of trace class operators. Moreover whenever the function F_T has a limit at infinity, all Dixmier traces return the limit as their value. This leads to the notion of *measurable operator* (see [11] for the type I case). We refer to [25] for the definitive discussion of the notion of measurable operator in the semifinite setting. All Dixmier traces take the same value on measurable operators and we use the notation $⨍$ in this case. That is, if $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \tau)$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbf{R}_*^+)^*$ we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

We now introduce (a special case of) the analytic spectral flow formula of [3, 4]. This formula starts with a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and computes the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$, where $u \in \mathcal{A}$ is unitary with $[\mathcal{D}, u]$ bounded, in the case where $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is of dimension $p \geq 1$. Thus for any $n > p$ we have by Theorem 9.3 of [4]:

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{n/2}} \int_0^1 \tau(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-n/2}) dt, \tag{1}$$

with $C_{n/2} = \int_{-\infty}^{\infty} (1 + x^2)^{-n/2} dx$. This real number $sf(\mathcal{D}, u\mathcal{D}u^*)$ recovers the pairing of the K-homology class $[\mathcal{D}]$ of $\mathcal{A}$ with the $K_1(\mathcal{A})$ class $[u]$ [7]. There is a geometric way to view this formula. It is shown in [4] that the functional $X \mapsto \tau(X(1 + (\mathcal{D} + X)^2)^{-n/2})$ on $\mathcal{N}_{sa}$ determines an exact one-form on an affine space modelled on $\mathcal{N}_{sa}$. Thus (1) represents the integral of this one-form along the path $\{\mathcal{D}_t = (1-t)\mathcal{D} + tu\mathcal{D}u^*\}$ provided one appreciates that $\dot{\mathcal{D}}_t = u[\mathcal{D}, u^*]$ is a tangent vector to this path.

Next we remind the reader of the local index theorem in noncommutative geometry or at least the special case needed for this paper. The original type I version of this result is due to Connes-Moscovici [13]. There are two new proofs, one due to Higson [17] and one in [7]. The latter argument handles the case of semifinite spectral triples. In the simplest terms, the local index theorem provides a formula for the pairing of a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with the K-theory of $\overline{\mathcal{A}}$. The special case that we require in this paper is as follows.

Theorem 2.2 ([7])**.** *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd QC^∞ $(1, \infty)$-summable semifinite spectral triple, relative to $(\mathcal{N}, \tau)$. Then for $u \in \mathcal{A}$ unitary the pairing of $[u] \in K_1(\overline{\mathcal{A}})$ with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by*

$$\langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \lim_{r \to 0^+} r\tau(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-1/2-r}).$$

In particular, the limit on the right exists.

For more information on this result see [9, 7, 8, 13].

3. The Gauge Spectral Triple of a Graph C^*-Algebra

For a more detailed introduction to graph C^*-algebras we refer the reader to [1, 22, 29] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. **We will always assume that the graph is row-finite** which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite* which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2 \cdots \mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i < n$. The maps r, s extend to $E^* := \bigcup_{n \geq 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$, we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from V). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \quad \text{and} \quad p_v = \sum_{\{e : s(e) = v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

It is proved in [22, Theorem 1.2] that there is a universal C^*-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1} S_{\mu_2} \dots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2 \cdots \mu_n$ is a path in E^n. Since the Cuntz-Krieger relations imply that the projections $S_e S_e^*$ are also mutually orthogonal, we have $S_e^* S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [22, Lemma 1.1]). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{2}$$

The algebraic relations and the density of $\text{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role throughout the paper. We adopt the conventions that vertices are paths of length 0, that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (2) are non-empty; we recover S_μ, for example, by taking $v = r(\mu)$, so that $S_\mu S_v^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{zS_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\epsilon/3$ argument using (2) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action*. Because S^1 is compact, averaging over γ with respect to normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi}\int_{S^1} \gamma_z(a)\, d\theta \quad \text{for} \quad a \in C^*(E), \quad z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From Equation (2), it is easy to see that a graph C^*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, formulas which involve sums of projections may contain infinite sums. To interpret these, we use strict convergence in the multiplier algebra of $C^*(E)$:

Lemma 3.1 ([29]). *Let E be a row-finite graph, let A be a C^*-algebra generated by a Cuntz-Krieger E-family $\{T_e, q_v\}$, and let $\{p_n\}$ be a sequence of projections in A. If $p_nT_\mu T_\nu^*$ converges for every $\mu, \nu \in E^*$, then $\{p_n\}$ converges strictly to a projection $p \in M(A)$.*

4. Kasparov modules for graph algebras

The Kasparov modules considered here are for C^*-algebras with trivial grading.

Definition 4.1. *An odd Kasparov A-B-module consists of a countably generated ungraded right B-C^*-module, with $\phi : A \to End_B(E)$ a $*$-homomorphism, together with $P \in End_B(E)$ such that $a(P - P^*)$, $a(P^2 - P)$, $[P, a]$ are all compact endomorphisms. Alternatively we may take $V = 2P - 1$ in favour of P such that $a(V - V^*)$, $a(V^2 - 1)$, $[V, a]$ are all compact endomorphisms for all $a \in A$.*

By [21, Lemma 2, Section 7], the pair (ϕ, P) determines a $KK^1(A, B)$ class, and every class has such a representative. The equivalence relations on pairs (ϕ, P) that give KK^1 classes are unitary equivalence $(\phi, P) \sim (U\phi U^*, UPU^*)$ and homology: $(\phi_1, P_1) \sim (\phi_2, P_2)$ if $P_1\phi_1(a) - P_2\phi_2(a)$ is a compact endomorphism for all $a \in A$. The latter may be recast in terms of operator homotopies as well, see [21, Section 7].

We recall the construction of an odd Kasparov module for certain graph C^*-algebras from [26]. For E a row finite directed graph, we set $A = C^*(E)$, $F = C^*(E)^\gamma$, the fixed point algebra for the $U(1)$ gauge action. The algebras A_c, F_c are defined as the finite linear span of the generators.

Right multiplication makes A into a right F-module, and similarly A_c is a right module over F_c. We define an F-valued inner product $(\cdot|\cdot)_R$ on both these modules by

$$(a|b)_R := \Phi(a^*b),$$

where Φ is the canonical expectation $A \to F$.

Definition 4.2. *Let X be the right F C^*-module obtained by completing A (or A_c) in the norm*

$$\|x\|_X^2 := \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

The algebra A acting by multiplication on the left of X provides a representation of A as adjointable operators on X. We let X_c be the copy of $A_c \subset X$. The $\mathbb{T}^1$ action on X_c is unitary and extends to a strongly continuous unitary action on X.

For each $k \in \mathbf{Z}$, the spectral projection onto the k-th spectral subspace of the $\mathbb{T}^1$ action is the operator Φ_k on X by

$$\Phi_k(x) = \frac{1}{2\pi}\int_{\mathbb{T}^1} z^{-k}\gamma_z(x)d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that on generators we have

$$\Phi_k(S_\alpha S_\beta^*) = \begin{cases} S_\alpha S_\beta^* & |\alpha| - |\beta| = k \\ 0 & |\alpha| - |\beta| \neq k \end{cases}. \tag{3}$$

We quote the following result from [26].

Lemma 4.3. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbf{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k\in\mathbf{Z}}\Phi_k$ converges to the identity operator on X.*

The unbounded operator of the next proposition is the generator of the $\mathbb{T}^1$ action on X. We refer to Lance's book, [24, Chapters 9,10], for information on unbounded operators on C^*-modules.

Proposition 4.4. *[cf [26]] Let X be the right C^*-F-module of Definition 4.2. Define $X_\mathcal{D} \subset X$ to be the linear space*

$$X_\mathcal{D} = \{x = \sum_{k\in\mathbf{Z}} x_k \in X : \|\sum_{k\in\mathbf{Z}} k^2(x_k|x_k)_R\| < \infty\}.$$

For $x \in X_\mathcal{D}$ define

$$\mathcal{D}x = \sum_{k\in\mathbf{Z}} kx_k.$$

Then $\mathcal{D} : X_\mathcal{D} \to X$ is a is self-adjoint, regular operator on X.

Remark On the generators of the graph C^*-algebra we have the formula

$$\mathcal{D}(S_\alpha S_\beta^*) = (|\alpha| - |\beta|)S_\alpha S_\beta^*.$$

The operator $\mathcal{D}$ gives us an unbounded Kasparov module, but we work with the bounded Kasparov module we obtain from the phase of $\mathcal{D}$. We need a preparatory Lemma.

Lemma 4.5. *Assume that the directed graph E is locally finite and has no sources. For all $a \in A$ and $k \in \mathbf{Z}$, $a\Phi_k \in End_F^0(X)$, the compact right endomorphisms of X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

In fact we show that for $k \geq 0$, with $|v|_k$ =the number of paths of length k ending at $v \in E^0$,

$$\Phi_k = \sum_{|\mu|=k} \Theta_{S_\mu, S_\mu}, \qquad \Phi_{-k} = \frac{1}{|v|_k} \sum_{|\mu|=k} \Theta_{S_\mu^*, S_\mu^*},$$

where the sums converge strictly in $End_F(X)$ and $\Theta_{x,y} z = x(y|z)_R$. In the particular case of the Cuntz algebra, which we examine later, we have

$$\Phi_{-k} = \frac{1}{n^k} \sum_{|\mu|=k} \Theta_{S_\mu^*, S_\mu^*}. \tag{4}$$

Theorem 4.6. *Suppose that the graph E is locally finite and has no sources, and let X be the right F module of Definition 4.2. Let $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then (X, V) is an odd Kasparov module for A-F and so defines an element of $KK^1(A, F)$.*

The next theorem presents a general result about the Kasparov product in the odd case.

Theorem 4.7. *Let (Y, T) be an odd Kasparov module for the C^*-algebras A, B. Then the Kasparov product of $K_1(A)$ with the class of (Y, T) is represented by*

$$\langle [u], [(Y, T)] \rangle = [\ker PuP] - [\mathrm{coker} PuP] \in K_0(B),$$

where P is the non-negative spectral projection for T.

Note that the above construction and Theorems 4.6 and 4.7 do not require a trace.We also remark that this pairing was studied in [26], as well as the relation to the semifinite index. To discuss all of this we must first build a semifinite spectral triple: this is our next task.

5. Semifinite spectral triples for graph algebras

In order to obtain such a spectral triple, we require that there exists on $A = C^*(E)$ a faithful, semifinite, norm lower-semicontinuous, gauge invariant trace. In [26] we studied the question of the existence of such traces, and showed that they were in one-to-one correspondence with 'graph traces' on E. These are positive real-valued functions defined on the vertices of E which reflect the structure of the graph. The most notable of the various necessary conditions for the existence of such a trace is that, in the graph E, no loop should have an exit. See [26] for more information.

We will begin with the right F_c module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that E is locally finite and has no sources. This ensures, by Lemma 4.5, that for all $a \in A$ the endomorphisms $a\Phi_k$ of X are compact endomorphisms. We also assume that there exists a faithful, semifinite, norm lower-semicontinuous, gauge invariant trace τ on $C^*(E)$ which by [26] is finite on A_c.

We define a $\mathbf{C}$-valued inner product on X_c:

$$\langle x, y \rangle := \tau((x|y)_R) = \tau(\Phi(x^* y)) = \tau(x^* y).$$

We define the Hilbert space $\mathcal{H} = L^2(X, \tau)$ to be the completion of X_c for $\langle\cdot,\cdot\rangle$. We need some additional information about this situation in order to construct a spectral triple. All of the following statements are proved in [26].

First the C^*-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$. Also the endomorphisms $\{\Phi_k\}_{k\in\mathbf{Z}}$ define mutually orthogonal projections on $\mathcal{H}$. For any $K \subset \mathbf{Z}$ the sum $\sum_{k\in K}\Phi_k$ converges strongly to a projection in $\mathcal{B}(\mathcal{H})$. In particular, $\sum_{k\in\mathbf{Z}}\Phi_k = Id_{\mathcal{H}}$, and so for all $x \in \mathcal{H}$ the sum $\sum_k \Phi_k x$ converges in norm to x.

Next the unbounded self adjoint operator needed for our spectral triple is the operator $\mathcal{D}$ with domain X_c extended to a closed densely defined self-adjoint operator on $\mathcal{H}$. Then the technical details needed for a smooth algebra are handled by the following results.

Lemma 5.1. *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on X_c, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leq \Big||\alpha| - |\beta|\Big|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_\infty = \Big||\alpha| - |\beta|\Big|$.

Corollary 5.2. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is*

$$A_c \subseteq \bigcap_{n\geq 0} \text{dom } \delta^n.$$

Definition 5.3. *Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By* [30, Lemma 16]*, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.*

The next Lemma addresses key facts necessary for dealing with summability issues of spectral triples for nonunital algebras. We will not discuss this here, but merely point out that the Lemma allows us to apply the results of [31]. For more details on the application in this setting see [26].

Lemma 5.4. *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in F \subset \mathcal{A}$ and $a \in \mathcal{A}$ satisfy $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is A. In particular, $\mathcal{A}$ is quasi-local.*

We now come to the two key definitions of this Section.

Definition 5.5. *Let $End_F^{00}(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (End_F^{00}(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.*

We need an appropriate trace on $\mathcal{N}$. Our object is to define one that is naturally related to τ.

Definition 5.6. *Let $T \in \mathcal{N}$ and $\mu \in E^*$. Let $|v|_k =$ the number of paths of length k with range v, and define for $|\mu| \neq 0$*

$$\omega_\mu(T) = \langle S_\mu, TS_\mu \rangle + \frac{1}{|r(\mu)|_{|\mu|}} \langle S_\mu^*, TS_\mu^* \rangle.$$

For $|\mu| = 0$, $S_\mu = p_v$, for some $v \in E^0$, set

$$\omega_\mu(T) = \langle S_\mu, TS_\mu \rangle = \langle p_v, Tp_v \rangle = \tau(p_v T p_v).$$

Define

$$\tilde{\tau} : \mathcal{N}_+ \to [0, \infty], \quad by \quad \tilde{\tau}(T) = \lim_{L \nearrow} \sum_{\mu \in L \subset E^*} \omega_\mu(T)$$

where L is in the net of finite subsets of E^*.

The immediate consequence of this definition is that the functional $\tilde{\tau}$ satisfies: for $T, S \in \mathcal{N}_+$ and $\lambda \in \mathbf{R}$, $\lambda \geq 0$

$$\tilde{\tau}(T + S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda \tilde{\tau}(T)$$

where we adopt the convention $0 \cdot \infty = 0$. Now we have the main property of $\tilde{\tau}$.

Proposition 5.7. *The functional $\tilde{\tau} : \mathcal{N}_+ \to [0, \infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$End_F^{00}(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \text{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, and

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x \rangle = \tau(y^* x), \quad x, y \in X_c.$$

We are now in a position to see that we have the semifinite spectral triple that we promised.

Theorem 5.8. *Let E be a locally finite graph with no sources, and let τ be a faithful, semifinite, gauge invariant, lower semicontinuous trace on $C^*(E)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞, $(1, \infty)$-summable, odd, local, semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$). For all non-zero $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class. If $v \in E^0$ has no sinks downstream*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$.

Finally we can compute our numerical index pairing.

Proposition 5.9. *Let E be a locally finite graph with no sources and a faithful graph trace g, and τ_g the corresponding faithful, semifinite, norm lower semicontinuous, gauge invariant trace on $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 5.8 with $K_1(A)$ can be computed as follows. Let P be the positive spectral projection for $\mathcal{D}$, and perform the C^* index pairing of Theorem 4.7:*

$$K_1(A) \times KK^1(A, F) \to K_0(F), \qquad [u] \times [(X, P)] \to [\ker PuP] - [\text{coker} PuP].$$

Then we have

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \tilde{\tau}_g(\ker PuP) - \tilde{\tau}_g(\text{coker} PuP) = \tilde{\tau}_g([\ker PuP] - [\text{coker} PuP]).$$

In fact this result is a special case of a much more general relationship which we describe in the next section.

6. The General Relationship Between Semifinite Index Theory and KK-Theory

A finitely summable spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ (in the usual $\mathcal{B}(\mathcal{H})$ sense) represents a K-homology class $[(\mathcal{A},\mathcal{H},\mathcal{D})]\in K^*(\overline{\mathcal{A}})$ for the C^*-algebra $\overline{\mathcal{A}}$, [11, 3]. The local index theorem in this case computes the pairing between this class and the K-theory $K_*(\overline{\mathcal{A}})$ in terms of the pairing of cyclic homology and cohomology via Chern characters.

For a semifinite spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ defined relative to $(\mathcal{N},\tau)$, it turns out that there is always a KK-class $[(\mathcal{A},\mathcal{H},\mathcal{D})]\in KK^*(\overline{\mathcal{A}},B)$, where B is constructed from the data of the spectral triple. Thus there is a $K_*(B)$-valued index given by the Kasparov product with the K-theory of $\overline{\mathcal{A}}$:

$$K_*(\overline{\mathcal{A}})\times KK^*(\overline{\mathcal{A}},B)\to K_*(B).$$

If the spectral triple is odd (even) and we pair with odd (even) K-theory of $\overline{\mathcal{A}}$ we obtain an index in $K_0(B)$. Let us just consider the odd-odd pairing for a moment. Let P be the non-negative spectral projection of $\mathcal{D}$, $u\in\mathcal{A}$ unitary, and denote by $\mathrm{Index}(PuP)$ the class in $K_0(B)$ given by Theorem 4.7. Then it turns out that PuP defines a Fredholm operator in the von Neumann algebra $P\mathcal{N}P$, and so we may take the trace of its kernel and cokernel projections. Then the analytic pairing yields a number, the spectral flow, and under some additional hypotheses, it is given by

$$sf_\tau(\mathcal{D},u\mathcal{D}u^*)=\tau_*(\mathrm{Index}(PuP))=\tau(\ker(PuP))-\tau(\mathrm{coker}(PuP)).$$

This statement holds under the condition that certain residue traces on $\mathcal{A}$ are faithful. An analogous statement holds in the even case. These results are proved in [19].

We may translate all this to cyclic theory via Chern characters:

$$\begin{array}{ccccc} K_1(\overline{\mathcal{A}}) & \times & KK^1(\overline{\mathcal{A}},B) & \to & K_0(B)\\ \downarrow Ch & & \downarrow Ch & & \downarrow Ch\\ H_1(\mathcal{A}) & \times & H^1(\mathcal{A},\mathcal{B}) & \to & H_0(\mathcal{B})\end{array}$$

Since the Chern characters respect the pairing, we may pair in K-theory and apply the Chern character, or vice versa. If, as is the case in the applications, we may regard the kernel and cokernel projections of PuP as living in an algebra $\mathcal{K}(B)$ Morita equivalent to B, and with $\mathcal{K}(B)\subset\mathcal{N}$, then we may regard the trace τ as a cyclic zero cocycle on $\mathcal{K}(B)$, and compute the pairing. Again, we compute the analytic spectral flow:

$$\begin{aligned}\langle Ch(u)\times Ch(\mathcal{A},\mathcal{H},\mathcal{D}),\tau\rangle &= \langle Ch(u\times(\mathcal{A},\mathcal{H},\mathcal{D})),\tau\rangle\\ &= \tau(\ker(PuP))-\tau(\mathrm{coker}(PuP))\\ &= sf_\tau(\mathcal{D},u\mathcal{D}u^*).\end{aligned}$$

Thus the semifinite index theorem is computing a 'zero-order' part of a more general index pairing in bivariant cyclic theory, where τ may be replaced by a more general cocycle.

7. Graph Algebras as Noncommutative Manifolds

In [28] we consider natural generalisations of Connes' axioms for noncommutative manifolds, [12], which simultaneously address semifinite triples and nonunital algebras. The axiom set we use is modelled on that of [32] where the spin manifold theorem is proven.

Most of the axioms admit straightforward generalisations, and we will not dwell on them here, other than to say that most are automatically satisfied by the gauge spectral triple of a locally finite graph E with no sources and such that $C^*(E)$ carries a faithful, semifinite, norm lower semi-continuous, gauge invariant trace. In particular, the graph E cannot have a loop with an exit, [26]. The most interesting axioms are those which impose further constraints on the graph.

The first, and most restrictive, axiom we consider is orientability. This axiom requires the existence of a Hochschild cycle c of the same degree as the (integral) dimension, which in this case is 1, such that $\pi_{\mathcal{D}}(c) = 1$ (see below). We define

$$c = \sum_{e\in E^1} S_e^* \otimes S_e.$$

Here the sum is over all edges in the graph. If E is the graph with a single vertex and single edge, then $C^*(E) \cong C(\mathbb{T}^1)$ and c is the usual volume form. In general, the convergence of the above sum (in the tensor product of the multiplier algebra with itself) and the condition $b(c) = 0$ are both guaranteed by the following 'single entry hypothesis'

every vertex receives precisely one edge, and emits at least one.

Coupled with the fact that no loop may have an exit, we see that the graphs we may consider are (disjoint unions) of two types:

infinite directed trees and a single loop comprised of N vertices and edges

The latter type are isomorphic to $M_N(C(\mathbb{T}^1))$ while the former are always nonunital and AF.

The Hochschild cycle c may be interpreted in several ways, but the most appealing to us is as a limit of compactly supported Hochschild 1-cycles, as in [30]. This is what one would expect of the (class of) a volume form in de Rham theory.

The representation of c, $\pi_{\mathcal{D}}(c)$, is by definition

$$\pi_{\mathcal{D}}(c) = \sum_{e\in E^1} S_e^*[\mathcal{D}, S_e] = \sum_{e\in E^1} p_{r(e)}$$

where $S^*[\mathcal{D}, S_e] = p_{r(e)}$ is a straightforward calculation. Since for each vertex v there is a unique edge e with range v, we can show that the sum on the right converges strictly to the identity in the multiplier algebra of $C^*(E)$, and strongly to the identity on $\mathcal{H}$. Thus the orientability condition is satisfied for directed trees and finite loops.

The next axiom to consider is finiteness. Essentially this asks that the smooth domain of $\mathcal{D}$, $\mathcal{H}_\infty = \cap_{m\geq 1}\mathrm{dom}\mathcal{D}^m$, be a finite projective module over $\mathcal{A}$. Thus $\mathcal{H}_\infty$ should be thought of as a module of smooth sections of a 'noncommutative vector bundle'.

Semifiniteness does not affect us here, but nonunitality does; see [30]. In particular, there are no issues to consider here for the N-point loop examples, only the directed trees. We take the approach that we should be able to recover the 'continuous sections vanishing at infinity' from these 'smooth sections'.

We summarise what our axiom of finiteness says:

1) $\mathcal{H}_\infty$ should be a continuous $\mathcal{A}$-module,

2) $\mathcal{H}_\infty$ embeds continuously as a dense subspace in the C^*-module X,

3) X is the completion of pA^N for some N ($A^N = A \oplus \ldots \oplus A$, N copies) and some projection p in the $N \times N$ matrices over a unitization of $A = C^*(E)$,

4) the Hermitian product $\mathcal{H}_\infty \ni x, y \to x^*y$ should have range in $\mathcal{A}$ (acting on the right).

This gives a relationship between 'decay at infinity' in the C^*-module sense and the Hilbert space sense. In particular, the trace must be bounded below, which in turn means that the graph can have only finitely many branchings. This is then intimately related to the K-theory of the graph algebra.

Proposition 7.1. *Suppose that the locally finite directed graph E has no sinks, no loops and satisfies the single entry condition. The $\mathcal{A}$-module $\mathcal{H}_\infty$ satisfies 1) and 3). The module $\mathcal{H}_\infty$ satisfies 2) if and only if the K-theory of $A = C^*(E)$ is finitely generated. In this case the Hilbert space $\mathcal{H}$ also satisfies the conditions 2). If the $K_*(A)$ is finitely generated then condition 4) holds.*

Thus when we consider the directed tree examples, we must restrict to those with finitely many branchings. It is interesting to observe that the name 'finiteness' for this axiom really refers to the finite projective module. It is thus surprising to see this axiom influence also the finiteness of the rank of the K-theory.

The final axiom we mention is Poincaré Duality. While it was shown in [32] that this axiom could be replaced by 'closedness' and a 'spin^c' condition (versions of both are satisfied by the gauge spectral triples of graph algebras), Poincaré Duality is still an important structural feature to examine.

It turns out that the algebra $A = C^*(E)$ does satisfy Poincaré Duality in K-theory, though one needs a suitable nonunital formulation, [30]. However, more is true. The fixed point subalgebra F for the gauge action also satisfies Poincaré Duality. This is true both for the N-point loops and the directed trees (with finitely many branchings). In essence, Poincaré Duality for *both* these algebras arises as a consequence of the other axioms.

In [28] we also consider the examples of higher rank graphs, or k-graphs, studied from this point of view in [27], where (k, ∞)-summable semifinite spectral triples were obtained under similar hypotheses on the k-graph as in the graph case. Again, most of the axioms are satisfied immediately, with orientation and finiteness placing similar constraints on the algebra (or k-graph). We have not considered Poincaré Duality for these higher rank graphs, but expect phenomena similar to that for the 1-graph case.

8. The Cuntz Algebra

8.1. **The KMS state for the gauge action.** The Cuntz algebra does not possess a faithful gauge invariant trace. It does however have a unique KMS state relative to the gauge action which is given by the faithful gauge invariant state $\tau \circ \Phi : O_n \to \mathbf{C}$ where $\Phi : O_n \to F$ is the expectation and $\tau : F \to \mathbf{C}$ the unique faithful normalised trace on the gauge invariant subalgebra.

Since the Cuntz algebra is the graph algebra of a locally finite graph with no sources, the generator of the gauge action $\mathcal{D}$ acting on the right C^*-F-module X gives us a Kasparov module $(X, \mathcal{D})$. As with tracial graph algebras, we take this class as our starting point.

The first difficulty we encounter is that there are no unitaries to pair with, since $K_1(O_n) = 0$. We require a new approach.

Let $\mathcal{H} = L^2(O_n)$ where the inner product is defined by

$$\langle a, b\rangle = (\tau \circ \Phi)(a^* b).$$

Then $\mathcal{D}$ extends to a self-adjoint unbounded operator on $\mathcal{H}$, [28], and we denote this closure by $\mathcal{D}$ from now on. The representation of O_n on $\mathcal{H}$ (by left multiplication) is bounded and nondegenerate, and the dense subalgebra $\mathrm{span}\{S_\mu S_\nu^*\}$ is in the smooth domain of the derivation δ. Thus we see that the central algebraic structures of the gauge spectral triple on a tracial graph algebra are mirrored in this construction.

What differs significantly from the tracial situation is the analytic information. We begin by obtaining some information about the trace on F, the corresponding state on O_n and the associated modular theory.

First of all the trace $\tau : F \to \mathbf{C}$ satisfies

$$\tau(S_\mu S_\nu^*) = \delta_{\mu\nu} \frac{1}{n^{|\mu|}}.$$

In terms of the associated state on O_{nc} we have some additional structure:

Lemma 8.1. *The algebra $O_{nc} = \mathrm{span}\{S_\mu S_\nu^*\}$ with the inner product arising from the state $\tau \circ \Phi$ is a modular Hilbert algebra.*

Let S first denote the operator $a \mapsto a^*$ defined on O_{nc}. The adjoint $F = S^*$ of S can be explicitly calculated on O_{nc} and satisfies:

$$F(S_\mu S_\nu^*) = n^{(|\mu|-|\nu|)} S_\nu S_\mu^*.$$

In particular, F is densely defined so that S is closable and we use S to denote the closure of S restricted to O_{nc}, and also F will denote the closure of F restricted to O_{nc}. Then S has a polar decomposition as

$$S = J\Delta^{1/2}, \quad \Delta = S^* S,$$

where J is an antilinear map, $J^2 = 1$. The Tomita-Takesaki modular theory, [20], shows that

$$\Delta^{-it} O_n'' \Delta^{it} = O_n'', \quad J O_n'' J^* = (O_n'')',$$

where O_n'' is the weak closure of O_n acting on $L^2(O_n, \tau\circ\Phi)$.

Lemma 8.2. *Writing $S : a \to a^*$ as $S = J\Delta^{1/2}$, we have*

$$\Delta S_\mu S_\nu^* = n^{|\nu|-|\mu|} S_\mu S_\nu^*, \qquad \forall \mu,\nu \in \mathbf{N}^n.$$

Thus the group of modular automorphisms is given by

$$\sigma_t(S_\mu S_\nu^*) = \Delta^{it} S_\mu S_\nu^* \Delta^{-it} = n^{it(|\nu|-|\mu|)} S_\mu S_\nu^*.$$

Corollary 8.3. *If $\mathcal{D}$ is the generator of the gauge action of $\mathbb{T}^1$ on O_n, we have*

$$\Delta = e^{-\mathcal{D}\log n} \text{ or } e^{it\mathcal{D}} = e^{-\frac{it}{\log n}\log\Delta}.$$

Hence the flows generated by $\log\Delta$ and $\mathcal{D}$ are the same, up to a constant rescaling of $-\log n$. More precisely

$$\Delta^{it} = e^{it\log\Delta} = e^{-(it\log n)\mathcal{D}}.$$

To continue, we recall the underlying right C^*-F-module, X, which is the completion of O_n for the norm $\|x\|_X^2 = \|\Phi(x^*x)\|_F$. Endomorphisms of the module X which preserve X_c (the copy of O_{nc} inside X) extend uniquely to bounded operators on the Hilbert space $\mathcal{H}$, [26].

8.2. **An attempted semifinite spectral triple.** We follow our previous strategy (Section 5) to uncover the difficulty.

Proposition 8.4. *Let $\mathcal{N}$ be the von Neumann algebra*

$$\mathcal{N} = (End_F^{00}(X_c))'',$$

where we take the commutant inside $\mathcal{B}(\mathcal{H})$. Then $\mathcal{N}$ is semifinite and there exists a faithful, semifinite, normal trace $\tilde\tau : \mathcal{N} \to \mathbf{C}$ such that for all rank one endomorphisms $\Theta_{x,y}$ of X_c,

$$\tilde\tau(\Theta_{x,y}) = (\tau\circ\Phi)(y^*x), \quad x,y \in X_c.$$

In addition, $\mathcal{D}$ is affiliated to $\mathcal{N}$.

Rather than explain the proof here, we simply observe for the reader's benefit that to check the trace property (on endomorphisms) only requires that τ is a trace on F, not all of O_n. Here is the formal calculation for rank one operators.

$$\begin{aligned}\tilde\tau(\Theta_{w,z}\Theta_{x,y}) &= \tilde\tau(\Theta_{w(z|x),y}) = \tau((y|w(z|x))) \\ &= \tau((y|w)(z|x)) = \tau((z|x)(y|w)) \\ &= \tilde\tau(\Theta_{x(y|w),z}) = \tilde\tau(\Theta_{x,y}\Theta_{w,z}).\end{aligned}$$

However this trace is not what we need for defining summability. To see this we do some calculations. For $k \geq 0$

$$\tilde\tau(\Phi_k) = \tilde\tau(\sum_{|\rho|=k}\Theta_{S_\rho,S_\rho}) = \tau(\sum_{|\rho|=k}(S_\rho|S_\rho)) = \tau(\sum_{|\rho|=k}S_\rho^*S_\rho) = \sum_{|\rho|=k}1 = n^k.$$

Similarly, for $k < 0$

$$\tilde\tau(\Phi_k) = n^k.$$

With respect to this trace it is not hard to see that we cannot expect $\mathcal{D}$ to satisfy any summability criterion. To get summability we are forced to define a new functional on $\mathcal{N}$ by

$$\tau_\Delta(T) := \tilde{\tau}(\Delta T).$$

Since $\tilde{\tau}$ is a faithful semifinite normal trace on $\mathcal{N}$, and Δ is a positive invertible operator affiliated to $\mathcal{N}$ we may show that τ_Δ is a faithful semifinite normal weight on $\mathcal{N}$.

To understand what we have done here we do some further computations: for $k \geq 0$

$$\tau_\Delta(\Phi_k) = \tau(\sum_{|\rho|=k}(S_\rho|\Delta S_\rho)) = \sum_{|\rho|=k} n^{-k}\tau(S_\rho^* S_\rho) = 1 \ \text{ and } \ \tau_\Delta(\Phi_{-k}) = 1.$$

However, τ_Δ is not a trace on finite rank endomorphisms.

$$\begin{aligned}
\tau_\Delta(\Theta_{x,y}\Theta_{w,z}) &= \tau_\Delta(\Theta_{x(y|w),z}) = \tilde{\tau}(\Delta\Theta_{x(y|w),z}) = \tau((z|\Delta x(y|w))) \\
&= \tau((z|\Delta x)(y|w)) \qquad \text{since } \Delta \text{ is linear over } F \\
&= \tau((y|w)(z|\Delta x)) \qquad \text{since } \tau \text{ is a trace on } F \\
&= \tau((y|w)(\Delta z|x)) \qquad \text{since } \Delta \text{ is self-adjoint on } X \\
&= \tilde{\tau}(\Theta_{w,\Delta z}\Theta_{x,y}) \\
&= \tilde{\tau}(\Delta\Delta^{-1}\Theta_{w,z}\Delta\Theta_{x,y}) = \tau_\Delta(\Delta^{-1}\Theta_{w,z}\Delta\Theta_{x,y}).
\end{aligned}$$

Lemma 8.5. *The modular automorphism group $\sigma_t^{\tau_\Delta}$ of τ_Δ is inner and given by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it}T\Delta^{-it}$. The weight τ_Δ is a KMS weight for the group $\sigma_t^{\tau_\Delta}$, and*

$$\sigma_t^{\tau_\Delta}|_{O_n} = \sigma_t^{\tau\circ\Phi}.$$

We cannot construct a semifinite spectral triple for $\mathcal{N}$. The best we can do is work with $\mathcal{M} = \mathcal{N}^\sigma$ the fixed point algebra for the modular automorphism group of τ_Δ. Then any $m \in \mathcal{M}$ commutes with the spectral projections of Δ or

$$[\Delta, m] = 0.$$

Also observe that since F and the projections Φ_k are invariant, they belong to $\mathcal{M}$. Finally the weight τ_Δ restricts to a normal, faithful, semifinite trace on $\mathcal{M}$.

The reward for having sacrificed a trace on $\mathcal{N}$ for a trace on $\mathcal{M}$ is the following.

Proposition 8.6. *We have $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M},\tau_\Delta)$, and for all $f \in F$ and all Dixmier traces $\tau_{\Delta,\omega}$*

$$\tau_{\Delta,\omega}(f(1+\mathcal{D}^2)^{-1/2}) = 2\tau(f).$$

Corollary 8.7. *For all $a, b \in O_{nc}$,*

$$lim_{s\to 1}(s-1)\tau_\Delta(ab(1+\mathcal{D}^2)^{-s/2}) = 2\tau(ab) = 2\tau(\sigma_i(b)a) = \tau_{\Delta,\omega}(\sigma_i(b)a(1+\mathcal{D}^2)^{-1/2}).$$

Moreover, the functional $a \to \tau_{\Delta,\omega}(a(1+\mathcal{D}^2)^{-1/2})$ is a KMS_1 state on O_{nc}.

Before continuing we present a set of data which appear to be essential if we are to describe in general the situation presented here for the Cuntz algebras; compare [15, Definition 2.1].

A modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to a semifinite von Neumann algebra $\mathcal{N}$ and faithful KMS state τ on $\mathcal{A}$ with respect to an action σ of the circle, would appear to need the following data.

The algebra $\mathcal{N}$ acts on the GNS Hilbert space $\mathcal{H}$ associated to τ and

0) the $*$-algebra $\mathcal{A}$ is faithfully represented in $\mathcal{N}$,

1) there is a faithful normal semifinite weight ϕ on $\mathcal{N}$ such that the modular automorphism group of ϕ is an inner automorphism group $\tilde{\sigma}$ of $\mathcal{N}$ with $\tilde{\sigma}|_{\mathcal{A}} = \sigma$,

2) ϕ restricts to a faithful semifinite trace on $\mathcal{M} = \mathcal{N}^\sigma$,

3) $[\mathcal{D}, a]$ extends to a bounded operator (in $\mathcal{N}$) for all $a \in \mathcal{A}$ and for λ in the resolvent set of $\mathcal{D}$, $f(\lambda - \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{M}, \phi)$, where $f \in \mathcal{A}^\sigma$, and $\mathcal{K}(\mathcal{M}, \phi)$ is the ideal of compact operators in $\mathcal{M}$ relative to ϕ. In particular, $\mathcal{D}$ is affiliated to $\mathcal{M}$.

The triple is even if there exists $\gamma = \gamma^*$, $\gamma^2 = 1$ such that $\gamma\mathcal{D} + \mathcal{D}\gamma = 0$ and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$. Otherwise it is odd.

Observe that this data implies that $\mathcal{D}$ commutes with the modular automorphism group in the strong sense that its spectral projections are invariant.

We will refer to any triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfying this data as a modular spectral triple recognising that as more examples are studied modifications or extensions may be required. In particular, this definition does not address actions of the reals that do not factor through the circle.

Of course the triple $(O_{nc}, \mathcal{H}, \mathcal{D})$ along with $\mathcal{N}, \tau_\Delta$ constructed in this Section, is a modular spectral triple.

8.3. **Modular unitaries.** As the Cuntz algebra is not contained in $\mathcal{M}$ we cannot use the semifinite spectral triple immediately. We need some additional structure.

Definition 8.8. *Let $\mathcal{A}$ be a $*$-algebra and $\sigma : \mathcal{A} \to \mathcal{A}$ an algebra automorphism such that*

$$\sigma(a)^* = \sigma^{-1}(a^*).$$

then we say that σ is a regular automorphism, [23].

Clearly for a modular spectral triple the automorphism $\sigma(a) = \Delta^{-1} a \Delta$ is regular.

Definition 8.9. *Let u be a unitary in a matrix algebra over $\mathcal{A}$, and $\sigma : \mathcal{A} \to \mathcal{A}$ a regular automorphism with fixed point algebra $F = \mathcal{A}^\sigma$. We say that u satisfies the* **modular condition** *with respect to σ if both the operators*

$$u\sigma(u^*), \quad u^*\sigma(u)$$

are in (a matrix algebra over) the algebra F. We denote by U_σ the set of modular unitaries obtained by taking the union over all matrix algebras over $\mathcal{A}$..

We are of course thinking of the case $\sigma(a) = \Delta^{-1} a \Delta$, where Δ is the modular operator for some weight on A. Hence the use of the terminology 'modular unitaries'. For unitaries in matrix algebras over $\mathcal{A}$ we use the regular automorphism $\sigma \otimes Id_n$ to state the modular condition, where Id_n is the identity of $M_n(\mathbf{C})$.

Definition 8.10. *Let u_t be a continuous path of modular unitaries such that $u_t\sigma(u_t^*)$ and $u_t^*\sigma(u_t)$ are also continuous paths in F. Then we say that u_t is a modular homotopy, and say that u_0 and u_1 are modular homotopic.*

Lemma 8.11. *The binary operation on modular homotopy classes in U_σ*

$$[u] + [v] := [u \oplus v]$$

is abelian.

We can now also see why the usual proof that the inverse of u is u^* in $K_1(A)$ is not available to us. This usual proof is as follows. Observe that $u \oplus v = (u \oplus 1)(1 \oplus v) \sim (1 \oplus u)(1 \oplus v) = (1 \oplus uv)$. Then we see that addition in K_1 arises from multiplication of unitaries, and so $[u] + [u^*] = [uu^*] = [1] = 0$. However, while the homotopy from $u \oplus 1$ to $1 \oplus u$ is a modular homotopy in U_σ by the last Lemma, the homotopy from $(u \oplus 1)(1 \oplus v)$ to $(1 \oplus u)(1 \oplus v)$ is not in general. The multiplication on the right by $(1 \oplus v)$ breaks the modular condition. In particular, the product of two modular unitaries need not be a modular unitary. What we do have in this situation is stated in the following result.

Lemma 8.12. *If $u \in M_l(F)$ is unitary then $u \oplus u^* \sim 1$. If $u \in M_l(F)$ then $-[u] = [u^*]$ in $K_1(A, \sigma)$, with $K_1(A, \sigma)$ defined below.*

We now formalise the above discussion. Compare the following with [18, Definition 4.8.1]

Definition 8.13. *Let $K_1(A, \sigma)$ be the abelian semigroup with one generator $[u]$ for each unitary $u \in M_l(A)$ satisfying the modular condition and with the following relations:*

1) $[1] = 0$,
2) $[u] + [v] = [u \oplus v]$,
3) *If $u_t,\ t \in [0, 1]$ is a continuous paths of unitaries in $M_l(A)$ satisfying the modular condition then $[u_0] = [u_1]$.*

Of course we can make this into a group by the Grothendieck construction, but as yet see no compelling reason to do so.

Example For $S_\mu \in O_{nc}$ we write $P_\mu = S_\mu S_\mu^*$. Then for each μ, ν we have a unitary

$$u_{\mu,\nu} = \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix}.$$

It is simple to check that this a self-adjoint unitary satisfying the modular condition.

Lemma 8.14. *For all μ, ν there is a modular homotopy*

$$u_{\mu,\nu} \sim u_{\nu,\mu}.$$

Example More generally, if σ is a regular automorphism of an algebra A with fixed point algebra F, $v \in A$ is a partial isometry with range and source projections in F, and furthermore $v\sigma(v^*),\ v^*\sigma(v) \in F$, then

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

is a modular unitary, as the reader may check. The proof of Lemma 8.14 applies to these unitaries to show that $u_v \sim u_{v^*}$.

There are a number of structural results we can prove about this situation. For example it is not hard to see that the centre of $\mathcal{M}$, denoted $\mathcal{Z}(\mathcal{M})$, contains the von Neumann algebra generated by the spectral projections of Δ. A more crucial matter is worth emphasising in the next result.

Lemma 8.15. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple such that $\mathcal{D}$ commutes with $F = \mathcal{A}^\sigma$. Let $u \in \mathcal{A}$ be a unitary. Then $uQu^* \in \mathcal{M}$ for all spectral projection Q of $\mathcal{D}$, if and only if u is modular.*

Proof. First, uQu^* is a projection in $\mathcal{N}$. For one direction we have

$$\begin{aligned} \sigma(uQu^*) &= \sigma(u)Q\sigma(u^*), \qquad Q \in \mathcal{M} \\ &= uu^*\sigma(u)Q\sigma(u^*) \\ &= uQu^*\sigma(u)\sigma(u^*), \qquad u^*\sigma(u) \in F \\ &= uQu^*. \end{aligned}$$

Hence uQu^* is invariant, and so in $\mathcal{M}$. On the other hand if

$$uQu^* = \sigma(uQu^*) = \sigma(u)Q\sigma(u^*)$$

then we have

$$Q = u^*\sigma(u)Q\sigma(u^*)u = Q + [u^*\sigma(u), Q]\sigma(u^*)u.$$

As $\sigma(u^*)u$ is invertible, we see that $[u^*\sigma(u), Q] = 0$. Since $u^*\sigma(u) \in \mathcal{A}$, and commutes with all Q, it lies in $F = \mathcal{M} \cap \mathcal{A}$. □

The truly important aspect of this lemma is that modular unitaries conjugate Δ to an element of $\mathcal{M}$, and so $u\Delta u^*$ commutes with Δ.

9. The Local Index formula for the Cuntz Algebras

To obtain a pairing between modular K_1 and modular spectral triples, we are going to use the spectral flow formula of [4]. Before we can do this effectively, we need to address an important detail. The modular unitaries do not lie in the algebra $\mathcal{M}$. Consequently the formula in [4] for spectral flow actually contains two additional terms which measure the spectral asymmetry of the end-points. These eta-type correction terms cancel when the end-points are unitarily equivalent, however the argument demonstrating this fails in the present situation because modular unitaries lie outside $\mathcal{M}$. It eventuates, by a different argument, that for modular unitaries of the form $u = u_v$ discussed above, and under the hypotheses of the theorem below, these correction terms do cancel out. For $r > 0$ this gives us

$$sf(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{1/2+r}} \int_0^1 \phi(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-1/2-r})dt.$$

We are now in a position to apply the reasoning in [7], to obtain a residue formula to compute the index pairing between $(O_{nc}, \mathcal{H}, \mathcal{D})$ and $K_1(O_n, \sigma)$ for these particular unitaries. Let $\mathcal{D}_k := \mathcal{D} \otimes Id_k$ and $\phi_k = \phi \otimes \mathrm{Tr}_k$.

Theorem 9.1. *Let $(O_{nc}, \mathcal{H}, \mathcal{D})$ be the QC^∞, $(1, \infty)$-summable, modular spectral triple relative to $(\mathcal{N}, \tau_\Delta)$, constructed previously. Then for any modular unitary of the form u_v with $u_v \in M_k(O_{nc})$, with v a partial isometry with range and source projections in $M_{k/2}(F)$ and $v\sigma(v^*)$, $v^*\sigma(v) \in M_{k/2}(F)$, and any Dixmier trace $\phi_{k,\omega}$ we have*

$$sf_{\phi_k}(\mathcal{D}_k, u_v\mathcal{D}_k u_v) = lim_{r\to 0} r\phi_k(u_v[\mathcal{D}_k, u_v](1 + \mathcal{D}_k^2)^{-1/2-r}) = \frac{1}{2}\phi_{k,\omega}(u_v[\mathcal{D}_k, u_v](1 + \mathcal{D}_k^2)^{-1/2}).$$

The functional

$$M_k(O_{nc}) \otimes M_k(O_{nc}) \ni a_0 \otimes a_1 \to lim_{r\to 0} r\phi_k(a_0[\mathcal{D}_k, a_1](1 + \mathcal{D}_k^2)^{-1/2-r})$$

is a twisted b, B-cocycle. Moreover, the spectral flow depends only on the modular homotopy class of u_v.

Following the proof of the local index theorem in [7], one can also obtain a twisted resolvent cocycle although we will not discuss this here.

Theorem 9.2. *Let $(O_{nc}, \mathcal{H}, \mathcal{D})$ be the modular spectral triple of the Cuntz algebra, and u a modular unitary of the form $u_{\mu,\nu}$. Then*

$$\begin{aligned} sf_{\phi_2}(\mathcal{D}_2, u\mathcal{D}_2 u^*) &= (|\mu| - |\nu|)\left(\frac{1}{n^{|\nu|}} - \frac{1}{n^{|\mu|}}\right) \in (n-1)\mathbf{Z}[1/n] \\ &\geq 0. \end{aligned}$$

Thus we see a wholly new kind of index pairing for the Cuntz algebra. The index pairings above would be zero if we were employing a trace, as all the unitaries we consider are self-adjoint, and so represent zero in ordinary K_1.

Modular K_1, and its pairing with twisted cyclic, represents a new way to obtain data about algebras without (faithful) traces, by using KMS states instead. The procedure we have outlined for the Cuntz algebra is in fact quite general, and we have recently applied it to the Haar state of $SU_q(2)$ with similar success, [10].

We expect to find many applications of these new invariants, both in mathematics and physics.

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