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## On Pogorelov estimates for Monge-Ampere type equations

### Abstract

In this paper, we prove interior second derivative estimates of Pogorelov type for a general form of Monge-Ampère equation which includes the optimal transportation equation. The estimate extends that in a previous work with Xu-Jia Wang and assumes only that the matrix function in the equation is regular with respect to the gradient variables, that is it satisfies a weak form of the condition introduced previously by Ma, Trudinger and Wang for regularity of optimal transport mappings. We also indicate briefly an application to optimal transportation.

### Keywords

type, ampere, equations, monge, pogorelov, estimates

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## ON POGORELOV ESTIMATES FOR MONGE-AMPÈRE TYPE EQUATIONS

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ABSTRACT. In this paper, we prove interior second derivative estimates of Pogorelov type for a general form of Monge-Ampère equation which includes the optimal transportation equation. The estimate extends that in a previous work with Xu-Jia Wang and assumes only that the matrix function in the equation is regular with respect to the gradient variables, that is it satisfies a weak form of the condition introduced previously by Ma, Trudinger and Wang for regularity of optimal transport mappings. We also indicate briefly an application to optimal transportation.

1. **Introduction.** There has been considerable research activity in recent years devoted to fully nonlinear, elliptic second order partial differential equations of the form [13],

$$\mathcal{F}[u] := F\{D^2u - A(\cdot, Du)\} = B(\cdot, u, Du), \quad (1)$$

in domains  $\Omega$  in Euclidean  $n$ -space,  $\mathbb{R}^n$ , as well as their extensions to Riemannian manifolds. Here the functions  $F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $A : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ ,  $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are given and the resultant operator  $\mathcal{F}$  is well-defined classically for functions  $u \in C^2(\Omega)$ . As customary  $Du$  and  $D^2u$  denote respectively the gradient vector and Hessian matrix of second derivatives of  $u$ , while we also use  $x, z, p, r$  to denote points in  $\Omega, \mathbb{R}, \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n$  respectively with corresponding partial derivatives denoted, when there is no ambiguity, by subscripts.

Equations of the form (1) arise in applications, notably in optimal transportation and conformal geometry. In particular, for  $F(r) = \det r$ , we obtain a *Monge-Ampère equation* of the general form

$$\mathcal{F}[u] := \det\{D^2u - A(\cdot, Du)\} = B(\cdot, u, Du). \quad (2)$$

Unless indicated otherwise we will assume the matrix  $A$  is symmetric, while for  $A \equiv 0$ , (2) reduces to the standard Monge-Ampère equation. The operator  $\mathcal{F}$  in (2) is elliptic, (degenerate elliptic), with respect to  $u$  whenever

$$D^2u - A(\cdot, Du) > 0 \quad (\geq 0), \quad (3)$$

which implies  $B > 0$  ( $\geq 0$ ). In such a case, we call  $u$  an elliptic, (degenerate elliptic), solution of (2).

The key estimates for classical solutions of equations of the form (2) are bounds for second derivatives as higher order estimates and regularity follow from the fully

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nonlinear theory [2]. For the standard Monge-Ampère equation, such estimates were originally proved by Pogorelov [9], but the corresponding estimates for (2) are in general not true, as demonstrated by the Lewy-Heinz example [10]. See [15] for further introduction and background.

The regularity of solutions of equations of the form (2) depends on the behaviour of the matrix  $A$  with respect to the  $p$  variables. Let  $\mathcal{U} \subset \Omega \times \mathbb{R}^n$ , such that its projection on  $\Omega$  is  $\Omega$  itself. We say that  $A$  is *regular* in  $\mathcal{U}$  if

$$D_{p_k p_l}^2 A_{ij} \xi_i \xi_j \eta_k \eta_l \geq 0 \quad (4)$$

in  $\mathcal{U}$ , for all  $\xi, \eta \in \mathbb{R}^n$  with  $\xi \cdot \eta = 0$ ; and *strictly regular* in  $\mathcal{U}$  if there exists a constant  $a_0 > 0$  such that

$$D_{p_k p_l}^2 A_{ij} \xi_i \xi_j \eta_k \eta_l \geq a_0 |\xi|^2 |\eta|^2 \quad (5)$$

in  $\mathcal{U}$ , for all  $\xi, \eta \in \mathbb{R}^n$  with  $\xi \cdot \eta = 0$ . Conditions (4), (5) were introduced in [8], [16] and called A3w, A3 respectively. The interior  $C^2$  estimate has been obtained in [8] under the strong condition (5), and global second derivative bounds were obtained later in [16] under the weak condition (4).

In this paper, we establish interior estimates of Pogorelov type for more general forms of equation (2) under natural conditions on the matrix function  $A$ . For these estimates we will assume  $A, B$  are  $C^2$  smooth,  $A$  is regular in an appropriate set  $\mathcal{U}$  and  $B$  is positive and non-decreasing in  $u$ .

It is convenient to express our interior estimates in terms of a degenerate elliptic strict supersolution  $u_0 \in C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$ , which satisfies

$$\mathcal{F}[u_0] \leq B(\cdot, u_0, Du_0) - \delta \quad \text{in } \Omega, \quad (6)$$

for some positive constant  $\delta$ . For the standard Monge-Ampère equation one takes  $u_0 \equiv 0$  or equivalently any affine function.

In the general case, while the proof follows a similar approach to the original Pogorelov proof, it is much more complicated. As well we need to assume a kind of global barrier condition, called *A-boundedness* in [13], namely that there exists a function  $\varphi \in C^2(\bar{\Omega})$  satisfying

$$[D_{ij}\varphi(x) - D_{p_k} A_{ij}(x, p) D_k \varphi(x)] \xi_i \xi_j \geq |\xi|^2 \quad (7)$$

for all  $\xi \in \mathbb{R}^n$ ,  $x, p \in \mathcal{U}$ . Condition (7) is trivially satisfied in the standard Monge-Ampère case as seen by taking  $\varphi(x) = |x|^2$ . As indicated in [6], when the diameter of  $\Omega$  is sufficiently small, a similar function  $\varphi$  is readily constructed for bounded  $\mathcal{U}$ . See also Remark 2.

In order to formulate our estimate, we denote sets

$$\mathcal{U}[u] = \{(x, u(x), Du(x)) \mid x \in \Omega\}$$

and

$$\mathcal{U}[u, u_0] = \{(x, u(x), tDu(x) + (1-t)Du_0(x)) \mid x \in \Omega, 0 \leq t \leq 1\}.$$

Our main result can now be stated,

**Theorem 1.1.** *Let  $u \in C^4(\Omega) \cap C^{0,1}(\bar{\Omega})$  be an elliptic solution of (2) satisfying  $u = u_0$  on  $\partial\Omega$ , where  $u_0 \in C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$  is a supersolution of (2) satisfying (6). Suppose  $A$  is regular in  $\mathcal{U}[u, u_0]$  with (7) satisfied in  $\mathcal{U}[u]$  and  $B_z \geq 0$ . Then we have for any  $\Omega' \Subset \Omega$*

$$\sup_{\Omega'} |D^2 u| \leq C, \quad (8)$$

where the constant  $C$  depends on  $n, A, B, \Omega, \Omega', u_0$ , and  $\sup_{\Omega} (|u| + |Du|)$ .

**Remark 1.** If  $A(\cdot, 0) \equiv 0$ , then as in the standard Monge-Ampère case we may take  $u_0 = 0$ . This situation also embraces the optimal transportation case which we discuss in the last section. More generally we could assume  $u_0$  is a degenerate elliptic solution of the homogeneous equation  $\mathcal{F}[u] = 0$ . The result also extends to non strict supersolutions  $u_0$ ; see Remark 3.

Theorem 1.1 is proved in Section 2. In Section 3, we indicate its application to optimal transportation. In particular we prove that strictly  $c$ -convex potentials are smooth when the initial and target densities are appropriately smooth.

**2. Pogorelov estimate.** In this section we will prove Theorem 1.1. The result extends the Pogorelov estimates established in [6] for the case of constant boundary values. A similar result for standard Monge-Ampère equations was established in [14], in particular the  $C^{1,1}$  regularity of homogeneous solutions was also obtained in [14]. Although our proof is obtained through modifications of the corresponding global bound in [16] and our previous interior estimates in [6], which used stronger conditions on  $A$  or  $u$ , for completeness we present here the detailed proof.

By  $B_z \geq 0$  and the comparison principle we automatically have  $u < u_0$  in  $\Omega$ . First we prove an interior estimate of the form,

$$(u_0 - u)^\tau |D^2 u| \leq C, \quad (9)$$

for appropriate constants  $\tau$  and  $C$ , depending on  $n, A, B, u_0$  and  $|u|_1$ .

Let  $v$  be the auxiliary function given by

$$v = \tau \log \eta + \log(w_{ij} \xi_i \xi_j) + \frac{1}{2} \beta |Du|^2 + e^{\kappa \varphi} \quad (10)$$

where  $\varphi$  is the barrier function in (7),  $w_{ij} = D_{ij}u - A_{ij}$ ,  $\eta = u_0 - u$ , and  $\beta, \kappa, \tau$  are positive constants to be determined. By suitable normalization, we may assume that  $\max \eta \leq 1$  and  $\varphi \geq 0$ .

Suppose  $v$  attains its maximum at  $\bar{x} \in \Omega$  and  $\xi = (1, 0, \dots, 0)$ . We may assume that the matrix  $\{w_{ij}\}$  is diagonal at  $\bar{x}$ , and denote  $w_{\xi\xi} = w_{ij} \xi_i \xi_j$ . Hence  $Dv(\bar{x}) = 0$ ,  $D^2 v(\bar{x}) \leq 0$ , and

$$L(v)(\bar{x}) \leq 0. \quad (11)$$

Here the linear operator  $L$  is defined by

$$L := w^{ij}(\bar{x})(D_{ij} + b_k^{ij} D_k) - (D_{p_k} \tilde{B}) D_k, \quad (12)$$

where  $\{w^{ij}\}$  is the inverse matrix of  $\{w_{ij}\}$ ,  $b_k^{ij} := -D_{p_k} A_{ij}$  and  $\tilde{B} = \log B$ .

By differentiating we have, at  $\bar{x}$ ,

$$\begin{aligned} D_i v &= \frac{\tau D_i \eta}{\eta} + \frac{D_i w_{11}}{w_{11}} + \beta D_k u D_{ki} u + \kappa e^{\kappa \varphi} D_i \varphi, \\ D_{ii} v &= \frac{\tau D_{ii} \eta}{\eta} - \frac{\tau (D_i \eta)^2}{\eta^2} + \frac{D_{ii} w_{11}}{w_{11}} - \frac{(D_i w_{11})^2}{w_{11}^2} + \\ &\quad \beta \sum_{k=1}^n ((D_{ik} u)^2 + D_k u D_{ikk} u) + \kappa e^{\kappa \varphi} (\kappa (D_i \varphi)^2 + D_{ii} \varphi). \end{aligned} \quad (13)$$

Hence

$$\begin{aligned}
Lv(\bar{x}) &= \frac{\tau}{\eta} L\eta - \frac{\tau}{\eta^2} \sum_{i=1}^n \frac{(D_i\eta)^2}{w_{ii}} + \frac{1}{w_{11}} Lw_{11} \\
&\quad - \frac{1}{w_{11}^2} \sum_{i=1}^n \frac{(D_i w_{11})^2}{w_{ii}} + \beta \sum_{k=1}^n D_k u L u_k \\
&\quad + \beta \sum_{i,k=1}^n \frac{1}{w_{ii}} (D_{ik} u)^2 + \kappa e^{\kappa\varphi} L\varphi + \kappa^2 e^{\kappa\varphi} \sum_{i=1}^n \frac{(D_i\varphi)^2}{w_{ii}}.
\end{aligned} \tag{14}$$

In the following we estimate each term on the right hand side of (14). First by (7) we have

$$\begin{aligned}
L\varphi(\bar{x}) &\geq \sum_{i=1}^n w^{ii}(\bar{x}) - \sum_{k=1}^n (D_{p_k} \tilde{B}) D_k \varphi(\bar{x}) \\
&\geq \frac{1}{2} \sum_{i=1}^n w^{ii}.
\end{aligned} \tag{15}$$

Note that  $|(D_{p_k} \tilde{B}) D_k \varphi(\bar{x})|$  is bounded, the second inequality follows since  $\sum_{i=1}^n w^{ii}$  is as large as we want, otherwise the proof is finished.

To estimate  $Lw_{11}$  and  $Lu_k$ , we differentiate equation (2) to get

$$\begin{aligned}
&w^{ii} [D_{ii} u_\xi - D_\xi A_{ii} - (D_{p_k} A_{ii}) D_k u_\xi] \\
&= D_\xi \tilde{B} + (D_z \tilde{B}) u_\xi + (D_{p_k} \tilde{B}) D_k u_\xi,
\end{aligned} \tag{16}$$

at  $\bar{x}$ , where  $\xi$  is a unit vector. A further differentiation yields

$$\begin{aligned}
&w^{ii} [D_{ii} u_{\xi\xi} - D_{\xi\xi} A_{ii} - (D_{p_k p_l} A_{ii}) D_k u_\xi D_l u_\xi - (D_{p_k} A_{ii}) D_k u_{\xi\xi} \\
&\quad - 2(D_{\xi p_k} A_{ii}) D_k u_\xi] - w^{ii} w^{jj} (D_\xi w_{ij})^2 \\
&= D_{\xi\xi} \tilde{B} + (D_{zz} \tilde{B}) u_\xi^2 + (D_{p_k p_l} \tilde{B}) D_k u_\xi D_l u_\xi \\
&\quad + 2(D_{\xi z} \tilde{B}) u_\xi + 2(D_{\xi p_k} \tilde{B}) D_k u_\xi + 2(D_{z p_k} \tilde{B}) (D_k u_\xi) u_\xi \\
&\quad + (D_z \tilde{B}) u_{\xi\xi} + (D_{p_k} \tilde{B}) D_k u_{\xi\xi}.
\end{aligned} \tag{17}$$

Letting  $\xi = e_k$ , the  $k$ th unit vector, we obtain from (16),

$$\begin{aligned}
Lu_k &= w^{ii} [D_{ii} u_k - D_{p_l} A_{ii} D_l u_k] - (D_{p_l} \tilde{B}) D_l u_k \\
&= w^{ii} D_k A_{ii} + D_k \tilde{B} + (D_z \tilde{B}) u_k.
\end{aligned} \tag{18}$$

Hence

$$\sum_k D_k u L u_k \geq -C \sum_i w^{ii},$$

where the constant  $C$  depends on  $n, A, B$  and  $\|u\|_{C^1}$ .

To estimate  $Lw_{11}$ , we first calculate  $Lu_{11}$ . Choosing  $\xi = e_1$  in (17), we get

$$\begin{aligned}
Lu_{11} &= w^{ii} [D_{ii} u_{11} - D_{p_k} A_{ii} D_k u_{11}] - (D_{p_k} \tilde{B}) D_k u_{11} \\
&\geq w^{ii} w^{jj} (D_1 w_{ij})^2 + w^{ii} A_{ii,kl} u_{k1} u_{l1} - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\},
\end{aligned}$$

where  $A_{ij,kl} = D_{p_k p_l}^2 A_{ij}$ . For the second term above, by the regularity assumption (4) and noticing that  $\{w_{ij}\}$  is diagonal, we have

$$\begin{aligned} w^{ii} A_{ii,kl} u_{k1} u_{l1} &= w^{ii} A_{ii,kl} (w_{k1} + A_{k1})(w_{l1} + A_{l1}) \\ &= w^{ii} [A_{ii,11} w_{11}^2 + 2A_{ii,1k} w_{11} A_{k1} + A_{ii,kl} A_{k1} A_{l1}] \\ &\geq -C w^{ii} (1 + w_{jj}). \end{aligned}$$

Hence

$$Lu_{11} \geq w^{ii} w^{jj} (D_1 w_{ij})^2 - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\},$$

where again the constant  $C$  depends on  $n$  and  $\|A\|_{C^2}$ . Since  $L$  is linear, from (12) and (18) we then obtain

$$\begin{aligned} Lw_{11} &= Lu_{11} - LA_{11} \\ &\geq Lu_{11} - Cw^{ii}(1 + w_{jj}) \\ &\geq w^{ii} w^{jj} (D_1 w_{ij})^2 - C\{(1 + w_{ii})w^{ii} + (w_{ii})^2\}. \end{aligned} \tag{19}$$

Indeed, the first inequality in (19) is from the following estimate

$$\begin{aligned} LA_{11} &= w^{ij} [D_{ij} A_{11} + D_{ip_k} A_{11} (w_{kj} + A_{kj}) + D_{jp_k} A_{11} (w_{ki} + A_{ki}) + \\ &\quad D_{p_k p_l} A_{11} (w_{ki} + A_{ki})(w_{kj} + A_{kj}) - D_{p_k} A_{ij} (D_k A_{11} + D_{p_l} A_{11} (w_{kl} + A_{kl}))] \\ &\quad - (D_{p_k} \tilde{B})(D_k A_{11} + D_{p_l} A_{11} (w_{kl} + A_{kl})) \\ &\leq C + C(w^{ii} + w^{ii} w_{jj}) + w^{ij} D_{p_k p_l} A_{11} w_{ki} w_{kj} \\ &\leq C\{w^{ii}(1 + w_{jj}) + w_{ii}\} \end{aligned}$$

where the constant  $C$  depends only on  $n$  and  $\|A\|_{C^2}$ , and it is clear that  $\sum_i w_{ii} > C$ , otherwise the proof is finished.

Next we estimate the term  $L\eta$  as follows

$$\begin{aligned} L\eta &= w^{ii} [D_{ii}\eta - (D_{p_k} A_{ii}) D_k \eta] - (D_{p_k} \tilde{B}) D_k \eta \\ &= w^{ii} [D_{ii} u_0 - w_{ii} - A_{ii}(x, Du) - (D_{p_k} A_{ii}(x, Du)) D_k \eta] \\ &\quad - (D_{p_k} \tilde{B}) D_k \eta \\ &\geq -Cn + w^{ii} [A_{ii}(x, Du_0) - A_{ii}(x, Du) - (D_{p_k} A_{ii}(x, Du)) D_k \eta]. \end{aligned} \tag{20}$$

Using the Taylor formula, for some  $\theta \in (0, 1)$ , we have

$$\begin{aligned} &A_{ii}(x, Du_0) - A_{ii}(x, Du) - \sum_k (D_{p_k} A_{ii}(x, Du)) D_k \eta \\ &= \sum_k D_{p_k} A_{ii}(x, \hat{p}) D_k \eta - \sum_k D_{p_k} A_{ii}(x, Du) D_k \eta \\ &= \frac{\theta}{2} \sum_{k,l} A_{ii,kl}(x, \bar{p}) D_k \eta D_l \eta \end{aligned}$$

where  $\hat{p} = (1 - \theta)Du + \theta Du_0$ ,  $\bar{p} = (1 - \bar{\theta})Du + \bar{\theta} Du_0$  and  $\bar{\theta} \in (0, \theta)$ . Thus, we can apply (4) to control the second term in (20),

$$\begin{aligned} \frac{\theta}{2} w^{ii} A_{ii,kl}(x, \bar{p}) D_k \eta D_l \eta &\geq \frac{\theta}{2} w^{ii} A_{ii,ii} (D_i \eta)^2 + \theta w^{ii} A_{ii,ik} D_i \eta D_k \eta \\ &\geq -C w^{ii} (D_i \eta)^2 - C w^{ii} D_i \eta. \end{aligned} \tag{21}$$

And then, we obtain

$$L\eta \geq -Cn - C w^{ii} D_i \eta - C w^{ii} (D_i \eta)^2.$$

Obviously

$$\begin{aligned} \sum_{i,k} \frac{1}{w_{ii}} (D_{ik}u)^2 &= \sum_i \frac{1}{w_{ii}} (w_{ii} + A_{ii})^2 + \sum_{k \neq i} \frac{1}{w_{ii}} A_{ik}^2 \\ &\geq w_{ii} - Cw^{ii}. \end{aligned}$$

Hence from (14), when  $|\eta| \leq |u_0 - u| \leq 1$ , we obtain

$$\begin{aligned} Lv &\geq \frac{\tau C}{\eta} (-n - w^{ii} D_i \eta - w^{ii} (D_i \eta)^2) - \frac{\tau}{\eta^2} \sum_i \frac{(D_i \eta)^2}{w_{ii}} + \frac{1}{w_{11}} \sum_{i,j} \frac{(D_1 w_{ij})^2}{w_{ii} w_{jj}} \\ &\quad - \frac{1}{w_{11}^2} \sum_i \frac{(D_i w_{11})^2}{w_{ii}} - \beta C w^{ii} + \kappa^2 e^{\kappa \varphi} \sum_{i=1}^n \frac{(D_i \varphi)^2}{w_{ii}} + \kappa e^{\kappa \varphi} w^{ii} \\ &\quad + \beta w_{ii} - C \left\{ \frac{1}{w_{11}} [(1 + w_{ii}) w^{ii} + w_{ii}^2] \right\} \tag{22} \\ &\geq -\frac{\tau C n}{\eta} - \frac{\tau C}{\eta^2} \sum_i \frac{(D_i \eta)^2}{w_{ii}} + \frac{1}{w_{11}} \sum_{i,j} \frac{(D_1 w_{ij})^2}{w_{ii} w_{jj}} - \frac{1}{w_{11}^2} \sum_i \frac{(D_i w_{11})^2}{w_{ii}} \\ &\quad + \kappa^2 e^{\kappa \varphi} \sum_{i=1}^n \frac{(D_i \varphi)^2}{w_{ii}} + (\kappa e^{\kappa \varphi} - \beta C) w^{ii} + \beta w_{ii} - C(w_{ii} + w^{ii}), \end{aligned}$$

where in the second step, we have used the inequality  $ab \geq -\varepsilon a^2 - C_\varepsilon b^2$  to get  $\frac{1}{\eta} w^{ii} D_i \eta \geq -\varepsilon w^{ii} - \frac{C_\varepsilon (D_i \eta)^2}{\eta^2 w_{ii}}$  for any small  $\varepsilon > 0$ .

Next we estimate  $\frac{\tau}{\eta^2} \sum_{i=1}^n \frac{(D_i \eta)^2}{w_{ii}}$ . Observe that  $\frac{(D_1 \eta)^2}{w_{11}}$  is under control. It suffices to consider  $\sum_{i=2}^n \frac{(D_i \eta)^2}{w_{ii}}$ . From (13),

$$\begin{aligned} \frac{D_i \eta}{\eta} &= -\frac{1}{\tau} \left( \frac{D_i w_{11}}{w_{11}} + \beta D_k u [w_{ki} - A_{ki}] + \kappa e^{\kappa \varphi} D_i \varphi \right), \\ \left( \frac{D_i \eta}{\eta} \right)^2 &= \frac{1}{\tau^2} \left( \frac{D_i w_{11}}{w_{11}} + \beta D_i u w_{ii} - \beta D_k u A_{ki} + \kappa e^{\kappa \varphi} D_i \varphi \right)^2 \\ &\leq \frac{4}{\tau^2} \left[ \left( \frac{D_i w_{11}}{w_{11}} \right)^2 + \beta^2 (D_i u)^2 w_{ii}^2 + \beta^2 (D_k u A_{ki})^2 + \kappa^2 e^{2\kappa \varphi} (D_i \varphi)^2 \right] \\ &\leq \frac{4}{\tau^2} \left( \frac{D_i w_{11}}{w_{11}} \right)^2 + \frac{C}{\tau^2} (\beta^2 w_{ii}^2 + \beta^2) + \frac{4}{\tau^2} \kappa^2 e^{2\kappa \varphi} (D_i \varphi)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \tau C \sum_{i=2}^n \frac{1}{w_{ii}} \left( \frac{D_i \eta}{\eta} \right)^2 &\leq \frac{4C}{\tau} \sum_{i=2}^n \frac{1}{w_{ii}} \left( \frac{D_i w_{11}}{w_{11}} \right)^2 + \frac{\beta^2 C}{\tau} w_{ii} \\ &\quad + \frac{\beta^2 C}{\tau} w^{ii} + \frac{4C}{\tau} \kappa^2 e^{2\kappa \varphi} \sum_{i=1}^n \frac{(D_i \varphi)^2}{w_{ii}}. \end{aligned}$$

We choose the constant  $\tau$  sufficiently large, say,

$$\tau = \beta^2 + 8C e^{\kappa \sup \varphi}.$$



Then we have the estimate

$$\begin{aligned} \tau C \sum_{i=2}^n \frac{1}{w_{ii}} \left( \frac{D_i \eta}{\eta} \right)^2 &\leq \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} + \frac{1}{2} \kappa^2 e^{\kappa\varphi} \sum_{i=1}^n \frac{(D_i \varphi)^2}{w_{ii}} \\ &\quad + C(w_{ii} + w^{ii}). \end{aligned} \quad (23)$$

Therefore by (22), at  $\bar{x}$

$$\begin{aligned} Lv &\geq -\frac{\tau C n}{\eta} - \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} + \frac{1}{w_{11}} \sum_{i,j} \frac{(D_1 w_{ij})^2}{w_{ii} w_{jj}} \\ &\quad + (\kappa e^{\kappa\varphi} - C\beta) w^{ii} + (\beta - C) w_{ii} - \frac{1}{w_{11}^2} \sum_i \frac{(D_i w_{11})^2}{w_{ii}}. \end{aligned} \quad (24)$$

Using the Pogorelov term  $\frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2$ , we have

$$\begin{aligned} &\frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} \sum_{i=1}^n \frac{(D_i w_{11})^2}{w_{ii}} - \frac{1}{2w_{11}^2} \sum_{i=2}^n \frac{(D_i w_{11})^2}{w_{ii}} \\ &= \frac{1}{w_{11}} w^{ii} w^{jj} (D_1 w_{ij})^2 - \frac{1}{w_{11}^2} w^{11} (D_1 w_{11})^2 - \frac{3}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 \\ &\geq \frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 + \frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} [(D_1 w_{1i})^2 - (D_i w_{11})^2] \\ &\geq -C w^{ii} / w_{11}^2, \end{aligned}$$

where the last inequality follows from  $w_{ij} = u_{ij} - A_{ij}$  and the following estimate

$$\begin{aligned} &\frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_1 w_{1i} - D_i w_{11})(D_1 w_{1i} + D_i w_{11}) \\ &= \frac{2}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i A_{11} - D_1 A_{1i})(2D_i w_{11} + D_i A_{11} - D_1 A_{1i}) \\ &\geq -\frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 - \frac{6}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i A_{11} - D_1 A_{1i})^2 \\ &\geq -\frac{1}{2} \frac{1}{w_{11}^2} \sum_{i=2}^n w^{ii} (D_i w_{11})^2 - C w^{ii} / w_{11}^2. \end{aligned}$$

Therefore from (24) we obtain

$$Lv \geq (\kappa e^{\kappa\varphi} - C\beta) w^{ii} + (\beta - C) w_{ii} - \frac{\tau C n}{\eta}. \quad (25)$$

Choose  $\beta > C + 1$  and  $\kappa$  large enough such that  $\kappa > C\beta$ . We then obtain

$$\eta \sum_i w_{ii}(\bar{x}) \leq C,$$

from which the desired estimate (9) readily follows.

To complete the proof of Theorem 1.1, we need to establish a positive lower bound for the difference  $(u_0 - u)$  in subdomains  $\Omega' \Subset \Omega$ . Let  $x_0 \in \Omega'$  and for  $0 < \rho < d'/2$ ,  $\sigma > 0$ , where  $d' = \text{dist}(\Omega', \partial\Omega)$ , set

$$\psi(x) = -\sigma(\rho^2 - |x - x_0|^2), \quad v_0 = u_0 + \psi.$$

We have  $D_{ij}\psi = 2\sigma\delta_{ij}$ ,  $|D\psi| \leq 2\sigma\rho$  in  $B_\rho(x_0)$ , so the matrix

$$\{D_{ij}v_0 - A_{ij}(x, Dv_0)\} \geq \sigma I \quad (26)$$

is positive definite when the radius  $\rho$  is chosen sufficiently small, in terms of  $\sigma, n$ ,  $\|A\|_{C^2}$  and  $\sup_{\Omega''} |Du_0|$ , where  $\Omega'' = \{\text{dist}(x, \partial\Omega) < d'/2\}$ .

Next we also have,

$$\mathcal{F}[v_0] - B(\cdot, v_0, Dv_0) < \delta, \quad (27)$$

in  $B_\rho(x_0)$  for  $\sigma$  sufficiently small, in terms of  $\delta, n$ ,  $\|A\|_{C^2}$  and  $\|u_0\|_{C^2(\Omega'')}$ , so that by the comparison principle, we obtain  $u_0 - u \geq -\psi$  in  $B_\rho(x_0)$ , and hence,

$$\inf_{\Omega'} (u_0 - u) \geq \sigma\rho^2. \quad (28)$$

Consequently from (28), the proof of Theorem 1.1 is finished.

**Remark 2.** For the Monge-Ampère equation on manifolds, (7) is a natural condition for existence of global smooth solutions, called existence of a geodesic convex function on  $\Omega$  by Hong in [3]. In the next section, we present a selection of examples in Euclidean space where both conditions (4) and (7) are satisfied.

We remark also that (7) with  $\mathcal{U} = \mathcal{U}[u]$  is also necessary for the domain  $\Omega$  to support an elliptic solution  $u$  of an equation of the form (2), when  $A$  is regular. To see this we suppose  $u_0 = 0$ ,  $A(\cdot, 0) = 0$ , as in the optimal transportation case, set  $\varphi = e^{Ku}$  and calculate, taking  $|\xi| = 1$  and  $Du = (D_1u, 0, \dots, 0)$ ,

$$\begin{aligned} & [D_{ij}\varphi - D_{p_k}A_{ij}(\cdot, Du)D_k\varphi]\xi_i\xi_j \\ &= e^{Ku}[K^2(Du \cdot \xi)^2 + K(D_{ij}u - D_{p_k}A_{ij}(\cdot, Du)D_ku)\xi_i\xi_j] \\ &\geq e^{Ku}[K(D_{ij}u - A_{ij}(\cdot, Du))\xi_i\xi_j \\ &\quad + (D_1u)^2(K\xi_1)^2 + KA_{ij,11}(\cdot, \theta Du)(D_1u)^2\xi_i\xi_j], \quad \theta \in (0, 1), \\ &\geq e^{Ku}[K(D_{ij}u - A_{ij}(\cdot, Du))\xi_i\xi_j + (D_1u)^2(K\xi_1)^2 - CK|\xi_1|^2(D_1u)^2], \end{aligned} \quad (29)$$

using Taylor's formula together with the regularity condition (4). Choosing  $K$  sufficiently large, we then obtain (7) from the ellipticity of  $u$ .

**Remark 3.** As earlier remarked the condition that  $u_0$  is a strict supersolution may be relaxed to the non-strict case, that is  $\delta = 0$  in (6). In this case the estimate (9) will continue to hold so the second derivative estimate (8) follows as before from a positive lower bound for the difference  $(u_0 - u)$  in subdomains  $\Omega' \Subset \Omega$ . By refining our previous argument and using the weak Harnack inequality for uniformly elliptic linear operators [2], we can establish such bounds, for example in the cases where  $u_0 \in C^2(\Omega)$  or  $\inf \mathcal{F}[u_0] > 0$  in  $\Omega$ .

**Remark 4.** Under a further structural assumption on the matrix function  $A$ ,

$$|A(x, p)| \leq \mu_0(1 + |p|^2), \quad \text{for all } (x, p) \in \Omega \times \mathbb{R}^n, \quad (30)$$

we can control the gradient of elliptic solutions  $u \in C^2(\Omega)$  of equation (2) in terms of their boundary gradients. This result is analogous to the gradients of convex functions being maximized on the boundary in the standard Monge-Ampère case.

Define the function  $v$  by  $v = |De^{\kappa u}|$  for some  $\kappa > 0$ . Suppose  $v$  attains its maximum at  $x \in \Omega$ . Applying the operator  $D_i u D_i$  to  $v$ , we obtain thus

$$\begin{aligned} 0 &= D_i u D_i |\kappa e^{\kappa u} Du| \\ &= D_i u \left( \kappa^2 e^{\kappa u} D_i u |Du| + \kappa e^{\kappa u} \frac{D_j u D_{ij}^2 u}{|Du|} \right) \\ &= \kappa^2 e^{\kappa u} |Du|^3 + \kappa e^{\kappa u} \frac{D_{ij}^2 u D_i u D_j u}{|Du|}. \end{aligned} \quad (31)$$

Using the ellipticity condition (3), we have

$$0 \geq \kappa^2 |Du|^4 + \kappa A_{ij} D_i u D_j u.$$

By the structural condition (30), we have the bound

$$|A_{ij} D_i u D_j u| \leq C |Du|^2 (1 + |Du|^2).$$

Hence, from (31) one can obtain that

$$0 \geq \kappa^2 |Du|^4 - \kappa C |Du|^2 (1 + |Du|^2). \quad (32)$$

Without loss of generality we assume at the point  $x$ ,  $|Du| \geq 1$ . Choose  $\kappa$  sufficiently large, the quantity on the right hand side of (32) will be positive. The contradiction then gives us the desired estimate,

$$\sup_{\Omega} |Du| \leq C \quad (33)$$

where  $C$  depends on the constant in (30),  $\|u\|_{L^\infty(\Omega)}$  and  $\sup_{\partial\Omega} |Du|$ .

We also remark that in the prescribed Jacobian and optimal transportation cases, the condition (30) is not necessary as the nonvanishing of the Jacobian determinant of the associated mapping automatically enables the gradient to be controlled by its boundary values.

**3. Optimal transportation.** Let  $\Omega$  and  $\Omega^*$  be bounded domains in  $\mathbb{R}^n$  and  $\rho, \rho^*$  be two probability densities in  $\Omega, \Omega^*$  respectively. Let  $c \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$  be a cost function. The optimal transportation problem is to find a measure preserving mapping  $T$  from  $\Omega$  to  $\Omega^*$ , (that is a Borel measurable mapping which pushes forward the measure with density  $\rho$  to that with density  $\rho^*$ ), which maximizes the cost functional,

$$\mathcal{C}(T) = \int_{\Omega} \rho(x) c(x, T(x)) dx,$$

over the set  $\mathcal{T}$  of measure preserving mappings  $T$  from  $\Omega$  to  $\Omega^*$ . Note that, we consider maximization problems rather than minimization to fit the exposition in our previous sections. It is trivial to pass between them replacing  $c$  by  $-c$ .

Through the Kantorovich dual problem, the optimal mapping  $T$  can be determined by a potential function  $u$  as

$$T_u(x) = c_x^{-1}(x, Du(x)). \quad (34)$$

We will assume that for each  $x \in \Omega, p \in \mathbb{R}^n$  there exists a unique  $y$  such that  $c_x(x, y) = p$ , together with the corresponding condition for  $x$  replaced by  $y \in \Omega^*$ , (A1). As well we assume

$$|\det c_{x,y}| \geq c_0 \quad (35)$$

on  $\Omega \times \Omega^*$  for some constant  $c_0 > 0$ , (A2).

Furthermore, when all the data is smooth and  $u \in C^2(\overline{\Omega})$ ,  $u$  is a classical (degenerate elliptic) solution of (2) associated with the boundary condition  $T(\Omega) = \Omega^*$ , where the matrix  $A$  is given by

$$A(x, p) = c_{xx}(x, T(x, p)) \quad (36)$$

and the function  $B$  is given by

$$B(x, u, Du) = |\det c_{x,y}| \frac{\rho(x)}{\rho^*(T(x))}. \quad (37)$$

In the case of optimal transportation, equation (2) arises from the prescription of the Jacobian determinant of the mapping  $T_u$  in (34), namely

$$|\det DT_u| = \frac{\rho(x)}{\rho^*(T(x))},$$

see [13, 18]. Note that when  $\frac{\rho}{\rho^*}$  is bounded from below,  $u$  is an elliptic solution and  $T_u$  maps interior and boundary points of  $\Omega$  to interior and boundary points respectively of  $\Omega^*$ . The boundary condition then becomes a nonlinear oblique boundary condition [16].

It is also well known that the potential  $u$  is  $c$ -convex in  $\Omega$ , namely for each  $x_0 \in \Omega$ , there exists  $y_0 \in \mathbb{R}^n$  such that

$$\varphi_0(x) := u(x_0) + c(x, y_0) - c(x_0, y_0) \leq u(x) \quad (38)$$

for all  $x \in \Omega$ .  $\varphi_0$  is called the  $c$ -support of  $u$  at  $x_0$ . When the equality holds only at  $x = x_0$ ,  $u$  is also called strictly  $c$ -convex at  $x_0$ . We say  $u$  is strictly  $c$ -convex in  $\Omega$  if it is strictly  $c$ -convex at all  $x \in \Omega$ .

Another important fact is that the matrix

$$\{D_{ij}\varphi_0 - A_{ij}(x, D\varphi_0)\} \equiv 0,$$

as the mapping determined by  $\varphi_0$  maps the whole domain to a fixed point. Hence, one can see that  $\varphi_0$  is indeed a degenerate elliptic solution of the homogeneous equation (2),  $\mathcal{F}[\varphi_0] = 0$ .

We can now state the main result in this section,

**Theorem 3.1.** *Suppose  $\rho \in C^2(\Omega)$ ,  $\rho^* \in C^2(\Omega^*) \cap C^{1,1}(\overline{\Omega}^*)$ ,  $\rho, \rho^*$  have positive upper and lower bounds, and  $A$  in (36) is regular in  $\Omega \times \mathbb{R}^n$ . Then, if the domain  $\Omega^*$  is  $c^*$ -convex with respect to  $\Omega$ , the strictly  $c$ -convex potential function  $u$  is  $C^3$  smooth in  $\Omega$ .*

**Remark 5.** When  $A$  is strictly regular, Theorem 3.1 was proved in [8]. The  $c^*$ -convexity of  $\Omega^*$  is necessary and also permits the removal of the dependence on  $Du$  in the constant in (8) as we have  $T(\Omega) \subset \overline{\Omega}^*$ , see [17]. Moreover it was shown by approximation in [8] that if  $\Omega^*$  is not  $c^*$ -convex with respect to  $\Omega$ , there exist smooth, positive mass distributions such that the potential function is not  $C^1$  smooth. On the other hand, Loeper [7] proved that the regularity of  $A$  is also necessary for the smoothness of potential functions.

*Proof.* With the *a priori* estimates established in Section 2, we need only to show that the potential function can be locally approximated by smooth ones. For this purpose, we adopt the method in [8] to show that  $u$  is smooth in any sufficiently small ball  $B_r \Subset \Omega$ .

Consider the approximating Dirichlet problems

$$\begin{aligned} \det \{D^2w - A(x, Dw)\} &= |\det c_{x,y}| \frac{\rho}{\rho^*} \quad \text{in } B_r, \\ w &= u_m \quad \text{on } \partial B_r, \end{aligned} \quad (39)$$

where  $\{u_m\}$  is a sequence of smooth functions converging uniformly to  $u$ . The existence of smooth solutions  $w = w_m$  of (39) follows from the same argument as in [8]. Namely, by the semi-convexity of  $u$ , there exists a constant  $C$  such that the function  $\tilde{u} = u + C|x|^2$  is convex, it follows that  $u$  is locally Lipschitz continuous in  $\Omega$  and twice differentiable almost everywhere. For  $h > 0$  sufficiently small, we let

$$\tilde{u}_h(x) = h^{-n} \int_{\Omega} \rho \left( \frac{x-y}{h} \right) \tilde{u}(y) dy, \quad (40)$$

denote the mollification of  $\tilde{u}$ , where  $\rho \in C^\infty(\mathbb{R}^n)$  is a symmetric mollifier satisfying  $\rho \geq 0$ ,  $\int \rho = 1$  and  $\text{supp} \rho \subset B_1(0)$ . Setting

$$u_m = \tilde{u}_{h_m} - C|x|^2, \quad (41)$$

where  $h_m \rightarrow 0$ , we then have  $u_m \in C^\infty(\Omega)$  and  $u_m \rightarrow u$  uniformly on compact subsets of  $\Omega$ . Moreover, for sufficiently large  $k$  and small  $r$ , the functions

$$v = v_m = u_m + k(|x|^2 - r^2) \quad (42)$$

will be sub barriers for (39), namely  $D^2v > A(x, Dv)$  and

$$\begin{aligned} \det \{D^2v - A(x, Dv)\} &\geq B(x, v, Dv) \quad \text{in } B_r, \\ v &= u_m \quad \text{on } \partial B_r. \end{aligned}$$

Consequently by the classical comparison principle [2], we have

$$w_m \geq v_m \quad \text{in } B_r,$$

and by the monotonicity lemma [8],

$$T_{w_m}(B_r) \subset T_{v_m}(\overline{B_r}).$$

The preceding arguments yield *a priori* bounds for solutions and their gradients of the Dirichlet problem (39). To conclude the existence of globally smooth solutions  $w_m$  by the method of continuity [2], we need global second derivative bounds. It was proved in [16] that *a priori* bounds for second derivatives are reduced to their boundary estimates, (see Theorem 3.1 in [16]). The latter can be established by the method in [1, 19], or more simply using the method introduced in [11, 12] for the Monge-Ampère equation. The key observation again is that functions of the form  $k(|x|^2 - r^2)$  provide appropriate barriers for large  $k$  and small  $r$ .

For a  $c$ -convex function  $w$  and a positive constant  $h > 0$ , we denote

$$S_{h,w}^0(x_0) := \{x \in \Omega : w(x) - \varphi_0(x) < h\}, \quad x_0 \in \Omega, \quad (43)$$

the sub-level set of  $w$ , where  $\varphi_0$  is the  $c$ -support at  $x_0$  defined in (38). Since  $u$  is strictly  $c$ -convex, there exists a small constant  $h > 0$  such that for  $m$  large enough  $S_{t,w_m}^0(x_0) \subset B_r(x_0)$  for all  $t \leq h$  and  $x_0 \in \Omega$ . In particular,  $\text{diam}(S_{t,w_m}^0(x_0)) \rightarrow 0$  as  $t \rightarrow 0$  for all  $x_0 \in \Omega$ .

In order to apply Theorem 1.1 to  $w_m$  in  $S_{h,m}^0 := S_{h,w_m}^0(x_0)$ , we need to verify the conditions in Theorem 1.1.

First, set  $w_0 = \varphi_0 + h$ . It follows that  $w_0$  is a degenerate elliptic supersolution of (2),  $0 < w_0 - w_m \leq h$  in  $S_{h,m}^0$  and  $w_m = w_0$  on  $\partial S_{h,m}^0$ . Obviously from the smoothness of the cost function  $c$ , we have  $w_0 \in C^2(\Omega)$ .

Next, we verify (7), the  $A$ -boundedness of  $S_{h,m}^0$ . Differentiate (36), one has

$$D_{p_k} A_{ij}(x, p) = c^{k,l} c_{ij,l}(x, T(x, p))$$

where  $\{c^{i,j}\} = \{c_{i,j}\}^{-1}$ . From the boundedness of  $\Omega^*$ , we have all the terms  $D_{p_k} A_{ij}$  are bounded in  $S_{h,m}^0$ . Hence, (7) is satisfied by letting  $\varphi(x) = |x|^2$  when  $\text{diam}(S_{h,m}^0)$  is small enough, (see [6] as well).

Therefore, for any subset  $U \Subset S_{h,m}^0(x_0)$ , applying Theorem 1.1 we obtain that  $\sup_U |D^2 w_m| \leq C$  where  $C$  is independent of  $m$ . Using a standard covering argument, one has  $\sup_{U'} |D^2 w_m| \leq C$  for all  $U' \Subset B_r(x_0)$ . Further regularity follows from the theory of elliptic equations in [2], we obtain the following a priori estimate

$$\|w_m\|_{C^3(U')} \leq C, \quad U' \Subset B_r(x_0),$$

where the constant  $C$  is independent of  $m$ . Note that the dependence on  $\sup_\Omega |u|$  is removed because  $B$  is independent of  $u$  as in (37).

We finally infer the existence of locally smooth elliptic solution  $w$  of the Dirichlet problem (39) with  $w = u$  on  $\partial B_r$ . From the comparison principle in [8], and since the regularity condition (4) implies that  $u$  is also a generalized solution in subdomains of  $B_r$ , ([4, 17]), we also have that  $w = u$  in  $B_r$ . Thus  $u \in C^3(U')$  in any compact subset  $U' \Subset B_r(x_0)$ . Since  $x_0$  was arbitrary, we conclude that  $u \in C^3(\Omega)$ . The proof of Theorem 3.1 is then finished.  $\square$

**Remark 6.** Note that if the potential  $u \in C^1(\Omega)$  then we need only assume  $\rho^* \in C^2(\Omega)$  in the hypothesis of Theorem 3.1. In this case the second derivative interior estimate is already done in [6].

**Remark 7.** Theorem 3.1 also applies to generalized solutions of the optimal transportation equation, without boundary conditions. The concept of generalized solution, introduced in [8], extends that of Aleksandrov and Bakelman for the standard Monge-Ampère equation, namely we call a  $c$ -convex function  $u$  a generalized solution of equation (2), with (36), (37), in  $\Omega$ , if

$$\int_E \rho = \int_{T_u(E)} \rho^*,$$

for all Borel set  $E \subset \Omega$ , where  $T_u$  now denotes the  $c$ -normal mapping of  $u$ .

By the same argument as above we conclude that a strictly  $c$ -convex generalized solution of (2) in a domain  $\Omega$  is  $C^3$  smooth if  $\rho > 0$ ,  $\rho^* > 0$ ,  $\rho \in C^2(\Omega)$ ,  $\rho^* \in C^2(\mathbb{R}^n)$ , and  $A$  in (36) is regular in  $\Omega \times \mathbb{R}^n$ . Note that it is shown in [8] that a potential  $u$  is a generalized solution under conditions (A1) and (A2). We remark also that a generalized solution is a viscosity solution if and only if  $A$  is regular, [5, 18]. In a future work we treat the extension of the regularity result to more general forms of (2).

*Examples:* We conclude with three examples of regular cost functions satisfying the global barrier condition (7).

**Example 1,** ([8, 16]).

Let  $\mathcal{M}_f, \mathcal{M}_g$  be two graphs given by  $X = (x, f(x)), Y = (y, g(y))$  where  $f, g \in C^2(\mathbb{R}^n)$ . Then the cost function

$$\tilde{c}(x, y) := |X - Y|^2$$

is equivalent to the cost function

$$c(x, y) = -(x \cdot y + f(x)g(y)).$$

By direct computation [8], we obtain

$$\begin{aligned} c_{i,j} &= -\delta_{ij} - f_i g_j, \\ \det c_{i,j} &= (-1)^n (1 + \nabla f \cdot \nabla g) \neq 0, \quad \text{if } \nabla f \cdot \nabla g \neq -1, \\ c^{i,j} &= -\left(\delta_{ij} - \frac{f_i g_j}{1 + \nabla f \cdot \nabla g}\right), \\ c_{i,j,k} &= -f_{ij} g_k, \quad c_{i,jk} = -f_i g_{jk}, \\ D_{p_k} A_{ij} &= \sum_m c^{m,k} c_{ij,m} = \sum_m \left(\delta_{m,k} - \frac{f_m g_k}{1 + \nabla f \cdot \nabla g}\right) f_{ij} g_m \\ &= f_{ij} \left(g_k - \frac{\nabla f \cdot \nabla g g_k}{1 + \nabla f \cdot \nabla g}\right) = f_{ij} g_k \frac{1}{1 + \nabla f \cdot \nabla g}. \end{aligned}$$

By choosing  $\varphi = f$  in (7), we then have

$$\begin{aligned} D_{ij} \varphi - D_{p_k} A_{ij} D_k \varphi &= f_{ij} - f_{ij} \frac{\nabla f \cdot \nabla g}{1 + \nabla f \cdot \nabla g} \\ &= \frac{f_{ij}}{1 + \nabla f \cdot \nabla g}. \end{aligned}$$

Consequently if  $f$  is locally uniformly convex (or concave) satisfying  $\nabla f \cdot \nabla g > -1$  (or  $< -1$ ), respectively then (7) is satisfied with  $\varphi = f$ , the defining function. If only  $f$  is convex then we can choose  $\varphi = f + \epsilon h$  for any locally uniformly convex  $h$  and  $0 < \epsilon, \epsilon \sup \nabla g \cdot \nabla h < 1$ .

Moreover, as shown in [8], if  $f$  and  $g$  are both convex (or concave) with  $\nabla f \cdot \nabla g > -1$ , then  $c$  is regular and strictly regular when both locally uniformly convex (or concave). Examples of such defining functions include  $\sqrt{1 + |x|^2}$ ,  $\epsilon|x|^2$  for sufficiently small  $\epsilon > 0$ , and many others.

**Example 2** (Power costs), [16].

$$c(x, y) = \frac{1}{m} |x - y|^m, \quad m \neq 0; \quad \log |x - y|, \quad m = 0.$$

For  $m < 1$  and  $x \neq y$ , we have

$$\begin{aligned} Y(x, p) &= x - |p|^{\frac{2-m}{m-1}} p, \\ A(x, p) &= A(p) = |p|^{\frac{m-2}{m-1}} I + (m-2) |p|^{-\frac{m}{m-1}} p \otimes p. \end{aligned} \tag{44}$$

The matrix  $A$  is regular for  $-2 \leq m < 1$  and strictly regular for  $-2 < m < 1$ .

By direct computation, we obtain

$$\begin{aligned} D_{p_k} A_{ij} &= \frac{m-2}{m-1} |p|^{-\frac{m}{m-1}} p_k \delta_{ij} - \frac{m(m-2)}{m-1} |p|^{\frac{2-3m}{m-1}} p_k p_i p_j \\ &\quad + (m-2) |p|^{-\frac{m}{m-1}} (p_i \delta_{jk} + p_j \delta_{ik}), \end{aligned}$$

and thus

$$|D_{p_k} A_{ij}| \leq C_m |p|^{\frac{1}{1-m}}, \quad \forall i, j, k.$$

Assume  $\Omega, \Omega^*$  are disjoint and  $d := \text{dist}(\Omega, \Omega^*)$ . From (44),

$$|p| \leq d^{m-1}, \quad (m < 1),$$

hence,

$$|D_{p_k} A_{ij}| \leq C_m d^{-1}, \quad \forall i, j, k.$$

By choosing  $\varphi = |x|^2$  in (7), we then have

$$D_{ij}\varphi - D_{p_k} A_{ij} D_k\varphi \geq 2\delta_{ij} - C_m d^{-1} \text{diam}(\Omega), \quad \forall i, j.$$

Therefore, when  $d$  is sufficiently large such that  $C_m d^{-1} \text{diam}(\Omega) \ll 1/2$ , the barrier condition (7) is then satisfied.

**Example 3** (Non-isotropic power costs).

$$c(x, y) = - \sum_{i=1}^n \frac{1}{m_i} |x_i - y_i|^{m_i}, \quad m_i \geq 2, \quad \forall i.$$

By direct calculations, we have

$$y_i = x_i + |p_i|^{\frac{2-m_i}{m_i-1}} p_i, \quad \forall i, \tag{45}$$

$$A_{ij}(x, p) = A_{ij}(p) = \delta_{ij} (1 - m_i) |p_i|^{\frac{m_i-2}{m_i-1}}, \quad \forall i, j,$$

for  $p_i > 0$  or  $p_i < 0$  that is  $y_i > x_i$  or  $y_i < x_i$ . The matrix  $A$  is regular, i.e. satisfies (4). Indeed, for any  $\xi, \eta \in \mathbb{R}^n$ , we have

$$D_{p_k p_l}^2 A_{ij} \xi_i \xi_j \eta_k \eta_l = \sum_{i=1}^n \frac{m_i - 2}{m_i - 1} |p_i|^{-\frac{m_i}{m_i-1}} \xi_i^2 \eta_i^2 \geq 0.$$

Without loss of generality, we can assume that  $y_i > x_i$  for all  $1 \leq i \leq n$ ,  $x \in \Omega, y \in \Omega^*$  and there exists  $R > 0$  such that  $\Omega \subset B_R(0)$ .

Set  $\varphi = \frac{1}{2} \sum_{i=1}^n (x_i + R)^2$ . By (45) we have

$$\begin{aligned} D_{ij}\varphi - D_{p_k} A_{ij} D_k\varphi &= \delta_{ij} + \delta_{ij} (m_i - 2) |p_i|^{-\frac{m_i}{m_i-1}} p_i D_i\varphi \\ &= \delta_{ij} + \delta_{ij} (m_i - 2) \frac{(y_i - x_i)}{|x_i - y_i|^2} (x_i + R) \\ &\geq \delta_{ij}, \quad (m_i \geq 2, \quad \forall i.) \end{aligned}$$

Therefore, the barrier condition (7) is satisfied. Note that we can also verify (4) and (7) for other values of the exponents  $m_i$ . Indeed if we take more generally

$$c(x, y) = - \sum_{i=1}^n \frac{c_i}{m_i} |x_i - y_i|^{m_i},$$

substituting  $\log|x_i - y_i|$  for  $\frac{1}{m_i}|x_i - y_i|^{m_i}$  when  $m_i = 0$ , then (4) and (7) are satisfied for  $y_i > x_i$  and constants  $c_i > 0$ , when  $m_i \geq 2$  or  $m_i < 1$ , and  $c_i < 0$ , when  $1 < m_i \leq 2$ .

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