Marching schemes for inverse scattering problems in waveguides with curved boundaries

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Marching schemes for inverse scattering problems in waveguides with curved boundaries

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Abstract

A marching scheme is developed for inverse scattering problems of the Helmholtz equation in waveguides with curved boundaries. We implement a local orthogonal transform to transform the irregular waveguide in physical plane into a regular rectangle in computing plane. Then the modified Helmholtz system in computational domain is piecewise solved through a second order numerical marching scheme, and we propose a spectral projector based on the truncated local propagating eigenfunction expansion to regularize the marching scheme. In the end, the marching scheme is verified by extensive numerical experiments, and it is shown that the scheme is efficient, stable and accurate in rapidly varying waveguides with curved boundaries, even when there are a variable number of propagating modes in the main propagation direction.

Keywords: Cauchy problem; Propagating mode; Marching method; Waveguide; Local orthogonal transform

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1 Introduction

Large-scale wave propagation problems widely exist in many scientific areas, e.g. acoustics, electro-magnetism, seismic migration and other applications, where we often need to solve the Helmholtz equation in a very large scale range-dependent waveguide with curved boundaries or interfaces\cite{1, 2, 3, 4}. For these large-scale problems with curved boundaries, direct methods are very expensive for they result in very large indefinite linear systems. In contrast to this, marching methods are usually more acceptable in the sense of efficiency and storage space.

To marching compute these problems, we need to flatten the waveguides with some mathematical treatments in the first. The ‘staircase’ approximation has once been popularly adopted. But it often leads to marching computing in a very small range step size, and the marching computing is caught in a huge amount of computation trap. Alternatively, the local orthogonal transform \cite{5, 6} is often more feasible than the staircase approximation. For waveguides with internal interfaces, Zhu and Li developed an analytical local orthogonal coordinate transform and derived a modified Helmholtz equation \cite{7, 8}. Then, a numerical local orthogonal transform method (NLOTM) was also proposed to remove the divisible condition required by the analytical local orthogonal transform \cite{9}.

Then, some marching schemes can be constructed on the modified Helmholtz equation in computational plane after implementing a local orthogonal transform.

Generally, marching methods can be categorized into two categories: the first-order methods and the second-order methods. Different with first-order methods \cite{10, 11, 12, 13}, where first-order approximations are implemented to approximate the Helmholtz equation, second-order methods stick rather with the Helmholtz equation and can deal with backscattering. The operator marching method (OMM)\cite{5, 14, 15} is an efficient second-order method in slowly varying waveguides. However, as shown in the later part of the work, it is not suitable for solving wave propagation in complex waveguides with variable number of propagating modes.

The marching methods in \cite{16, 17, 18} are also second-order methods. But unlike the OMM, they are only restricted to inverse scattering problems or Cauchy problems \cite{15, 19, 20} in conjunction with inverse problems in waveguides. Natterer and Wübbeling utilize the fast Fourier transform to filter the marching solution with a carefully determined bandwidth \cite{17}. While in \cite{18}, an algorithm for downward extrapolation is presented to suppress only the evanescent waves with a spectral projector. For convenience, we call the marching method with the fast Fourier transform.
“the Fourier marching method” and the marching method with spectral projector “the spectral projector marching method”. Both the two marching methods are not restricted to small propagation angles, and can deal easily with backscattering.

Inverse scattering problems or Cauchy problems of the Helmholtz equation are highly ill posed, which leads to great illnesses arising in every marching step of marching methods for Cauchy problems. Therefore, some mathematical treatments have to be implemented for obtaining a physically meaningful solution. As shown by Natterer and Wübbeling [17], the stabilization of the Fourier marching method can be achieved simply by suppressing the evanescent waves, and the corresponding low-pass filtered solution will be very close to the true solution, provided that the parameter to cut off the frequency is correctly chosen. However, they also point out the error estimate needs to be formulated much tighter under appropriate conditions, since the error bound is exponentially grown and only a loss in bandwidth can make the exponent not too big. In fact, a similar method has been used to compute Cauchy problems of the Maxwell equations by Vöegeler in [21] in 2003, where the stability is maintained through restricting the solutions to spatial frequencies slightly lower than a cut off frequency.

Sandberg and Beylkin [18] more clearly demonstrate the causes for the instability of Cauchy problems. They attribute the instability of an elliptic equation to the unwanted amplification of evanescent waves, and their marching method projects the marching solution into a subspace composed only by propagating eigenfunctions. For practical use of the method, a simple matrix polynomial recursion is then used to accelerate the computing of the spectral projector, which avoids the expensive construction of eigensystems[22]. In addition, for three-dimensional problems, [23] also points out that a reasonable speed for computing spectral projectors can be obtained through using PLR (Partitioned Low Rank) representation of matrices.

However, it should be noticed that these work only deals with Cauchy problems in regular waveguides. In practical applications, more waveguides are irregular. To this end, this work mainly concentrate on general Cauchy problems in large scale complex waveguides with curved boundaries or interfaces. It is our purpose to develop an efficient and stable marching method for Cauchy problems in such irregular waveguides. The strategy is planed as follows: we first implement a local orthogonal transform to transform the irregular waveguides into a regular domain; then we build our marching scheme on the modified Helmholtz system in the computational plane. We also attempt to provide a general theoretical proof for basic principles applied to marching methods associated with Cauchy problems of the Helmholtz
equation in irregular waveguides.

The paper is arranged as follows. The basic mathematical formulations are described in Section 2. In Section 3, we derive the stability and accuracy condition for Cauchy problems of the modified Helmholtz equation, and propose the marching scheme for waveguides with curved boundaries. Section 4 presents some numerical results in various irregular waveguides. We conclude our work with some discussions in Section 5.

2 Mathematical formulation

In this section, we first present the basic mathematical formulation of the Cauchy problem in waveguides with curved boundaries. Then a local orthogonal transform is implemented to flatten the curved boundaries, and the modified Helmholtz equation is obtained correspondingly. In the end, we introduce the transverse operator of the modified Helmholtz equation and its characteristic problem on the transformed computational domain.

2.1 The Cauchy problem

The initial value problem or Cauchy problem of the Helmholtz equation in $\mathbb{R}^n$ takes the following form

$$\begin{align*}
\triangle u + \kappa^2 u &= 0 \text{ in } x_n > 0 \\
u &= f, \quad \frac{\partial u}{\partial n} = g \text{ on } x_n = 0 \\
u|_{x_1=0} &= 0, \quad \frac{\partial u}{\partial n}|_{x_i=h_i} = 0 \\
(i = 1, 2, \ldots, n-1)
\end{align*}$$

(1)

where $x_n$ plays the role of the parameter $t$, $n$ is the normal vector of the boundary $x_i = h_i(x_1, x_{i-1}, x_{i+1}, \ldots, x_n)$, which is the $n-1$-dimensional surfaces in $x_i$ direction.

For simplicity, we consider the following two-dimensional Helmholtz equation

$$\begin{align*}
u_{xx} + \nu_{zz} + \kappa^2(x, z)u &= 0 \\
u &= f, \quad \frac{\partial u}{\partial n} = g \text{ on } x = 0 \\
u|_{z=0} &= 0, \quad \frac{\partial u}{\partial n}|_{z=h(x)} = 0
\end{align*}$$

(2)
where $x$ plays the role of the time variable $t$, $f$ and $g$ are initial conditions for Dirichlet and Neumann respectively.

\[
\begin{align*}
  u(0, z) &= f(z), \\
  u_x(0, z) &= g(z), \\
  \kappa_0(z) &= \\kappa(x, z) \\
  u(L, z) &= ?, \\
  \kappa_\infty(z) &= 
\end{align*}
\]

Fig. 1. The Cauchy problem sketch for a waveguide with curved bottom.

We assume that the problem is range independent (i.e. wavenumber, interfaces and boundaries are independent of $x$-direction) for $x \leq 0$ and $x \geq L$.

Suppose no wave come from $+\infty$, the exact boundary condition (radiation condition) at $x = L$ is $u_x = i\sqrt{\partial_z^2 + \kappa_\infty^2} u$, where $i = \sqrt{-1}$ and the square root operator is defined in [24].

We concentrate on solving the equation for $0 \leq x \leq L$ since the Helmholtz equation can be easily solved by separable variable method for $x \leq 0$ or $x \geq L$.

For a forward problem (with the inverse scattering or Cauchy problem as its inverse problem), the boundary condition (BC) on $x$-direction is generally imposed as

\[
\begin{align*}
  u(0, z) &= f(z), \\
  u_x(L, z) &= i\sqrt{\partial_z^2 + \kappa_\infty^2} u. \\
\end{align*}
\]  
\tag{3}

While, for the Cauchy problem, both the initial Drichlet and Neumann conditions need to be given at $x = 0$ simultaneously (2). To provide a proper $g(z)$ for the comparison with the OMM solution, we may utilize the OMM and BC(3) to reconstruct the exact DtN map

\[
Q_0 : \left. \frac{\partial u}{\partial x} \right|_{x=0} = Q_0 u,
\]  
\tag{4}

and let $g(z) = Q_0 u(x, z)$. In practical problems, the $g(z)$ can be measured by some measuring instruments.

Then we have the following exact boundary conditions for the Cauchy problem

\[
\begin{align*}
  u(0, z) &= f(z), \\
  u_x(0, z) &= g(z), \\
  u(x, 0) &= 0, \\
  \left. \frac{\partial u}{\partial n} \right|_{z=h(x)} &= 0. \\
\end{align*}
\]  
\tag{5}
where $f, g \in C^1[0, 1]$ should satisfy corresponding compatibility conditions.

## 2.2 The local orthogonal transform

We flatten the curved boundaries by the local orthogonal transform [25]. For simplicity, we only discuss the numerical marching scheme for the Cauchy problem in waveguides with one curved bottom, which are typical in ocean acoustics.

Suppose the local orthogonal transform be represented by

\[
\begin{align*}
\hat{x} &= F(x, z), \\
\hat{z} &= G(x, z),
\end{align*}
\]

with the function $F$ being determined according to the orthogonal condition

\[
\frac{\partial F}{\partial x} \cdot \frac{\partial G}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial G}{\partial z} = 0.
\]

The local orthogonal transforms transform Eq.(2) in physical plane into the modified Helmholtz equation

\[
V_{\hat{x}\hat{x}} + \alpha(\hat{x}, \hat{z})V_{\hat{z}\hat{z}} + \beta(\hat{x}, \hat{z})V_{\hat{z}} + \gamma(\hat{x}, \hat{z})V = 0
\]

\[
V = \tilde{f}, \quad \frac{\partial V}{\partial n} = \tilde{g} \text{ on } \hat{x} = 0
\]

\[
V|_{\hat{z}=0} = 0, \quad V_{\hat{z}} + \frac{2h'(x)^2 - (h(x) - 1)h''(x)}{2[1 + h'(x)^2]}V|_{\hat{z}=1} = 0
\]

We demand the coefficient of the $V_{\hat{z}}$ be zero in order to remove $V_{\hat{z}}$. Then $W(x, z)$ is governed by $2W_{\hat{z}}F_z + WF_{zz} + 2W_xF_x + WF_{xx} = 0$ [6]. Coefficients of (6) are obtained as follows:

\[
\begin{align*}
\alpha(\hat{x}, \hat{z}) &= \frac{G_z^2 + G_x^2}{F_z^2 + F_x^2}, \\
\beta(\hat{x}, \hat{z}) &= \frac{2W_zG_z + WG_{zz} + 2W_xG_x + WG_{xx}}{W(F_z^2 + F_x^2)}, \\
\gamma(\hat{x}, \hat{z}) &= \frac{W_{xx} + WG_{zz} + \kappa^2 W}{W(F_z^2 + F_x^2)}.
\end{align*}
\]

The coefficients for one layered medium can be found in [25]. As for stratified waveguides with curved interfaces or boundaries, we can perform the analytic local orthogonal transform[6, 7, 8, 26] or the numerical transform[9] to flatten the stratified computing domain with curved boundaries or interfaces.
2.3 The transverse operator and its characteristic problem

The transverse operator of the modified Helmholtz equation (6) is
\[ D(\hat{x}) = \alpha(\hat{x}, \hat{z}) \frac{\partial^2}{\partial \hat{z}^2} + \beta(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}} + \gamma(\hat{x}, \hat{z}) \]
where \( \hat{x} \) is fixed.

The characteristic problem of \( D(\hat{x}) \) is defined as
\[ D(\hat{x}) \phi(\hat{x}, \hat{z}) = \lambda \phi(\hat{x}, \hat{z}), \quad 0 < \hat{z} < 1. \]  
(8)

Correspondingly, we have
\[ \phi(\hat{x}, 0) = 0, \quad \left. \frac{d\phi(\hat{x}, \hat{z})}{d\hat{z}} \right|_{\hat{z}=1} = -\frac{2h'(x)^2 - (h(x) - 1)h''(x)}{2[1 + h'(x)^2]} \phi(\hat{x}, 1), \]  
(9)
according to the top and bottom condition of (6), where the bottom boundary \( z = h(x) \) is transformed into \( \hat{z} = 1 \).

It is noticed that, the eigenvalues \( \lambda \) of \( D(\hat{x}) \) are real since the transverse operator \( D(\hat{x}) \) with (9) can be transformed into a symmetric matrix, and the corresponding eigenfunctions are orthogonal [6].

The square root operator \( D^{1/2}(\hat{x}) = \sqrt{\alpha(\hat{x}, \hat{z})} \frac{\partial^2}{\partial \hat{z}^2} + \beta(\hat{x}, \hat{z}) \frac{\partial}{\partial \hat{z}} + \gamma(\hat{x}, \hat{z}) \) is defined to satisfy
\[ D^{1/2}(\hat{x}) \phi(\hat{x}, \hat{z}) = \sqrt{\lambda} \phi(\hat{x}, \hat{z}), \quad 0 < \hat{z} < 1, \]  
(10)
where \( \lambda > 0 \) is called propagating mode, \( \lambda < 0 \) is called leaky or evanescent mode.

Let \( \{\phi_j(\hat{x}, \hat{z})\}_{j=1}^{\infty} \) be a system of eigenfunctions of \( D(\hat{x}) \). The eigenfunction space is \( L^2(0, 1) \), i.e.
\[ \int_0^1 |\phi_j(\hat{x}, \hat{z})|^2 d\hat{z} < +\infty, \quad j = 1, 2, \ldots, \infty. \]

While, we note that \( D(\hat{x}) \) is asymmetric and \( \{\phi_j(\hat{x}, \hat{z})\}_{j=1}^{\infty} \) may not be orthogonal for some complex waveguides (for example, lossy waveguides). To this end, some techniques in [27] may be used to treat the situations.

3 The marching method

To guarantee the stability, we first derive the stable representation condition for the Cauchy problem of the modified Helmholtz equation (6). Then, a numerical marching scheme is developed on the modified Helmholtz system according to the stability condition.
3.1 The representation of the marching solution

We denote the representation of a marching solution as a closed-form formula approximating to the real solution of the modified Helmholtz equation in a given interval.

To establish an efficient and stable numerical marching scheme, we need look for the best representation for the marching solution before carrying out the marching computing.

Suppose that the waveguide in computational domain is divided into $M$ $\hat{x}$-independent piecewise segments, and the discrete points $\hat{x}_m (m = 0,1, \ldots, M + 1)$ satisfy

$$0 = \hat{x}_0 < \hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_m < \hat{x}_{m+1} < \cdots \hat{x}_{M+1} = L < +\infty.$$  

Each interval $[\hat{x}_m, \hat{x}_{m+1}]$ corresponds to a range-independent segment approximatively. The approximated modified Helmholtz equation on $[\hat{x}_m, \hat{x}_{m+1}]$ is

$$V_{\hat{x}\hat{x}} + \alpha(\hat{x}_{m+\frac{1}{2}}, \hat{z})V_{\hat{x}\hat{z}} + \beta(\hat{x}_{m+\frac{1}{2}}, \hat{z})V_{\hat{z}\hat{z}} + \gamma(\hat{x}_{m+\frac{1}{2}}, \hat{z})V = 0, \quad (11)$$

which governs the wave propagation from $\hat{x}_m$ to $\hat{x}_{m+1}$. Without loss of generality, we take $m = 0$, and only consider the wave propagation from $\hat{x}_0$ to $\hat{x}_1$.

At $\hat{x} \in [\hat{x}_0, \hat{x}_1]$, we decompose the wave field as right- and left-going waves according to [5]

$$V(\hat{x}, \hat{z}) = V^{(+)}(\hat{x}, \hat{z}) + V^{(-)}(\hat{x}, \hat{z}), \quad (12)$$

where $V^{(+)}(\hat{x}, \hat{z})$ and $V^{(-)}(\hat{x}, \hat{z})$ satisfy the following two one-way equations

$$V^{(+)}_{\hat{x}}(\hat{x}, \hat{z}) = iD_{\hat{z}\hat{z}}V^{(+)}(\hat{x}, \hat{z}), \quad V^{(-)}_{\hat{x}}(\hat{x}, \hat{z}) = -iD_{\hat{z}\hat{z}}V^{(-)}(\hat{x}, \hat{z}) \quad (13)$$

respectively.

Integrate the above formulas from $\hat{x}_0$ to $\hat{x}(\hat{x}_0 < \hat{x} < \hat{x}_1)$, we have

$$\int_{\hat{x}_0}^{\hat{x}_1} \frac{dV^{(+)}}{V^{(+)}} = \int_{\hat{x}_0}^{\hat{x}_1} iD_{\hat{z}\hat{z}}(\hat{x}, \hat{z})d\hat{x}, \quad \int_{\hat{x}_0}^{\hat{x}_1} \frac{dV^{(-)}}{V^{(+)}} = \int_{\hat{x}_0}^{\hat{x}_1} -iD_{\hat{z}\hat{z}}(\hat{x}, \hat{z})d\hat{x}. \quad (14)$$

Then

$$V^{(+)}(\hat{x}, \hat{z}) = e^{iD_{\hat{z}\hat{z}}(\hat{x}, \hat{z})}(\hat{x} - \hat{x}_0) V^{(+)}(\hat{x}_0, \hat{z})$$

$$V^{(-)}(\hat{x}, \hat{z}) = e^{-iD_{\hat{z}\hat{z}}(\hat{x}, \hat{z})}(\hat{x} - \hat{x}_0) V^{(-)}(\hat{x}_0, \hat{z}). \quad (15)$$

Let $\{\phi_j(\hat{x}_{\frac{1}{2}}, \hat{z})\}$ be the eigenfunction system of $D(\hat{x}_{\frac{1}{2}})$, i.e.

$$D(\hat{x}_{\frac{1}{2}})\phi_j(\hat{x}_{\frac{1}{2}}, \hat{z}) = \lambda_j\phi_j(\hat{x}_{\frac{1}{2}}, \hat{z}), \quad j = 1, 2, \ldots, +\infty,$$
and correspondingly

$$D_{\hat{x}_2}^{\hat{x}_1} \phi_j(\hat{x}_1, \hat{\xi}) = \sqrt{\lambda_j} \phi_j(\hat{x}_1, \hat{\xi}), \ j = 1, 2, \ldots, +\infty.$$ 

Suppose

$$V^{(+)}(\hat{x}_0, \hat{\xi}) = \sum_{j=1}^{\infty} \alpha_j \phi_j(\hat{x}_1, \hat{\xi}), \quad (16)$$

$$V^{(-)}(\hat{x}_0, \hat{\xi}) = \sum_{j=1}^{\infty} \beta_j \phi_j(\hat{x}_1, \hat{\xi}), \quad (17)$$

where $\alpha_j$ and $\beta_j$ are the strength coefficients of the right- and left-going wave respectively. We note that the suppositions (16) and (17) are accurate if the interval $[\hat{x}_0, \hat{x}_1]$ is $\hat{x}$-independent.

Substitution of (16) and (17) into (15) yields

$$V^{(+)}(\hat{x}, \hat{\xi}) = e^{i D_{\hat{x}_2}^{\hat{x}_1} \hat{x}} \sum_{j=1}^{\infty} \alpha_j \phi_j(\hat{x}_1, \hat{\xi}) = \sum_{j=1}^{\infty} \alpha_j e^{i D_{\hat{x}_2}^{\hat{x}_1} (\hat{x} - \hat{x}_0)} \phi_j(\hat{x}_1, \hat{\xi}), \quad (18)$$

and

$$V^{(-)}(\hat{x}, \hat{\xi}) = e^{-i D_{\hat{x}_2}^{\hat{x}_1} \hat{x}} \sum_{j=1}^{\infty} \beta_j \phi_j(\hat{x}_1, \hat{\xi}) = \sum_{j=1}^{\infty} \beta_j e^{-i D_{\hat{x}_2}^{\hat{x}_1} (\hat{x} - \hat{x}_0)} \phi_j(\hat{x}_1, \hat{\xi}), \quad (19)$$

respectively.

Then, from (12), it follows that

$$V(\hat{x}, \hat{\xi}) = \sum_{j=1}^{\infty} (\alpha_j e^{i \sqrt{\lambda_j} (\hat{x} - \hat{x}_0)} + \beta_j e^{-i \sqrt{\lambda_j} (\hat{x} - \hat{x}_0)}) \phi_j(\hat{x}_1, \hat{\xi}), \quad (20)$$

which is an exact representation of the marching solution for the modified Helmholtz equation in an $\hat{x}$-independent interval $[\hat{x}_0, \hat{x}_1]$.

However, (20) is unstable when it is directly used in a marching scheme for Cauchy problems. For marching computing the Cauchy problem, we need to control the exponentially growing error propagation along the main direction of wave propagation.
Our strategy is to truncate (20) with the $p$ propagating modes

$$V(\hat{x}, \hat{z}) = \sum_{j=1}^{p} (\alpha_j e^{i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)} + \beta_j e^{-i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)})\phi_j(\hat{x}, \hat{z}),$$  \hspace{1cm} (21)$$

which physically represents the waves that can be transmitted in $[\hat{x}_0, \hat{x}_1]$.

In the following, we will proof (21) be the best tradeoff between accuracy and stability for the Cauchy problem (6).

We define that a representation of $V(\hat{x}, \hat{z})$ is stable in $[\hat{x}_0, \hat{x}_1]$, if the representation of $V(\hat{x}, \hat{z})$ is bounded for any $[\hat{x}_0, \hat{x}_1] \subset [-\infty, +\infty]$. Then we have the following stability condition for the Cauchy problem of the modified Helmholtz equation.

**Theorem 3.1.** Let $V(\hat{x}, \hat{z})$ be a solution to (6) in $[\hat{x}_0, \hat{x}_1]$. Then $V(\hat{x}, \hat{z})$ is stable if and only if it is approximated by (21).

**Proof.** (i) For a solution approximated by (21), we have

$$\|V(\hat{x}, \hat{z})\| \leq \sum_{j=1}^{p} (|\alpha_j| ||e^{i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)}|| + |\beta_j| ||e^{-i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)}||) \|\phi_j(\hat{x}, \hat{z})\|,$$  \hspace{1cm} (22)$$

where $\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)$ is real, and $||e^{\pm i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)}|| = 1$, since the $p$ modes are propagating modes. We define $C = \sup_j |\phi_j(\hat{x}_1, \hat{z})|$ to establish

$$\|V(\hat{x}, \hat{z})\| \leq \sum_{j=1}^{p} (|\alpha_j| + |\beta_j|) C,$$  \hspace{1cm} (23)$$

where $\alpha_j$ and $\beta_j$ are constants determined by (16) and (17). Let $K = \sup_j (|\alpha_j| + |\beta_j|)$. Then it follows that

$$\|V(\hat{x}, \hat{z})\| \leq pKC.$$  \hspace{1cm} (24)$$

Therefore, $V(\hat{x}, \hat{z})$ in $[\hat{x}_0, \hat{x}_1]$ is bounded for any fixed number $p$ of propagating modes. In other words, $V(\hat{x}, \hat{z})$ in $[\hat{x}_0, \hat{x}_1]$ is stable, if it is approximated by (21).

(ii) Suppose a leaky eigenfunction $\phi_k(\hat{z})(k > p)$ be incorporated into a stable solution $V(\hat{x}, \hat{z})$ in $[\hat{x}_0, \hat{x}_1]$. We have

$$V(\hat{x}, \hat{z}) = (\alpha_k e^{i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)} + \beta_k e^{-i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)})\phi_k(\hat{x}, \hat{z})$$  \hspace{1cm} (25)$$

$$+ \sum_{j=1, j \neq k}^{+\infty} (\alpha_j e^{i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)} + \beta_j e^{-i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)})\phi_j(\hat{x}, \hat{z}),$$
according to (20), which gives rise to

\[
\|V(\hat{x}, \hat{z})\|^2 = \langle V(\hat{x}, \hat{z}), V^*(\hat{x}, \hat{z}) \rangle
\]
\[
= \|\alpha_k e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)} + \beta_k e^{-i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}\|^2 \|\phi_k(\hat{x}, \hat{z})\|^2
\]
\[
+ \sum_{j=1, j \neq k}^{+\infty} \|\alpha_j e^{i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)} + \beta_j e^{-i\sqrt{\lambda_j}(\hat{x} - \hat{x}_0)}\|^2 \|\phi_j(\hat{x}, \hat{z})\|^2.
\]

Letting \(\|\phi_k(\hat{x}, \hat{z})\|^2 \geq c\) for a constant \(c > 0\) and omitting the second part of the right side, we obtain

\[
\|V(\hat{x}, \hat{z})\|^2 \geq \|\alpha_k e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)} + \beta_k e^{-i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}\|^2 c^2
\]
\[
= \|\alpha_k e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)} + 2\alpha_k \beta_k + \beta_k^2 e^{-2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}\|^2 c^2
\]
\[
\geq \left[|\beta_k|^2 e^{-2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}(2|\alpha_k \beta_k| + |\alpha_k|^2 e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)})\right] c.
\]

Then

\[
\|V(\hat{x}, \hat{z})\| \geq \sqrt{|\beta_k|^2 e^{-2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}(2|\alpha_k \beta_k| + |\alpha_k|^2 e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)})} c
\]

Here it should be noticed that, \(-2i\sqrt{\lambda_k}\) is positive real and \(2i\sqrt{\lambda_k}\) is negative, since \(\lambda_k < 0\). Correspondingly, \(\|e^{-2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}\|\) exponentially increases in \([\hat{x}_0, \hat{x}_1]\), and \(2|\alpha_k \beta_k| + |\alpha_k|^2 \|e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}\| \geq 2|\alpha_k \beta_k| + |\alpha_k|^2 \) in \([\hat{x}_0, \hat{x}_1]\). For any positive \(\Gamma\), and an enough wide interval \([\hat{x}_0, \hat{x}_1]\), there exists an enough large \(\hat{x}\), subject to

\[
\|V(\hat{x}, \hat{z})\| \geq \sqrt{|\beta_k|^2 e^{-2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)}(2|\alpha_k \beta_k| + |\alpha_k|^2 e^{2i\sqrt{\lambda_k}(\hat{x} - \hat{x}_0)})} c \geq \Gamma,
\]

which leads to a contradiction to the stability of \(V(\hat{x}, \hat{z})\).

Therefore, any stable solution should be composed only by the \(p\) local propagating eigenfunctions without any leaky mode involved, i.e. (21) holds. □

If the bottom is flat, Theorem 3.1 is essentially equivalent to Theorem 25 in [28], where the Helmholtz equation can be seen as a special case of the modified Helmholtz equation.

In the context of ultrasonic imaging, the idea of removing only evanescent waves has also been suggested by Natterer and Wübbeling in [16, 17]. However, the error estimation in [17] has an exponential growth of the error bound. As discussed by Natterer, the error bound can be formulated much tighter under appropriate conditions, since the growth could not be observed in numerical experiments. Comparatively speaking, Theorem 3.1 presents a more precise description to the stability and accuracy of a marching scheme for Cauchy problems of the modified Helmholtz equation.
3.2 The numerical marching scheme

Our marching scheme is a second-order marching scheme through repeatedly projecting with a truncated local propagating eigenfunction expansion.

The numerical marching scheme is derived by discretizing the 2-D equation (6) on a cartesian grid. We work on the grid \((m\tau, lh)\), \(m = 1, 2 \cdots M + 1, l = 1, 2, \cdots, N\), with \(\tau, h > 0\) being the step sizes in direction \(\hat{x}\) and \(\hat{z}\), respectively. We denote by \(V_{m,l}\) the approximation to \(V(m\tau, lh)\) and by \(V_m\) the vector with components \(V_m, 1, \cdots, V_m, N\). Denote the discretized matrices of the operators \(D, D^2\) as \(A, \sqrt{A}\) respectively. \(D\) in \([\hat{x}_{m-1}, \hat{x}_m]\) is denoted as \(D_m\), its matrix approximation is \(A_m\). Here the order of these matrices is \(N \times N\).

3.2.1 The truncated local propagating eigenfunction expansion

In [17], Natterer puts \(H = L_2(\mathbb{R}_{n-1})\), \(K = \{v \in H : \hat{v}(\xi) = 0 \text{ for } |\xi| \geq k\}\), where \(\hat{v}\) is the \(n - 1\)-dimensional Fourier transform of \(v\). The orthogonal projection \(P\) of \(H\) onto \(K\) is given by

\[
(Pv)(\xi) = \begin{cases} \\
\hat{v}(\xi), |\xi| \leq k, \\
0, \text{ otherwise}. \end{cases}
\]  

(30)

where \(k\) is the cut off frequency, which is very crucial to guarantee the stability of the marching computing.

Different with the regularization strategy (30), we propose a more direct way to regularize the marching solution according to the local eigenfunction expansion (21), where we need not to determine the cut off frequency any more. What we only need to know is the number of propagating modes in a piecewise local interval, and that is much easier to be correctly determined than the cut off frequency.

The essence of our strategy is similar to the downward extrapolation of [18], except that: 1. we compute the spectral projector through a direct computing of the eigensystem, while [18] computes the spectral projector with a polynomial iteration of transverse matrix; 2. we regularize the marching solutions while [18] regularizes the transverse matrix.

Let \(\hat{z}\) be discretized by \(N\) points, with \(\hat{z}_j = jh, j = 1, 2, \cdots, N, h = \frac{1}{N+1/2}\). The top and bottom boundary conditions can be written as

\[
V(\hat{x}, 0) = 0, \quad V_z(\hat{x}, 1) + \eta(x)V = 0,
\]  

(31)

where

\[
\eta(x) = \frac{2h'(x)^2 - h(x)h''(x)}{2[1 + h'(x)^2]}.
\]
We discretize (31) with

\[
V(\hat{x}, \hat{z}_1) = 0, \quad \frac{V(\hat{x}, \hat{z}_{N+1}) - V(\hat{x}, \hat{z}_N)}{h} + \eta(x) \frac{V(\hat{x}, \hat{z}_{N+1}) + V(\hat{x}, \hat{z}_N)}{2} = 0, \quad (32)
\]

where \( \hat{z}_1 = 0, \hat{z}_N = 1 - \frac{h}{2}, \hat{z}_{N+1} = \hat{z}_N + h \) is the prolongation of \( \hat{z}_N \) with respect to \( \hat{z} = 1 \).

According to (32), we have the following second-order approximations of \( \partial \hat{z} \) and \( \partial^2 \hat{z} \) respectively

\[
d_1 = \frac{1}{2h} \begin{pmatrix}
0 & 1 \\
-1 & 0 & 1 \\
\ddots & \ddots & \ddots \\
-1 & 0 & 1 \\
-1 & d_{NN}
\end{pmatrix},
\]

\[
d_2 = \frac{1}{h^2} \begin{pmatrix}
-2 & 1 \\
1 & -2 & 1 \\
\ddots & \ddots & \ddots \\
1 & -2 & 1 \\
1 & d_{2NN}
\end{pmatrix},
\]

where

\[
d_{NN} = \frac{2 - \eta(x)h}{2 + \eta(x)h}, \quad d_{2NN} = \frac{-2 + 3\eta(x)h}{2 + \eta(x)h}.
\]

Then, the transverse operator \( D \) of the modified Helmholtz equation is approximated by the \( N \times N \) tridiagonal matrix

\[
A = \alpha(\hat{x}, \hat{z})d_2 + \beta(\hat{x}, \hat{z})d_1 + \text{diag}(\hat{x}, \hat{z}) = \begin{pmatrix}
b_1 & c_1 \\
 a_2 & b_2 & c_2 \\
 & \ddots & \ddots & \ddots \\
a_{N-1} & b_{N-1} & c_{N-1} \\
 a_N & b_N
\end{pmatrix}
\]

with \( c_i = a_{i+1} > 0 (i = 1, 2, \cdots, N - 1) \).

Make the following matrix decomposition

\[
A = V \Lambda V^{-1} \quad (33)
\]

with \( V = [\phi_1, \phi_2, \cdots, \phi_N], \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), A\phi_j = \lambda_j \phi_j \).

Two types of methods can be applied to obtain (33). The first type is related to matrix iteration, such as the Rayleigh quotient iteration[29, 30, 31]; the other is related to matrix factorization, such as the QR method.
For slowly varying waveguides, the Rayleigh quotient iteration is generally competitive with matrix factorization methods [5, 26]. But, the iteration method may lose some propagating modes if the waveguide is rapidly varying, especially the boundary is highly oscillatory. A less efficient but more robust alternative is to compute the eigensystem through special QR methods designed for real tridiagonal matrix. One can refer to [6] for the concrete eigensystem computing scheme.

In each step of the marching computing, we need to project the waves in \([\hat{x}_{m-1}, \hat{x}_{m+1}]\) into the eigenfunction space in \([\hat{x}_m, \hat{x}_{m+1}]\).

Let

\[
V(\hat{x}, \hat{z}) = \sum_{j=1}^{N} \gamma_j \phi_j(\hat{x}_{m+\frac{1}{2}}, \hat{z})
\]

be the discrete form of (20) in \((\hat{x}_{m-1}, \hat{x}_{m+1})\), where \(\gamma_j = \alpha_j e^{i \sqrt{\lambda_j (\hat{x} - \hat{x}_m)}} + \beta_j e^{-i \sqrt{\lambda_j (\hat{x} - \hat{x}_m)}}\), \(\phi_j(\hat{x}_{m+\frac{1}{2}}, \hat{z}) (j = 1, \ldots, N)\) are the eigenvectors of the transverse matrix \(A_m\). According to Theorem 3.1, (34) should be truncated by

\[
V(\hat{x}, \hat{z}) = \sum_{j=1}^{p} \gamma_j \phi_j(\hat{x}_{m+\frac{1}{2}}, \hat{z}),
\]

where \(p\) is the number of propagating modes in \([\hat{x}_m, \hat{x}_{m+1}]\).

Therefore, the projection \(P\) in \([\hat{x}_{m-1}, \hat{x}_{m+1}]\) is defined as the following map

\[
P_{m+1} : \text{span}\{V_{m+1}\} \rightarrow \text{span}\{\bar{V}_{m+1}\}
\]

\[
V(\hat{x}, \hat{z}) \rightarrow U(\hat{x}, \hat{z}),
\]

(36)

with \(V_{m+1} = [\phi_1, \phi_2, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_N]\), \(\bar{V}_{m+1} = [\phi_1, \phi_2, \ldots, \phi_p]\), where the eigenfunctions \(\phi_1, \phi_2, \ldots, \phi_p\) correspond to the \(p\) propagating modes in \([\hat{x}_m, \hat{x}_{m+1}]\), and \(\phi_{p+1}, \phi_{p+2}, \ldots, \phi_N\) correspond to the \(N - p\) evanescent modes.

We denote the projection (36) as

\[
U(\hat{x}, \hat{z}) = P_{m+1} V(\hat{x}, \hat{z}),
\]

(37)

where \(\hat{x} \in [\hat{x}_{m-1}, \hat{x}_{m+1}]\).

In sum, the projection is more advantageous in the sense that it doesn’t need to determine the cut-off frequency, while the determination of \(k\) in (30) often yields a loss in bandwidth to obtain a better error bound [17].
### 3.2.2 The second-order marching scheme

The second-order marching scheme is initialized by:

1. truncate the initial conditions \( V(0, \hat{z}) = \tilde{f}(\hat{z}) \) and \( V_{\hat{x}}(0, z) = \tilde{g}(\hat{z}) \) by the projection \((36)\) with the propagating eigenfunctions at \( \hat{x} = \hat{x}_{1/2} \), and utilize the two truncated initial conditions to compute two projected solutions \( U_0 \) at \( \hat{x} = \hat{x}_0 \) and \( U_{-1} \) at \( \hat{x} = \hat{x}_{-1} \) respectively;

2. compute the solution \( V_1 \) at \( \hat{x} = \hat{x}_1 \) from \( U_0 \) and \( U_{-1} \) by a second-order difference scheme of the modified Helmholtz equation.

Every marching step \( m \rightarrow m + 1, m = 1, 2, \ldots, M \) of the marching scheme consists of the following treatments:

3. compute \( U_m \) and \( U_{m-1} \) through projecting the marching solutions \( V_m \) at \( \hat{x} = \hat{x}_m \) and \( V_{m-1} \) at \( \hat{x} = \hat{x}_{m-1} \) respectively by the projection \((36)\) with the propagating eigenfunctions at \( \hat{x}_{m+1/2} \);

4. compute the solution \( V_{m+1} \) at \( \hat{x} = \hat{x}_{m+1} \) from \( U_m \) at \( \hat{x} = \hat{x}_m \) and \( U_{m-1} \) at \( \hat{x} = \hat{x}_{m-1} \) by a second-order difference scheme of the modified Helmholtz equation.

The concrete marching step for \( m \rightarrow m + 1, m = 0, 1, 2, \ldots, M \) in the \( \hat{x} \)-direction is as follows.

First, truncate the solution \( V_{m-1} \) and \( V_m \) by

\[
U(\hat{x}_{m-1}, \hat{z}) = P_{m+1}V(\hat{x}_{m-1}, \hat{z}), \quad U(\hat{x}_m, \hat{z}) = P_{m+1}V(\hat{x}_m, \hat{z}).
\]

Then compute \( V_{m+1} \) according to

\[
\frac{V_{m+1,l} - 2U_{m,l} + U_{m-1,l}}{\tau^2} + \alpha_{m,l} \frac{V_{m,l+1} - 2U_{m,l} + U_{m,l-1}}{h^2} + \beta_{m,l} \frac{U_{m,l+1} - U_{m,l-1}}{2h} + \gamma_{m,l}U_{m,l} = 0.
\]

The recursion \((39)\) is initiated by

\[
U_0 = P_0\tilde{f}, \quad V_1 - U_{-1} = 2\tau P_0\tilde{g}
\]

where \( \tilde{f}, \tilde{g} \) are discrete forms of the functions \( f, g \) in \((5)\), \( P_0 \) is the projection at \( \hat{x} = \hat{x}_0 \).

For \( m = 0, 1, 2, \ldots, M \), the boundary condition on the bottom gives

\[
\frac{U_{m,N+1} - U_{m,N-1}}{2h} + \eta(x)\frac{U_{m,N+1} + U_{m,N-1}}{2} = 0.
\]
Substitution of (41) and (40) into (39) gives

\[
V_{1,l} = \begin{cases} 
0, & l = 0; \\
(1 + \frac{\tau^2}{h^2}\alpha_{0,l} - \frac{\tau^2}{2}\gamma_{0,l})P_0\hat{f}(\hat{z}_l) - \frac{1}{2}(\frac{\tau^2}{h^2} + \frac{\tau^2}{2h})P_0\hat{f}(\hat{z}_{l+1}) - \frac{1}{2}(\frac{\tau^2}{h^2} - \frac{\tau^2}{2h})P_0\hat{f}(\hat{z}_{l-1}) + \tau P_0\tilde{g}(\hat{z}_l), & 1 \leq l \leq N - 1; \\
(1 + \frac{\tau^2}{h^2}\alpha_{0,N} - \frac{\tau^2}{2}\gamma_{0,N})P_0\hat{f}(\hat{z}_N) - \frac{\tau^2}{h^2}\alpha_{0,N}P_0\hat{f}(\hat{z}_{N-1}) + \tau P_0\tilde{g}(\hat{z}_N), & l = N,
\end{cases}
\]  

(42)

where we abbreviate this formula as \( V_1 = \text{initialize}(\hat{f}(\hat{z}), \tilde{g}(\hat{z})) \).

Substitution of (41) into (39) gives

\[
V_{m+1,l} = \begin{cases} 
0, & l = 0; \\
(2 + \frac{\tau^2}{h^2}\alpha_{m,l} - \frac{\tau^2}{2}\gamma_{m,l})U_{m,l} - \frac{\tau^2}{h^2}\alpha_{m,l}U_{m,l+1} - \frac{\tau^2}{2h}\beta_{m,l}U_{m+1,l}, & 1 \leq l \leq N - 1; \\
(2 + \frac{\tau^2}{h^2}\alpha_{m,N} - \frac{\tau^2}{2}\gamma_{m,N})U_{m,N} - 2\frac{\tau^2}{h^2}\alpha_{m,N}U_{m,N-1} - U_{m-1,N}, & l = N,
\end{cases}
\]  

(43)

where we abbreviate this formula as \( V_{m+1} = \text{marching}(V_m, V_{m-1}) \).

In summary, we have the following Algorithm 1 for the Cauchy problem of the modified Helmholtz equation (6).

### 3.2.3 The variable number of propagating modes

For waveguides with curved boundaries, eigenvalues and eigenfunctions are dependent on the \( \hat{x} \)-direction, and the number of propagating modes may suddenly change at some places where the range-dependence is strong. This phenomenon has serious effects to some marching methods, and may lead to failure of the methods.

Suppose the number of propagating modes be \( p = n + 1 \) in \( [\hat{x}_0, \hat{x}_1] \), \( p = n \) in \( [\hat{x}_1, \hat{x}_2] \). It can be seen as the \( n + 1 \)-th mode in \( [\hat{x}_0, \hat{x}_1] \) is totally reflected in \( [\hat{x}_1, \hat{x}_2] \).
Algorithm 1 Marching Scheme for the Cauchy problem

**Input:** \( V(0, \hat{z}) = \tilde{f}(\hat{z}), V_{\hat{x}}(0, z) = \tilde{g}(\hat{z}) \)

**Output:** \( V(\hat{x}_{M+1}, \hat{z}), \) with \( \hat{z} = \hat{z}_0, \hat{z}_1, \cdots, \hat{z}_N \)

1: \( V_1 = \text{initialize}(\tilde{f}(\hat{z}), \tilde{g}(\hat{z})) \)
2: \( m = 1; \)
3: repeat
4: \( U_{m-1} = P_{m+1}V_{m-1}, U_m = P_{m+1}V_m; \)
5: \( V_{m+1} = \text{marching}(U_m, U_{m-1}); \)
6: \( m = m + 1; \)
7: until \( (m = M + 1) \)

According to Theorem 3.1, for obtaining both accuracy and stability simultaneously, we have to truncate the solution with the \( n + 1 \) propagating modes in \( [\hat{x}_0, \hat{x}_1] \), while with the \( n \) propagating modes in \( [\hat{x}_1, \hat{x}_2] \).

In fact, smaller \( p \) leads to inaccuracy, while larger \( p \) leads to instability. Only correctly determined \( p \) can deal with the problems caused by the variable number of propagating modes. To this end, our marching scheme is designed to adaptively adjust its truncating number in the marching computing process. And therefore our marching method is theoretically more accurate and stable for rapidly varying waveguides with curved boundaries.

As for other marching methods, we take the operator marching method as an example, which is an efficient second order method in slowly varying waveguides. However, the OMM for the modified Helmholtz equation will fail inevitably for variable number \( p \) of propagating modes. We present a simple explanation here. Let \( N_0 \) be the number for the OMM to truncate the transverse operator matrix. If \( p < N_0 \), that means total reflection happens in the marching computing process. The computation of reflection operator suffers great illnesses, which will lead to a severely unstable operator marching process. While if \( N_0 < p \), the marching computing will be enough stable, but the accuracy can not be guaranteed, since there must be some important propagating waves omitted somewhere in the waveguide. Therefore, the OMM in its current form is in fact unsuitable to solve the problems in a waveguide with the variable number of propagating modes. One can refer [32] for the detailed analysis.

In the end, we also point that the eigensystems only have to be piecewise computed in each interval for those rapidly varying areas, especially with variable number of propagating modes. As for slowly varying waveguides, the eigensystems are not
necessary to be computed repeatedly in each marching step. We can reduce the frequency of computing eigensystems to obtain a significantly improved efficiency in slowly varying part of the waveguide.

4 Numerical results

Two numerical examples are chosen to verify the marching scheme for waveguides with curved boundaries. In the first example, we consider the waveguide which is slowly varying with constant number \( p \) propagating modes. While, the second one is a typical rapidly varying waveguide with the variable \( p \) propagating modes causing by the oscillation of the bottom.

Both the two examples compute the wave field at \( x = L \) from the Cauchy and Dirichlet data at \( x = 0 \), where some propagating eigenfunction is supposed to be the incident wave. For a comparison with the OMM solutions, the Cauchy data for the initial Neumann condition is imposed according to the computation results of the OMM [6, 24, 25, 26] with the boundary condition (3).

**Example 1.** We consider the Helmholtz equation in a waveguide given by

\[
\kappa(x, z) = \kappa_0, h(x) = 1 - \varepsilon e^{-\sigma(x - \frac{L}{2})^2}
\]

where

\[
\kappa_0 = 10, L = 10, \varepsilon = 0.1, \sigma = 20, 0 \leq z \leq h(x), 0 \leq x \leq 10.
\]

The parameters for matrix approximation are \( N_0 = 4, N = 200 \), where \( N \) is the number of points to discretize the \( \hat{z} \) variable, \( N_0 \) is the number to truncate the \( N \times N \) matrices that approximate the operators in the OMM [6]. Suppose the incident wave \( u_0 \) at \( x = 0 \) is \( V_0^{(i)} \) (corresponding to the \( i \)-th propagating mode at \( x = 0 \)), \( i = 1, 2, 3 \).

In this example, we consider a constant \( p = 3 \) propagating modes in the waveguide with a slowly varying bottom. The solutions are computed with a range step size \( \tau = \frac{L}{2048} (M = 2048) \) (Fig.2).

Graphically, these reconstructed solutions are accurately obtained for \( \tau = \frac{L}{2048} \) by our method. Here, it should also be noticed that the OMM solutions can be exactly obtained with much larger range step sizes, which is a great advantage of the OMM in slowly varying waveguides.

However, although it is not a large range step method, our marching method with local propagating eigenfunction expansion is also computationally efficient, since the computations in every step is smaller than the OMM.
Fig. 3 demonstrates that the marching scheme can approximately keep a second-order convergence, where the reference solution is the OMM solution obtained with \( \tau = \frac{L}{2048} \).

In sum, our marching method is efficient and accurate in slowly varying waveguides. But more important, we point out that the advantage of our marching method is that it can deal with strong backscattering in waveguides caused by the curved boundaries or interfaces, especially when there is a variable number of propagating modes—where the traditional OMM fails inevitably (See the following Example 2).

Fig. 2. Comparison of \( u(L, \hat{z}) \) between the OMM solutions (dashed lines) and our marching solutions (solid lines) in Example 1.

Fig. 3. Relative errors \( u_3(L, \hat{z}) \) in Example 1.
Example 2. We consider the Helmholtz equation in a waveguide given by

\[ \kappa(x, z) = \kappa_0, \quad h(x) = 1 - \varepsilon e^{-\sigma \left(\frac{x}{L} - \frac{1}{2}\right)^2} \]

where \( \kappa_0 = 20, \quad L = 10, \quad \varepsilon = 0.2, \quad \sigma = 20, \quad 0 \leq z \leq h(x), \quad 0 \leq x \leq 10. \)

The parameters for matrix approximation are \( N_0 = 7, N = 200. \) Suppose the incident wave \( u_0 \) at \( x = 0 \) is \( V_0^{(i)} \) (corresponding to the \( i \)-th propagating mode at \( x = 0 \)), \( i = 1, 2, \ldots, 6. \)

![Fig. 4. The variation of the number \( p \) of propagating modes.](image)

The number of propagating modes in Example 2 changes with the range direction in the waveguide. Roughly speaking, \( p = 5 \) in the central part of the waveguide (Fig.4), while only \( p = 6 \) outside the central part.

The solutions are computed with the range step size \( \tau = \frac{L}{2048} \) by the OMM and our marching method respectively (Fig.5). Numerical experiments show that a stable OMM solution can only be obtained in a truncating number smaller than the number of propagating modes. Therefore, the OMM solutions are presented with different truncating number \( N_0 = \begin{cases} 5, & \text{for } V_0^{(i)}(i = 1, 2, \ldots, 5) \\ 6, & \text{for } V_0^{(6)} \end{cases} \). While to obtain the propagation of incident wave \( u_0 = V_0^{(6)} \), we have to use the sixth propagating mode and we set \( N_0 = 6. \)

As shown by Fig.5, the OMM solutions for \( u_1, \ldots, u_4 \) closely agree with the solutions obtained by our marching scheme. But, a relative large deviation appears
Fig. 5. Comparison of $u(L, \hat{z})$ between the OMM solutions (dashed lines) and our marching solutions (solid lines) in Example 2.

Fig. 6. Relative errors $u_3(L, \hat{z})$ in Example 2.

in $u_5$, where the omitted sixth propagating mode cause relative large deviations in the OMM solution $u_5$. What’s more, the OMM solution $u_6$ obtained with $N_0 = 6$ and $\tau = \frac{L}{2048}$ is totally blown up in the last two sub-figures.

Fig.6 demonstrates that our method can approximately keep a second-order convergence, where the reference solution is obtained with $\tau = \frac{L}{2560}$ by our marching method, since the OMM can not present a proper solution any more. As shown, the convergence speed is a little slower than that of Example 1. However, that should be only attributed to that the waveguide is more complex here, which even leads to
the failure of the OMM.

![Graph showing comparison of u_6 between OMM solution (dashed lines) and our marching solution (solid line).](image)

Fig. 7. Comparison of u_6 between OMM solution (dashed lines) and our marching solution (solid line).

As our marching solution is not clearly demonstrated in the last two sub-figures of Fig. 7, we further compare the OMM solution with our marching solution for u_6 with various range step sizes larger than \( \tau = \frac{L}{2048} \). Fig. 7 shows that the OMM solution of u_6 is stable as shown by the sub-figures with range step sizes \( \tau = \frac{L}{512}, \frac{L}{892}, \frac{L}{1024} \) respectively.

However, these stable solutions can not be accurate, since the OMM involves the sixth mode (unwanted evanescent mode) in the central part of the waveguide. In addition, the experimental results also verify that these OMM solutions obtained with \( \tau = \frac{L}{512}, \frac{L}{892}, \frac{L}{1024} \) are completely different, and we can’t discover any convergence trend in these solutions.

In Fig. 8, we present a comparison of our marching solutions obtained in various range steps. Roughly speaking, the solutions converge to the solution in range step size \( \tau = \frac{L}{1024} \), and the convergence trend only appears for \( \tau \geq \frac{L}{1024} \). If \( \tau < \frac{L}{1024} \), the solutions begin to become inaccurate and unstable due to the side effects of accumulated strong reflections for the sixth propagating mode. In fact, as analyzed in [32], one can never expect to obtain more accurate marching solutions in rapidly varying waveguides by unlimitedly decreasing the range step sizes.
5 Conclusions

An efficient numerical marching scheme is proposed for inverse scattering problems of the Helmholtz equation in waveguides with curved boundaries or interfaces. A local orthogonal transform is implemented to flatten the waveguide, and we obtain a modified Helmholtz system correspondingly. Then the numerical marching scheme is developed on the modified Helmholtz system, and a spectral projector based on the truncated local propagating eigenfunction expansion is utilized to regularize the marching scheme.

The numerical comparison with the OMM indicates our marching method is the same computationally efficient and accurate as the OMM in slowly varying waveguides with curved boundaries. More importantly, our marching method can be efficiently implemented in rapidly-varying waveguides with a variable number of propagating modes, while the OMM fails inevitably in this case.

Compared with the Fourier marching method [17] and the spectral projector marching method [18], our marching method is competitive for computing wave propagation in rapidly varying waveguides with curved boundaries or interfaces. We present the accurate and stable representation of marching solution for the modified Helmholtz equation, and prove that the evanescent waves is the source of instability of the Cauchy problem. The theoretical result has more explanatory power than the error estimate in [17], where there is an unobserved exponential growth of the error bound. In our marching scheme, we construct a spectral projector of the modified Helmholtz equation through projecting the marching solution into an approximated
subspace composed by local propagating eigenfunctions. The spectral projector is essentially similar to the spectral projector of [18], but it is in a completely different formulation. By coupling the spectral projector, we develop an efficient marching method in an entirely different way to solve the Cauchy problems of the modified Helmholtz equation.

In sum, we provides an efficient marching method for computing inverse scattering problems in waveguides with curved boundaries or interfaces. In addition, in its framework, it is possible to develop marching schemes for wave propagation problems in more complex waveguides or higher dimensional space. And, we may develop some new iterative procedures based on our marching scheme to solve inverse problems in range-dependent waveguides with curved boundaries or interfaces. These are interesting topics to be discussed further in the future work.

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