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Abstract

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REGULARITY IN OPTIMAL TRANSPORTATION

JIAKUN LIU

(joint work with Neil S. Trudinger and Xu-Jia Wang)

1. INTRODUCTION

In this talk, we give some estimates for solutions to the Monge-Ampère equation arising in optimal transportation. The Monge-Ampère equation under consideration has the following type

$$(1.1) \quad \det\{D^2u(x) - A(x, Du)\} = f(x) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $A = \{A_{ij}\}$ is an $n \times n$ symmetric matrix defined in $\Omega \times \mathbb{R}^n$.

In optimal transportation, u is the potential function, the matrix A and the right hand side f are given by

$$(1.2) \quad A(x, Du) = D_x^2 c(x, T_u(x)),$$

$$(1.3) \quad f = |\det\{D_{xy}^2 c\}| \frac{\rho}{\rho^* \circ T_u},$$

where $c(\cdot, \cdot)$ is the cost function, $T_u : x \rightarrow y$ is the optimal mapping determined by $Du(x) = D_x c(x, y)$, and ρ, ρ^* are mass distributions respectively in the initial domain Ω and the target domain Ω^* .

We assume that the cost function $c \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the following conditions:

(A1) For any $x, p \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $D_x c(x, y) = p$; and for any $y, q \in \mathbb{R}^n$, there is a unique $x \in \mathbb{R}^n$ such that $D_y c(x, y) = q$.

(A2) For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\det\{D_{xy}^2 c(x, y)\} \neq 0$,

(A3) For any $x, p \in \mathbb{R}^n$, and any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$(1.4) \quad A_{ij,kl}(x, p) \xi_i \xi_j \eta_k \eta_l > c_0 |\xi|^2 |\eta|^2,$$

where $A_{ij,kl} = D_{p_k p_l}^2 A_{ij}$ and A is given by (1.2).

Under above conditions on the cost function, the optimal mapping is uniquely determined by the corresponding potential function. Therefore, it suffices to study the regularity of potential functions, i.e. regularity of elliptic solutions of (1.1).

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The materials presented here will partly appear in my PhD thesis.

2. REGULARITY RESULTS

In the special case when the cost function is the Euclidean distance squared, the regularity of potential functions has been obtained by Caffarelli, Urbas and many other mathematicians. Our goal is to establish the corresponding regularity results for general cost functions satisfying conditions (A1)–(A3), assuming the mass distributions are merely measurable or Hölder continuous, [1, 2, 3].

The first one is the $C^{1,\alpha}$ regularity for potentials, [1]. The similar result was previously obtained by Loeper. We give a completely different proof and our exponent is optimal when the inhomogeneous term $f \in L^\infty$.

Theorem 2.1. *Let u be a potential function to the optimal transportation problem. Assume the cost function c satisfies conditions A1, A2, A3, Ω^* is c -convex with respect to Ω , and $f \geq 0$, $f \in L^p(\Omega)$ for some $p \in (\frac{n+1}{2}, +\infty]$. Then $u \in C^{1,\alpha}(\bar{\Omega})$, where $\alpha = \frac{\beta(n+1)}{2n^2+\beta(n-1)}$ and $\beta = 1 - \frac{n+1}{2p}$.*

Especially when $p = \infty$, our Hölder exponent $\alpha = \frac{1}{2n-1}$ is optimal.

The second result is the Hölder and more general continuity estimates for second derivatives, when the inhomogeneous term is Hölder and Dini continuous, together with corresponding regularity results for potentials, [2].

Theorem 2.2. *Assume the cost function c satisfies (A1)–(A3) and f satisfies $C_1 \leq f \leq C_2$ for some positive constants $C_1, C_2 > 0$. Let $u \in C^2(\Omega)$ be an elliptic solution of (1.1). Then for all $x, y \in \Omega_\delta$, we have the estimate*

$$(2.1) \quad |D^2u(x) - D^2u(y)| \leq C \left[d + \int_0^d \frac{\omega_f(r)}{r} + d \int_d^1 \frac{\omega_f(r)}{r^2} \right],$$

where $d = |x - y|$, $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$, $C > 0$ depends only on $n, \delta, C_1, C_2, A, \sup|Du|$, and the modulus of continuity of Du . It follows that:

- (i) *If f is Dini continuous, then the modulus of continuity of D^2u can be estimated by (2.1) above;*
- (ii) *If $f \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, then*

$$(2.2) \quad \|u\|_{C^{2,\alpha}(\Omega_\delta)} \leq C \left[1 + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)} \right];$$

- (iii) *If $f \in C^{0,1}(\Omega)$, then*

$$(2.3) \quad |D^2u(x) - D^2u(y)| \leq Cd[1 + \|f\|_{C^{0,1}} |\log d|] \quad \forall x, y \in \Omega_\delta.$$

From the estimates in Theorem 2.2, we conclude the corresponding regularity results for potentials, which are semi-convex, almost everywhere elliptic solutions of equation (1.1).

Corollary 2.1. *Assume the cost function c satisfies (A1)–(A3) and ρ, ρ^* are Dini continuous, and uniformly bounded and positive, in Ω, Ω^* respectively. Then if the target domain Ω^* is c^* -convex, with respect to Ω , any potential function $u \in C^2(\Omega)$ and is an elliptic solution of (1.1), satisfying the estimates (2.1), (2.2) and (2.3),*

with f given by (1.3), where C depends on $n, \delta, C_1, C_2, \Omega, \Omega^*$ and c . Consequently if the densities ρ, ρ^* are Hölder continuous and Ω, Ω^* are c, c^* -convex with respect to each other, then the optimal mapping T_u is a $C^{1,\alpha}$ diffeomorphism from Ω to Ω^* for some $\alpha > 0$.

The last result is the following $W^{2,p}$ estimate, which we have been working on most currently, [3].

Theorem 2.3. *Assume the cost function c satisfies (A1)–(A3). Let u be a (elliptic) weak solution of (1.1) on the domain $\Omega := S_{h,u}^0(x_0)$ defined by a sub-level set, and $u = \varphi + h$ on $\partial\Omega$, where φ is the c -support of u at x_0 (see the definitions in [3]).*

Then if f is continuous and pinched by two positive constants $C_1, C_2 > 0$ such that $C_1 \leq f \leq C_2$. We have $D^2u \in L^p(\Omega_{1/4})$ for any $1 \leq p < \infty$ and

$$\|u\|_{W^{2,p}(\Omega_{1/4})} \leq C(p, \sigma)$$

with σ the modulus of continuity of f and $\Omega_{1/4}$ is the $1/4$ -dilation of Ω with respect to its center of mass.

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