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Approximate techniques for dispersive shock waves in nonlinear media

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Abstract

Many optical and other nonlinear media are governed by dispersive, or diffractive, wave equations, for which initial jump discontinuities are resolved into a dispersive shock wave. The dispersive shock wave smooths the initial discontinuity and is a modulated wavetrain consisting of solitary waves at its leading edge and linear waves at its trailing edge. For integrable equations the dispersive shock wave solution can be found using Whitham modulation theory. For nonlinear wave equations which are hyperbolic outside the dispersive shock region, the amplitudes of the solitary waves at the leading edge and the linear waves at the trailing edge of the dispersive shock can be determined. In this paper an approximate method is presented for calculating the amplitude of the lead solitary waves of a dispersive shock for general nonlinear wave equations, even if these equations are not hyperbolic in the dispersionless limit. The approximate method is validated using known dispersive shock solutions and then applied to calculate approximate dispersive shock solutions for equations governing nonlinear optical media, such as nematic liquid crystals, thermal glasses and colloids. These approximate solutions are compared with numerical results and excellent comparisons are obtained.

Keywords

techniques, approximate, media, dispersive, nonlinear, waves, shock

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Approximate techniques for dispersive shock waves in nonlinear media

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Abstract

Many optical and other nonlinear media are governed by dispersive, or diffractive, wave equations, for which initial jump discontinuities are resolved into a dispersive shock wave. The dispersive shock wave smooths the initial discontinuity and is a modulated wavetrain consisting of solitary waves at its leading edge and linear waves at its trailing edge. For integrable equations the dispersive shock wave solution can be found using Whitham modulation theory. For nonlinear wave equations which are hyperbolic outside the dispersive shock region, the amplitudes of the solitary waves at the leading edge and the linear waves at the trailing edge of the dispersive shock can be determined. In this paper an approximate method is presented for calculating the amplitude of the lead solitary waves of a dispersive shock for general nonlinear wave equations, even if these equations are not hyperbolic in the dispersionless limit. The approximate method is validated using known dispersive shock solutions and then applied to calculate approximate dispersive shock solutions for equations governing nonlinear optical media, such as nematic liquid crystals, thermal glasses and colloids. These approximate solutions are compared with numerical results and excellent comparisons are obtained.

keywords solitary waves, dispersive shock waves, nonlocal optical media, conservation laws

1 Introduction

Bores, also termed dispersive shock waves or collisionless shocks, depending on the application area, are a ubiquitous waveform in nonlinear wave systems. The term shock arises from supersonic gas dynamics in which a shock is a propagating sharp discontinuity, across which pressure, density and other physical quantities undergo a jump[1]. However, for wave systems in which there is dispersion or diffraction an initial discontinuity is resolved, due to the large derivatives involved, into a non-uniform oscillatory wavetrain. Usually, the leading edge of this wavetrain consists of solitary waves, while the trailing edge consists of linear waves. Bores were first observed as surface waves on fluids, the most famous being tidal bores, such as the Severn Bore in England and the tidal bore in the Bay of Fundy in Canada. These bores arise due to narrowing estuaries enhancing strong tides so that the tide breaks with dispersion then resolving the breaking wave form into a bore. Bores also occur in the atmosphere, the most well known being glory waves[2, 3, 4, 5], and as internal waves in the ocean[6]. Fluid bores can be of two types, viscous and undular bores, depending on whether viscosity or dispersion, respectively, dominate the evolution[1, 7, 8]. While bores were first observed in fluid systems, they occur in nonlinear optical media. Numerical and experimental studies have found that dispersive shocks can form in nonlinear crystals[9, 10, 11] and nonlinear thermal media[12, 13, 14].

The major theoretical advance for the description of undular bores, or dispersive shock waves or collisionless shocks, was the development of modulation theory which describes the evolution of non-steady, or modulated, nonlinear wavetrains[1, 15]. When the modulation equations form a hyperbolic system the underlying wavetrain is stable, while when the modulation equations are elliptic, the wavetrain is unstable. Hyperbolic modulation equations possess a simple wave solution which describes an undular bore. This undular bore solution was first found for the Korteweg-de Vries (KdV) equation[16, 17], which describes weakly nonlinear long waves in a fluid, based on the modulation equations for the KdV equation[1, 15]. The modulation equations for nonlinear wave equations which possess an inverse scattering solution can be set into Riemann invariant form, so that the simple wave solution describing an undular bore can be found explicitly[18]. When the governing nonlinear wave equation does not have an inverse scattering solution it is usually impossible to set the modulation equations in Riemann invariant form, so that the undular bore solution can be determined. In this case, a general method has been developed which can determine the solitary waves at the leading edge and the linear waves at the trailing edge of the bore[19, 20, 21]. However, this method relies on the nonlinear wave equation being hyperbolic outside of the bore region.

A broad class of nonlinear optical media have a response which is termed nonlocal, examples being thermal media[12, 13, 14, 22], thermal glasses[23, 24] and nematic liquid crystals[25, 26]. For such nonlocal media the optical beam evolution is coupled to an elliptic equation for the medium response. The medium response is nonlocal as it extends far beyond the beam waist. A consequence of this elliptic response of the medium is that when a bore (dispersive shock) forms the governing equations are not hyperbolic outside of the bore region. The method of El[19, 20, 21] then cannot be used to derive the leading and trailing edges of the bore solution.

In the present work an approximate method will be discussed which can determine the amplitude of the solitary waves at the leading edge of an undular bore. This method does not rely on the existence of an inverse scattering solution or the governing equations being hyperbolic outside the bore region. The bore is approximated by a train of equal amplitude, equally spaced solitary waves. This approximation is not valid in the early stages of bore evolution from an initial jump discontinuity, but becomes a good approximation as the bore develops. This is because as it develops more waves are generated in the bore, with the length of the leading edge of the bore, which can be well approximated by solitary waves, increasing. Hence, as a bore evolves it becomes dominated by solitary waves. Conservation equations for the governing equations are then used to determine the amplitude of these solitary waves. The validity of this approximate method will be determined by comparing its predictions with the known bore solutions of the KdV equation, the Benjamin-Ono equation, the modified KdV (mKdV) equation and the nonlinear Schrödinger (NLS) equation. The first three of these equations arise in water wave theory, while the last one arises in water wave theory, fibre optics and nonlinear optics. It is found that the approximate theory gives a good approximation for the amplitude of the solitary wave at the leading edge of a bore, with the error varying between 0% for the KdV equation to 30% for the Benjamin-Ono equation. With this validation of the approximate method, it is then used to find the amplitude of the leading solitary waves of bores for nonlocal equations governing nonlinear optical beam propagation. The modulation equations describing optical beam propagation in many nonlocal media, such as nematic liquid crystals, are elliptic, so that periodic wave solutions are modulationally unstable. A simple wave bore solution is then not expected to exist. However, if the nonlocality is large enough the onset of modulational(MI) instability is delayed[27], so a bore-type solution exists for experimental length scales[28]. The approximate method is found to give solutions in good agreement with numerical solutions of these nonlocal equations.

2 Integrable equations

To develop and validate the approximate method for determining the amplitude of the lead waves in a bore, or dispersive shock, let us first consider the standard integrable equations which have known bore solutions determined from modulation theory, these equations being the KdV, mKdV, Benjamin-Ono and NLS equations. These are all integrable systems for which Whitham modulation theory provides dispersive shock solutions. These known dispersive shock solutions then provide test cases against which the approximate method can be validated.

2.1 Korteweg-de Vries equation

The simplest equation for which to develop the approximate method is the Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

which is the generic nonlinear dispersive wave equation having an exact solution via the inverse scattering transform[1, 29]. This equation arises in a large number of application areas, including water waves, both surface and internal[1], and plasma physics. The dispersive shock wave solution for the KdV equation[16, 17] has been derived from the modulation equations for this equation[1, 15]. The simplest initial condition which will lead to the development of a dispersive shock wave is the jump initial condition

$$u = \begin{cases} A, & x < 0, \\ 0, & x > 0, \end{cases} \tag{2}$$

where A is the jump amplitude.

To find an approximate solution for the dispersive shock wave generated by the initial condition (2) we shall approximate it by a uniform train of KdV solitons[30, 31], which have the form

$$u = A_s \operatorname{sech}^2 \sqrt{\frac{A_s}{2}}(x - 2A_s t). \tag{3}$$

This approximation is appropriate for large time as then the bore consists of a large number of individual waves dominated by solitary waves extending from its leading edge[16, 17]. The approximation is not valid near the trailing edge of the bore, where it consists of linear waves. However, this trailing edge region is small in comparison with the leading edge portion for large time.s

The method determines the amplitude A_s of the solitons generated by the bore. For an initial-boundary value problem[30, 32] all the mass and energy created at the boundary is converted into solitary waves. Hence, the number of solitary waves N and their spacing can be determined. However, for the initial condition (2), which gives an initial value problem on the infinite line $-\infty < x < \infty$, mass and energy can be generated at a different rate to the creation of solitary waves. Hence, the number of solitary waves generated cannot be easily found, but the amplitude of these solitary waves can be.

The KdV equation (1) has the mass and energy conservation equations

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(3u^2 + u_{xx}\right) = 0, \tag{4}$$

$$\frac{\partial}{\partial t}u^2 + \frac{\partial}{\partial x}\left(4u^3 + 2uu_{xx} - u_x^2\right) = 0. \tag{5}$$

Integrating these conservation laws between $x = \pm\infty$ gives

$$\frac{d}{dt} < u >= 3A^2 \quad \text{and} \quad \frac{d}{dt} < u^2 >= 4A^3, \tag{6}$$

on using the initial condition (2) to determine the flux contributions at $x = \pm\infty$. Here $< Q >$ denotes the average

$$< Q >= \int_{-\infty}^{\infty} Q \, dx. \tag{7}$$

Taking the ratio of the two averaged conservation equations (6) and integrating gives

$$4A < u >= 3 < u^2 >, \tag{8}$$

on assuming that there are no solitons initially. For a single solitary wave we have

$$< u >= 2\sqrt{2A_s} \quad \text{and} \quad < u^2 >= \frac{4\sqrt{2}}{3} A_s^{3/2}. \tag{9}$$

Substituting these expressions into (8) gives the relation $A_s = 2A$ for the amplitude of the lead soliton in terms of the jump height. This is the same expression as that given by modulation theory for the KdV equation[16, 17].

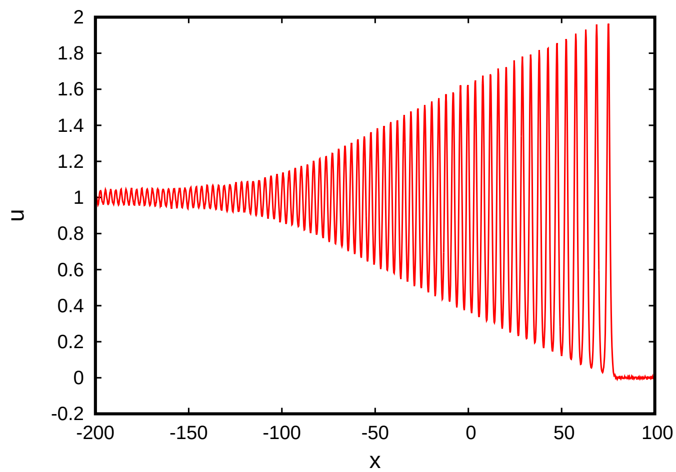

Figure 1: (Color online) Numerical solution of the KdV equation (1) at $t = 20$ for the initial condition (2) with $A = 1$.

If all the mass and energy of the initial condition is converted directly into solitary waves, then the simple theory developed in this section can also be used to determine the number $N(t)$ of waves in the bore at time t. Since the amplitude of the solitons in the bore has been assumed to be constant

$$\frac{d}{dt} < u >= M \frac{dN}{dt}, \tag{10}$$

where M is the mass of a single soliton. The mass conservation expressions in (6) and (9) and the soliton amplitude relation $A_s = 2A$ then give

$$N = \frac{3}{4} A^{3/2} t. \tag{11}$$

Figure 1 shows the free surface height, u versus x, for the KdV equation (1). Shown is the numerical solution of (1) at $t = 20$ for the initial condition (2) with $A = 1$. The figure shows a typical bore solution of the KdV equation. There is a linear amplitude variation between the solitons at the front of the bore and linear waves at the rear. The amplitude of the lead soliton in the numerical bore corresponds very closely to the theoretical prediction of 2. For this example the formula (11) predicts $N = 15$ waves in the bore. This is not in agreement with the figure, which has about 40 waves in the bore. However, all the waves in the numerical bore do not have the same amplitude, a key assumption of the approximate theory. If the mass of N solitons is redistributed so that the solitons have a linearly decreasing amplitude, this then gives $\frac{3}{2}N = 23$ waves in the bore at time $t = 20$. Hence, about half of the mass being generated from the initial condition is being converted into KdV solitons.

2.2 mKdV and Benjamin-Ono equations

The preceding analysis shows that treating the bore as a train of equal amplitude solitons gives the exact value for the amplitude of the lead soliton of the bore and a good approximation for the number of waves in the bore. Let us now apply this approximate method to the mKdV and Benjamin-Ono equations, both of which have an inverse scattering solution and for both of which there is a dispersive shock wave solution from the modulation equations for each equation. The mKdV and Benjamin-Ono equations are

$$\frac{\partial u}{\partial t} + 12u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{12}$$

and

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} + PV\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u_{yy}}{y - x}\, dy = 0, \tag{13}$$

respectively. In the Benjamin-Ono equation PV denotes the Cauchy principal value of the integral. The mKdV equation arises in water wave theory, for instance waves on the interface of a two layer fluid when the depths of the two layers are nearly equal. The Benjamin-Ono equation arises for waves in a two layer fluid when one of the layers is much deeper than the other[33]. The soliton solution of the mKdV equation is

$$u = A_s \operatorname{sech} \sqrt{2}A_s(x - 2A_s^2 t) \tag{14}$$

and that for the Benjamin-Ono equation is

$$u = \frac{A_s}{1 + \frac{A_s^2}{4}\left(x - \frac{A_s}{2}t\right)^2}. \tag{15}$$

The application of the approximate theory to the mKdV and Benjamin-Ono equations is the same as that discussed in the previous subsection for the KdV equation, so the details will not be given and only the final result will be quoted.

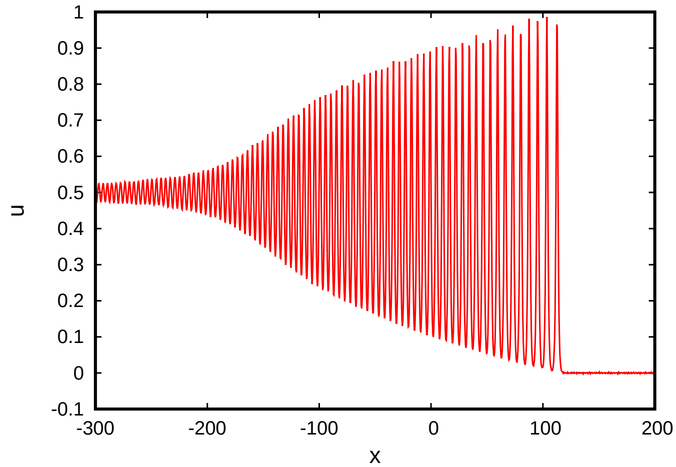

Figure 2: (Color online) Numerical solution of the mKdV equation (12) at $t = 60$ for the initial condition (2) with $A = 0.5$.

Modulation theory for the mKdV equation[34] gives the amplitude of the leading soliton generated from the initial condition (2) as $A_s = 2A$, the same as for the KdV case. The approximate theory based on the mass and energy conservation equations for the mKdV equation (12) gives this lead soliton amplitude as

$$A_s = \frac{3\pi}{4} A \approx 2.356\ldots A. \tag{16}$$

Modulation theory for the Benjamin-Ono equation[35] gives the amplitude of the leading soliton generated from the initial condition (2) as $A_s = 4A$, while the approximate theory gives

$$A_s = \frac{8}{3} A. \tag{17}$$

The error in the lead soliton amplitude for the mKdV equation is 18% and for the Benjamin-Ono equation is 33%.

Figure 2 shows the surface elevation, u versus x, for the mKdV equation (12). Shown is the numerical solution of (12) at $t = 60$ for $A = 0.5$. The figure shows a typical bore solution of the mKdV equation. The wave amplitude varies in a quadratic manner through the bore[34], in contrast to the KdV bore, which has a linear amplitude variation. Again, the numerical amplitude of the lead wave corresponds closely with the modulation theory prediction of $2A = 1$. There are about 60 waves in this mKdV bore, compared with the theoretical prediction of $N = 20$ waves, which was found using a similar method to that described for the KdV equation. In this case, about one third of the mass of the initial condition is being converted into mKdV solitons.

2.3 NLS equation

The details of approximating a dispersive shock wave by a train of equal amplitude solitary waves are different for NLS-type equations than for the KdV-type equations discussed above. The NLS equation is

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0. \tag{18}$$

This equation arises in water wave stability theory[1] and nonlinear optics[36]. As it has an inverse scattering solution[29], it has modulation equations from which a bore solution can be determined[28]. However, the NLS equation (18) is focusing, so that wavetrains show MI. The modulation equations for the focusing NLS equation are then elliptic, so no simple wave solution exists. However, the focusing NLS equation (18) does have a bore-type solution before MI sets in. This is because the modulation equations for the NLS equation (18) are hyperbolic in the soliton and linear wave limits[28]. A bore-type solution can then be constructed with a soliton at the leading edge and linear waves at the trailing edge[28]. The step initial condition

$$u = \begin{cases} Ae^{ikx}, & x < 0, \\ 0, & x \geq 0. \end{cases} \tag{19}$$

will be used to generate a dispersive shock wave. This initial condition will generate a bore until MI takes over[28]. Modulation theory for the NLS equation[28] gives the amplitude of the lead soliton of the dispersive shock wave as $A_s = 2A$.

The soliton solution of the NLS equation (18) is

$$u = A_s \operatorname{sech} A_s(x - kz)\, e^{iA_s^2 z/2 + ik(x-kz)}. \tag{20}$$

As for the KdV-type equations, mass and energy conservation equations for the NLS equation (18) will be used to find an approximation for the amplitude of the lead soliton of the dispersive shock wave generated by the initial condition (19). These mass and energy conservation equations are

$$i\frac{\partial}{\partial z}|u|^2 + \frac{1}{2}\frac{\partial}{\partial x}\left(u^* u_x - u u_x^*\right) = 0 \tag{21}$$

and

$$i\frac{\partial}{\partial z}\left(|u_x|^2 - |u|^4\right) + \frac{1}{2}\frac{\partial}{\partial x}\left[u_x^* u_{xx} - u_x u_{xx}^* - 2|u|^2\left(u^* u_x - u u_x^*\right)\right] = 0, \tag{22}$$

respectively. Here the * superscript denotes the complex conjugate. Integrating these conservation equations between $x = \pm\infty$ and using the initial condition (19) gives

$$\frac{d}{dz} < M >= kA^2, \quad \frac{d}{dz} < H >= A^2 k\left(k^2 - 2A^2\right). \tag{23}$$

from the mass and energy conservation equations, respectively. Here M and H are the integrated mass and energy densities. The NLS soliton solution (20) gives

$$< M >= 2A_s \quad \text{and} \quad < H >= -\frac{2}{3}A_s^3 + 2k^2 A_s. \tag{24}$$

The conservation relations (23) then give

$$A_s = \sqrt{6}A \approx 2.45\ldots A. \tag{25}$$

The approximate lead soliton amplitude then differs from the exact amplitude $2A$ by 22.5%.

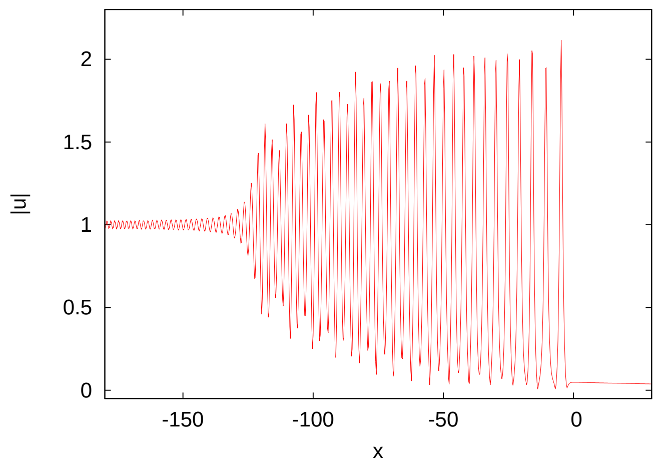

Figure 3: (Color online) Numerical solution of the NLS equation (18) at $z = 50$ for the initial condition (19) with $A = 1$ and $k = 0$.

Figure 3 shows the wave amplitude, $|u|$ versus x, for the NLS equation (18). Shown is the numerical solution of (19) at $z = 50$ for $A = 1$ and $k = 0$. The lead wave has amplitude 2, as predicted by modulation theory. For this value of z the NLS bore is qualitatively similar to the KdV and mKdV bores. However, the modulation equations form an elliptic system and there is no hyperbolic expansion fan solution. The NLS solution (20), which shows that all solitons are stationary for $k = 0$, provides an insight into the behaviour of the NLS bore; the individual waves do not completely separate and are not ordered by amplitude.

In summary, the approximate method developed in this work gives an amplitude of the lead solitary wave of a dispersive shock wave which differs from the modulation theory value by between 0% and 33%, with 20% being a typical difference. This approximate method will now be used to find the amplitude of the lead solitary wave in dispersive shock waves for which the governing equations have no modulation equations which can be put into Riemann invariant form, or which are not hyperbolic in the dispersionless limit[28].

3 Nematic liquid crystals

The approximate method developed above will now be used to find the amplitude of the lead solitary wave in a dispersive shock wave in a nematic liquid crystal. Solitary wave in nematic liquid crystals are termed nematicons[25], so this terminology will be used in the present section. The equations governing the propagation of an optical beam in a nematic liquid crystal are[26, 37, 38]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + 2\theta\, u = 0, \quad \nu\nabla^2\theta - 2q\theta = -2|u|^2. \tag{26}$$

Here u is the envelope of the electric field of the optical beam, θ is the rotation of the nematic molecules due to the optical beam, ν is the elasticity of the nematic medium and q is related to the intensity of the external field which pre-tilts the nematic molecules. In the normal experimental regime ν is large[39], $O(100)$, so that the nematic is termed a nonlocal medium in that the nematic response extends far beyond the waist of the beam. This nonlocal response means that the nematic equations (26) are not hyperbolic in the non-dispersive limit, so that the method of El[19, 20, 21] to determine the leading and trailing edges of a bore cannot be used. The method developed in the present work is then the only one which can be used to give an approximation to the lead wave of the dispersive shock wave. The nematicon equations (26) are focusing, so that their modulation equations would be elliptic and so no simple wave dispersive shock wave solution exists. As for the NLS equation of Section 2.3 a bore-type solution can be found before MI sets in[28]. In this context, the large nonlocality ν can delay the onset of MI to such an extent that it is not observed over experimental nematic cell lengths unless the optical power is raised above the usual low milliwatt levels[27, 40, 41].

As for the NLS equation, the simplest initial condition to generate a dispersive shock wave is the jump initial condition

$$u = \begin{cases} Ae^{ikx}, & x < 0, \\ 0, & x > 0, \end{cases} \qquad \theta = \begin{cases} \frac{A^2}{q}, & x < 0, \\ 0, & x > 0 \end{cases} \tag{27}$$

The initial condition for θ has been chosen to satisfy the director equation, the second of (26), in $x < 0$.

As for the KdV-type equations and the NLS equation, mass and energy conservation laws for the nematicon equations (26) will be used to determine the amplitude of the lead nematicon of a dispersive shock wave. These conservation equations are

$$i\frac{\partial}{\partial z}(|u|^2) + \frac{1}{2}\frac{\partial}{\partial x}(u^* u_x - u u_x^*) = 0, \tag{28}$$

$$\begin{aligned} &i\frac{\partial}{\partial z}(|u_x|^2 - 4\theta|u|^2 + \nu\theta_x^2 + 2q\theta^2) + \frac{1}{2}\frac{\partial}{\partial x}(u_x^* u_{xx} \\ &-u_x u_{xx}^* - 4\theta u^* u_x + 4\theta u u_x^* - 4i\nu\theta_x\theta_z) = 0, \end{aligned} \tag{29}$$

respectively. All the previous equations have an exact soliton solution. However, the nematicon equations (26) have no such exact solitary wave solution. In this case, a variational approach has been found to give a good approximation to the steady nematicon[31]. This variational approximation is based on the trial functions

$$u = A_s \operatorname{sech} \frac{x - kz}{w} \, e^{i(kx+\sigma z)}, \quad \theta = \alpha \operatorname{sech}^2 \frac{x - kz}{\beta}. \tag{30}$$

for the electric field u and director angle θ. Here α is the amplitude of the director response, w and β are the widths of the two beams in the electric field and the nematic and σ is the propagation constant. It is found that the amplitude and width of the nematicon are determined by

$$\alpha = \frac{4}{3w(I - wI_w)}, \tag{31}$$

$$\frac{32\nu\alpha}{15\beta} + \frac{16}{3} q\alpha\beta - A_s^2 I = 0, \tag{32}$$

$$\frac{16\nu\alpha^2}{15\beta^2} - \frac{8}{3} q\alpha^2 + \alpha A_s^2 I_\beta = 0, \tag{33}$$

$$\sigma = -\frac{k^2}{2} - \frac{1}{6w^2} + \frac{\alpha I}{4w}. \tag{34}$$

I is the integral

$$I(w, \beta) = \int_{-\infty}^{\infty} \operatorname{sech}^2(\frac{x}{\beta}) \operatorname{sech}^2(\frac{x}{w}) \, dx, \tag{35}$$

which cannot be evaluated unless $w = \beta$, which is the case in the local limit $\nu = 0$. Integrating the mass and energy conservation equations (28) and (29) between $x = \pm\infty$ gives

$$\frac{d}{dz} < M >= kA^2, \quad \frac{d}{dz} < H >= kA^2 \left(k^2 - 4\frac{A^2}{q} \right) \tag{36}$$

on using the initial condition (27) to evaluate the flux terms at $x = \pm\infty$. The variational approximation (30) then gives the mass and energy densities as

$$< M >= 2A_s^2 w, \quad < H >= \frac{2A_s^2}{3w} + \frac{16\nu\alpha^2}{15\beta} + 2k^2 A_s^2 w + \frac{8}{3} q\alpha^2\beta - \alpha A_s^2 I. \tag{37}$$

As for the NLS equation of Section 2.3, taking the ratio of these conservation relations gives the equation

$$\frac{2A_s^2}{3w} + \frac{16\nu\alpha^2}{15\beta} + \frac{8}{3} q\alpha^2\beta - \alpha A_s^2 I = -\frac{8A^2}{q} A_s^2 w. \tag{38}$$

The amplitude of the lead nematicon of the bore is found by solving (38) together with (31)–(34). This represents a set of transcendental equations which must be solved numerically.

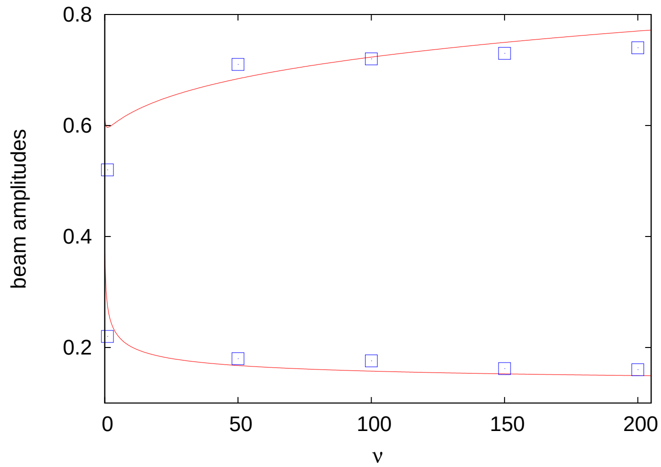

Figure 4: (Color online) Beams amplitudes versus ν. Shown are A_s (upper solid line) and α (lower solid line) from the approximate theory. Numerical estimates for the average maximum amplitude (squares). The other parameters are $A = 0.25$, $q = 1$ and $k = 0$.

Figure 4 displays the beam amplitudes, A_s and α, versus ν. The other parameters are $A = 0.25$, $q = 1$ and $k = 0$. Shown are the predictions of the approximate theory and the numerical results. The numerical estimate is the maximum amplitude in the bore averaged from the z position at which the first nematicon has formed until the z value at which MI dominates. An averaging process is needed as there is some oscillation in the profile amplitude while the bore develops.

For small ν the undular bore is qualitatively similar to that for the NLS equation (see figure 3), while for large ν the nematicons which are generated interact with each other nonlocally due to the broad response of the nematic causing a wide potential well enclosing all the nematicons in the bore, see figure 4[31]. Hence, for small ν the maximum amplitude does not vary much once it is fully formed, while for large ν the maximum amplitude varies with z since the waves interact.

The general trend is that, as ν increases, the electric field amplitude increases and that of the director decreases. For small ν the theoretical prediction for the electric field amplitude overestimates the numerical amplitude by about 20%, which is consistent with the NLS limit (as $\nu \to 0$) discussed in Section 2.3. For larger ν the comparison between the approximate theory and numerical solutions is excellent, with differences of less than 5%.

4 Colloids

Let us now consider optical beam propagation in another nonlinear medium, a colloid. The system of equations governing the evolution of the beam is similar to that for beam propagation in a nematic liquid crystal of Section 3. These equations are[42, 43]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + (\eta - \eta_0)\, u = 0, \quad |u|^2 = g(\eta) - g_0,$$
$$\text{with} \;\; g(\eta) = \frac{3-\eta}{(1-\eta)^3} + \ln \eta, \quad g_0 = g(\eta_0). \tag{39}$$

Again, u is the envelope of the electric field of the optical beam. The concentration of colloidal particles is given by η, termed the packing fraction. The refractive index of the colloid depends on this concentration through the equation of state $g(\eta)$. η_0 is the background packing fraction of the medium, in the absence of solitary waves. As for the NLS equation of Section 2.3, a dispersive shock wave is generated by the jump initial condition

$$u = \begin{cases} Ae^{ikx}, & x < 0, \\ 0, & x > 0, \end{cases} \qquad \eta = \begin{cases} \eta_m, & x < 0, \\ \eta_0, & x > 0. \end{cases} \tag{40}$$

To satisfy the colloid equations (39) the particle concentration in $x < 0$, η_m, must satisfy $A^2 = g(\eta_m) - g_0$.

As for the nematic equations of Section 3 there is no exact solitary wave solution of the colloid equations (39). Again, a variational method can be used to obtain a good approximation to this solitary wave[44], based on the trial functions

$$u = A_s \operatorname{sech} \frac{x - kz}{w}\, e^{i\sigma z + ikx}, \quad \eta = \eta_0 + \alpha \operatorname{sech}^2 \frac{x - kz}{\beta} \tag{41}$$

for the electric field u and the particle concentration η. This variational approximation gives that the steady solitary wave is determined by[44]

$$\begin{aligned} &\sigma = -\frac{1}{6w^2} + \frac{\alpha\Omega_1}{w}, \\ &1 - 3\alpha w(\Omega_1 - w\frac{\partial \Omega_1}{\partial w}) = 0, \\ &4A_s^2\alpha(\Omega_1 - \beta\frac{\partial\Omega_1}{\partial\beta}) - \beta(\alpha\frac{d\Xi_1}{d\alpha} - \Xi_1) - 4\beta(\alpha\frac{d\Theta_1}{d\alpha} - \Theta_1) = 0, \\ &4\alpha A_s^2\frac{\partial\Omega_1}{\partial\beta} - \Xi_1 - 4\Theta_1 + 4\alpha\,(1 + g_0) = 0. \end{aligned} \tag{42}$$

Here

$$\Omega_1(w, \beta) = \int_0^\infty \operatorname{sech}^2 \frac{\zeta}{\beta} \operatorname{sech}^2 \frac{\zeta}{w}\, d\zeta,$$

$$\Xi_1(\alpha) = 2\int_0^\infty \left[\frac{4-2\eta_0-2\alpha\,\mathrm{sech}^2\,\zeta}{(1-\eta_0-\alpha\,\mathrm{sech}^2\,\zeta)^2} - \frac{4-2\eta_0}{(1-\eta_0)^2}\right]\,d\zeta, \tag{43}$$

$$\Theta_1(\alpha) = \int_0^\infty \left[\eta_0\ln(1+\frac{\alpha}{\eta_0}\,\mathrm{sech}^2\,\zeta) + \alpha\,\mathrm{sech}^2\,\zeta\ln(\eta_0+\alpha\,\mathrm{sech}^2\,\zeta)\right]\,d\zeta.$$

The amplitude of the lead solitary wave of the dispersive shock wave generated by the initial condition (40) will again be determined by mass and energy conservation equations. The mass conservation equation for (39) is

$$i\frac{\partial}{\partial z}|u|^2 + \frac{1}{2}\frac{\partial}{\partial x}\left(u^*u_x - uu_x^*\right) = 0, \tag{44}$$

and the energy conservation equation is

$$\begin{aligned} & i\frac{\partial}{\partial z}\left[|u_x|^2 - 2\left(\eta-\eta_0\right)|u|^2 + \frac{4-2\eta}{\left(1-\eta\right)^2} - \frac{4-2\eta_0}{\left(1-\eta_0\right)^2}\right. \\ & \left. +\ 2\eta\ln\eta - 2\eta_0\ln\eta_0 - 2\left(\eta-\eta_0\right)\left(1+g_0\right)\right] \\ & +\ \frac{1}{2}\frac{\partial}{\partial x}\left[u_x^*u_{xx} - u_xu_{xx}^* - 2\left(\eta-\eta_0\right)\left(u^*u_x - uu_x^*\right)\right] = 0. \end{aligned} \tag{45}$$

These mass and energy equations are integrated between $x = \pm\infty$ using the initial jump (40) to give

$$\frac{d}{dz} < M >= kA^2, \quad \frac{d}{dz} < H >= kA^2[k^2 - 2(\eta_m - \eta_0)]. \tag{46}$$

Dividing the conservation results (46) and integrating, on noting that there are no solitary waves initially, gives an equation for the amplitude of the lead solitary wave of the dispersive shock wave

$$< H >= \left[k^2 - 2(\eta_m - \eta_0)\right] < M > . \tag{47}$$

The approximate solitary wave solution (41) is now used to calculate the mass M and energy H, resulting in

$$\begin{aligned} < M > &= 2A_s^2 w, \\ < H > &= \frac{2}{3}\frac{A_s^2}{w} + 2k^2A_s^2w - 4\alpha A_s^2\Omega_1 + \beta\Xi_1 + 4\beta\Theta_1 - 4\alpha\beta(1+g_0), \end{aligned} \tag{48}$$

so that the final equation determining the amplitude of the lead solitary wave of the dispersive shock wave is

$$\frac{2}{3}\frac{A_s^2}{w} - 4\alpha A_s^2\Omega_1 + \beta\Xi_1 + 4\beta\Theta_1 - 4\alpha\beta(1+g_0) + 4A_s^2w(\eta_m - \eta_0) = 0, \tag{49}$$

The amplitude of the lead colloidal solitary wave in the bore is found by solving (42) and (49). This represents a set of transcendental equations which must be solved numerically. The solution of these transcendental equations shows that three qualitatively different solitary wave amplitude A_s versus jump height

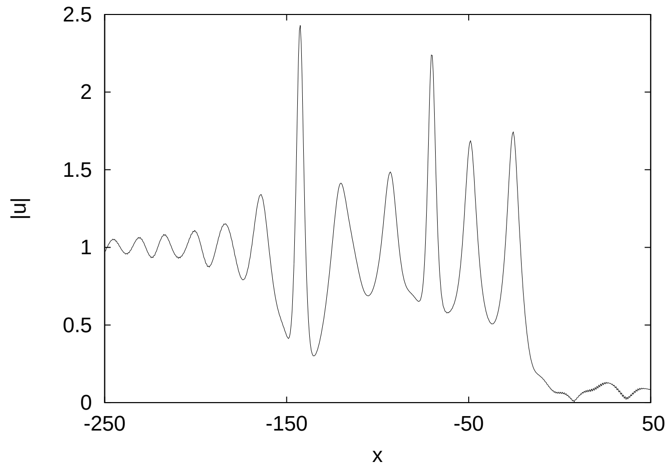

Figure 5: (Color online) Numerical solution of the colloid equation (39) at $z = 1000$ for the initial condition (40). The initial and background packing fractions are $\eta_m = 2.43 \times 10^{-2}$ and $\eta_0 = 1 \times 10^{-2}$. The other parameters are $A = 1$ and $k = 0$.

A diagrams are possible, depending on the background packing fraction[45]. For large background packing fractions a single stable solution branch occurs. At moderate values an S-shaped response curve results, with multiple solution branches, while for small values the upper solution branch separates from the middle unstable branch. Hence, for low to moderate values of the background packing fraction the dispersive shock bifurcates from the low to the high power branch as the jump height is increased. These multiple steady-state response diagrams, also typically found in combustion applications, are unusual in applications involving solitary waves[45].

Figure 5 shows the electric field amplitude, $|u|$ versus x, for the colloid equations (39). Shown is the numerical solution for the initial condition (40) at $z = 1000$. The parameter values are the initial and background packing fractions $\eta_m = 2.43 \times 10^{-2}$ and $\eta_0 = 1 \times 10^{-2}$, respectively. Also $A = 1$ and $k = 0$. The packing fraction η is not shown as its profile is qualitatively the same as that for $|u|$. For this propagation distance seven large solitary waves have formed, with the largest wave, with $a = 2.43$, sixth from the front of the bore. As for nematic bores, the waves interact with each other in a complicated manner and they are not ordered by amplitude. The maximum amplitude in the bore, averaged over z, is 2.76. This compares well with the prediction of the approximate solution, $a = 2.44$, which is within 12% of the numerical prediction.

5 Conclusions

An approximate method for determining the amplitude of the lead solitary wave in an undular bore has been developed. The method, which is based on conservation laws, is benchmarked using integrable equations for which exact results are known. It was then applied to a range of equations governing nonlinear optical media for which exact results are not possible. The method has a wide applicability, giving accurate results for systems with stable undular bore solutions and also for applications, from nonlinear optics, governed by focusing equations. Focusing equations are subject to MI, but our approximate method gives accurate predictions for the bore which develops at short propagation distances before the onset of MI. The method should also be applicable to other nonlinear optical media.

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