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MODULAR INDEX INVARIANTS OF MUMFORD CURVES

A. CAREY, M. MARCOLLI, A. RENNIE

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1. Introduction

Mumford curves generalize the Tate uniformization of elliptic curves and provide p-adic analogues of the uniformization of Riemann surfaces, [26]. The type of p-adic uniformization considered for these curves is a close analogue of the Schottky uniformization of complex Riemann surfaces, where instead of a Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ acting on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, one has a p-adic Schottky group acting on the boundary of the Bruhat–Tits tree and on the Drinfeld p-adic upper half plane.

The analogy between Mumford curves and Schottky uniformization of Riemann surfaces was a key ingredient in the results of Manin on Green functions of Arakelov geometry in terms of hyperbolic geometry [23], motivated by the analogy with earlier results of Drinfeld–Manin for the case of p-adic Schottky groups [15]. Manin's computation of the Green function for a Schottky-uniformized Riemann surface in terms of geodesics lengths in the hyperbolic handlebody uniformized by the same Schottky group provides a geometric interpretation of the missing "fibre at infinity" in Arakelov geometry in terms the tangle of bounded geodesics inside the hyperbolic 3-manifold. In order to make this result compatible with Deninger's description of the Gamma factors of L-functions as regularized determinants and with Consani's archimedean cohomology [8], this formulation of Manin was reinterpreted in terms of noncommutative geometry by Consani–Marcolli [9]. In particular, the model proposed in [9] for the "fibre at infinity" uses a noncommutative space which describes the action of the Schottky group Γ on its limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ via the crossed product C^*-algebra $C(\Lambda_\Gamma) \rtimes \Gamma$. This is a particular case of a Cuntz–Krieger algebra given by the graph C^*-algebra of the finite graph Δ_Γ/Γ, with Δ_Γ the Cayley graph of $\Gamma \simeq \mathbb{Z}^{*g}$.

Following the same analogy between Schottky uniformization of Riemann surfaces and p-adic uniformization of Mumford curves, Consani and Marcolli extended their construction [9] to the case of Mumford curves, [10, 11]. More interesting graph C^*-algebras appear in the p-adic case than in the archimedean setting, namely the ones associated to the graph Δ_Γ/Γ, which is the dual graph of the specialization of the Mumford curve and to Δ'_Γ/Γ, which is the dual graph of the closed fibre of the minimal smooth model of the curve. After these results of Consani–Marcolli, the construction was further refined in [13] and extended to some classes of higher rank buildings generalizing the rank-one case of Schottky groups acting on Bruhat–Tits trees. The relation between Schottky uniformizations, noncommutative geometry, and graph C^*-algebras was further analyzed in [12, 14].

The main question in this approach is how much of the algebraic geometry of Mumford curves can be recovered by means of the noncommutative geometry of certain C^*-algebras associated to the action of the Schottky group on the Bruhat–Tits tree, on its limit set, and on the Drinfeld upper half plane, and conversely how much the noncommutative geometry is determined by algebro-geometric information coming from the Mumford curve.

In this paper we analyze another aspect of this question, based on recent results on circle actions on graph C^*-algebras and associated KMS states and modular index theory, [3]. In particular, we first show how numerical information like the Schottky invariants given by the translation lengths of a given set of generators of the Schottky group can be recovered from the modular index invariants of the graph C^*-algebra determined by the action of the p-adic Schottky group on the Bruhat–Tits tree. We then analyze the relation between graph weights for this same graph C^*-algebra and theta functions of the Mumford curve. Unlike the previous results on Mumford curves and noncommutative geometry, which concentrated on the use of the graph C^*-algebra associated to the *finite* graph Δ'_Γ/Γ or Δ_Γ/Γ, here we use the full infinite graph $\Delta_{\mathbb{K}}/\Gamma$, where $\Delta_{\mathbb{K}}$ is the Bruhat–Tits tree, with boundary at infinity $\partial\Delta_{\mathbb{K}}/\Gamma = X_\Gamma(\mathbb{K})$, the $\mathbb{K}$-points of the Mumford curve, with $\mathbb{K}$ a finite extension of $\mathbb{Q}_p$. The fact of working with an infinite graph requires a more subtle analysis of the modular index theory and a setting for the graph weights, where the main information is located inside the finite graph Δ'_Γ/Γ and is propagated along the infinite trees in $\Delta_{\mathbb{K}}/\Gamma$ attached to the vertices of Δ'_Γ/Γ, towards the conformal boundary $X_\Gamma(\mathbb{K})$. The graph weights are solutions of a combinatorial equation at the vertices of the graph, which can be thought of as governing a momentum flow through the graph. We prove that for graphs such as $\Delta_{\mathbb{K}}/\Gamma$ faithful graph weights are the same as gauge invariant, norm lower semicontinuous faithful semifinite functionals on the graph C^*-algebra. This is the basis for constructing KMS states associated to graph weights, which are then used to compute modular index invariants for these type III geometries.

The theta functions of Mumford curves in turn can be described as in [33] in terms of Γ-invariant currents on the Bruhat–Tits tree $\Delta_{\mathbb{K}}$ and corresponding signed measures of total mass zero on the boundary. We show that this description of theta functions leads to an inhomogeneous version of the equation defining graph weights, or equivalently to a homogeneous version, but where the weights are allowed to have a sign instead of being positive and are also required to be integer valued. We also show that there is an isomorphism between the abelian group of Γ-invariant currents on the Bruhat-Tits tree and linear functionals on the K_0 of the graph C^*-algebra of the quotient graph $C^*(\Delta_{\mathbb{K}}/\Gamma)$. This implies that theta functions of the Mumford curve define functionals on the K-theory of the graph algebra, with the only ambiguity given by the action of $\mathbb{K}^*$.

Finally, we discuss how to use the spectral flow to construct solutions of the inhomogeneous graph weights equations and how to use these to construct theta functions in case where one has more than one (positive) graph weight on $\Delta_{\mathbb{K}}/\Gamma$.

It would be interesting, in a similar manner, to explore how other invariants associated to type III noncommutative geometries, such as the approach followed by Connes–Moscovici in [7] and by Moscovici in [25], may relate to the algebraic geometry of Mumford curves in the specific case of the algebras $C^*(\Delta_{\mathbb{K}}/\Gamma)$.

2. Mumford Curves

We recall here some well known facts from the theory of Mumford curves, which we need in the rest of the paper. The results mentioned in this brief introduction can be found for instance in [26], [22], [17] and were also reviewed in more detail in [10].

2.1. The Bruhat–Tits tree.

Let $\mathbb{K}$ denote a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O} = \mathcal{O}_{\mathbb{K}} \subset \mathbb{K}$ be its ring of integers, with $\mathfrak{m} \subset \mathcal{O}$ the maximal ideal. The finite field $k = \mathcal{O}/\mathfrak{m}$ of cardinality $q = \#\mathcal{O}/\mathfrak{m}$ is called the residue field.

Let $\Delta^0_{\mathbb{K}}$ denote the set of equivalence classes of free rank 2 $\mathcal{O}$-modules, with the equivalence relation

$$M_1 \sim M_2 \Leftrightarrow \exists \lambda \in \mathbb{K}^*, \quad M_1 = \lambda M_2.$$

The group $\mathrm{GL}_2(\mathbb{K})$ acts on $\Delta^0_{\mathbb{K}}$ by $gM = \{gm \mid m \in M\}$. This descends to an action of $\mathrm{PGL}_2(\mathbb{K})$, since $M_1 \sim M_2$ for M_1 and M_2 in the same $\mathbb{K}^*$-orbit. Given $M_2 \subset M_1$, one has $M_1/M_2 \simeq \mathcal{O}/\mathfrak{m}^l \oplus \mathcal{O}/\mathfrak{m}^k$, for some $l, k \in \mathbb{N}$. The action of $\mathbb{K}^*$ preserves the inclusion $M_2 \subset M_1$, hence one has a well defined metric

$$d(M_1, M_2) = |l - k|. \tag{2.1}$$

The Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{K})$ is the infinite graph with set of vertices $\Delta^0_{\mathbb{K}}$, and an edge connecting two vertices M_1, M_2 whenever $d(M_1, M_2) = 1$. It is an infinite tree with vertices of valence $q+1$ where $q = \#\mathcal{O}/\mathfrak{m}$. The group $\mathrm{PGL}_2(\mathbb{K})$ acts on $\Delta_{\mathbb{K}}$ by isometries. The boundary $\partial\Delta_{\mathbb{K}}$ is naturally identified with $\mathbb{P}^1(\mathbb{K})$.

2.2. p-adic Schottky groups and Mumford curves.

A Schottky group Γ is a finitely generated, discrete, torsion-free subgroup of $\mathrm{PGL}_2(\mathbb{K})$ whose nontrivial elements $\gamma \neq 1$ are all *hyperbolic*, *i.e.* such that the eigenvalues of γ in $\mathbb{K}$ have different valuation. The group Γ acts freely on the tree $\Delta_{\mathbb{K}}$. Hyperbolic elements γ have two fixed points $z^{\pm}(\gamma)$ on the boundary $\mathbb{P}^1(\mathbb{K})$. For an element $\gamma \neq 1$ in Γ the *axis* $L(\gamma)$ of γ is the unique geodesic in the Bruhat–Tits tree $\Delta_{\mathbb{K}}$ with endpoints the two fixed points $z^{\pm}(\gamma) \in \mathbb{P}^1(\mathbb{K}) = \partial\Delta_{\mathbb{K}}$.

Let $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{K})$ be the closure in $\mathbb{P}^1(\mathbb{K})$ of the set of fixed points of the elements $\gamma \in \Gamma \setminus \{1\}$. This is called the *limit set* of Γ. Only in the case of $\Gamma = (\gamma)^{\mathbb{Z}} \simeq \mathbb{Z}$, with a single hyperbolic generator γ, one has $\#\Lambda_\Gamma < \infty$. This special case, as we see below, correponds to Mumford curves of genus one. In general, for higher genus (higher rank Schottky groups), the limit set is uncountable (typically a fractal). The *domain of discontinuity* for the Schottky group Γ is the complement $\Omega_\Gamma(\mathbb{K}) = \mathbb{P}^1(\mathbb{K}) \smallsetminus \Lambda_\Gamma$. The quotient $X_\Gamma := \Omega_\Gamma/\Gamma$ gives the analytic model (via uniformization) of an algebraic curve X defined over $\mathbb{K}$ (*cf.* [26] p. 163).

For a p-adic Schottky group $\Gamma \subset \mathrm{PGL}(2, \mathbb{K})$ there is a smallest subtree $\Delta'_\Gamma \subset \Delta_{\mathbb{K}}$ that contains the axes $L(\gamma)$ of all the elements $\gamma \neq 1$ of Γ. The set of ends of Δ'_Γ in $\mathbb{P}^1(\mathbb{K})$ is the limit set Λ_Γ of Γ. The group Γ acts on Δ'_Γ with quotient Δ'_Γ/Γ a finite graph. There is also a smallest tree Δ_Γ on which Γ acts, with vertices a subset of vertices of the Bruhat–Tits tree. The tree Δ'_Γ contains extra vertices with respect to Δ_Γ. These come from vertices of $\Delta_{\mathbb{K}}$ that are not vertices of Δ_Γ, but which lie on paths in Δ_Γ. (Δ_Γ is not a subtree of $\Delta_{\mathbb{K}}$, while Δ'_Γ is.) The quotient Δ_Γ/Γ is also a finite graph. Both the graphs Δ'_Γ/Γ and Δ_Γ/Γ have algebro-geometric significance: Δ'_Γ/Γ is the dual graph of the closed fibre of the minimal smooth model of the algebraic curve X over $\mathbb{K}$; Δ_Γ/Γ is the dual graph of the specialization of the curve X. The latter is a k-split degenerate, stable curve, with k the residue field of $\mathbb{K}$.

The set of $\mathbb{K}$-points $X_\Gamma(\mathbb{K})$ of the Mumford curve is identified with the ends of the graph $\Delta_\mathbb{K}/\Gamma$. With a slight abuse of notation we sometimes say that X_Γ is the boundary of $\Delta_\mathbb{K}/\Gamma$ and write

$$\partial\Delta_\mathbb{K}/\Gamma = X_\Gamma. \tag{2.2}$$

The graph $\Delta_\mathbb{K}/\Gamma$ contains the finite subgraph Δ'_Γ/Γ. Infinite trees depart from the vertices of the subgraph Δ'_Γ/Γ with ends on the boundary at infinity X_Γ. We assume that the base point v belongs to Δ'_Γ so that all these trees are oriented outward from the finite graph Δ'_Γ/Γ. All the nontrivial topology resides in the graph Δ'_Γ/Γ from which one can read off the genus of the curve.

So far, the use of methods of noncommutative geometry in the context of Mumford curves and Schottky uniformization (*cf.* [9], [10], [11], [12], [13]) concentrated on the finite graphs Δ'_Γ/Γ and Δ_Γ/Γ and noncommutative spaces associated to the dynamics of the action of the Schottky group on its limit set. Here we consider the full infinite graph $\Delta_\mathbb{K}/\Gamma$ of (2.2). In fact, we will show that it is precisely the presence in $\Delta_\mathbb{K}/\Gamma$ of the infinite trees attached to the vertices of the finite subgraph Δ'_Γ/Γ that makes it possible to construct interesting KMS states on the associated graph C^*-algebra and hence to apply the techniques of modular index theory to obtain new invariants of a K-theoretic nature for Mumford curves.

2.3. **Directed graphs and their algebras.**

There are different ways to introduce a structure of directed graph on the finite graphs Δ_Γ/Γ, Δ'_Γ/Γ and on the infinite graph $\Delta_\mathbb{K}/\Gamma$.

One possibility, considered for instance in [12], is not to prescribe an orientation on the graphs. This means that one keeps for each edge the choice of both possible orientations. The associated directed graph has then double the number of edges to account for the two possible orientations. This approach has the problem that it makes the graph C^*-algebras more complicated and the combinatorics correspondingly more involved than strictly necessary, so we will not follow it here.

Another way to make the graphs of Mumford curves into directed graphs is by the choice of a projective coordinate $z \in \mathbb{P}^1(\mathbb{K})$ (*cf.* [10], [11]). The choice of the coordinate z determines uniquely a base point $v \in \Delta^0_\mathbb{K}$, given by the origin of three non-overlapping paths with ends the points 0, 1 and ∞ in $\mathbb{P}^1(\mathbb{K})$. The choice of v gives an orientation to the tree $\Delta_\mathbb{K}$ given by the outward direction from v. This gives an induced orientation to any fundamental domains of the Γ-action in $\Delta_\mathbb{K}$, Δ'_Γ and Δ_Γ containing the base vertex v, which one can use to obtain all the possible induced orientations on the quotient graphs.

There is still another possibility of orienting the tree $\Delta_\mathbb{K}$ in a way that is adapted to the action of Γ, and this is the one we adopt here. It is described in Lemma 2.1 below.

Suppose we are given the choice of a projective coordinate $z \in \mathbb{P}^1(\mathbb{K})$ and assume that the corresponding vertex v is in fact a vertex of Δ'_Γ. Let $\{\gamma_1, \ldots, \gamma_g\}$ be a set of generators for Γ. An orientation of $\Gamma\backslash\Delta_\mathbb{K}$ is then obtained from a Γ-invariant orientation of $\Delta_\mathbb{K}$ as follows.

Lemma 2.1. *Consider the chain of edges in $\Delta_\mathbb{K}$ connecting the base vertex v to $\gamma_i v$. Then there is a choice of orientation of these edges that induces an orientation on the quotient graph and that extends to a Γ-invariant orientation of $\Delta_\mathbb{K}$.*

Proof. Consider the chain of edges between v and $\gamma_i v$. If all of them have distinct images in the quotient graph orient them all in the direction away from v and towards $\gamma_i v$. If there is more than one edge in the path from v to $\gamma_i v$ that maps to the same edge in the quotient graph, orient the first one that occurs from v to $\gamma_i v$ and the others consistently with the induced orientation of the corresponding edge in the quotient graph. Similarly, orient the edges between v and $\gamma_i^{-1} v$ in the direction pointing towards v, with the same caveat for edges with the same image in the quotient. Propagate this orientation across the tree Δ'_Γ by repeating the same procedure with the edges between $\gamma_i^{\pm 1} v$ and $\gamma_j \gamma_i^{\pm 1} v$ and between $\gamma_i^{\pm 1} v$ and $\gamma_j^{-1} \gamma_i^{\pm 1} v$ and so on. Continuing in this way, one obtains an orientation of the tree Δ'_Γ compatible with the induced orientation on the quotient graph Δ'_Γ/Γ. One then orients the rest of the tree $\Delta_{\mathbb{K}}$ away from the subtree Δ'_Γ. □

An example of the orientations obtained in this way on the tree Δ_Γ and on the quotient graph for the genus two case is given in Figures 7, 8, 9 below.

2.4. **Graph algebras for Mumford curves.** For a more detailed introduction to graph C^*-algebras we refer the reader to [1, 20, 29] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. *We will always assume that the graph is row-finite*, which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite* which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2 \cdots \mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \le i < n$. The maps r, s extend to $E^* := \bigcup_{n\ge 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$, we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \ge w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from V). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \quad\text{and}\quad p_v = \sum_{\{e : s(e) = v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

It is proved in [20, Theorem 1.2] that there is a universal C^*-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2 \cdots \mu_n$ is a path in E^n. Since the Cuntz-Krieger relations imply that the projections $S_e S_e^*$ are also mutually orthogonal, we have $S_e^* S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [20, Lemma 1.1]). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{2.3}$$

The algebraic relations and the density of $\operatorname{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role throughout the paper. We adopt the conventions that vertices are paths of length 0, that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (2.3) are non-empty; we recover S_μ, for example, by taking $\nu = r(\mu)$, so that $S_\mu S_\nu^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{zS_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and

$\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\epsilon/3$ argument using (2.3) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action.* Because S^1 is compact, averaging over γ with respect to normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi}\int_{S^1} \gamma_z(a)\, d\theta \quad \text{for} \quad a \in C^*(E), \quad z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From Equation (2.3), it is easy to see that a graph C^*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, formulas which involve sums of projections may contain infinite sums. To interpret these, we use strict convergence in the multiplier algebra of $C^*(E)$:

Lemma 2.2 ([20])**.** *Let E be a row-finite graph, let A be a C^*-algebra generated by a Cuntz-Krieger E-family $\{T_e, q_v\}$, and let $\{p_n\}$ be a sequence of projections in A. If $p_n T_\mu T_\nu^*$ converges for every $\mu, \nu \in E^*$, then $\{p_n\}$ converges strictly to a projection $p \in M(A)$.*

The directed graph $\Delta_{\mathbb{K}}/\Gamma$ we obtain from a Mumford curve, with the orientation of Lemma 2.1, is locally finite, has no sources and contains a subgraph Δ'_Γ/Γ with no sources and with the following two properties. If v is any vertex in $\Delta_{\mathbb{K}}/\Gamma$ there exists a path in $\Delta_{\mathbb{K}}/\Gamma$ with range v and source contained in Δ'_Γ/Γ, and for any path with source outside M, the range is outside M. For such a graph we can define a new circle action by restricting the gauge action to the subgraph. The properties of this action turn out to be crucial for us.

The reason may be found in [3], where the existence of a Kasparov A-A^σ module for a circle action σ on A was found to be equivalent to a condition on the spectral subspaces $A_k = \{a \in A : \sigma_z(a) = z^k a\}$. The condition, called the *spectral subspace condition* in [3], states that for all $k \in \mathbb{Z}$, $A_k A_k^*$, always an ideal in A^σ, is in fact a complemented ideal in A^σ. Thus we must have $A^\sigma = A_k A_k^* \oplus G_k$ for some other ideal G_k. It turns out that the graphs arising from Mumford curves allow us to define a circle action for which the spectral subspaces satisfy the spectral subspace condition.

Definition 2.3. *Let E be a locally finite directed graph with no sources, $M \subset E$ a subgraph with no sources and such that*
1) for any $v \in E^0$ there is a path μ with $s(\mu) \in M$ and $r(\mu) = v$
2) for all paths ρ with $s(\rho) \notin M$ we have $r(\rho) \notin M$.
Then we say that E has zhyvot M, and that M is a zhyvot of E.

The zhyvot action $\sigma : \mathbb{T} \to Aut(C^*(E))$ is defined by

$$\sigma_z(S_e) = \begin{cases} \gamma_z(S_e) & e \in M^1 \\ S_e & e \notin M^1 \end{cases} \qquad \sigma_z(p_v) = p_v, \quad v \in E^0,$$

where γ is the usual gauge action. If μ is a path in E, let $|\mu|_\sigma$ be the non-negative integer such that $\sigma_z(S_\mu) = z^{|\mu|_\sigma} S_\mu$.

Remark The zhyvot of a graph need not be unique.

Example In the case of Mumford curves, the finite graph Δ'_Γ/Γ gives a zhyvot for the infinite graph $\Delta_{\mathbb{K}}/\Gamma$. There are other possible choices of a zhyvot for the same graph $\Delta_{\mathbb{K}}/\Gamma$, which are interesting from the point of view of the geometry of Mumford curves. In particular, in the theory of Mumford curves, one considers the reduction modulo powers $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{\mathbb{K}}$, which provides infinitesimal neighborhoods of order n of the closed fiber. For each $n \geq 0$, we consider a subgraph $\Delta_{\mathbb{K},n}$ of the Bruhat-Tits tree $\Delta_{\mathbb{K}}$ defined by setting

$$\Delta^0_{\mathbb{K},n} := \{v \in \Delta^0_{\mathbb{K}} : \, d(v, \Delta'_\Gamma) \leq n\},$$

with respect to the distance (2.1), with $d(v, \Delta'_\Gamma) := \inf\{d(v, \tilde{v}) : \, \tilde{v} \in (\Delta'_\Gamma)^0\}$, and

$$\Delta^1_{\mathbb{K},n} := \{w \in \Delta^1_{\mathbb{K}} : \, s(w), r(w) \in \Delta^0_{\mathbb{K},n}\}.$$

Thus, we have $\Delta_{\mathbb{K},0} = \Delta'_\Gamma$ and $\Delta_{\mathbb{K}} = \cup_n \Delta_{\mathbb{K},n}$. For all $n \in \mathbb{N}$, the graph $\Delta_{\mathbb{K},n}$ is invariant under the action of the Schottky group Γ on Δ, and the finite graph $\Delta_{\mathbb{K},n}/\Gamma$ gives the dual graph of the reduction $X_{\mathbb{K}} \otimes \mathcal{O}/\mathfrak{m}^{n+1}$. Thus, we refer to the $\Delta_{\mathbb{K},n}$ as *reduction graphs*. They form a directed family with inclusions $j_{n,m} : \Delta_{\mathbb{K},n} \hookrightarrow \Delta_{\mathbb{K},m}$, for all $m \geq n$, with all the inclusions compatible with the action of Γ. Each of the quotient graphs $\Delta_{\mathbb{K},n}/\Gamma$ also gives a zhyvot for $\Delta_{\mathbb{K}}/\Gamma$. In the following we will concentrate on the case where $M = \Delta'_\Gamma/\Gamma$ but one can equivalently work with the reduction graphs.

Given a graph E with zhyvot M and $k \geq 0$ define

$$F_k := \overline{\operatorname{span}}\{S_\mu S_\nu^* : \, |\mu|_\sigma = |\nu|_\sigma \geq k\},$$

$$G_k := \overline{\operatorname{span}}\{S_\mu S_\nu^* : \, 0 \leq |\mu|_\sigma = |\nu|_\sigma < k, \text{ and either } r(\mu) = r(\nu) \notin M \text{ or } r(\mu) = r(\nu) \text{ is a sink in } M\}.$$

Observe that in the definition of G_k, the sinks need not be sinks of the full graph E, just sinks of the subgraph M.

Notation Given a path $\rho \in E^*$, we let $\underline{\rho}$ denote the initial segment of ρ and let $\overline{\rho}$ denote the final segment; in all cases the length of these segments will be clear from context. We always have $\rho = \underline{\rho}\overline{\rho}$.

Lemma 2.4. *Let E be a locally finite directed graph with no sources and zhyvot $M \subset E$. Let $F = C^*(E)^\sigma$ be the fixed point algebra for the zhyvot action. Then*

$$F = F_k \oplus G_k, \qquad k = 1, 2, 3, \ldots.$$

Proof. We first check using generators that $F_k G_k = G_k F_k = \{0\}$; once we have shown that $F_k + G_k = F$ this will also show that F_k and G_k are both ideals (that they are subalgebras follows from similar, but simpler, calculations to those below).

Fix $k \geq 1$. Let $S_\mu S_\nu^* \in G_k$ so that $0 \leq |\mu|_\sigma = |\nu|_\sigma < k$ and either $r(\mu) = r(\nu) \notin M$ or is a sink of M. Let $S_\rho S_\tau^* \in F_k$ so that $|\rho|_\sigma = |\tau|_\sigma \geq k$. Then

$$S_\mu S_\nu^* S_\rho S_\tau^* = \begin{cases} S_\mu S_{\overline{\rho}} S_\tau^* \delta_{\nu,\underline{\rho}} & |\nu| \leq |\rho| \\ S_\mu S_{\overline{\nu}}^* S_\tau^* \delta_{\underline{\nu},\rho} & |\nu| \geq |\rho| \end{cases},$$

where $|\cdot|$ denotes the usual length of paths. When $|\nu| \geq |\rho|$, the product is nonzero if and only if $\underline{\nu} = \rho$ but

$$|\underline{\nu}|_\sigma \leq |\nu|_\sigma < |\rho|_\sigma,$$

so this can not happen. When $|\nu| \leq |\rho|$, the product is nonzero if and only if $\nu = \underline{\rho}$, but the range of $\nu \notin M$ or is a sink of M while $|\nu|_\sigma = |\underline{\rho}|_\sigma$ implies that $|\underline{\rho}|_\sigma < |\rho|_\sigma$, and so $r(\underline{\rho}) \in M$ and is not a sink of M. Hence the product is zero, and $G_k F_k = \{0\}$. The computation $F_k G_k = \{0\}$ is entirely analogous, so we omit it.

To see that $F_k + G_k = F$, we need only show that the generators $S_\mu S_\nu^*$ with $0 \le |\mu|_\sigma = |\nu|_\sigma < k$ and $r(\mu) = r(\nu) \in M$ is not a sink, are sums of elements from F_k and G_k, all other generators having been accounted for.

So let $0 \le n = |\mu|_\sigma = |\nu|_\sigma < k$ and recall that $|\rho| \preceq k$ if $|\rho| = k$ or $|\rho| < k$ and $r(\rho)$ is a sink. Then

$$S_\mu S_\nu^* = \sum_{\rho\in E^*,\ s(\rho)=r(\mu),\ |\rho|\preceq k-n+1} S_\mu S_\rho S_\rho^* S_\nu^*.$$

If $0 \le |\rho|_\sigma < k-n$ then we must have $r(\rho) \notin M$ or $r(\rho)$ a sink of M. This is because $|\rho|_\sigma \le |\rho|$, and if $r(\rho)$ is not a sink, we have strict inequality since $|\rho| = k-n+1$. Hence if $r(\rho)$ is not a sink, $r(\rho) \notin M$. On the other hand if $r(\rho)$ is a sink of E, then either $r(\rho) \notin M$ or $r(\rho)$ is a sink of M.

Thus for $0 \le |\rho|_\sigma < k-n$ we have $S_\mu S_\rho S_\rho^* S_\nu^* \in G_k$, while if $k-n \le |\rho|_\sigma \le k-n+1$, we have $S_\mu S_\rho S_\rho^* S_\nu^* \in F_k$.

Finally, to see that $F = F_k \oplus G_k$ for each $k \ge 0$, observe that we can split the sequence

$$0 \to F_k \xrightarrow{i} F \to G_k \to 0$$

using the homomorphism $\phi_k : F \to F_k$ defined by

$$\phi_k(f) = P_k f P_k, \qquad P_k = \sum_{|\mu|_\sigma=k} S_\mu S_\mu^*.$$

Checking that ϕ_k is a homomorphism and has range F_k is an exercise with the generators. □

Proposition 2.5. *Let E be a locally finite directed graph without sources and with zhyvot M. For $k \in \mathbb{Z}$ let $A_k = \{a \in C^*(E) : \sigma_z(a) = z^k a\}$ denote the spectral subspaces for the zhyvot action. Then*

$$A_k A_k^* = \begin{cases} F_k & k \ge 0 \\ F & k \le 0 \end{cases}.$$

Remark In particular, the spectral subspace assumptions of [3] are satisfied for the zhyvot action on a graph with a zhyvot.

Proof. With $|\mu|$ denoting the ordinary length of paths in E, we have the product formula

$$(S_\mu S_\nu^*)(S_\sigma S_\rho^*)^* = S_\mu S_\nu^* S_\rho S_\sigma^* = \begin{cases} S_\mu S_{\overline{\rho}} S_\sigma^* \delta_{\nu,\underline{\rho}} & |\nu| \le |\rho| \\ S_\mu S_{\overline{\nu}}^* S_\sigma^* \delta_{\underline{\nu},\rho} & |\nu| \ge |\rho| \end{cases}, \tag{2.4}$$

where $\underline{\rho}$ is the initial segment of ρ of appropriate length, and $\overline{\rho}$ is the final segment. If $S_\mu S_\nu^*$, $S_\sigma S_\rho^*$ are in A_k, $k \ge 0$, then

$$|\mu|_\sigma - |\nu|_\sigma = k = |\gamma|_\sigma - |\rho|_\sigma,$$

so that $|\gamma|_\sigma \ge k$ and $|\mu|_\sigma \ge k$. Together with Equation (2.4), this shows that for $k \ge 0$ we have $A_k A_k^* \in F_k$. Conversely, if $S_\alpha S_\beta^* \in F_k$, so $|\alpha|_\sigma = |\beta|_\sigma \ge k$, we can factor

$$S_\alpha S_\beta^* = S_{\underline{\alpha}} S_{\overline{\alpha}} S_\beta^* = S_{\underline{\alpha}} (S_\beta S_{\overline{\alpha}}^*)^* \in A_k A_k^*.$$

For $k \le 0$ we of course have $A_k A_k^* \in F$, and so we need only show that for any $S_\alpha S_\beta^* \in F$, $S_\alpha S_\beta^* \in A_k A_k^*$.

Here we use the final property of zhyvot graphs, namely that we can find a path $\lambda \in E^*$ with $s(\lambda) \in M$ and $r(\lambda) = r(\alpha) = r(\beta)$. Moreover, because M has no sources, we can take $|\lambda|_\sigma$ as great as we like. Thus we can write

$$S_\alpha S_\beta^* = S_\alpha S_\lambda^* S_\lambda S_\beta^* = (S_\alpha S_\lambda^*)(S_\beta S_\lambda^*)^*.$$

Choosing $|\lambda|_\sigma = |\alpha|_\sigma + |k|$ shows that $S_\alpha S_\beta^* \in A_k A_k^*$. □

This allows us to recover some known structure of the fixed point algebra of a graph algebra for the usual gauge action, and to understand via the spectral subspace condition of [3] exactly why the assumptions of [27] were required to construct a Kasparov module.

Corollary 2.6. *Let E be a locally finite directed graph without sources. Then the fixed point algebra for the usual gauge action decomposes as*

$$F = F_k \oplus G_k, \quad k = 1, 2, 3, \dots$$

where

$$F_k = \overline{\mathrm{span}}\{S_\mu S_\nu^* : |\mu| = |\nu| \geq k\}, \quad G_k = \overline{\mathrm{span}}\{S_\mu S_\nu^* : 0 \leq |\mu| = |\nu| < k,\ r(\mu) = r(\nu) \text{ is a sink}\}.$$

Proof. This follows from Proposition 2.5 since E is a graph with zhyvot E. □

3. Schottky invariants of Mumford Curves and Field Extensions

3.1. **Schottky lengths and valuation.** Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$ be a p-adic Schottky group acting by isometries on the Bruhat–Tits tree $\Delta_\mathbb{K}$. As recalled in §2.2 above, a hyperbolic element $\gamma \in \Gamma$ determines a unique axis $L(\gamma)$ in $\Delta_\mathbb{K}$, which is the infinite path of edges connecting the two fixed points $z^\pm(\gamma) \in \Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{K}) = \partial\Delta_\mathbb{K}$. The element γ acts on $L(\gamma)$ by a translation of length $\ell(\gamma)$.

To a given set of generators $\{\gamma_1, \cdots \gamma_g\}$ of Γ one can associate the translation lengths $\ell(\gamma_i)$. We refer to the collection of values $\{\ell(\gamma_i)\}$ as the *Schottky invariants* of $(\Gamma, \{\gamma_i\})$. For example, in the case of genus one, one can assume the generator of Γ is given by a matrix for the form

$$\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

with $|q| < 1$ so that the fixed points are $z^+(\gamma) = 0$ and $z^-(\gamma) = \infty$. The element γ acts on the axis $L(\gamma)$ as a translation by a length $\ell(\gamma) = \log|q|^{-1} = v_\mathfrak{m}(q)$ equal to the number of vertices in the closed graph (topologically a circle) Δ'_Γ/Γ. We see clearly that, even in the simple genus one case, knowledge of the Schottky invariant $\ell(\gamma)$ does not suffice to recover the curve. This is clear from the fact that the Schottky length only sees the valuation of $q \in \mathbb{K}^*$. Nonetheless, the Schottky lengths give useful computable invariants.

3.2. **Field extensions.** In the following section, where we derive explicit KMS states associated to the infinite graphs given by the quotients $\Delta_\mathbb{K}/\Gamma$, we also discuss the issue of how the invariants we construct in this way for Mumford curves behave under field extensions of $\mathbb{K}$. To this purpose, we recall here briefly how the graphs $\Delta_\mathbb{K}$ and Δ'_Γ are affected when passing to a field extension (*cf.* [22]). This was also recalled in more detail in [10].

Let $\mathbb{L} \supset \mathbb{K}$ be a field extension with finite degree, $[\mathbb{L} : \mathbb{K}] < \infty$, and let $e_{\mathbb{L}/\mathbb{K}}$ be its ramification index. Let $\mathcal{O}_\mathbb{L}$ and $\mathcal{O}_\mathbb{K}$ denote the respective rings of integers. There is an embedding of the sets of

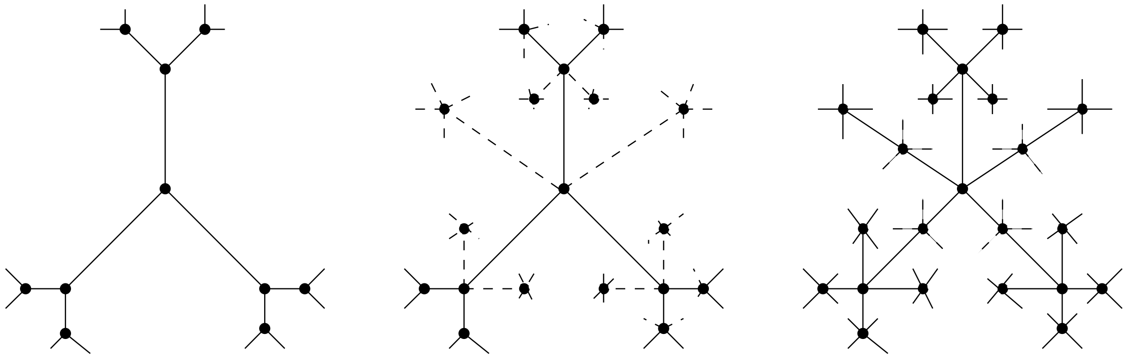

FIGURE 1. The tree $\Delta_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{Q}_2$ and $\Delta_{\mathbb{L}}$ for a field extension with $f = 2$ and $e_{\mathbb{L}/\mathbb{K}} = 2$

vertices $\Delta^0_{\mathbb{K}} \hookrightarrow \Delta^0_{\mathbb{L}}$ obtained by assigning to a free $\mathcal{O}_{\mathbb{K}}$-module M of rank 2 the free $\mathcal{O}_{\mathbb{L}}$-module of the same rank given by $M \otimes_{\mathcal{O}_{\mathbb{K}}} \mathcal{O}_{\mathbb{L}}$. This operation preserves the equivalence relation. However, the embedding $\Delta^0_{\mathbb{K}} \hookrightarrow \Delta^0_{\mathbb{L}}$ obtained in this way is not isometric, as one can see from the isomorphism $(\mathcal{O}_{\mathbb{K}}/\mathfrak{m}^r) \otimes \mathcal{O}_{\mathbb{L}} \simeq \mathcal{O}_{\mathbb{L}}/\mathfrak{m}^{re_{\mathbb{L}/\mathbb{K}}}$. This can be corrected by modifying the metric on the graphs $\Delta_{\mathbb{L}}$, for all extensions $\mathbb{L} \supset \mathbb{K}$: if one uses the $\mathbb{K}$-normalized distance

$$d_{\mathbb{K}}(M_1, M_2) := \frac{1}{e_{\mathbb{L}/\mathbb{K}}} d_{\mathbb{L}}(M_1, M_2), \tag{3.1}$$

on $\Delta^0_{\mathbb{L}}$, one obtains an isometric embedding $\Delta^0_{\mathbb{K}} \hookrightarrow \Delta^0_{\mathbb{L}}$.

Geometrically, the relation between the Bruhat–Tits trees $\Delta_{\mathbb{K}}$ and $\Delta_{\mathbb{L}}$ is described by the following procedure that constructs $\Delta_{\mathbb{L}}$ from $\Delta_{\mathbb{K}}$ given the values of $e_{\mathbb{L}/\mathbb{K}}$ and $[\mathbb{L} : \mathbb{K}]$. The rule for inserting new vertices and edges when passing to a field extension $\mathbb{L} \supset \mathbb{K}$ is the following.

(1) $e_{\mathbb{L}/\mathbb{K}} - 1$ new vertices $\{v_1, \ldots, v_{e_{\mathbb{L}/\mathbb{K}}-1}\}$ are inserted between each pair of adjacent vertices in $\Delta^0_{\mathbb{K}}$. Let $\Delta^0_{\mathbb{L},\mathbb{K}}$ denote the set of all these additional vertices.
(2) $q^f + 1$ edges depart from each vertex in $\Delta^0_{\mathbb{K}} \cup \Delta^0_{\mathbb{L},\mathbb{K}}$, with $f = \frac{1}{e_{\mathbb{L}/\mathbb{K}}}[\mathbb{L} : \mathbb{K}]$. Each such edge has length $\frac{1}{e_{\mathbb{L}/\mathbb{K}}}$.
(3) Each new edge attached to a vertex in $\Delta^0_{\mathbb{K}} \cup \Delta^0_{\mathbb{L},\mathbb{K}}$ is the base of a number of homogeneous tree of valence $q^f + 1$. The number is determined by the property that in the resulting graph the vertex from which the trees stem also has to have valence $q^f + 1$. The Bruhat–Tits tree $\Delta_{\mathbb{L}}$ is the union of $\Delta_{\mathbb{K}}$ with the additional inserted vertices $\Delta^0_{\mathbb{L},\mathbb{K}}$ and the added trees stemming from each vertex.

This procedure is illustrated in Figure 1, which we report here from [10].

Suppose we are given a p-adic Schottky group $\Gamma \subset \mathrm{PGL}_2(\mathbb{K})$. Since all nontrivial elements of Γ are hyperbolic (the eigenvalues have different valuation), one can see that the two fixed points of any nontrivial element of Γ are in $\mathbb{P}^1(\mathbb{K}) = \partial\Delta_{\mathbb{K}}$. Thus, the limit set Λ_Γ is contained in $\mathbb{P}^1(\mathbb{K})$.

When one considers a finite extension $\mathbb{L} \supset \mathbb{K}$ and the corresponding Mumford curve $X_\Gamma(\mathbb{L}) = \Omega_\Gamma(\mathbb{L})/\Gamma$ with $\Omega_\Gamma(\mathbb{L}) = \mathbb{P}^1(\mathbb{L}) \smallsetminus \Lambda_\Gamma$, one can see this as the boundary of the graph $\Delta_{\mathbb{L}}/\Gamma$. Notice that the subtree

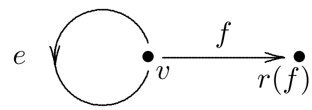

FIGURE 2. A loop with exit

$\Delta'_{\Gamma,\mathbb{L}}$ of $\Delta_{\mathbb{L}}$ and the subtree $\Delta'_{\Gamma,\mathbb{K}}$ of $\Delta_{\mathbb{K}}$, both of which have boundary Λ_Γ only differ by the presence of the additional $e_{\mathbb{L}/\mathbb{K}}-1$ new vertices in between any two adjacent vertices of $\Delta'_{\Gamma,\mathbb{K}}$, while no new direction has been added (the limit points are the same). In particular, this means that the finite graph $\Delta'_{\Gamma,\mathbb{L}}/\Gamma$ is obtained from $\Delta'_{\Gamma,\mathbb{K}}/\Gamma$ by adding $e_{\mathbb{L}/\mathbb{K}}-1$ vertices on each edge. The infinite graph $\Delta_{\mathbb{K}}/\Gamma$ is obtained by adding to each vertex of the finite graph $\Delta'_{\Gamma,\mathbb{K}}/\Gamma$ a finite number (possibly zero) of infinite homogeneous trees of valence $q+1$ with base at that vertex. Given the finite graph $\Delta'_{\Gamma,\mathbb{K}}/\Gamma$, the number of such trees to be added at each vertex is determined by the requirement that the valence of each vertex of $\Delta_{\mathbb{K}}/\Gamma$ equals $q+1$. The infinite graph $\Delta_{\mathbb{L}}/\Gamma$ is obtained from the graph $\Delta_{\mathbb{K}}/\Gamma$ by replacing the homogeneous trees of valence $q+1$ starting from the vertices of $\Delta'_{\Gamma,\mathbb{K}}/\Gamma$ with homogeneous trees of valence q^f+1 stemming from the vertices of $\Delta'_{\Gamma,\mathbb{L}}/\Gamma$, so that each resulting vertex of $\Delta_{\mathbb{L}}/\Gamma$ has valence q^f+1.

We analyze the effect of field extensions from the point of view of KMS weights and modular index theory in §4.1 below.

4. Graph KMS Weights on Directed Graphs

Let E be a row finite graph, and $C^*(E)$ the associated graph C^*-algebra.

Definition 4.1. *A graph weight on E is a pair of functions $g : E^0 \to [0,\infty)$ and $\lambda : E^1 \to [0,\infty)$ such that for all vertices v*

$$g(v) = \sum_{s(e)=v} \lambda(e) g(r(e)).$$

A graph weight is called faithful if $g(v) \neq 0$ for all $v \in E^0$. If $\sum_{v\in E^0} g(v) = 1$, we call (g,λ) a graph state.

Remark If $\lambda(e) = 1$ for all $e \in E^1$, we obtain the definition of a graph trace, [32].

Example Suppose e is a simple loop in a graph, with exit f at the vertex v, and that there are no other loops, and no other exits from v, as in Figure 2. Set

$$g(v) = \lambda(e)g(v) + \lambda(f)g(r(f)).$$

Then $g(v) = \frac{\lambda(f)}{1-\lambda(e)} g(r(f))$.

Remark A graph weight is in fact specified by a single function $h : E^* \to [0,\infty)$. For paths v of length zero, i.e. vertices, $h(v) = g(v)$ and for paths μ of length $k \geq 1$, $h(\mu) = \lambda(\mu_1)\lambda(\mu_2)\cdots\lambda(\mu_k)$. We retain the (g,λ) notation but extend the definition of λ by $\lambda(\mu) = \prod_{i=1}^k \lambda(\mu_i)$.

Recall that a path $|\mu|$ has length $|\mu| \preceq k$ if $|\mu| = k$ or $|\mu| < k$ and $r(\mu)$ is a sink.

We then have the following result, which can be proved by induction.

Lemma 4.2. *If (g,λ) is a graph weight on E, then*
$$g(v) = \sum_{s(\mu)=v,\ |\mu|\preceq k} \lambda(\mu)g(r(\mu)),$$
where for a path $\mu = e_1\cdots e_j$, $j \le k$, $\lambda(\mu) = \prod \lambda(e_j)$.

We then define a functional $\phi_{g,\lambda}$ associated to a graph weight (g,λ) as follows.

Definition 4.3. *Given (g,λ) on E a graph weight, define $\phi_{g,\lambda} : \mathrm{span}\{S_\mu S_\nu^* : \mu,\nu \in E^*\} \to \mathbb{C}$ by*
$$\phi_{g,\lambda}(S_\mu S_\nu^*) := \delta_{\mu,\nu}\lambda(\nu)\phi_{g,\lambda}(p_{r(\nu)}) := \lambda(\nu)\delta_{\mu,\nu}g(r(\nu)).$$

This yields the following useful results.

Proposition 4.4. *Let $A_c = \mathrm{span}\{S_\mu S_\nu^* : \mu,\nu \in E^*\}$, and let (g,λ) be a faithful graph weight on E. Then A_c with the inner product*
$$\langle a,b\rangle := \phi_{g,\lambda}(a^*b)$$
is a modular Hilbert algebra (or Tomita algebra).

Proof. To complete the definition of modular Hilbert algebra, we must supply a complex one parameter group of algebra automorphisms σ_z and verify a number of conditions set out in [31]. So for $z \in \mathbb{C}$ define
$$\sigma_z(S_\mu S_\nu^*) = \left(\frac{\lambda(\mu)}{\lambda(\nu)}\right)^z S_\mu S_\nu^*.$$
Extending by linearity we can define σ_z on all of A_c. To verify the algebra automorphism property, it suffices to show that
$$\sigma_z(S_\mu S_\nu^* S_\rho S_\kappa^*) = \sigma_z(S_\mu S_\nu^*)\sigma_z(S_\rho S_\kappa^*).$$
To do this we introduce some notation. If ρ is a path we write $\rho = \underline{\rho}\overline{\rho}$ where $\underline{\rho}$ is the initial segment of ρ (of length to be understood from context) and $\overline{\rho}$ for the final segment. First we compute the product on the left hand side.
$$S_\mu S_\nu^* S_\rho S_\kappa^* = \begin{cases} \delta_{\nu,\underline{\rho}} S_\mu S_{\overline{\rho}} S_\kappa^* & |\nu| \le |\rho| \\ \delta_{\underline{\nu},\rho} S_\mu S_{\overline{\nu}}^* S_\kappa^* & |\nu| \ge |\rho| \end{cases}.$$
So
$$\sigma_z(S_\mu S_\nu^* S_\rho S_\kappa^*) = \begin{cases} \left(\frac{\lambda(\mu\overline{\rho})}{\lambda(\kappa)}\right)^z \delta_{\nu,\underline{\rho}} S_\mu S_{\overline{\rho}} S_\kappa^* & |\nu| \le |\rho| \\ \left(\frac{\lambda(\mu)}{\lambda(\kappa\underline{\nu})}\right)^z \delta_{\underline{\nu},\rho} S_\mu S_{\overline{\nu}}^* S_\kappa^* & |\nu| \ge |\rho| \end{cases}.$$
On the right hand side we have
$$\sigma_z(S_\mu S_\nu^*)\sigma_z(S_\rho S_\kappa^*) = \left(\frac{\lambda(\mu)\lambda(\rho)}{\lambda(\nu)\lambda(\kappa)}\right)^z \begin{cases} \delta_{\nu,\underline{\rho}} S_\mu S_{\overline{\rho}} S_\kappa^* & |\nu| \le |\rho| \\ \delta_{\underline{\nu},\rho} S_\mu S_{\overline{\nu}}^* S_\kappa^* & |\nu| \ge |\rho| \end{cases}$$
$$= \begin{cases} \left(\frac{\lambda(\mu)\lambda(\overline{\rho})}{\lambda(\kappa)}\right)^z \delta_{\nu,\underline{\rho}} S_\mu S_{\overline{\rho}} S_\kappa^* & |\nu| \le |\rho| \\ \left(\frac{\lambda(\mu)}{\lambda(\overline{\nu})\lambda(\kappa)}\right)^z \delta_{\underline{\nu},\rho} S_\mu S_{\overline{\nu}}^* S_\kappa^* & |\nu| \ge |\rho| \end{cases}$$
and this is easily seen to be the same as the left hand side whenever the product is nonzero. Observe we have used the fact that
$$\lambda(\rho) = \lambda(\underline{\rho})\lambda(\overline{\rho}).$$

We need to show that $\langle a, b\rangle = \phi_{g,\lambda}(a^*b)$ does define an inner product. Let $a \in A_c$ and let $p \in A_c$ be a finite sum of vertex projections such that $pa = ap = a$ (p is a local unit for a). Then, since $g(v) > 0$ for all $v \in E^0$,

$$b \mapsto \frac{\phi_{g,\lambda}(pbp)}{\phi_{g,\lambda}(p)}$$

is a state on $pC^*(E)p$, and so positive. Hence

$$\phi_{g,\lambda}(a^*a) \geq 0.$$

To show that the inner product is definite requires more care. First observe that if $\Psi : A_c \to \mathrm{span}\{P_\mu = S_\mu S_\mu^*\}$ is the expectation on to the diagonal subalgebra, then $\phi_{g,\lambda} = \phi_{g,\lambda} \circ \Psi$. So we consider $a \in A_c$ and write $\Psi(a^*a) = \sum_\mu c_\mu P_\mu - \sum_\nu c_\nu P_\nu$. Here the c_μ, $c_\nu > 0$ and none of the paths μ is repeated in the sum. The average $\Psi(a^*a)$ is a positive operator, so if $\Psi(a^*a)$ is non-zero, all the P_ν in the negative part must be subprojections of $\sum_\mu P_\mu$ (otherwise $\Psi(a^*a)$ would have some negative spectrum). Since we are in a graph algebra, the Cuntz-Krieger relations tell us we can write

$$\sum_\mu P_\mu = \sum_\mu \sum_\rho P_{\mu\rho}$$

for some paths ρ extending the various μ, and that moreover all the P_ν appear as some $P_{\mu\rho}$. Thus

$$\Psi(a^*a) = \sum_\mu c_\mu P_\mu - \sum_\nu c_\nu P_\nu = \sum_\mu c_\mu \sum_\rho P_{\mu\rho} - \sum_\nu c_\nu P_\nu = \sum_\mu \sum_\rho d_{\mu\rho} P_{\mu\rho},$$

where the $d_{\mu\rho}$ are necessarily positive. Now we can compute

$$\phi_{g,\lambda}(a^*a) = \phi_{g,\lambda}(\sum_\mu \sum_\rho d_{\mu\rho} P_{\mu\rho}) = \sum_\mu \sum_\rho d_{\mu\rho} \lambda(\mu\rho) g(r(\mu\rho)) > 0.$$

So now we come to verifying the various conditions defining a modular Hilbert algebra. First, we need to consider the action of A_c on itself by left multiplication. This action is multiplicative,

$$\langle ba, a\rangle := \phi_{g,\lambda}(a^*b^*a) = \langle a, b^*a\rangle,$$

and continuous

$$\langle ba, ba\rangle = \phi_{g,\lambda}(a^*b^*ba) \leq \|b^*b\| \langle a, a\rangle,$$

where $\|\cdot\|$ denotes the C^*-norm coming from $C^*(E)$. As $A_c^2 = A_c$, the density of A_c^2 in A_c is trivially fulfilled. Also for all real t, $(1 + \sigma_t)(S_\mu S_\nu^*) = (1 + (\lambda(\mu)/\lambda(\nu))^t) S_\mu S_\nu^*$, and so it is an easy check to see that $(1 + \sigma_t)(A_c)$ is dense in A_c for all real t. Also

$$\langle \sigma_{\bar{z}}(S_\mu S_\nu^*), S_\rho S_\kappa^*\rangle = \left(\frac{\lambda(\mu)}{\lambda(\nu)}\right)^z \langle S_\mu S_\nu^*, S_\rho S_\kappa^*\rangle$$

is plainly analytic in z (the reason for $\sigma_{\bar{z}}$ is that our inner product is conjugate linear in the first variable). Since a finite sum of analytic functions is analytic, $\langle \sigma_{\bar{z}}(a), b\rangle$ is analytic for all $a, b \in A_c$.

The remaining items to check are the compatibility of σ_z with the inner product and involution, and all of these we can check for monomials $S_\mu S_\nu^*$. The first item to check is

$$\begin{aligned}(\sigma_z(S_\mu S_\nu^*)^*) &= \left(\frac{\lambda(\mu)}{\lambda(\nu)}\right)^{\bar z} S_\nu S_\mu^* \\ &= \left(\frac{\lambda(\nu)}{\lambda(\mu)}\right)^{-\bar z} S_\nu S_\mu^* \\ &= \sigma_{-\bar z}((S_\mu S_\nu^*)^*).\end{aligned}$$

Next we require $\langle \sigma_z(a), b\rangle = \langle a, \sigma_{\bar z}(b)\rangle$. So we compute

$$\begin{aligned}\langle \sigma_z(S_\mu S_\nu^*), S_\rho S_\kappa^*\rangle &= \left(\frac{\lambda(\mu)}{\lambda(\nu)}\right)^{\bar z} g(r(\kappa)) \begin{cases} \delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}}\lambda(\kappa\overline{\mu}) & |\mu| \geq |\rho| \\ \delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa}\lambda(\kappa) & |\mu| \leq |\rho| \end{cases} \\ &= g(r(\kappa)) \begin{cases} \left(\frac{\lambda(\mu)}{\lambda(\kappa\overline{\mu})}\right)^{\bar z} \delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}}\lambda(\kappa\overline{\mu}) & |\mu| \geq |\rho| \\ \left(\frac{\lambda(\underline{\rho})}{\lambda(\nu)}\right)^{\bar z} \delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa}\lambda(\kappa) & |\mu| \leq |\rho| \end{cases} \\ &= g(r(\kappa)) \begin{cases} \left(\frac{\lambda(\underline{\mu})}{\lambda(\kappa)}\right)^{\bar z} \delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}}\lambda(\kappa\overline{\mu}) & |\mu| \geq |\rho| \\ \left(\frac{\lambda(\underline{\rho})}{\lambda(\underline{\kappa})}\right)^{\bar z} \delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa}\lambda(\kappa) & |\mu| \leq |\rho| \end{cases} \\ &= g(r(\kappa)) \begin{cases} \left(\frac{\lambda(\rho)}{\lambda(\kappa)}\right)^{\bar z} \delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}}\lambda(\kappa\overline{\mu}) & |\mu| \geq |\rho| \\ \left(\frac{\lambda(\underline{\rho})\lambda(\overline{\rho})}{\lambda(\underline{\kappa})\lambda(\overline{\rho})}\right)^{\bar z} \delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa}\lambda(\kappa) & |\mu| \leq |\rho| \end{cases} \\ &= \langle S_\mu S_\nu^*, \sigma_{\bar z}(S_\rho S_\kappa^*)\rangle,\end{aligned}$$

the last line following (when $|\mu| \leq |\rho|$) since the final segments of ρ and κ must agree if the inner product is nonzero. The final condition to check is that $\langle \sigma_1(a^*), b^*\rangle = \langle b, a\rangle$.

First we compute

$$\begin{aligned}\langle \sigma_1(S_\mu S_\nu^*), S_\rho S_\kappa^*\rangle &= \frac{\lambda(\mu)}{\lambda(\nu)} \begin{cases} \delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa}\lambda(\kappa)g(r(\kappa)) & |\mu| \leq |\rho| \\ \delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}}\lambda(\kappa\overline{\mu})g(r(\mu)) & |\mu| \geq |\rho| \end{cases} \\ &= \begin{cases} \lambda(\rho)g(r(\kappa))\delta_{\mu,\underline{\rho}}\delta_{\nu\overline{\rho},\kappa} & |\mu| \leq |\rho| \\ \lambda(\mu)g(r(\mu))\delta_{\underline{\mu},\rho}\delta_{\nu,\kappa\overline{\mu}} & |\mu| \geq |\rho| \end{cases}.\end{aligned}$$

Next we have

$$\begin{aligned}\langle S_\kappa S_\rho^*, S_\nu S_\mu^*\rangle &= \begin{cases} \delta_{\underline{\kappa},\nu}\delta_{\rho,\mu\overline{\kappa}}\lambda(\mu\overline{\kappa})g(r(\kappa)) & |\kappa| \geq |\nu| \\ \delta_{\kappa,\underline{\nu}}\delta_{\rho\overline{\nu},\mu}\lambda(\mu)g(r(\mu)) & |\kappa| \leq |\nu| \end{cases} \\ &= \begin{cases} \lambda(\rho)g(r(\kappa))\delta_{\underline{\kappa},\nu}\delta_{\rho,\mu\overline{\kappa}} & |\kappa| \geq |\nu| \\ \lambda(\mu)g(r(\mu))\delta_{\kappa,\underline{\nu}}\delta_{\rho\overline{\nu},\mu} & |\kappa| \leq |\nu| \end{cases}\end{aligned}$$

Now for the inner product to be nonzero, we must have $|\rho| + |\nu| = |\kappa| + |\mu|$, and so $|\mu| \leq |\rho| \Leftrightarrow |\nu| \leq |\kappa|$. Comparing the Kronecker deltas in the corresponding cases then yields the desired equality for monomials, and the general case follows by linearity. □

Theorem 4.5. *Let E be a locally finite directed graph. Then there is a one-to-one correspondence between gauge invariant norm lower semicontinuous faithful semifinite functionals on $C^*(E)$ and faithful graph weights on E.*

Proof. This is proved similarly to [27, Proposition 3.9] where the tracial case is considered.

First suppose that (g, λ) is a faithful graph weight on E. Then $(A_c, \phi_{g,\lambda})$ is a modular Hilbert algebra. Since the left representation of A_c on itself is faithful, each p_v, $v \in E^0$, is represented by a non-zero projection. Let the representation be π.

The gauge invariance of $\phi_{g,\lambda}$ shows that for all $z \in \mathbb{T}$, the map $\gamma_z : A_c \to A_c$ extends to a unitary $U_z : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is the completion of A_c in the Hilbert space norm. It is easy to show that $U_z\pi(a)U_{\bar{z}}(b) = \pi(\gamma_z(a))(b)$ for $a, b \in A_c$. Hence $U_z\pi(a)U_{\bar{z}} = \pi(\gamma_z(a))$ and so $\alpha_z(\pi(a)) := U_z\pi(a)U_{\bar{z}}$ gives a point norm continuous action of $\mathbb{T}$ on $\pi(A_c)$ implementing the gauge action.

We may thus invoke the gauge invariant uniqueness theorem [1] to deduce that the representation extends to a faithful representation of $C^*(E)$.

Now $\pi(C^*(E)) \subset \pi(A_c)'' = \overline{\pi(A_c)}^{u.w.}$, the ultra-weak closure. Then [31, Theorem 2.5] shows that the functional $\phi_{g,\lambda}$ extends to a faithful, normal semifinite weight $\psi_{g,\lambda}$ on the left von Neumann algebra of A_c, $\pi(A_c)''$.

Restricting the extension $\psi_{g,\lambda}$ to $C^*(E)$ gives a faithful weight. It is norm semifinite since it is defined on A_c which is dense in $C^*(E)$. Finally, if $a_j \to a$ in norm, then the a_j converge ultra-weakly as well, so $\liminf \psi_{g,\lambda}(a_j) \geq \psi_{g,\lambda}(a)$, which shows that the restriction of $\psi_{g,\lambda}$ to $C^*(E)$ is norm lower semicontinuous.

To get the gauge invariance of $\psi_{g,\lambda}$ we recall that $T \in \pi(A_c)''$ is in the domain of $\psi_{g,\lambda}$ if and only if $T = \pi(\xi)\pi(\eta)^*$ for left bounded elements $\xi, \eta \in \mathcal{H}$. Then $\psi_{g,\lambda}(T) = \psi_{g,\lambda}(\pi(\xi)\pi(\eta)^*) := \langle \xi, \eta \rangle$. As $U_z\xi$ and $U_z\eta$ are also left bounded we have

$$\begin{aligned}
\psi_{g,\lambda}(U_zTU_{\bar{z}}) &= \psi_{g,\lambda}(U_z\pi(\xi)\pi(\eta)^*U_{\bar{z}}) = \psi_{g,\lambda}(U_z\pi(\xi)(U_z\pi(\eta))^*) \\
&= \psi_{g,\lambda}(\pi(\gamma_z(\xi))\pi(\gamma_z(\eta))^*) = \langle U_z\xi, U_z\eta \rangle \\
&= \langle \xi, \eta \rangle = \psi_{g,\lambda}(T).
\end{aligned}$$

So $\psi_{g,\lambda}$ is α_z invariant, and $a \mapsto \psi_{g,\lambda}(\pi(a))$ defines a faithful semifinite norm lower semicontinuous gauge invariant weight on $C^*(E)$.

Conversely, suppose that ϕ is a faithful semifinite norm lower semicontinuous weight on $C^*(E)$ which is gauge invariant. Define

$$g(v) = \phi(p_v), \qquad \lambda(e) = \frac{\phi(S_eS_e^*)}{\phi(S_e^*S_e)}.$$

It is readily checked that (g, λ) is a faithful graph weight. □

In order to make contact with the index theory for KMS weights set out in [3], we require the action associated to our graph weight to be a circle action satisfying the spectral subspace condition, namely that $A_kA_k^*$ should be complemented in the fixed point algebra F.

A sufficient condition to obtain a circle action is that $\lambda(e) = \lambda^{n_e}$ for every edge $e \in E^1$, where now $n : E^1 \to \mathbb{Z}$, and $\lambda \in (0, 1)$. In fact we will simplify matters further and deal here just with a function of the form $n_e \in \{0, 1\}$ for all $e \in E^1$. While this is rather restrictive, it suffices for the examples we consider here. We call such functions *special graph weights*. In order for our special graph weight to accurately reflect the properties of the zhyvot action on our graph, we will also require that $n_e = 1$ if and only if $e \in M^1$.

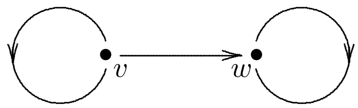

FIGURE 3. Graph of $SU_q(2)$

Also all the graphs we wish to consider are graphs with *finite* zhyvots, with the rest of the graph being composed of trees. Since it is easy to construct faithful graph traces (i.e. special graph weights with $n_e \equiv 0$) on (unions of) trees given just the values of the trace on the root(s), [27], it seems we need only worry about constructing a graph state on the zhyvot.

However, there is a subtlety: *neglecting the trees can affect the existence of special graph weights.*

Example Graph states on $SU_q(2)$. Recall that for $0 \leq q < 1$ the C^*-algebra $SU_q(2)$ is (isomorphic to) the graph C^*-algebra of the graph in Figure 3, [18].

We want to solve

$$g(v) = \lambda^{n_1} g(v) + \lambda^{n_2} g(w), \quad g(w) = \lambda^{n_3} g(w).$$

First $\lambda^{n_3} = 1$, so for $\lambda \neq 1$ (which we aren't interested in), $n_3 = 0$. Then

$$g(v) = \frac{\lambda^{n_2}}{1 - \lambda^{n_1}} g(w).$$

Imposing the requirement that we have a graph state, $g(v) + g(w) = 1$, we get

$$g(v) = \frac{\lambda^{n_2}}{1 - \lambda^{n_1} + \lambda^{n_2}}, \quad g(w) = \frac{1 - \lambda^{n_1}}{1 - \lambda^{n_1} + \lambda^{n_2}}.$$

Observe that if $n_1 = n_2$ we have

$$g(v) = 1 - \lambda^{n_1}, \quad g(w) = \lambda^{n_1}.$$

In this case we get the Haar state by setting $\lambda = q^{2/n_1}$, [6].

Observe that for $\lambda = 1$ the only nonzero graph trace vanishes on w, and we get the usual trace on the top circle with the kernel of $\phi_g = C(S^1) \otimes \mathcal{K}$. For $\lambda > 1$, we get the same family as before by replacing (n_1, n_2) by $(-n_1, -n_2)$. For a special graph state we must have $n_1 = 1$ and $n_3 = 0$. For n_2 we may choose either value.

So it seems we can not obtain a special graph weight with $n_e = 1$ for all edges in the zhyvot. However, if we add trees to the graph, the loop on the vertex w will acquire exits, and then it is easy to construct special graph weights with $n_e = 1$ precisely when e is an edge in the zhyvot.

Lemma 4.6. *Let M be a finite graph and label the vertices $v_1, \ldots, v_n$ so that the sinks, if any, are $v_{r+1}, \ldots, v_n$. Let $p_{jk} \in \mathbb{N} \cup \{0\}$ be the number of edges from v_j to v_k. Then M has a faithful special graph state (g, λ, n) for $\lambda \in (0,1)$ and $n : E^1 \to \{1\} \subset \mathbb{N}$ if and only if the matrix*

$$\begin{pmatrix} (\lambda p_{jk})_{r\times r} & (\lambda p_{jk})_{r\times n-r} \\ 0_{n-r\times r} & Id_{n-r\times n-r} \end{pmatrix}$$

has an eigenvector $(x_1, \ldots, x_n)^T$ with eigenvalue 1 and $x_j > 0$ for $j = 1, \ldots, n$.

Proof. The equations defining a special graph weight for $\lambda \in (0,1)$ are

$$g(v_j) = \sum_{k=1}^{n} \lambda p_{jk} g(v_k) \quad j = 1, \dots, r, \qquad g(v_j) = \sum_{k=1}^{n} \delta_{jk} g(v_k) \quad j = r+1, \dots, n.$$

This gives the necessary and sufficient condition for the existence of a special graph weight $\tilde{g}$ with $\tilde{g}(v_j) = x_j$. To get a state we normalise the eigenvector. □

The lemma can obviously be generalized to deal with general graph states on finite graphs. Moreover we note that work in progress is extending the modular index theory to quasi-periodic actions of $\mathbb{R}$, and a modified version of the above lemma will give existence criteria in the quasi-periodic case also.

Corollary 4.7. *Let E be a locally finite directed graph without sources, and with finite zhyvot $M \subset E$. Let (g, λ, n) be a special graph weight on E for $\lambda \in (0,1)$, $n|_{M^1} \equiv 1$ and $n|_{E^1 \setminus M^1} \equiv 0$. Then $\phi_{g,\lambda}$ extends to a positive norm lower semi-continuous gauge invariant (usual gauge action) functional on $C^*(E)$. The functional $\phi_{g,\lambda}$ is faithful iff (g, λ) is faithful. We have the formula*

$$\phi_{g,\lambda}(ab) = \phi_{g,\lambda}(\sigma(b)a), \quad a, b \in A_c,$$

where $\sigma(S_\mu S_\nu^*) = \frac{\lambda(\nu)}{\lambda(\mu)} S_\mu S_\nu^*$ is a densely defined regular automorphism of $C^*(E)$. In particular, $\phi_{g,\lambda}$ is a KMS weight on $C^*(E)$ for the (modified) zhyvot action

$$\sigma_t(S_\mu S_\nu^*) = \left(\frac{\lambda(\mu)}{\lambda(\nu)}\right)^{it} S_\mu S_\nu^* = \lambda^{(|\mu|_\sigma - |\nu|_\sigma)it} S_\mu S_\nu^*.$$

Proof. The formula $\phi_{g,\lambda}(ab) = \phi_{g,\lambda}(\sigma(b)a)$ follows from Proposition 4.4. Together with the norm lower semicontinuity and the gauge invariance coming from Theorem 4.5, we see that $\phi_{g,\lambda}$ is a KMS weight on $C^*(E)$. □

4.1. **The effect of field extensions.** Suppose that we start with the infinite graph $\Delta_{\mathbb{K}}/\Gamma$ and we pass to the graph $\Delta_{\mathbb{L}}/\Gamma$, for $\mathbb{L}$ a finite extension of $\mathbb{K}$, by the procedure described in Section 3. As we have seen, this procedure consists of inserting $e_{\mathbb{L}/\mathbb{K}} - 1$ new vertices along edges and attaching infinite trees to the old and new vertices, so that the resulting valence of all vertices is the desired $q^f + 1$.

Here we show that, if we have constructed a special graph weight for $\Delta_{\mathbb{K}}/\Gamma$, then we obtain corresponding special graph weights on all the $\Delta_{\mathbb{L}}/\Gamma$ for finite extensions $\mathbb{L} \supset \mathbb{K}$. The special graph weight for $\Delta_{\mathbb{L}}/\Gamma$ is obtained from that of $\Delta_{\mathbb{K}}/\Gamma$ by solving explicit equations.

Proposition 4.8. *Let E be a locally finite directed graph with no sources and with finite zhyvot M. Suppose that (g, λ, n) is a faithful special graph weight on E with $n|_M \equiv 1$ and $n|_{E \setminus M} \equiv 0$, and $\lambda \in (0,1)$. Let F be the graph obtained from E by inserting some new vertices along edges of M and attaching any positive number of trees to the new vertices and any number of trees to the vertices of E. Then F has finite zhyvot $\tilde{M}$, with $\tilde{M}^0 = \{v \in F^0 : v \in M^0 \text{ or } v = r(e),\ e \in F^1,\ s(e) \in M^0\}$ and $\tilde{M}^1 = \{e \in F^1 : r(e) \in \tilde{M}^0\}$, and a faithful special graph weight $(\tilde{g}, \lambda, \tilde{n})$ with the same value of λ and $\tilde{n}|_{\tilde{M}} \equiv 1$, $\tilde{n}|_{F \setminus \tilde{M}} \equiv 0$.*

Proof. It is clear that F is a graph and that $\tilde{M}$ is a zhyvot for F, since we can not introduce sources when vertices are only introduced splitting an existing edge into two, since one of them has range the new vertex. Since extending a faithful graph state on the zhyvot $\tilde{M}$ to any graph obtained by adding trees to vertices is possible, we need only be concerned with building a new special graph state on the zhyvot.

The problem turns out to be local, and we refer to Figure 4 for the notation we shall use.

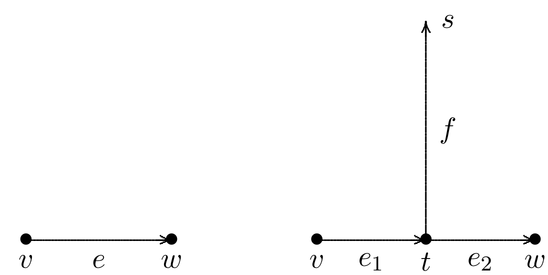

Figure 4. Inserting a vertex

We suppose we have an edge e with $s(e) = v$ and $r(e) = w$ in a graph with special graph weight (g, λ, n). So $g(v) = \lambda(e)g(w) + R$, where $R = \sum_{s(f)=v,\ f\neq e} \lambda(e)g(r(f))$.

We now introduce a new vertex t splitting e into two edges e_1, e_2 with $s(e_1) = v$, $r(e_1) = t$, $s(e_2) = t$, $r(e_2) = w$. We also introduce a new edge f with $s(f) = t$, $r(f) = s$ for some other vertex s. We observe that we could add several edges $f_1, \ldots, f_n$ with source t, and we indicate the modifications required in this case below.

We want to construct a special graph weight $\tilde{g}$ without changing our parameter λ, or the values of the graph weight where it is already defined. Thus we would like to solve

$$\tilde{g}(v) = \lambda\tilde{g}(t) + R, \quad \tilde{g}(t) = \lambda\tilde{g}(w) + \lambda\tilde{g}(s), \quad \tilde{g}(v) = g(v), \quad \tilde{g}(w) = g(w).$$

A solution to the above equations is as follows. Define $\tilde{g} = g$ on all previously existing vertices, and on the vertex $s = r(f)$ set $\tilde{g}(s) = \frac{1-\lambda}{\lambda}g(w)$. Then the above equations are satisfied and we obtain $\tilde{g}(t) = g(w)$. If we have multiple edges $f_1, \ldots, f_n$ then replacing $\tilde{g}(s)$ by $\sum_j \tilde{g}(r(f_j))$ we have a solution provided

$$\sum_j \tilde{g}(r(f_j)) = \frac{1-\lambda}{\lambda}g(w),$$

and thus we may just set the value of $\tilde{g}(r(f_j))$ to be $\frac{1}{n}\frac{1-\lambda}{\lambda}g(w)$.

Finally, define $\tilde{n}$ by making it identically one on edges in $\tilde{M}$ and identically zero on other edges. Observe that f is not an edge in the zhyvot. □

5. Modular Index Invariants of Mumford Curves

We have seen that we can associate directed graphs to Mumford curves. These graphs consist of a finite graph along with trees emanating out from some or all of its vertices. Though we do not have a general existence result, generically we can construct "special" graph weights on such graphs.

From this we can construct both an equivariant Kasparov module $(X, \mathcal{D})$ and a modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as in [3]. Here the equivariance is with respect to the 'modified zhyvot action' introduced in Corollary 4.7. To compute the index pairing using the results of [3], we need only be able to compute traces of operators of the form $p\Phi_k$ where $p \in F$ is a projection and the Φ_k are spectral projections of the $\mathbb{T}$ action (or of $\mathcal{D}$ or of Δ).

In the specific case of Mumford curves, the modular index pairings we would like to compute are with the modular partial isometries arising from loops in the central graph corresponding to the action on $\Delta_{\mathbb{K}}$ of each one of a chosen set of generators $\{\gamma_1, \ldots, \gamma_g\}$ of the Schottky group Γ. These correspond to the fundamental closed geodesics in $\Delta_{\mathbb{K}}/\Gamma$, by analogy to the fundamental closed geodesics in the hyperbolic 3-dimensional handlebody $\mathbb{H}/\Gamma$ considered in [23], [9]. The lengths of these fundamental closed geodesics are the Schottky invariants of $(\Gamma, \{\gamma_1, \ldots, \gamma_g\})$ introduced above.

We introduce some notation so that we may effectively describe these projections. The zhyvot of the graph we denote by M. Since outside of M our graph is a union of trees, we may and do suppose that the restriction of our graph weight to the exterior of M is a graph trace. That is, for all $v \notin M$ we have

$$g(v) = \sum_{s(e)=v} g(r(e)),$$

and so for $e \notin M$, $\sigma_t(S_e) = S_e$.

In [3] we showed how to construct a Kasparov module for $A = C^*(E)$ and $F = A^\sigma$. We let $\Phi : A \to F$ be the expectation given by averaging over the circle action, and define an inner product on A with values in F by setting

$$(a|b) := \Phi(a^*b).$$

We denote the C^*-module completion by X, and note that it is a full right F-module. There is an obvious action of A by left multiplication, and this action is adjointable.

On the dense subspace $A_c \subset X$ we define an unbounded operator $\mathcal{D}$ by defining it on generators and extending by linearity. We set

$$\mathcal{D}S_\mu S_\nu^* := (|\mu|_\sigma - |\nu|_\sigma)\, S_\mu S_\nu^*,$$

so that up to a factor of $\log(\lambda)$, $\mathcal{D}$ is the generator of the zhyvot action. Observe that for a path μ contained in the exterior of M we have $|\mu|_\sigma = 0$. The closure of $\mathcal{D}$ is self-adjoint, regular, and for all $a \in A_c$ the endomorphism of X given by $a(1 + \mathcal{D}^2)^{-1}$ is a compact endomorphism.

It is proved in [3] that $({}_AX_F, \mathcal{D})$ is an equivariant Kasparov module for A-F (with respect to the zhyvot action) and so it defines a class in $KK^{1,\mathbb{T}}(A, F)$.

Similarly, if we set $\mathcal{H} := \mathcal{H}_{\phi_{g,\lambda}}$ to be the GNS space of A associated to the weight $\phi_{g,\lambda}$, we obtain an unbounded operator $\mathcal{D}$ (with the same definition on $A_c \subset \mathcal{H}$). The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is not quite a spectral triple.

The compact endomorphisms of the C^*-module X, $End^0_F(X)$, act on $\mathcal{H}$ in a natural fashion, [3], and we define a von Neumann algebra by $\mathcal{N} = (End^0_F(X))''$. There is a natural trace $\mathrm{Tr}_{\phi_{g,\lambda}}$ on $\mathcal{N}$ satisfying

$$\mathrm{Tr}_{\phi_{g,\lambda}}(\Theta_{x,y}) = \phi_{g,\lambda}((y|x))$$

for all x, $y \in X$. We define a weight $\phi_{\mathcal{D}}$ on $\mathcal{N}$ by

$$\phi_{\mathcal{D}}(T) := \mathrm{Tr}_{\phi_{g,\lambda}}(\lambda^{\mathcal{D}} T), \quad T \in \mathcal{N}.$$

Then the modular group of $\phi_{\mathcal{D}}$ is inner, and we let $\mathcal{M} \subset \mathcal{N}$ denote the fixed point algebra of the modular action. Then $\phi_{\mathcal{D}}$ restricts to a trace on $\mathcal{M}$ and it is shown in [3] that

$$f(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \phi_{\mathcal{D}}), \quad f \in F.$$

Using this information it is shown in [3] that there is a pairing between $(A_c, \mathcal{H}, \mathcal{D})$ and homogenous (for the zhyvot action) partial isometries $v \in A_c$ with source and range projections in F. The pairing is given by the spectral flow

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) \in \mathbb{R},$$

this being well-defined since $v\mathcal{D}v^* \in \mathcal{M}$. The numerical spectral flow pairing and the equivariant KK pairing are compatible.

In order to compute the spectral flow, we need explicit formulae for the spectral projections of $\mathcal{D}$ both as an operator on X and as an operator on $\mathcal{H}$.

To this end, if $v \in E^0$ and $m > 0$ we set $|v|_m$ =the number of paths μ with $|\mu|_\sigma = m$ and $r(\mu) = v$. It is important that our graph is locally finite and has no sources so that $0 < |v|_m < \infty$ for all $v \in E^0$ and $m > 1$.

Proposition 5.1. *The spectral projections of $\mathcal{D}$ can be represented as follows:*

1) *For $m > 0$*

$$\Phi_m = \sum_{\substack{|\mu|_\sigma = m \\ s(\mu)\in M \\ r(\mu)\in M}} \Theta_{S_\mu, S_\mu}.$$

2) *For $m = 0$*

$$\Phi_0 = \sum_{v \in E^0} \Theta_{p_v, p_v}.$$

3) *For $m < 0$, $v \in E^0$*

$$p_v \Phi_m = \frac{1}{|v|_{|m|}} \sum_{\substack{|\mu|_\sigma = |m| \\ r(\mu) = v}} \Theta_{S^*_\mu, S^*_\mu}.$$

In all cases, for (a subprojection of) a vertex projection p_v, the operator $p_v\Phi_m$ is a finite rank endomorphism of the Kasparov module X and in the domain of $\phi_{\mathcal{D}}$ as an operator in $\mathcal{M} \subset \mathcal{N}$.

Proof. We first recall that the C^*-module inner product is given by
$$(x|y)_R = \Phi(x^*y).$$
Now let $S_\rho S_\gamma^* \in X$ and with $|\mu|_\sigma > 0$ consider
$$\begin{aligned}\Theta_{S_\mu,S_\mu} S_\rho S_\gamma^* &= S_\mu (S_\mu | S_\rho S_\gamma^*)_R \\ &= \delta_{|\mu|_\sigma, |\rho|_\sigma - |\gamma|_\sigma} S_\mu S_\mu^* S_\rho S_\gamma^* \\ &= \delta_{|\mu|_\sigma, |\rho|_\sigma - |\gamma|_\sigma} \delta_{\mu,\underline{\rho}} S_\rho S_\gamma^*.\end{aligned}$$
Hence this is nonzero only when $|\rho|_\sigma = |\gamma|_\sigma + |\mu|_\sigma \geq |\mu|_\sigma$ and $\underline{\rho} = \mu$. Thus when $|\rho|_\sigma - |\gamma|_\sigma = m > 0$
$$\sum_{\substack{|\mu|_\sigma = m \\ s(\mu)\in M \\ r(\mu)\in M}} \Theta_{S_\mu,S_\mu} S_\rho S_\gamma^* = \Theta_{\underline{\rho},\underline{\rho}} S_\rho S_\gamma^* = S_\rho S_\gamma^*,$$
and $\sum \Theta_{S_\mu,S_\mu}$ is zero on all other elements of X. Hence the claim for the positive spectral projections is proved, since finite sums of generators $S_\rho S_\gamma^*$ are dense in X. A similar argument proves the claim for the zero spectral projection.

For the negative spectral projections, we observe that
$$\begin{aligned}\Theta_{S_\mu^*,S_\mu^*} S_\rho, S_\gamma^* &= S_\mu^* \delta_{r(\mu),s(\rho)} \delta_{|\mu|_\sigma + |\rho|_\sigma, |\gamma|_\sigma} S_\mu S_\rho S_\gamma^* \\ &= \delta_{r(\mu),s(\rho)} \delta_{|\mu|_\sigma + |\rho|_\sigma, |\gamma|_\sigma} S_\rho S_\gamma^*.\end{aligned}$$
Summing over all paths μ with $|\mu|_\sigma = m > 0$ and $r(\mu) = s(\rho)$ gives
$$\sum_{\substack{|\mu|_\sigma = m \\ r(\mu)=s(\rho)}} \Theta_{S_\mu^*,S_\mu^*} S_\rho S_\gamma^* = \delta_{|\rho|_\sigma - |\gamma|_\sigma, -m} |s(\rho)|_m S_\rho S_\gamma^*.$$
Hence for a vertex $v \in E^0$
$$\begin{aligned}\frac{1}{|v|_{|m|}} \sum_{\substack{|\mu|_\sigma = |m| \\ r(\mu)=v}} \Theta_{S_\mu^*,S_\mu^*} S_\rho S_\gamma^* &= \delta_{|\rho|_\sigma - |\gamma|_\sigma, -m} \delta_{s(\rho),v} S_\rho S_\gamma^* \\ &= p_v \Phi_{-m} S_\rho S_\gamma^*.\end{aligned}$$
In all cases $p_v\Phi_k$ is a finite sum of rank one endomorphisms, and so finite rank. In particular they are expressed as finite rank endomorphisms of $\mathrm{dom}(\phi)^{1/2} \subset X$, since for a graph weight g, λ all the S_μ and S_μ^* lie in the domain of the associated weight ϕ. This ensures that these endomorphisms extend by continuity to the Hilbert space completion of $\mathrm{dom}(\phi)^{1/2}$ and by the construction of $\phi_{\mathcal{D}}$, each $p_v\Phi_k \in \mathcal{M} \subset \mathcal{N}$ has finite *trace*. Similar comments evidently apply to projections of the form $S_\mu S_\mu^*$ since this is a subprojection of $p_{s(\mu)}$. □

For large positive k, the computation of $\phi_{\mathcal{D}}(S_\mu S_\mu^* \Phi_k)$ is extremely difficult, and needs to be handled on a 'graph-by-graph' basis. However it turns out that we need only compute for $|k| \leq |\mu|_\sigma$, and this is completely tractable.

Lemma 5.2. *Let γ be a path in E with $s(\gamma), r(\gamma) \in M$ and $|\gamma|_\sigma > 0$. Then for all $k \in \mathbb{Z}$ with $|\gamma|_\sigma \geq |k|$ we have*
$$\phi_{\mathcal{D}}(S_\gamma S_\gamma^* \Phi_k) = \phi_{g,\lambda}(S_\gamma S_\gamma^*) = \lambda^{|\gamma|_\sigma} g(r(\gamma)).$$

For a path of length zero (i.e. a vertex v) in M and $k<0$ we have

$$\phi_{\mathcal{D}}(p_v\Phi_k)=\phi_{g,\lambda}(p_v)=g(v).$$

Proof. We begin with $|\gamma|_\sigma\geq k>0$. In this case the definitions yield

$$\begin{aligned}\phi_{\mathcal{D}}(S_\gamma S_\gamma^*\Phi_k)&=\sum_{\substack{|\mu|_\sigma=k\\ s(\mu)\in M\\ r(\mu)\in M}}\phi_{\mathcal{D}}(S_\gamma S_\gamma^*\Theta_{S_\mu,S_\mu})\\ &=\sum_{\substack{|\mu|_\sigma=k\\ s(\mu)\in M\\ r(\mu)\in M}}\lambda^k\phi_{g,\lambda}(S_\mu^*S_\gamma S_\gamma^*S_\mu)\\ &=\lambda^k\phi_{g,\lambda}(S_{\overline{\gamma}}S_{\overline{\gamma}}^*)\\ &=\lambda^{|\gamma|_\sigma}\phi_{g,\lambda}(S_{\overline{\gamma}}^*S_{\overline{\gamma}})=\lambda^{|\gamma|_\sigma}\phi_{g,\lambda}(p_{r(\gamma)})\\ &=\phi_{g,\lambda}(S_\gamma S_\gamma^*).\end{aligned}$$

So now consider $|\gamma|_\sigma>0$ or $\gamma=v$ for some vertex $v\in M$ and $k>0$. In the latter case, $S_\gamma S_\gamma^*=p_vp_v=p_v=S_\gamma^*S_\gamma$. Then

$$\begin{aligned}\phi_{\mathcal{D}}(S_\gamma S_\gamma^*\Phi_{-k})&=\frac{1}{|s(\gamma)|_k}\sum_{\substack{|\mu|_\sigma=k\\ r(\mu)=s(\gamma)}}\lambda^{-k}\phi_{g,\lambda}(S_\mu S_\gamma S_\gamma^*S_\mu^*)\\ &=\frac{1}{|s(\gamma)|_k}\sum_{\substack{|\mu|_\sigma=k\\ r(\mu)=s(\gamma)}}\lambda^{|\gamma|_\sigma}\phi_{g,\lambda}(p_{r(\gamma)})\\ &=\lambda^{|\gamma|_\sigma}\phi_{g,\lambda}(p_{r(\gamma)})\\ &=\phi_{g,\lambda}(S_\gamma S_\gamma^*).\end{aligned}$$

This completes the proof. □

We now have the necessary ingredients to compute the modular index pairing with S_γ where γ here denotes a loop contained in the finite graph $M=\Delta'_\Gamma/\Gamma$ and corresponding to an element in the chosen set of generators of the Schottky group. We suppose that $k=|\gamma|_\sigma$ is non-zero, so that the loop is non-trivial, and denote $\beta:=-\log\lambda$.

By [3, Lemma 4.10] and Lemma 5.2 we have

$$\begin{aligned}sf_{\phi_{\mathcal{D}}}(S_\gamma S_\gamma^*\mathcal{D},S_\gamma\mathcal{D}S_\gamma^*)&=-\sum_{j=0}^{k-1}e^{-\beta j}\mathrm{Tr}_\phi(S_\gamma S_\gamma^*\Phi_j)\\ &=-\sum_{j=0}^{k-1}\phi_{\mathcal{D}}(S_\gamma S_\gamma^*\Phi_j)\\ &=-k\phi_{g,\lambda}(S_\gamma S_\gamma^*)\\ &=-k\lambda^k g(r(\gamma))=-ke^{-\beta k}g(r(\gamma)).\end{aligned}$$

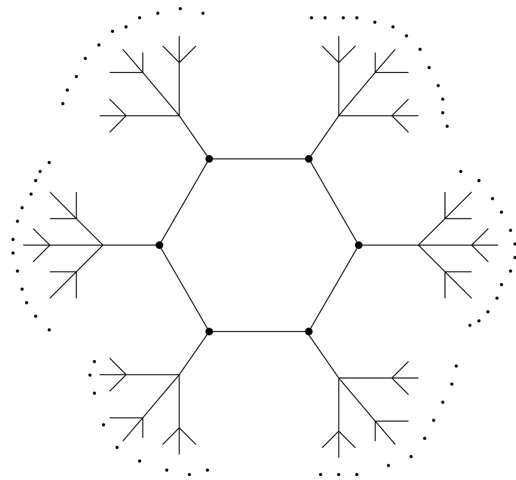

FIGURE 5. The genus one case.

Since we assume that (g, λ) are given as part of the data of our special graph weight, we can extract the integer k. Moreover, k determines the value of the index pairing.

Thus, we see that the Schottky invariants of the data $(\Gamma, \{\gamma_1, \ldots, \gamma_g\})$ can be recovered from the modular index pairing and in fact determine it, for a given graph weight (g, λ). This confirms the fact that the noncommutative geometry of the graph algebra $C^*(E) = C^*(\Delta_{\mathbb{K}}/\Gamma)$ maintains the geometric information related to the action of the Schottky group on the Bruhat–Tits tree $\Delta_{\mathbb{K}}$. This is still less information than being able to reconstruct the curve, since the Schottky invariants only depend on the valuation. We show explicitly in the next section how the construction of graph weights works in some simple examples of Mumford curves.

6. Low genus examples

We consider here the cases of the elliptic curve with Tate uniformization (genus one case) and the three genus two cases considered in [10]. In each of these examples we give an explicit construction of graph weights and compute the relevant modular index pairings, showing that one recovers from them the Schottky invariants. Notice that, for the genus two cases, the finite zhyvot graphs Δ_Γ/Γ are the same considered in [10], which we report here, though in the present setting we work with the infinite graphs $\Delta_{\mathbb{K}}/\Gamma$. We discuss here the graph weights equation on the zhyvot graph and on the infinite graph $\Delta_{\mathbb{K}}/\Gamma$.

Example: Genus 1 As a first application to Mumford curves we consider the simplest case of genus one. In this case, the Schottky uniformization is the Tate uniformization of p-adic elliptic curves. The p-adic Schottky group is just a copy of $\mathbb{Z}$ generated by a single hyperbolic element in $\mathrm{PGL}_2(\mathbb{K})$. In this case the graph $\Delta_{\mathbb{K}}/\Gamma$ will be always of the form illustrated in Figure 5, with a central polygon with n vertices and trees departing from its vertices. With our convention on the orientations, the edges are oriented in such a way as to go around the central polygon, while the rest of the graph, *i.e.* the trees stemming from the vertices of the polygon, are oriented away from it and towards the boundary $X_\Gamma = \partial\Delta_{\mathbb{K}}/\Gamma$.

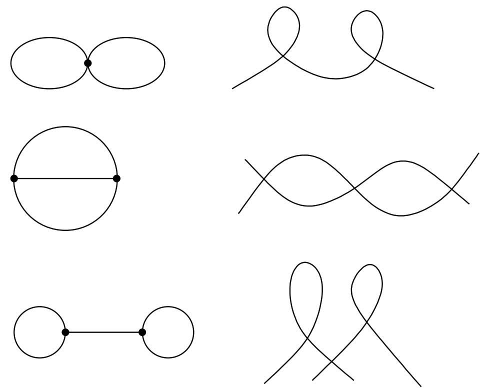

FIGURE 6. The graphs Δ_Γ/Γ for genus $g = 2$, and the corresponding fibers.

Label the vertices on the polygon by v_i, $i = 1, \ldots, n$. To get a special graph weight we need $0 < \lambda < 1$ and a function g on the vertices such that

$$g(v_i) = \lambda g(v_{i+1}) + B_i, \qquad B_i = \sum_{v_{i+1} \neq w = r(e),\ s(e) = v_i} g(w).$$

To simplify we suppose all the $g(v_i)$ are equal, $\sum g(v_i) = 1$ and all the B_i are equal. Then we obtain a special graph weight for any $\lambda < 1$ by setting

$$g(v_i) = \frac{1}{n}, \qquad B_i = \frac{1-\lambda}{n}.$$

For each i we can now define the various $g(w)$ appearing in the sum defining B_i by $g(w) = \frac{1}{m_i} g(w)$ where m_i is the number of such $g(w)$. This graph weight can be extended to the rest of the trees as a graph trace, and the associated $\mathbb{T}$ action is nontrivial on each S_e, $e \in M^1$ where M is just the central polygon. Hence choosing γ to be the (directed) path which goes once around the polygon (the choice of $r(\gamma) = s(\gamma)$ is irrelevant) gives

$$\langle [\gamma], \phi_{g,\lambda} \rangle = -\lambda^n.$$

Example: Genus 2 In the case of genus two, the possible graphs Δ_Γ/Γ and the corresponding special fibers of the algebraic curve are illustrated in Figure 6, which we reproduce from [10], see also [24].

We see more in detail the various cases. These are the same cases considered in [10].

Case 1: In the first case, the tree Δ_Γ is just a copy of the Cayley graph of the free group Γ on two generators as in Figure 7.

The graph algebra of this graph is the Cuntz algebra O_2. The only possible special graph state is $g(v) = 1$ for the single vertex and $\lambda = 1/2$. This corresponds to the usual gauge action and its unique KMS state, [5]. Once we add trees to this example, many more possibilities for the KMS weights appear.

Case 2: In the second case, the finite directed graph Δ_Γ/Γ is of the form illustrated in Figure 8. We label by $a = e_1$, $b = e_2$ and $c = e_3$ the oriented edges in the graph Δ_Γ/Γ, so that we have a corresponding set of labels $E = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$ for the edges in the covering Δ_Γ. A choice of generators for the group $\Gamma \simeq \mathbb{Z} * \mathbb{Z}$ acting on Δ_Γ is obtained by identifying the generators γ_1 and γ_2 of Γ with

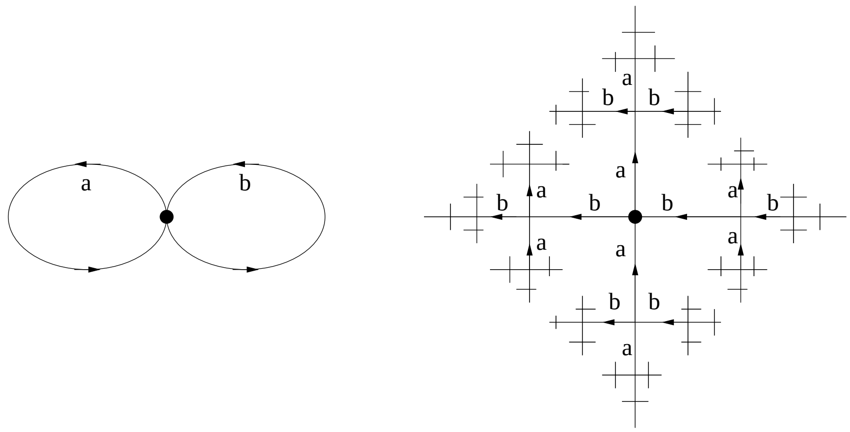

FIGURE 7. Genus two: first case

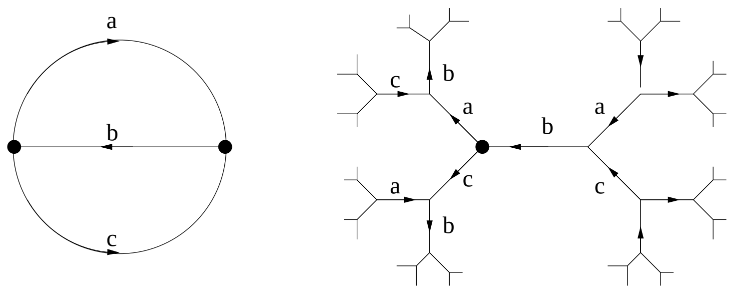

FIGURE 8. Genus two: second case

TABLE 1. graph states for Case 2

n_1	n_2	n_3	λ	$g(v)$	$g(w)$
0	0	0 or 1	—	—	—
0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
1	0	1	$\frac{-1+\sqrt{5}}{2}$	λ^2	λ
1	1	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{1+\sqrt{2}}$	$\frac{\sqrt{2}}{1+\sqrt{2}}$

the chains of edges ab and $a\bar{c}$, hence the orientation on the tree Δ_Γ and on the quotient graph is as illustrated in the figure.

There are four special graph states on this graph algebra (up to swapping the roles of the edges a and c). Let $v = s(b)$ and $w = r(b)$. Let $n_1 = n(b)$, $n_2 = n(a)$ and $n_3 = n(c)$ with each $n_j \in \{0, 1\}$. Then the various states are described in Table 1. To fit with our requirement that the zhyvot action 'sees' every edge in the zhyvot, we should adopt only the last choice of state. Of course, we are again neglecting the trees, and including them would give us many more options.

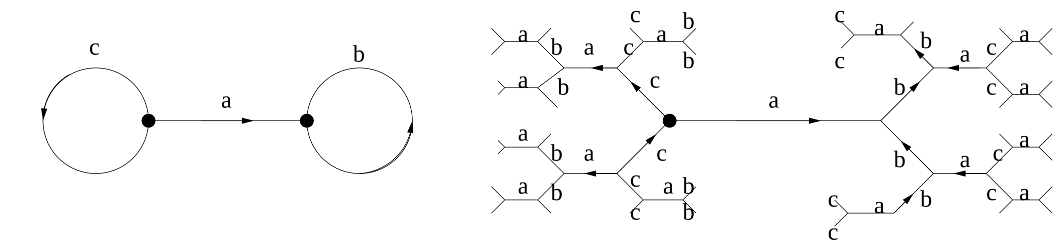

FIGURE 9. Genus two: third case

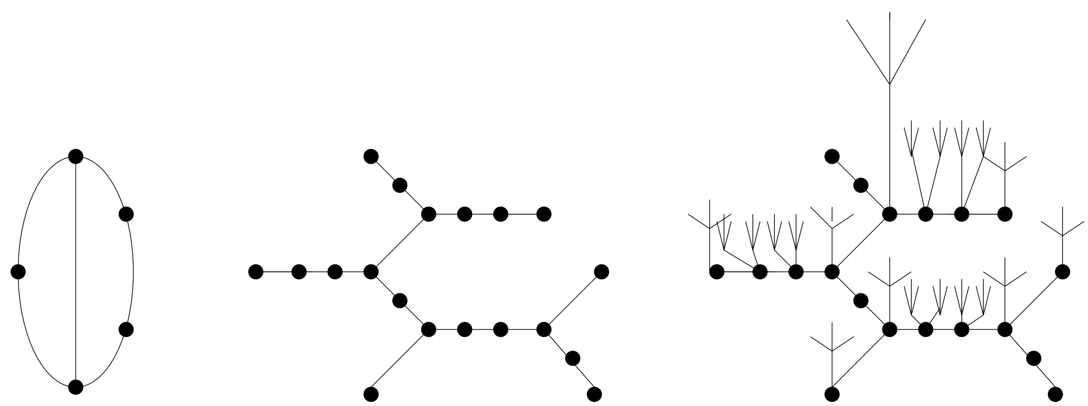

FIGURE 10. An example of a graph Δ'_Γ/Γ, the tree Δ'_Γ and its embedding in $\Delta_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{Q}_3$.

Case 3: In the third case the obtained oriented graph is the same as the graph of $SU_q(2)$ of Figure 3. We have already described the graph states for $SU_q(2)$. In this case, the inclusion of trees is necessary to obtain a special graph weight adapted to the zhyvot action. In fact, a choice of generators for the group $\Gamma \simeq \mathbb{Z} * \mathbb{Z}$ acting on Δ_Γ is given by $ab\bar{a}$ and c, so that the obtained orientation is as in the figure.

When we considers the tree Δ'_Γ instead of Δ_Γ one is typically adding extra vertices. The way the tree Δ'_Γ sits inside the Bruhat–Tits tree $\Delta_{\mathbb{K}}$ and in particular how many extra vertices of $\Delta_{\mathbb{K}}$ are present on the graph Δ'_Γ/Γ with respect to the vertices of Δ_Γ/Γ gives some information on the uniformization, *i.e.* it depends on where the Schottky group Γ lies in $\mathrm{PGL}_2(\mathbb{K})$, unlike the information on the graph Δ_Γ/Γ which is purely combinatorial. For example, in the genus two case of Figure 6 one can have graphs Δ'_Γ/Γ and $\Delta_{\mathbb{K}}/\Gamma$ of the form as in Figure 10.

Remark: Higher genus In the higher genus case one knows by [17, p.124] that any stable graph can occur as the graph Δ_Γ/Γ of a Mumford curve. By stable graph we mean a finite graph which is connected and such that each vertex that is not connected to itself by an edge is the source of at least three edges.

Thus, the combinatorial complexity of the graph is pretty much arbitrary. One can also assume, possibly after passing to a finite extension of the field $\mathbb{K}$, that there are infinite homogeneous trees attached to each vertex of the graph Δ_Γ/Γ.

We make the final remark that the restriction to circle actions in this paper is likely an artificial one. In work in progress, KMS index theory is extended to quasi-periodic actions of the reals. As the action associated to *any* graph weight will be quasi-periodic, this will hopefully allow us to prove general existence theorems for (quasi-periodic) graph weights compatible with the zhyvot action.

7. Jacobian and theta functions

We recall here briefly the relation between the Jacobian and theta functions of a Mumford curve and the group of currents on the infinite graph $\Delta_{\mathbb{K}}/\Gamma$.

Recall first that a current on a locally finite graph G is an integer valued function of the oriented edges of G that satisfies the following properties.

(1) Orientation reversal:

$$\mu(\bar{e}) = -\mu(e), \tag{7.1}$$

where $\bar{e}$ is the edge e with the reverse orientation.

(2) Momentum conservation:

$$\sum_{s(e)=v} \mu(e) = 0. \tag{7.2}$$

One denotes by $\mathcal{C}(G)$ the abelian group of currents on the graph G.

Suppose we are given a tree $\mathcal{T}$. Then the group $\mathcal{C}(\mathcal{T})$ can be equivalently described as the group of finitely additive measures of total mass zero on the space $\partial\mathcal{T}$ of ends of the tree by setting $m(U(e)) = \mu(e)$, where $U(e) \subset \partial\mathcal{T}$ is the clopen set of ends of the infinite half lines starting at the vertex $s(e)$ along the direction e. For $G = \mathcal{T}/\Gamma$, the group of currents $\mathcal{C}(G) = \mathcal{C}(\mathcal{T})^\Gamma$ can be identified with the group of Γ-invariant measures on $\partial\mathcal{T}$, *i.e.* finitely additive measures of total mass zero on $\partial\mathcal{T}/\Gamma$.

As above, we let $X = X_\Gamma$ be a Mumford curve, uniformized by the p-adic Schottky group Γ. We consider the above applied to the tree $\Delta_{\mathbb{K}}$ with the action of the Schottky group Γ and the infinite, locally finite quotient graph $\Delta_{\mathbb{K}}/\Gamma$ with $\partial\Delta_{\mathbb{K}}/\Gamma = X_\Gamma(\mathbb{K})$.

It is known (see [33], Lemma 6.3 and Theorem 6.4) that the Jacobian of a Mumford curve can be described, as an analytic variety, via the isomorphism

$$\mathrm{Pic}^0(X) \cong \mathrm{Hom}(\Gamma, \mathbb{K}^*)/c(\Gamma_{ab}), \tag{7.3}$$

where $\Gamma_{ab} = \Gamma/[\Gamma, \Gamma]$ denotes the abelianization, $\Gamma_{ab} \cong \mathbb{Z}^g$, with g the genus, and the homomorphism

$$c : \Gamma_{ab} \to \mathrm{Hom}(\Gamma_{ab}, \mathbb{K}^*) \tag{7.4}$$

is defined by the first map in the homology exact sequence

$$0 \to \mathcal{C}(\Delta_{\mathbb{K}})^\Gamma \xrightarrow{c} \mathrm{Hom}(\Gamma, \mathbb{K}^*) \to H^1(\Gamma, \mathcal{O}(\Omega_\Gamma)^*) \to H^1(\Gamma, \mathcal{C}(\Delta_{\mathbb{K}})) \to 0, \tag{7.5}$$

associated to the short exact sequence

$$0 \to \mathbb{K}^* \to \mathcal{O}(\Omega_\Gamma)^* \to \mathcal{C}(\Delta_{\mathbb{K}}) \to 0 \tag{7.6}$$

of Theorem 2.1 of [33], where $\mathcal{O}(\Omega_\Gamma)^*$ is the group of invertible holomorphic functions on $\Omega_\Gamma \subset \mathbb{P}^1$.

In the sequence (7.5), one uses the fact that $H^i(\Gamma) = 0$ for $i \geq 2$ and the identification

$$\mathcal{C}(\Delta_{\mathbb{K}})^{\Gamma} = H^0(\Gamma, \mathcal{C}(\Delta_{\mathbb{K}})) = \Gamma_{ab} = \pi_1(\Delta_{\mathbb{K}}/\Gamma)_{ab}, \tag{7.7}$$

see [33], Lemma 6.1 and Lemma 6.3. One can then use the short exact sequence

$$0 \to \mathcal{C}(\Delta_{\mathbb{K}}) \to \mathcal{A}(\Delta_{\mathbb{K}}) \xrightarrow{d} \mathcal{H}(\Delta_{\mathbb{K}}) \to 0, \tag{7.8}$$

where $\mathcal{C}(\Delta_{\mathbb{K}})$ is the group of currents on the Bruhat–Tits tree $\Delta_{\mathbb{K}}$, $\mathcal{A}(\Delta_{\mathbb{K}})$ is the group of integer values functions on the set of edges of $\Delta_{\mathbb{K}}$ satisfying $h(\bar{e}) = -h(e)$ under orientation reversal $e \mapsto \bar{e}$ and $\mathcal{H}(\Delta_{\mathbb{K}})$ is the group of integer valued functions on the set of vertices of $\Delta_{\mathbb{K}}$. The map d in (7.8) is given by

$$d : \mathcal{A}(\Delta_{\mathbb{K}}) \to \mathcal{H}(\Delta_{\mathbb{K}}), \quad d(h)(v) = \sum_{s(e)=v} h(e). \tag{7.9}$$

The long exact homology sequence associated to (7.8) is given by

$$0 \to \mathcal{C}(\Delta_{\mathbb{K}}/\Gamma) \to \mathcal{A}(\Delta_{\mathbb{K}}/\Gamma) \xrightarrow{d} \mathcal{H}(\Delta_{\mathbb{K}}/\Gamma) \xrightarrow{\Phi} H^1(\Gamma, \mathcal{C}(\Delta_{\mathbb{K}})) \to 0, \tag{7.10}$$

where one has $H^1(\Gamma, \mathcal{C}(\Delta_{\mathbb{K}})) \cong \mathbb{Z}$ and, under this identification, the last map in the exact sequence is given by

$$\Phi : \mathcal{H}(\Delta_{\mathbb{K}}/\Gamma) \to H^1(\Gamma, \mathcal{C}(\Delta_{\mathbb{K}})) = \mathbb{Z}, \quad \Phi(f) = \sum_{v \in (\Delta_{\mathbb{K}}/\Gamma)^0} f(v). \tag{7.11}$$

Moreover, one has an identification

$$H^1(\Gamma, \mathcal{O}(\Omega_\Gamma)^*) = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X),$$

the group of equivalence classes of holomorphic (hence by GAGA algebraic) line bundles on the curve X, and the last map in the exact sequence (7.5) is then given by the degree map $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$, whose kernel is the Jacobian $J(X) = \mathrm{Pic}^0(X)$, see [33] Lemma 6.3.

A theta function for the Mumford curve $X = X_\Gamma$ is an invertible holomorphic function $f \in \mathcal{O}(\Omega_\Gamma)^*$ such that

$$\gamma^* f = c(\gamma) f, \quad \forall \gamma \in \Gamma,$$

with $c \in \mathrm{Hom}(\Gamma, \mathbb{K}^*)$ the automorphic factor. The group $\Theta(\Gamma)$ of theta functions of the curve X is then obtained from the exact sequences (7.6) and (7.5) as ([33])

$$0 \to \mathbb{K}^* \to \Theta(\Gamma) \to \mathcal{C}(\Delta_{\mathbb{K}})^{\Gamma} \to 0. \tag{7.12}$$

More precisely, let $\mathbb{H}_{\mathbb{K}} = \mathbb{P}^1_{\mathbb{K}} \smallsetminus \mathbb{P}^1(\mathbb{K})$ be Drinfeld's p-adic upper half plane. It is well known (see for instance the detailed discussion given in [2] §I.1 and §I.2) that $\mathbb{H}_{\mathbb{K}}$ is a rigid analytic space endowed with a surjective map

$$\Lambda : \mathbb{H}_{\mathbb{K}} \to \Delta_{\mathbb{K}} \tag{7.13}$$

to the Bruhat–Tits tree $\Delta_{\mathbb{K}}$ such that, for vertices $v, w \in \Delta^0_{\mathbb{K}}$ with $v = s(e)$ and $w = r(e)$, for an edge $e \in \Delta^1_{\mathbb{K}}$, the preimages $\Lambda^{-1}(v)$ and $\Lambda^{-1}(w)$ are open subsets of $\Lambda^{-1}(e)$. The picture of the relation between $\mathbb{H}_{\mathbb{K}}$ and $\Delta_{\mathbb{K}}$ through the map Λ is given in Figure 11.

Given a theta function $f \in \Theta(\Gamma)$, the associated current $\mu_f \in \mathcal{C}(\Delta_{\mathbb{K}})^{\Gamma}$ obtained as in (7.12) is given explicitly by the growth of the spectral norm in the Drinfeld upper half plane when moving along an edge in the Bruhat–Tits tree, that is,

$$\mu(e) = \log_q \|f\|_{\Lambda^{-1}(r(e))} - \log_q \|f\|_{\Lambda^{-1}(s(e))}, \tag{7.14}$$

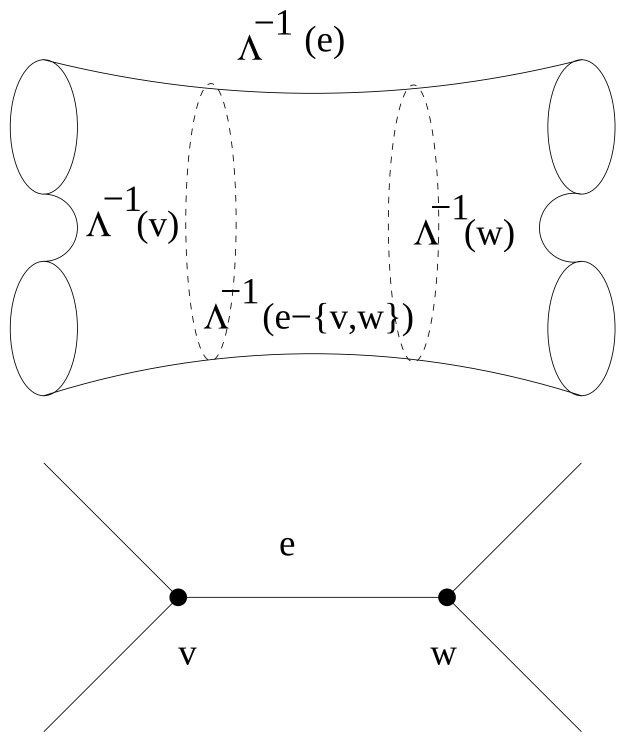

FIGURE 11. The p-adic upper half-plane and the Bruhat–Tits tree

with $q = \#\mathcal{O}/\mathfrak{m}$ and $\|f\|_{\Lambda^{-1}(v)}$ is the spectral norm

$$\|f\|_{\Lambda^{-1}(v)} = \sup_{z\in\Lambda^{-1}(v)} |f(z)|$$

with $|\cdot|$ the absolute value with $|\pi| = q^{-1}$, with π a uniformizer, that is, $\mathfrak{m} = (\pi)$.

The case of function fields over a finite field $\mathbb{F}_q$ of characteristic p is similar to the p-adic case, with $\mathbb{H}_{\mathbb{K}}$ and $\Delta_{\mathbb{K}}$ the Drinfeld upper half plane and the Bruhat–Tits tree in characteristic p (see for instance [16]).

7.1. **Graph weights, currents, and theta functions.** We now show how to relate theta functions on the Mumford curve to graph weights. The type of graph weights we consider here will in general not be special graph weights as those we considered in the previous sections. In fact, we will see that, when constructing currents from graph weights, we need to work with functions $\lambda(e)$ of the special form $\lambda(e) = N_{\bar{e}}^{-1}$, where N_e is defined as in (7.15) below. Along the outer trees of the graph $\Delta_{\mathbb{K}}/\Gamma$, the fact that these are homogeneous trees of valence $q+1$ (or q^f+1 for field extensions) and the orientation of these trees away from the zhyvot graph Δ'_Γ/Γ gives that $\lambda(e) = \lambda^{n_e}$ for $\lambda = q^{-1} \in (0,1)$ (or $\lambda = q^{-f}$ for field extensions) and $n_e = 1$, the expression for N_e inside Δ'_Γ/Γ depends on the orientation on Δ'_Γ described in Lemma 2.1. Similarly, as we see below, when we construct (inhomogeneous or signed) graph weights from currents, we use the function $\lambda(e) = N_e^{-1}$, which is again of the form q^{-1} (or q^{-f}) on the outer trees of $\Delta_{\mathbb{K}}/\Gamma$, but which also depends on the given orientation inside Δ'_Γ/Γ. It will be interesting to consider the quasi-periodic actions associated to these types of graph weights.

We first show that the same methods that produce graph weights can be used to construct real valued currents on the graph $\Delta_{\mathbb{K}}/\Gamma$.

Lemma 7.1. *Let (g,λ) be a graph weight on the infinite, locally finite graph $\Delta_{\mathbb{K}}/\Gamma$. For an oriented edge $e \in (\Delta_{\mathbb{K}}/\Gamma)^1$ let*

$$N_e := \#\{e' \in (\Delta_{\mathbb{K}}/\Gamma)^1 \,|\, s(e') = s(e)\}. \tag{7.15}$$

Then the function

$$\mu(e) := \lambda(e) g(r(e)) - \frac{1}{N_e} g(s(e)) \tag{7.16}$$

satisfies the momentum conservation equation (7.2). Moreover, if $g : (\Delta_{\mathbb{K}}/\Gamma)^0 \to [0,\infty)$ is a function on the vertices of the graph such that (g,λ) is a graph weight for $\lambda(e) = N_{\bar{e}}^{-1}$, then the function $\mu : (\Delta_{\mathbb{K}}/\Gamma)^1 \to \mathbb{R}$ given by (7.16) is a real valued current on $\Delta_{\mathbb{K}}/\Gamma$.

Proof. The first result is a direct consequence of the equation for graph weights

$$g(v) = \sum_{s(e)=v} \lambda(e) g(r(v)), \tag{7.17}$$

which gives the equation (7.2) for (7.16). The second statement also follows from (7.17), where in this case the resulting function $\mu : (\Delta_{\mathbb{K}}/\Gamma)^1 \to \mathbb{R}$ given by (7.16) also satisfies the orientation-reversed equation

$$\sum_{r(e)=v} \mu(e) = 0. \tag{7.18}$$

In particular, it also satisfies

$$\mu(\bar{e}) = \frac{1}{N_e} g(s(e)) - \frac{1}{N_{\bar{e}}} g(r(e)) = -\mu(e),$$

hence it defines a real valued current on $\Delta_{\mathbb{K}}/\Gamma$,

$$\mu \in \mathcal{C}(\Delta_{\mathbb{K}}/\Gamma) \otimes_{\mathbb{Z}} \mathbb{R}.$$

∎

Conversely, one can use theta functions on the Mumford curve to construct graph weights on the tree, which however do not have the positivity property. We introduce the following notions generalizing that of positive graph weight given in Definition 4.1.

Definition 7.2. *Given a graph E, we define an inhomogeneous graph weight to be a triple (g,λ,χ) of non-negative functions $g : E^0 \to \mathbb{R}_+ = [0,\infty)$, $\lambda, \chi : E^1 \to [0,\infty)$ satisfying*

$$g(v) + d\chi(v) = \sum_{s(e)=v} \lambda(e) g(r(e)), \tag{7.19}$$

where, as above, $d\chi(v) = \sum_{s(e)=v} \chi(e)$. A rational virtual graph weight is a pair (g,λ) of functions $g : E^0 \to \mathbb{Q}$ and $\lambda : E^1 \to \mathbb{Q}_+ = \mathbb{Q} \cap [0,\infty)$ such that there exist rational valued inhomogeneous graph weights $(g^{\pm},\lambda,\chi)$ with $g(v) = g^+(v) - g^-(v)$ for all $v \in E^1$.

A virtual graph weight satisfies the equation (7.17). The choice of the inhomogeneous weights $(g^{\pm},\lambda,\chi)$ that give the decomposition $g(v) = g^+(v) - g^-(v)$ is non-unique.

Lemma 7.3. *Let $f \in \Theta(\Gamma)$ be a theta function for the Mumford curve $X = X_\Gamma$, with $\Theta(\Gamma)$ as in (7.12). Then f defines an associated pair of rational valued functions (g, λ) on the tree $\Delta_{\mathbb{K}}$, with $\lambda(e) = N_e^{-1}$ and with $g : \Delta^0_{\mathbb{K}} \to \mathbb{Q}$ satisfying the equation (7.17).*

Proof. By (7.12) we know that the theta function $f \in \Theta(\Gamma)$ determines an associated integer valued current μ_f on the graph $\Delta_{\mathbb{K}}/\Gamma$, that is, an element in $\mathcal{C}(\Delta_{\mathbb{K}})^\Gamma = \mathcal{C}(\Delta_{\mathbb{K}}/\Gamma)$. We view μ_f as a current on the tree $\Delta_{\mathbb{K}}$ that is Γ-invariant. We also know that the current $\mu = \mu_f$ is given by the expression (7.14) in terms of the spectral norm on the Drinfeld p-adic upper half plane.

If we set

$$g(v) := \log_q \|f\|_{\Lambda^{-1}(v)}, \tag{7.20}$$

We see easily that the condition (7.2) for the current $\mu(e) = g(r(e)) - g(s(e))$ implies that the function $g : \Delta^0_{\mathbb{K}} \to \mathbb{Z}$ satisfies the weight equation (7.17) with $\lambda(e) = N_e^{-1}$. In fact, we have

$$\sum_{s(e)=v} g(r(e)) = N_v g(v),$$

with $N_v = \#\{e' : s(e') = v\} = N_e$, for all e with $s(e) = v$. □

Lemma 7.4. *The function $g : \Delta^0_{\mathbb{K}} \to \mathbb{Z}$ associated to a theta function in $f \in \Theta(\Gamma)$ is an integer valued rational virtual graph weight.*

Proof. The measure $\mu = \mu_f$ can be written (non-uniquely) as a difference

$$\mu(e) = \chi^+(e) - \chi^-(e), \tag{7.21}$$

with non-negative $\chi^\pm : \Delta^1_{\mathbb{K}} \to \mathbb{N} \cup \{0\}$ satisfying

$$\chi^\pm(\bar{e}) = \chi^\mp(e) \quad \text{and} \quad \sum_{s(e)=v} \chi^+(e) = \sum_{s(e)=v} \chi^-(e), \tag{7.22}$$

for all $e \in \Delta^1_{\mathbb{K}}$. One then considers the equations

$$g^\pm(r(e)) - g^\pm(s(e)) = \chi^\pm(e). \tag{7.23}$$

We first see that (7.23) determines unique solutions $g^\pm : \Delta^0_{\mathbb{K}} \to \mathbb{Q}_+$ with $g^\pm(v) = 0$ at the basepoint. In fact, suppose we are given a vertex $w \neq v$ in the tree. There is a unique path $P(v, w)$ in $\Delta_{\mathbb{K}}$ connecting the base vertex v to w. It is given by a sequence $P(v, w) = e_1, \dots, e_n$ of oriented edges. Let $v = v_0, \dots, v_n = w$ be the corresponding sequence of vertices. The equation (7.23) implies

$$g^\pm(w) = \sum_{j=1}^n \chi^\pm(e_j). \tag{7.24}$$

This determines uniquely the values of $g^\pm$ at each vertex in $\Delta_{\mathbb{K}}$. The solutions obtained in this way satisfy $g^+(v) - g^-(v) = g(v)$, where $g(v) = \log_q \|f\|_{\Lambda^{-1}(v)}$ as in Lemma 7.3. The $g^\pm$ satisfy by construction the inhomogeneous weight equation

$$g^\pm(w) + d\chi^\pm(v) = \sum_{s(e)=w} \lambda(e) g^\pm(r(e)),$$

for $\lambda(e) = N_e^{-1}$. Thus, the pair (g, λ) of Lemma 7.3 is a rational virtual weight. □

Notice that, even though the current μ_f is Γ-invariant by construction, and the function $\lambda(e) = N_e^{-1}$ is also Γ-invariant, the function $g : \Delta^0_{\mathbb{K}} \to \mathbb{Q}$ obtained as above is not in general Γ-invariant, hence it need not descend to a graph weight on $\Delta_{\mathbb{K}}/\Gamma$.

In fact, for $g(v) = \log_q \|f\|_{\Lambda^{-1}(v)}$, one sees that

$$g(\gamma v) = \log_q \|f\|_{\Lambda^{-1}(\gamma v)} = \log_q \|f \circ \gamma\|_{\Lambda^{-1}(v)} = \log_q |c(\gamma)| + \log_q \|f\|_{\Lambda^{-1}(v)} = g(v) + \log_q |c(\gamma)|,$$

where $f(\gamma z) = c(\gamma) f(z)$.

More generally, one has the following result.

Lemma 7.5. *Let (g, λ) be a rational virtual weight on the tree $\Delta_{\mathbb{K}}$, with $g : \Delta^0_{\mathbb{K}} \to \mathbb{Q}$ and with $\lambda(e) = N_e^{-1}$. Then the function g satisfies*

$$g(\gamma v) - g(v) = d\beta_\gamma(v), \tag{7.25}$$

where $d\beta_\gamma(v) = \sum_{s(e)=v} \beta_\gamma(e)$ and

$$\beta_\gamma(e) = \lambda(e)(g(\gamma r(e)) - g(s(e))). \tag{7.26}$$

This satisfies $\beta_\gamma(\bar{e}) = -\beta_{\gamma^{-1}}(\gamma e)$ and the 1-cocycle equation

$$d\beta_{\gamma_1\gamma_2}(v) = d\beta_{\gamma_1}(\gamma_2 v) + d\beta_{\gamma_2}(v). \tag{7.27}$$

Proof. First notice that, by the weight equation (7.17), the function g satisfies

$$g(\gamma v) - g(v) = d\alpha_\gamma(v), \tag{7.28}$$

where $d\alpha_\gamma(v) = \sum_{s(e)=v} \alpha_\gamma(e)$ with

$$\alpha_\gamma(e) = \lambda(e) \left(\, g(\gamma r(e)) - g(r(e)) \, \right). \tag{7.29}$$

Notice moreover that we have

$$\alpha_\gamma(e) = \beta_\gamma(e) - \mu(e),$$

for β_γ as in (7.26) and $\mu = \mu_f$ the Γ-invariant current (7.14). Since $d\mu(v) \equiv 0$ we have $d\alpha_\gamma(v) = d\beta_\gamma(v)$, which gives (7.25). One checks the expression for $\beta_\gamma(\bar{e})$ directly from (7.26), using the Γ-invariance of N_e and $\lambda(e)$. The 1-cocycle equation is also easily verified by

$$g(\gamma_1\gamma_2 v) - g(v) - g(\gamma_1\gamma_2 v) + g(\gamma_2 v) - g(\gamma_2 v) + g(v) = 0.$$

□

In general, the condition for a (virtual) graph weight on the tree $\Delta_{\mathbb{K}}$ to descend to a (virtual) graph weight on the quotient $\Delta_{\mathbb{K}}/\Gamma$ is that the functions (g, λ) satisfy

$$g(v) = \sum_{s(e)=\gamma v} \lambda(e) g(r(e)), \tag{7.30}$$

for all $\gamma \in \Gamma$. This is clearly equivalent to the vanishing of $d\beta_\gamma(v)$ and to the invariance $g(\gamma v) = g(v)$.

Another possible way of describing (rational) virtual graph weights, instead of using the inhomogeneous equations, is by allowing the function λ to have positive or negative sign, namely we consider $\lambda : E^1 \to \mathbb{Q}$ and look for non-negative solutions $g : E^0 \to \mathbb{Q}_+$ of the original graph weight equation (7.17).

A rational virtual weight (g,λ) defines a solution $(\tilde{g},\tilde{\lambda})$ as above, with $\tilde{\lambda}: E^1 \to \mathbb{Q}$ and $\tilde{g}: E^0 \to \mathbb{Q}_+$, by setting λ to be $\tilde{\lambda}(e) = \lambda(e)\,\mathrm{sign}(g(s(e)))\,\mathrm{sign}(g(r(e)))$ and $\tilde{g}(v) = \mathrm{sign}(g(v))g(v) = |g(v)|$. This definition has an ambiguity when $g(v)=0$, in which case we can take either $\mathrm{sign}(g(v)) = \pm 1$.

7.2. **Theta functions and K-theory classes.** Another useful observation regarding the relation of theta functions of the Mumford curve X_Γ to properties of the graph algebra of the infinite graph $\Delta_{\mathbb{K}}/\Gamma$, is the fact that one can associate to the theta functions elements in the K-theory of the boundary C^*-algebra $C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma$.

This is not a new observation: it was described explicitly in [30] and also used in [12], though only in the case of finite graphs. The finite graph hypothesis is used in [30] to obtain the further identification of the Γ-invariant $\mathbb{Z}$-valued currents on the covering tree with the first homology group of the graph.

In our setting, the graph $\Delta_{\mathbb{K}}/\Gamma$ consists of a finite graph Δ'_Γ/Γ together with infinite trees stemming from its vertices. We still have the same result on the identification with the K-theory group $K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma)$ of the boundary algebra, as well as with the first homology of the graph $\Delta_{\mathbb{K}}/\Gamma$, which is the same as the first homology of the finite graph Δ'_Γ/Γ.

Proposition 7.6. *There are isomorphisms*

$$\mathcal{C}(\Delta_{\mathbb{K}},\mathbb{Z})^\Gamma \cong H_1(\Delta_{\mathbb{K}}/\Gamma,\mathbb{Z}) \cong \mathrm{Hom}(K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma),\mathbb{Z}). \tag{7.31}$$

Proof. The first isomorphism follows directly from (7.7).

To prove the second identification

$$\mathcal{C}(\Delta_{\mathbb{K}},\mathbb{Z})^\Gamma \cong \mathrm{Hom}(K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma),\mathbb{Z}),$$

first notice that $\partial\Delta_{\mathbb{K}}$ is a totally disconnected compact Hausdorff space, hence $K_1(C(\partial\Delta_{\mathbb{K}})) = 0$ and in the exact sequence of [28] for the K-theory of the crossed product by the free group Γ one obtains an isomorphism of $K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma)$ with the coinvariants

$$C(\partial\Delta_{\mathbb{K}},\mathbb{Z})_\Gamma = C(\partial\Delta_{\mathbb{K}},\mathbb{Z})/\{f\circ\gamma - f \mid f \in C(\partial\Delta_{\mathbb{K}},\mathbb{Z})\},$$

where $C(\partial\Delta_{\mathbb{K}},\mathbb{Z})$ is the abelian group of locally constant $\mathbb{Z}$-valued functions on $\partial\Delta_{\mathbb{K}}$, *i.e.* finite linear combinations with integer coefficients of characteristic functions of clopen subsets. We then show that the abelian group $\mathcal{C}(\Delta_{\mathbb{K}},\mathbb{Z})^\Gamma$ of Γ-invariant currents on the tree $\Delta_{\mathbb{K}}$ can be identified with

$$\mathcal{C}(\Delta_{\mathbb{K}},\mathbb{Z})^\Gamma \cong \mathrm{Hom}(K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma),\mathbb{Z}) = H_1(\Delta_{\mathbb{K}}/\Gamma,\mathbb{Z}). \tag{7.32}$$

To see that a current $\mu \in \mathcal{C}(\Delta_{\mathbb{K}},\mathbb{Z})^\Gamma$ defines a homomorphism $\phi: C(\partial\Delta_{\mathbb{K}},\mathbb{Z})_\Gamma \to \mathbb{Z}$, we use the fact that we can view the current μ on the tree as a measure m of total mass zero on the boudary $\partial\Delta_{\mathbb{K}}$ by setting $m(V(e)) = \mu(e)$, where $V(e)$ is the subset of the boundary determines by all infinite paths starting with the oriented edge e. We then define the functional

$$\phi(f) = \int_{\partial\Delta_{\mathbb{K}}} f\,dm, \tag{7.33}$$

where the integration is defined by $\phi(\sum_i a_i\chi_{V(e_i)}) = \sum_i a_i\mu(e_i)$ on characteristic functions. To see that ϕ is defined on the coinvariants it suffices to check that it vanishes on elements of the form $f\circ\gamma - f$, for some $\gamma\in\Gamma$. This follows by change of variables and the invariance of the current μ,

$$\int f\circ\gamma\,dm = \int f\,dm\circ\gamma^{-1} = \int f\,dm.$$

Conversely, suppose we are given a homomorphism $\phi : C(\partial\Delta_{\mathbb{K}}, \mathbb{Z})_\Gamma \to \mathbb{Z}$. We define a map $\mu : \Delta^1_{\mathbb{K}} \to \mathbb{Z}$ by setting $\mu(e) = \phi(\chi_{V(e)})$, where $\chi_{V(e)}$ is the characteristic function of the set $V(e) \subset \partial\Delta_{\mathbb{K}}$. We need to show that this defines a Γ-invariant current on the tree. We need to check that the equation

$$\sum_{s(e)=v} \mu(e) = 0$$

and the orientation reversal condition $\mu(\bar{e}) = -\mu(e)$ is satisfied.

Notice that we have, for any given vertex $v \in \Delta^0_{\mathbb{K}}$, $\cup_{s(e)=v} V(e) = \partial\Delta_{\mathbb{K}}$. If we set

$$h(v) := \sum_{s(e)=v} \phi(\chi_{V(e)}),$$

we obtain a Γ-invariant $\mathbb{Z}$-valued function on the set of vertices $\Delta^0_{\mathbb{K}}$, *i.e.* a $\mathbb{Z}$-valued function on the vertices $(\Delta_{\mathbb{K}}/\Gamma)^0$. In fact, we have

$$\phi(f \circ \gamma) = \phi(f)$$

by the assumption that ϕ is defined on the coinvariants $C(\partial\Delta_{\mathbb{K}}, \mathbb{Z})_\Gamma$, hence

$$h(\gamma v) = \sum_{s(e)=\gamma v} \phi(\chi_{V(e)}) = \sum_{s(e)=v} \phi(\chi_{V(e)} \circ \gamma) = h(v).$$

Since by construction $h = d\mu$, with $\mu(e) = \phi(\chi_{V(e)})$ and $d : \mathcal{A}(\Delta^1_{\mathbb{K}}/\Gamma) \to \mathcal{H}(\Delta^0_{\mathbb{K}}/\Gamma)$ as in (7.10), it is in the kernel of the map Φ of (7.11). This means that

$$\Phi(h) = \sum_{v \in \Delta^0_{\mathbb{K}}/\Gamma} h(v) = 0,$$

but we know that

$$h(v) = \sum_{s(e)=v} \phi(\chi_{V(e)}) = \phi(\sum_{s(e)=v} \chi_{V(e)}) = \phi(\chi_{\partial\Delta_{\mathbb{K}}})$$

so that the condition $\Phi(h) = 0$ implies $h(v) = 0$ for all v, *i.e.* $\phi(\chi_{\partial\Delta_{\mathbb{K}}}) = 0$. This gives

$$\sum_{s(e)=v} \phi(\chi_{V(e)}) = 0$$

which is the momentum conservation condition for the measure μ. Moreover, the fact that the measure on $\partial\Delta_{\mathbb{K}}$ defined by $\mu(e) = \phi(\chi_{V(e)})$ has total mass zero also implies that

$$0 = \phi(\chi_{\partial\Delta_{\mathbb{K}}}) = \phi(\chi_{V(e)}) + \phi(\chi_{V(\bar{e})}),$$

hence $\mu(\bar{e}) = -\mu(e)$, so that μ is a current. The condition $\phi(f \circ \gamma) = \phi(f)$ shows that it is a Γ-invariant current. □

The results of this section relate the theta functions of Mumford curves to the K-theory of a C^*-algebra which is not directly the graph algebra $C^*(\Delta_{\mathbb{K}}/\Gamma)$ we worked with so far, but the "boundary algebra" $C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma$. However, it is known by the result of Theorem 1.2 of [21] that the crossed product algebra $C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma$ is strongly Morita equivalent to the algebra $C^*(\Delta_{\mathbb{K}}/\Gamma)$. In fact, we use the fact that $\Delta_{\mathbb{K}}$ is a tree and that the p-adic Schottky group Γ acts freely on $\Delta_{\mathbb{K}}$, so that $C^*(\Delta_{\mathbb{K}}) \rtimes \Gamma \simeq C^*(\Delta_{\mathbb{K}}/\Gamma) \otimes \mathcal{K}(\ell^2(\Gamma))$. Thus $C^*(\Delta_{\mathbb{K}}/\Gamma)$ is strongly Morita equivalent to $C^*(\Delta_{\mathbb{K}}) \rtimes \Gamma$. Moreover, $\Delta_{\mathbb{K}}$ is the universal covering tree of $\Delta_{\mathbb{K}}/\Gamma$ and Γ can be identified with the fundamental group so that the argument of Theorem 4.13 of [21] holds in this case and the result of Theorem 1.2 of

[21] applies. Thus the K-theory considered here can be also thought of as the K-theory of the latter algebra and we obtain the following result.

Corollary 7.7. *A theta function $f \in \Theta(\Gamma)$ defines a functional $\phi_f \in \mathrm{Hom}(K_0(C^*(\Delta_{\mathbb{K}}/\Gamma)), \mathbb{Z})$. Two theta functions $f, f' \in \Theta(\Gamma)$ define the same $\phi_f = \phi_{f'}$ if and only if they differ by the action of $\mathbb{K}^*$.*

Proof. The first statement follows from Proposition 7.6 and the identification $K_0(C^*(\Delta_{\mathbb{K}}/\Gamma)) = K_0(C(\partial\Delta_{\mathbb{K}}) \rtimes \Gamma)$ which follows from the strong Morita equivalence discussed above. The second statement is then a direct consequence of Proposition 7.6 and (7.12). □

Corollary 7.7 shows that there is a close relationship between the K-homology of $C^*(\Delta_{\mathbb{K}}/\Gamma)$ and theta functions. In the next section we make a first step towards constructing theta functions from graphical data and spectral flows.

8. Inhomogenous graph weights equation and the spectral flow

Let E be a graph with zhyvot M with a choice of (not necessarily special) graph weight (g, λ) adapted to the zhyvot action. We would like to construct an inhomogenous graph weight (G, λ, χ) from these data.

The motivation for the construction is as follows. In the case of a special graph weight, the spectral flow only sees edges and paths in the zhyvot M of E. Consequently

$$-\sum_{s(e)=v} sf_{\phi_{\mathcal{D}}}(S_e S_e^* \mathcal{D}, S_e \mathcal{D} S_e^*) = \sum_{\substack{s(e)=v \\ e\in M}} \lambda(e) g(r(e)) \le g(v).$$

Thus in some sense the spectral flow is trying to reproduce the graph weight, but misses information from edges not in M. Alternatively, one may think of restricting (g, λ) to the zhyvot and asking whether it is still a (special) graph weight. This is usually not the case, for exactly the same reason.

So to obtain (G, λ, χ) we begin with the ansatz that λ is the function given to us with our graph weight and that

$$G(v) = g(v) - \alpha(v),$$

so that we require $\alpha(v) \le g(v)$ for all $v \in E$. We now compute

$$\begin{aligned}
\sum_{s(e)=v} \lambda(e) G(r(e)) &= g(v) - \sum_{s(e)=v} \lambda(e)\alpha(r(e)) \\
&= g(v) - \alpha(v) + \alpha(v) - \sum_{s(e)=v} \lambda(e)\alpha(r(e)) \\
&= G(v) + \sum_{s(e)=v} \left(\frac{1}{N_e}\alpha(v) - \lambda(e)\alpha(r(e)) \right).
\end{aligned}$$

Hence to obtain an inhomogenous graph weight, we must have

$$\chi(e) = \frac{1}{N_e}\alpha(s(e)) - \lambda(e)\alpha(r(e)).$$

Here are some possible choices of α.

1) $\alpha = g$. This forces $G = 0$ and so we obtain an inhomogenous graph weight only when $\frac{1}{N_e} = \lambda(e)$.

2) $\alpha = c\,g$, $0 < c < 1$. Now $G \neq 0$ but we still need $\frac{1}{N_e} = \lambda(e)$ in order to obtain an inhomogenous graph weight.

3) $\alpha(v) = \sum_{\substack{s(e)=v\\ e\in M}} \lambda(e)g(r(e)) = -\sum_{s(e)=v} sf_{\phi_{\mathcal{D}}}(S_eS_e^*\mathcal{D}, S_e\mathcal{D}S_e^*)$. In this case we obtain

$$\chi(e) = \frac{1}{N_e}\sum_{\substack{s(f)=s(e)\\ f\in M}} \lambda(f)g(r(f)) - \lambda(e)\sum_{\substack{s(h)=r(e)\\ h\in M}} \lambda(h)g(r(h)).$$

This **may** be non-negative for certain values of λ.

4) $\alpha(v) = \begin{cases} g(v) & v \in M \\ 0 & v \notin M \end{cases}$ This gives

$$\chi(e) = \begin{cases} \frac{1}{N_e}g(s(e)) - \lambda(e)g(r(e)) & s(e),\, r(e) \in M \\ \frac{1}{N_e}g(s(e)) & s(e) \in M,\, r((e) \notin M \\ 0 & s(e),\, r(e) \notin M \end{cases}$$

Thus χ is non-negative provided

$$\frac{1}{N_e}g(s(e)) \geq \lambda(e)g(r(e)).$$

The choices 3) and 4) both yield triples (G, λ, χ) satisfying Equation (7.19), and all that remains to understand is the positivity of the function χ.

In fact it is easy to construct examples where 4) fails to give a non-negative function χ. However, 3) is more subtle. At present we have no way of deciding whether we can always find a g so that the function α in 3) is non-negative for λ associated to the zhyvot action. It would seem that passing to field extensions allows us to construct graph weights adapted to the (new) zhyvot so that both 3) and 4) fail. The reason is we may increase N_e while keeping $\lambda(e)$ constant.

We describe here another construction of inhomogenous graph weights adapted to the zhyvot action. This uses special graph weights, with $\lambda(e) \neq 1$ only on the edges inside the zhyvot, so it does not apply to the construction of theta functions, where one needs $\lambda(e) = N_e^{-1}$ (which is $q^{-1} \neq 1$ outside of the zhyvot), but we include it here for its independent interest.

Lemma 8.1. *Let (g, λ) be a graph weight on the graph E with zhyvot M such that $\lambda(e) \neq 1$ iff $e \in M^1$. With the notation of Section 5, define $\alpha_k : E^0 \to [0, \infty)$ by*

$$\alpha_k(v) = \phi_{\mathcal{D}}(p_v\Phi_k), \quad k = 0, 1, 2, \ldots.$$

Then $\alpha_{k-1}(v) \geq \alpha_k(v)$ for all $v \in E^0$.

Proof. We first observe that for $k \geq 1$,

$$\Phi_k = \sum_{|\mu|_\sigma = k} \Theta_{S_\mu, S_\mu}.$$

This follows easily by induction from $\Phi_0 = \sum_{v\in E^0}\Theta_{p_v,p_v}$ and the definition of the zhyvot action. Then

$$\begin{aligned}\phi_{\mathcal{D}}(p_v\Phi_k) &= \sum_{e\in M^1,|\mu|_\sigma=k-1,s(\mu)=v}\lambda(\mu)\lambda(e)Trace_\phi(\Theta_{S_\mu S_e,S_\mu S_e})\\ &= \sum_{e\in M^1,|\mu|_\sigma=k-1,s(\mu)=v}\lambda(\mu)\lambda(e)\phi(S_e^*S_\mu^*S_\mu S_e)\\ &\leq \sum_{e\in E^1,|\mu|_\sigma=k-1,s(\mu)=v}\lambda(\mu)\lambda(e)\phi(S_e^*S_\mu^*S_\mu S_e)\\ &= \sum_{e\in E^1,|\mu|_\sigma=k-1,s(\mu)=v}\lambda(\mu)\phi(S_eS_e^*S_\mu^*S_\mu)\\ &= \sum_{|\mu|_\sigma=k-1,s(\mu)=v}\lambda(\mu)\phi(p_{r(\mu)}S_\mu^*S_\mu)\\ &= \phi_{\mathcal{D}}(p_v\Phi_{k-1}).\end{aligned}$$

□

Theorem 8.2. *Let (g,λ) be a graph weight on the graph E with zhyvot M such that $\lambda(e)\neq 1$ iff $e\in M^1$. Then for all $k\geq 1$ the triple $(g-\alpha_k,\lambda,\chi_k)$ is an inhomogenous graph trace, where $\chi_k(e) = \lambda(e)(\alpha_{k-1}(r(e))-\alpha_k(r(e)))$.*

Proof. There are a couple of simple observations here. If $v\in E^0\setminus M^0$, then $\alpha_k(v)=0$ when $k>0$. This follows from Lemma 8.1. This means that for $k\geq 1$

$$\begin{aligned}\sum_{s(e)=v}\lambda(e)\alpha_k(r(e)) = \sum_{s(e)=v}\lambda(e)\phi_{\mathcal{D}}(p_{r(e)}\Phi_k) &= \sum_{s(e)=v}\lambda(e)\phi_{\mathcal{D}}(S_e^*S_e\Phi_k)\\ &= \sum_{s(e)=v}\phi_{\mathcal{D}}(S_e\Phi_kS_e^*)\\ &= \sum_{s(e)=v}\phi_{\mathcal{D}}(S_eS_e^*\Phi_{k+1})\\ &= \phi_{\mathcal{D}}(p_v\Phi_{k+1}) = \alpha_{k+1}(v),\end{aligned}$$

the last line following from the Cuntz-Krieger relations. Then the inhomogenous graph weight equation is simple.

$$\begin{aligned}\sum_{s(e)=v}\lambda(e)(g(r(e))-\alpha_k(r(e))) &= \sum_{s(e)=v}\lambda(e)(g(r(e))-\alpha_{k-1}(r(e))) + \sum_{s(e)=v}\lambda(e)(\alpha_{k-1}(r(e))-\alpha_k(r(e)))\\ &= (g(v)-\alpha_k(v)) + \sum_{s(e)=v}\lambda(e)(\alpha_{k-1}(r(e))-\alpha_k(r(e))).\end{aligned}$$

So the inhomogenous graph weight equation is satisfied if we set $\chi_k(e) = \lambda(e)(\alpha_{k-1}(r(e))-\alpha_k(r(e)))$, and both $g-\alpha_k$, $\chi_k\geq 0$ by Lemma 8.1. □

These inhomogenous weights are canonically associated to the decompositions $F = F_k\oplus G_k$ of the fixed point algebra arising from the zhyvot action.

8.1. **Constructing theta functions from spectral flows.** We show how some of the methods described above that produce inhomogeneous graph weights with $\lambda(e) = N_e^{-1}$ can be adapted to construct theta functions on the Mumford curve.

First notice that, given such a construction of a solutions of the inhomogeneous graph weights as above, we can produce a rational virtual graph weight in the following way.

Lemma 8.3. *Suppose we are given two graph weights g_1, g_2 on the infinite graph $\Delta_{\mathbb{K}}/\Gamma$, both with the same $\lambda(e) = N_e^{-1}$. Suppose we are also given inhomogeneous graph weights of the form $G_i(v) = g_i(v) - \alpha_i(v)$ as above, with $0 \le \alpha_i(v) \le g_i(v)$ at all vertices, and with $\chi_i(e) = \frac{1}{N_e}(\alpha_i(s(e)) - \alpha_i(r(e)))$. Then setting $\hat{G}_i(v) = g_i(v) - \alpha(v)$ with $\alpha(v) = \min\{\alpha_1(v), \alpha_2(v)\}$ gives two solutions of the inhomogeneous graph weight equation with the same $\chi(e) = \frac{1}{N_e}(\alpha(s(e)) - \alpha(r(e)))$ and $\lambda(e) = N_e^{-1}$.*

Proof. We have

$$\hat{G}_i(v) = g_i(v) - \alpha(v) = \sum_{s(e)=v} \frac{1}{N_e} g_i(r(e)) - \alpha(v) = \sum_{s(e)=v} \frac{1}{N_e} \hat{G}_i(r(e)) + \sum_{s(e)=v} \frac{1}{N_e}(\alpha(r(e)) - \alpha(s(e))$$

which shows that both $\hat{G}_i(v)$ are solutions of the inhomogeneous weight equation

$$\hat{G}_i(v) + d\chi(v) = \sum_{s(e)=v} \frac{1}{N_e} \hat{G}_i(r(e)),$$

where $\chi(e) = N_e^{-1}(\alpha(s(e)) - \alpha(r(e)))$. □

Thus, whenever we have multiple solutions for the graph weights on $\Delta_{\mathbb{K}}/\Gamma$ we can construct associated rational virtual graph weights by setting

$$\hat{G}(v) = \hat{G}_1(v) - \hat{G}_2(v). \tag{8.1}$$

If the graph weights g_i are rational valued, $g_i : E^0 \to \mathbb{Q}_+$, then the virtual graph weight $\hat{G}$ is also rational valued, $\hat{G} : E^0 \to \mathbb{Q}$.

Moreover, since the graph $\Delta_{\mathbb{K}}/\Gamma$ has a finite zhyvot with infinite trees coming out of its vertices, one can obtain a rational virtual graph weight that is integer valued. In fact, along the trees outside the zhyvot Δ'_Γ/Γ of $\Delta_{\mathbb{K}}/\Gamma$, the condition

$$\hat{G}(v) = \sum_{s(e)=v} \frac{1}{N_e} \hat{G}(r(e))$$

is satisfied by extending $\hat{G}(v)$ from the zhyvot by $\hat{G}(r(e)) = \hat{G}(s(e))$ along the trees. In this case, what remains is the finite graph, the zhyvot, which only involves finitely many denominators for a rational valued $\hat{G}(v)$, which means that one can obtain an integer valued solution. We will therefore assume that the rational virtual graph weights constructed as in (8.1) and Lemma 8.3 are integer valued, $\hat{G} : E^0 \to \mathbb{Z}$. This in particular includes the cases constructed using the spectral flow.

According to the results of §7.1 above, we then have the following result.

Proposition 8.4. *Suppose we are given a virtual graph weight $\hat{G} : E^0(\Delta_{\mathbb{K}}/\Gamma) \to \mathbb{Z}$ as above and a homomorphism $c : \Gamma \to \mathbb{K}^*$. Then there is a theta function f on the Mumford curve X_Γ satisfying $\log_q \|f\|_{\Lambda^{-1}(v)} = \tilde{G}(v)$ and $f(\gamma z) = c(\gamma) f(z)$, where $\tilde{G} : E^1(\Delta_{\mathbb{K}}) \to \mathbb{Z}$ is defined as $\tilde{G}(v) = \hat{G}(v)$ on a fundamental domain of the action of Γ on $\Delta_{\mathbb{K}}$ and extended to $\Delta_{\mathbb{K}}$ by $\tilde{G}(\gamma v) = \log_q |c(\gamma)| + \hat{G}(v)$.*

Proof. This follows from the identification of the group of theta functions $\Theta(\Gamma)$ with the extension (7.12) of the group $\mathcal{C}(\Delta_{\mathbb{K}})^{\Gamma}$ of currents on $\Delta_{\mathbb{K}}/\Gamma$ by $\mathbb{K}^*$, and identifying the current $\mu(e) = \hat{G}(r(e)) - \hat{G}(s(e))$ with $\mu_f(e) = \log_q \|f\|_{\Lambda^{-1}(r(e))} - \log_q \|f\|_{\Lambda^{-1}(s(e))}$. □

The last two results highlight the interest in determining whether non-negative α can be found for the function $\lambda(e) = N_e^{-1}$.

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