A study on quantitatively pricing various convertible bonds

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A study on quantitatively pricing various convertible bonds

Sha Lin

This thesis is presented as part of the requirements for the conferral of the degree:

Doctor of Philosophy

Supervisor:
Prof. Song-Ping Zhu & Dr. Xiaoping Lu

The University of Wollongong
SMAS School of Mathematics and Applied Statistics

March 19, 2019
Declaration

I, Sha Lin, declare that this thesis is submitted in partial fulfilment of the requirements for the conferral of the degree Doctor of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

Sha Lin

March 19, 2019
Abstract

Financial derivatives are becoming increasingly popular among investors as well as academic researchers. Among these, convertible bonds have received a large amount of attention since they are beneficial to both the issuer and holder in the sense that they help reduce the cash interest payment for the issuer while enabling the holder to reduce the risk of directly holding the underlying stocks. Despite the popularity of convertible bonds in practice, their pricing problems still remain challenging not only because most of them traded in real markets are of American-style, but also due to many additional features that could be introduced into vanilla convertible bonds to cater for different demands. Thus, convertible bonds are often priced with various numerical approaches because the predominant complexity arises from the determination of the bond price together with different free boundaries, which are introduced due to the incorporation of various additional features.

This thesis contributes to the literature significantly by pricing various types of convertible bonds under different models. In particular, Chapters 3-5 focus on deriving integral equation formulations for the prices of puttable, callable-puttable and resettable convertible bonds under the Black-Scholes model. The pricing of convertible bonds under stochastic volatility and interest rate models is then discussed in Chapter 6. Due to the additional stochastic source, analytical pricing formulae are no longer available, and an efficient predictor-corrector scheme is established to obtain the convertible bond prices as well as the optimal conversion boundary.
Acknowledgments

During my Ph.D. study, there are many people to whom I would like to express my sincere gratitude. Without them, this thesis could not be finished.

First of all, my supervisor, Prof. Song-Ping Zhu, has given me a lot of help. It is he who has guided me into the world of financial mathematics. It is he who encourages me whenever I feel frustrated. It is he who helps me find the right way when I am confused. It is also he who corrects various mistakes I make. All in all, Prof. Song-Ping Zhu is a responsible and qualified supervisor, and I am so proud to be one of his Ph.D. students.

My gratitude also goes to my co-supervisor, Dr. Xiaoping Lu, for her help and patient guidance. I would also like to thank Dr. Heather Jamieson and Ms. Ashlee Davis for their help in polishing my thesis. I am also grateful for my dear friends, Dr. Xinjiang He, Dr. Guiyuan Ma, Dr. Xiangchen Zeng, Dr. Ziwei Ke, Ms. Dong Yan, Dr. Chengbo Yang and Dr. Chi Chung Siu, who have made my studies at the University of Wollongong full of fun.

I am also grateful for the International Postgraduate Tuition Award from the University of Wollongong and the Chinese Government Scholarship from the China Scholarship Council which have provided me with financial support so that I can realize my dream of doing a Ph.D. here. I would like to thank the IMIA and the Australian Mathematical Society for their financial support that made it possible for me to attend academic conferences, including ANZIAM 2018 and QMF 2018.

Finally, I would like to thank my parents, sister and boyfriend for their endless support, understanding and encouragement.
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Chapter 1

Introduction and Literature review

1.1 Convertible bonds

In finance practice, a derivative is a contract whose value is dependent on the price of an underlying asset; this underlying asset could be a stock, index, or interest rate. The aims of issuing financial derivatives are to insure against price movements for hedging, increasing exposure to price movements for speculation or getting access to otherwise hard-to-trade assets or markets. Among these, the use for hedging purposes is the most important one, as the management of different types of risk caused by the price movement is always an ongoing topic for investors. Due to this, financial derivatives are becoming increasingly popular, ever since the establishment of the Chicago Board Options Exchange in 1973.

Financial derivatives can be classified into two main types according to how they are traded, i.e., exchange-traded and over-the-counter (OTC) derivatives. While exchange-traded derivatives, that is those that are traded on exchange markets, are standardized financial instruments and traded through a recognized intermediary, the OTC derivatives, representing those traded directly between two parties without going through any intermediary, provide more flexibility as these contracts can be customized according to the demand of the two parties. In fact, the OTC derivative market is actually the largest market for derivatives.

Among all the financial derivatives, convertible bonds have attracted a lot of attention and their trading volumes have experienced rapid growth. A convertible bond is actually one of the hybrid financial instruments, combining the attributes of fixed-income securities and equities. For the simplest convertible bond, the holder receives fixed-rate coupon payments as if holding a classical bond when the conversion has not been exercised, and he/she is also entitled to convert the bond into a predetermined number of stocks to maximize his/her benefit. Of course, more sophisticated convertible bond contracts with different embedded options and trig-
ering conditions have been established to cater for the different kinds of demand from issuers as well as investors. This has contributed to the rapid development of the convertible bond market.

This hybrid nature of convertible bonds actually favours a vast class of investors who want to gain more return than that provided by the classical bond while at the same time avoiding the high risk involved in directly holding the underlying stocks. The convertible bond also provides an opportunity for the investors to participate in both the fixed-income and equity market. On the other hand, the issuer can also take advantage of convertible bonds in the sense that selling convertible bonds enables the issuer to receive more payments compared with selling classical bonds. More importantly, this actually helps reduce cash interest payments, since once the bonds are converted into the stocks, the issuers no longer need to pay anything.

Although this particular hybrid nature of convertible bonds has made them extremely useful to both the issuer and investor, it has also led to a much more complicated pricing problem. Even with the simplest American-style convertible bonds, there is an additional optimal conversion boundary. This arises from the right of the holder to choose whether the bond is to be converted or not, which needs to be determined simultaneously with the bond price. When additional clauses are introduced into convertible bonds, the corresponding pricing problem can become even more complex.

Because of the tremendous complexity and diversity of convertible bonds, finding suitable mathematical tools to derive the solution of the pricing problem is very important and crucial. Although a large amount of research interest has been directed into this area, most of the developed approaches for the pricing of convertible bonds with additional features are numerical ones, even when the simplest Black-Scholes model is adopted. There are few existing results on how to price convertible bonds when the Black-Scholes model is unable to capture the features of the underlying asset. Thus, a more sophisticated model, such as stochastic volatility and stochastic interest rate models, must be adopted.

The aim of this thesis can be summarized from two aspects. On one hand, analytical pricing formulae for convertible bonds with different additional features under the Black-Scholes are presented, using the integral equation approach. This integral equation approach is superior to numerical methods because errors are usually introduced at very early stage of computation when numerical methods are adopted and these methods often suffer from inefficiency problems, making them difficult to apply in practice. On the other hand, when stochastic volatility and stochastic interest rate models are adopted, analytical approaches are no longer possible due to the additional dimension introduced by the newly added stochastic source. To overcome these difficulties we establish an efficient and accurate numerical approach
for the pricing of convertible bonds.

Before we move to conduct a review of the literature, several important and common types of convertible bonds need to be introduced, the details of which are presented in the next two subsections.

1.1.1 Vanilla convertible bonds

The most classical and simplest type of convertible bonds is the so-called standard (vanilla) convertible bond, which forms the basis of other more complex convertible bonds. The vanilla convertible bond can be treated as a straight bond plus a call option, and it gives the right to the holder to convert the bond into a preset number, which is named as the conversion ratio, of the stocks\(^a\). This means that the holder of the bond can choose to receive the face value of the bond or a certain number of the stocks at expiry.

It should be pointed out that similar to the option contract, there are also two different styles, according to whether the early exercise of the conversion is allowed or not. While a European-style convertible bond only allows the holder to convert the bond at expiry, the holder of an American-style one can convert the bond at any time during the lifetime of the bond. The latter case is much more complicated than the former one, as the pricing of the American-style convertible bond is actually a free boundary problem, where an unknown optimal conversion boundary of the holder always needs to be determined together with the bond price. This adds another degree of complexity to the corresponding pricing problem.

1.1.2 Additional features

As mentioned above, some additional features can be incorporated to formulate non-standard convertible bonds according to practical demand. In the following, some important and widely used features are illustrated. It should be noted that similar to the conversion feature, all the features introduced below can also be set as European-style or American-style, and this will be omitted when discussing these features.

Call feature

This is the right that enables the issuer to recall (repurchase) the bond at the predetermined call price. In practice, there are two styles of the call feature. The first allows the issuer to recall the bond at the call price after a certain date without

\(^a\)It should be remarked here that in the real market, the contract often specifies the conversion price instead of the conversion ratio, and there is also the relationship between these two values, i.e., Face value = Conversion ratio \times Conversion price.
imposing any other conditions, the so-called “hard-call” feature. The second one, named as “soft-call” feature, entitles the issuer to recall the bond when the underlying asset price satisfies a typical condition specified in the contract, a typical example of which is that the underlying price exceeds 120% of the conversion price for 15 days out of past consecutive 30 days. One can clearly observe that such a call feature actually protects the benefit of the issuer, and thus the formulated callable convertible bond would be worth less than the corresponding vanilla convertible bond.

It should be remarked here that although the soft-call convertible bonds have been popular in the Chinese market in recent years, very few researchers have considered the corresponding pricing problems. A very recent work produced by Ma et. al. evaluated the contract using the two-factor willow tree method. In fact, this soft-call feature can be treated as the moving window Parisian feature (see), which is much more complicated than the Parisian options considered in . This is not only because of the early exercise feature but also the moving window, which is introduced by the fact that we always need to consider past consecutive days to see if the call feature is activated or not. Such a complicated problem is left for future research.

Put feature

This is the right that allows the holder to sell the bond back to the issuer at the put price listed in the contract. Similar to the call feature introduced above, the put feature can also be classified into two different types, i.e., the hard-put and the soft-put feature. While the former gives the right to the holder to sell the bond at the put price before a certain date without any other conditions, the latter can only be activated if the underlying asset price satisfies a typical condition, such as the underlying price staying below 80% of the conversion price for 20 days out of past consecutive 30 days. Since this feature is in favor of the holder, it is expected that the value of the formulated puttable convertible bond would be higher than that of the corresponding vanilla convertible bond.

It should be pointed out that the call/put feature embedded in the convertible bond and the call/put options are very different. In fact, all the convertible bonds can be treated as different types of call options, and thus there is no put-call parity for convertible bonds.

Reset feature

With the reset feature, the conversion ratio/price can be reset to a new value depending on the evolution of the underlying asset price, and this is usually used in
the case that the conversion price will be decreased if the underlying asset does not perform well for a certain period. For example, the conversion price will be replaced by the current underlying price when the underlying price is below 60% of the conversion price after a year. This is not a right given to the holder, but it is beneficial to the holder as it reduces the cash flow of the issuer.

Contingent conversion

As is well known, there is a conversion price associated with all the convertible bonds, which is the amount that the holder needs to pay for each share of the stock when they chose to convert the bond into stocks. Being different from most of the convertible bonds, contingent convertible bonds, proposed by Flannery\cite{45}, do not specify the conversion price in the contract, and instead the actual stock price at conversion acts as the conversion price. Moreover, the conversion is no longer the right of the holder, but will take place automatically, once a pre-specified event leading to the conversion process called trigger happens. For example, if the current stock price of the issuer is $100, the contingent clause would be that the bond will be directly converted into stocks when the stock price falls below $50. It has been widely acknowledged that contingent convertible bonds have the potential to prevent systematic collapse of important financial institutions\cite{1}, and a bankruptcy can be fully prevented because of fast input of capital coming from the conversion.

Non-dilutive feature

The non-dilutive feature can be realized by the issuer through selling a standard convertible bond and hedging it through purchasing call options on its own stock with the same notional amount and maturity as specified in the convertible bond contract. This will cancel out the dilution in the case of a conversion taking place. The contract would often restrict the possibility of early conversion to fully prevent dilution. This has been introduced in the environment of lower interest rates, from which the issuers already benefit in the straight bond market, to encourage the issuance of convertible bonds.

1.2 Literature review

1.2.1 Pricing models

In 1973, Black & Scholes\cite{9} and Merton\cite{85} made a great contribution by proposing the so-called Black-Scholes model (B-S model) or the Black-Scholes-Merton model, assuming that the underlying asset price follows a geometric Brownian mo-
CHAPTER 1. INTRODUCTION AND LITERATURE REVIEW

This model is very popular as it could lead to simple pricing formulae for various important financial derivatives, such as European and barrier options, and thus it is still widely used in today’s finance markets.

However, some simplified assumptions, such as the constant interest rate and volatility, made in the B-S model are not consistent with real market observations. In particular, the distribution of the underlying asset price is usually asymmetric and exhibits the features like skewness and fat-tails, which are at odds with the assumption of the GBM. On the other hand, the implied volatility extracted from the real market option prices always shows a “smile” curve, which violates the assumption of the constant volatility in the B-S model.

There are two different types of modifications to the B-S model; the first is to replace the standard Brownian motion with another stochastic process, while the other is to add the stochastic interest rate or/and the stochastic volatility into the B-S model. Lévy processes are a typical example included in the former category, and they are very popular because they also possess the property of the independent and stationary increments as the Brownian motion does. For example, Merton considered a jump-diffusion model by utilizing a Gaussian distribution to model jumps of log-returns, while the Variance-Gamma model and CGMY model were respectively proposed by Madan and Carr, et. al., both of which are infinite activity Lévy processes. Of course, apart from the Lévy processes, there are also many other stochastic processes that have been applied to replace the Brownian motion, such as the fractional Brownian motion used by Necula to capture the long range dependence in asset prices.

On the other hand, the relaxation of the constant interest rate or volatility assumptions in the B-S model has also received a lot of attention. Thus, these two approaches will be elaborated in the following.

Stochastic interest rate models

In the context of the stochastic interest rate, several short-rate models have been established to describe the future evolution of the short rate. These models can be mainly divided into two main categories, depending on how many stochastic factors are used to determine the process of the interest rate.

The first category is the so-called one-factor short-rate model, where the interest rate is controlled by a single Brownian motion. For example, the Merton model assumes that the stochastic interest rate follows a Gauss-Wiener process, while the Vasicek model adopts an Ornstein-Uhlenbeck stochastic process for the interest rate. Other well-known models include the Rendleman-Bartter model, using another GBM for the interest rate, the Cox-Ingersoll-Ross model (CIR model), which is defined as a sum of squared Ornstein-Uhlenbeck processes, the Ho-Lee
model[53], defining the stochastic interest rate as a normal process, the Hull-White
model[56], which extends the Vasicek model by allowing the model parameters to
be time dependent, the Black-Karasinski model[8], a log-normal version of the Hull-
White model, and the Kalotay-Williams-Fabozzi model[66], which is a log-normal
analogue to the Ho-Lee model.

On the other hand, multi-factor short-rate models, under which the stochastic
interest rate is controlled by two or more Brownian motions, have also been pro-
posed to provide more flexibility in fitting real market data. The Longstaff-Schwartz
model[78] and the Chen model[24] are examples for the two-factor model and the
three-factor model, respectively. A general framework of multi-factor interest rate
models are established in[72], based on the assumption that the term structure of
interest rates is “embedded in a large macroeconomic system”.

Apart from the short-rate models, another framework to incorporate the stochastic
interest rate is the Heath-Jarrow-Morton framework (HJM)[51]. In fact, the model
under the HJM framework is very different from the short-rate models mentioned
above, since the HJM-type models are able to capture the full forward rate curve,
while the short-rate ones only yield a point on the rate curve.

Local and stochastic volatility models

Due to the phenomenon of the “volatility smile”, another popular approach in mod-
ifying the B-S model is to add a non-constant volatility. A natural choice for this
is to make the volatility a deterministic function of the underlying asset price and
time, formulating the local volatility model[35,41]. The constant elasticity of variance
(CEV) model[27] can be treated as a local volatility model, which is used by market
traders in the finance practice, especially for modeling equities and commodities.

On the other hand, an alternative approach belonging to this category is to make
the volatility of the underlying asset price another random variable, formulating the
stochastic volatility (SV) models. In particular, the volatility is assumed to follow a
log-normal distribution under the Hull-White model[55], while the SABR volatility
model proposed in[49] proves to be able to reproduce the smile effect of the volatility
smile[57]. Moreover, the popular Heston model[52] adopts the CIR process for the
variance of the underlying asset price, while a similar 3/2 model has recently gained a
lot of attention due to its attractive features[20], which assumes that the randomness
of the variance process varies with $v_t^{3/2}$. Being different from the Heston and 3/2
models, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH)
model[10] assumes that the randomness of the variance process varies with $v_t$. 
1.2.2 Convertible bonds pricing

It should be pointed out that the pricing of American-style financial derivatives is usually a very challenging problem\textsuperscript{[54,79]} as a result of the inherent characteristic of these contracts, that they can be exercised at any time before the expiry. Thus, for an American-style convertible bond, the buyer has an additional right to convert the bond into stocks earlier during the life of the contract, which formulates the corresponding pricing problem as a free boundary problem. This means that the optimal conversion price should be determined together with the bond price in the solution procedure. Mathematically, the unknown domain of the solution, resulted from this additional right, makes the pricing problem highly nonlinear and thus difficult to solve.

One of the earliest work for pricing convertible bonds was produced by Ingersoll\textsuperscript{[62]}, who took the firm value as the underlying variable under the B-S framework and derived a closed-form pricing formula for some special cases using the no-arbitrage theory. This approach was further extended in\textsuperscript{[15,73]} to consider more complex situations. However, these results were very restrictive as they did not incorporate the possibility of early conversion, which is a common feature of convertible bonds in practice. Brennan & Schwartz\textsuperscript{[12]} also worked under a firm-value based B-S model, but adopted a finite difference method for the pricing of American-Style convertible bonds. However, considering that firm values are usually not observable in real markets, this model is not suitable to be applied in practice as far as model calibration is concerned. Thus, just a few year later, McConnel & Schwartz\textsuperscript{[84]} modified the approach to propose a single-factor pricing model for a zero-coupon convertible bond with the stock price, which is available in real markets, being selected as the underlying variable.

Since then, a large amount of research interest has been led into the pricing of American-style vanilla convertible bonds under the stock-value based B-S framework. Of course, numerical approaches, such as the binomial tree method\textsuperscript{[31]}, Monte Carlo simulation\textsuperscript{[79]}, finite difference method\textsuperscript{[98]} and finite volume method\textsuperscript{[112]}, can be adopted. However, numerical methods often suffer from inaccuracy and inefficiency problems, which hinder their potential applications in finance practice. Therefore, a large number of authors have been focusing on searching for analytical solutions. In particular, Zhu\textsuperscript{[106]} presented an analytical solution in the form of a Taylor series expansion for the American-style vanilla convertible bond, using the Homotopy Analysis Method (HAM)\textsuperscript{[74]}, and this approach was further extended by Chan & Zhu\textsuperscript{[23]}, who successfully derived an approximation solution for the price of a convertible bond under the regime-switching model. An alternative semi-analytical approach that is often employed for financial derivative pricing is the integral equa-
tion method, and it has already been applied by Zhu & Zhang\textsuperscript{[111]} to formulate an integral equation representation for vanilla convertible bond prices, following Kim’s approach\textsuperscript{[68]}.

As mentioned above, the B-S model is sometimes inadequate to capture the main characteristics exhibited by the underlying asset price, and thus various more sophisticated models have been proposed to modify the B-S model. In terms of pricing the convertible bonds, stochastic interest rate models are among the most popular ones simply due to the nature of long lifetime for most of the convertible bonds in real markets. However, due to the additional stochastic source introduced, the analytical solution is only available in very special cases, a typical example of which is the closed-form pricing formula for a simple European-style convertible bond presented by Nyborg\textsuperscript{[88]}, who assumed that the holder can only convert the bond at maturity. When it comes to pricing American-style convertible bonds under stochastic interest rate models, numerical methods must be resorted to. For example, the finite difference method has been adopted by Brennan & Schwartz\textsuperscript{[12]}, Ayache et al.,\textsuperscript{[5]} and Andersen & Buffum\textsuperscript{[4]} among many others to value convertible bonds, while Barone-Adesi et al.,\textsuperscript{[6]} priced these contracts using the method of characteristics together with finite elements. Binomial/trinomial trees are also very popular in pricing convertible bonds with stochastic interest rates, and they have been discussed by a number of different authors, including Takahashi et al.,\textsuperscript{[97]}, David & Lischka\textsuperscript{[34]}, Hung & Wang\textsuperscript{[59]}, Carayannopoulos & Kalimipalli\textsuperscript{[18]} and Chambers & Lu\textsuperscript{[21]}. Recently, Lin & Zhu\textsuperscript{[77]} established a predictor-corrector method embedded with an ADI scheme for the pricing of convertible bonds when the interest rate or volatility is assumed to be stochastic, the results of which are presented in Chapter 6.

Nowadays, due to the different kinds of demands in practice, many useful clauses have been introduced into the vanilla convertible bond, examples of which include the call feature, put feature and reset feature, formulating different types of convertible bonds. For example, incorporating the call feature yields the so-called callable convertible bonds. Brennan & Schwartz\textsuperscript{[12]} was believed to be the first to discuss the pricing problem of these contracts in theory, while the corresponding solutions were derived in their later article using the finite difference method\textsuperscript{[13]}. Later on, the binomial tree method was also applied to obtain the price of a callable convertible bond by Bernini\textsuperscript{[7]}, while Yagi & Sawaki\textsuperscript{[103]} considered the pricing problem of callable convertible bonds using the game option defined in\textsuperscript{[67]}. On the other hand, when the put feature is taken into consideration, Nyborg\textsuperscript{[88]} presented the boundary conditions for the formulated puttable convertible bonds, while Lvov et al.\textsuperscript{[80]} and Ammann et al.\textsuperscript{[3]} numerically solved the pricing problem by using Monte Carlo simulations. Recently, an integral equation formulation for the puttable convertible
bond price was presented by Zhu et al.\textsuperscript{[100]}, which forms the main context of Chapter 3. Lin & Zhu\textsuperscript{[76]} went even further to apply the integral equation approach for convertible bond pricing when both call and put features are available, with details provided in Chapter 4. Finally, the pricing of resettable convertible bonds has not been investigated until very recently, and there are only few numerical results on this topic\textsuperscript{[43,70]}. However, the resettable convertible bonds are gaining attention from both market practitioners and academic researchers, because they are now widely used in the finance industry. To cope with the demand for the accurate and efficient pricing of resettable convertible bonds, Lin & Zhu\textsuperscript{[75]} developed an integral equation representation for the price of this relatively new type of convertible bonds, and this will be illustrated in Chapter 5.

1.3 Structure of thesis

In Chapter 2, we introduce some basic mathematical knowledge needed for pricing different kinds of American-style convertible bonds under various models. We start by recalling different models used to describe the underlying dynamics, and derive the PDEs accompanied by appropriate boundary conditions for convertible bond pricing. We also review some numerical methods and the integral equation approach, with some examples presented to make these approaches better understood. Some of these techniques will be extended to price different types of convertible bonds under various models in later chapters.

In Chapter 3, we adopt an integral equation approach to price American-style puttable convertible bonds. Upon applying an incomplete Fourier transform, an integral equation representation for the bond price is derived under the B-S model. To avoid numerical difficulty caused by the discontinuity along both free boundaries as well as the involvement of two first-order derivatives of the unknown optimal exercise prices in this formulation, a second integral equation formulation is further presented, after some manipulations of the first form. The effect of the put feature is investigated through numerical examples.

In Chapter 4, the call feature is further introduced into the American-style puttable convertible bonds to formulate the so-called callable-puttable convertible bonds, which makes the corresponding pricing problem even more complicated due to the tangled presence of callability, puttability, as well as conversion. Such complexity can be further shown when mathematically solving this problem, which involves the discussion of various different scenarios. Different integral equation formulations are presented by solving different PDE systems, and various properties of callable-puttable convertible bonds are numerically demonstrated.

In Chapter 5, we propose an integral equation approach for pricing American-style
resettable convertible bonds under the B-S model. This is a challenging problem because an unknown optimal conversion price needs to be determined together with the bond price. There is also the additional complexity that the value of the conversion ratio will change when the underlying price touches the reset price. Despite these difficulties, we still manage to present an integral equation formulation for the bond price, after successfully establishing the governing PDE system. The bond price turns out to be a non-monotonic increasing function of the underlying price, which is a unique feature that distinguishes it from other types of convertible bonds.

In Chapter 6, we move out of the B-S framework by allowing the volatility or interest rate to be stochastic when pricing American-style convertible bonds. The newly introduced stochastic source leads to a two-dimensional free boundary problem, for which a predictor-corrector scheme with the Douglas-Rachford method embedded in the correction step has been developed to solve the corresponding pricing PDE system. Numerical results are used to validate our approach as well as to show the influence of stochastic volatility and interest rate on the bond prices and the optimal conversion boundary.

Some concluding remarks are provided in Chapter 7 to summarize the main results presented in this thesis.
Chapter 2

Background of Mathematics

In this chapter, we introduce some mathematical models and techniques, which are the foundation of and will be utilized in this thesis.

In particular, some popular models for the underlying asset price when pricing the convertible bond are firstly discussed. The details on how to obtain the PDE governing the convertible bond prices from the stochastic differential equation (SDE) of the underlying asset price will then be illustrated, after which appropriate boundary and terminal conditions are given to formulate closed PDE systems. In addition, some mathematical techniques, including different numerical and analytical methods, are further investigated, which are the main tools in obtaining the value of the considered bond. It should be pointed out that these techniques can of course be used for pricing other financial derivatives, as long as their prices solve the similar PDE system as we discuss here.

2.1 The Black-Scholes model

One of the most popular and classical mathematical models to price the convertible bond is the so-called Black-Scholes model, which was established by Black & Scholes\[\text{[9]}\] and Merton\[\text{[85]}\] in 1973. It was initially proposed for evaluating options, and it can be regarded as the foundation of the financial derivative pricing theory. In particular, they assumed that the underlying price, $S_t$, should satisfy a GBM:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

(2.1)

where $\mu$ is the drift term, $\sigma$ is the constant volatility and $W_t$ is a standard Brownian motion. The B-S model was firstly applied for the pricing of convertible bonds by Ingersoll\[\text{[61]}\] and Brennan & Schwartz\[\text{[12]}\] in 1977, by taking the firm value as the underlying asset. However, this is not appropriate as the value of the firm is very difficult to obtain in the real market, and thus McConnel & Schwartz\[\text{[84]}\]
improved this method, by replacing the firm value with the stock price, which is more observable. Before we are able to derive the PDE governing the convertible bond prices, several important assumptions made under the B-S model should be pointed out. Firstly, there exists a risk-free asset, such as a bank account, whose value accumulates with a continuously compounded risk-free interest rate $r$. Secondly, the financial market is perfect in the sense that the market trading is continuous without transaction costs. Thirdly, short selling is permitted and all securities are perfectly divisible. Fourthly, there are no arbitrage opportunities, and all derivatives can be perfectly hedged with the underlying price and bank deposit. Under these assumptions, two methods that are equivalent to each other will be presented to derive the PDE for the bond prices from the SDE for the underlying asset price.

2.1.1 Derivation of the PDE

Let us begin with the Itô’s Lemma.

Lemma 2.1.1 (Itô’s Lemma). Assume $X_t$ is a random variable, and satisfies the following SDE

$$dX_t = A(X,t)dt + B(X,t)dW_t,$$

then any twice differentiable scalar function $F(X,t)$ follows the stochastic dynamics as follows

$$dF = B(X,t)\frac{\partial F}{\partial X}dW_t + \left(\frac{\partial F}{\partial t} + A(X,t)\frac{\partial F}{\partial X} + \frac{1}{2}B^2(X,t)\frac{\partial^2 F}{\partial X^2}\right)dt.$$  \hspace{1cm} (2.3)

If $V(S_t,t)$ denotes the value of the convertible bond, and the underlying asset price $S_t$ satisfies the SDE (2.1), applying Itô’s Lemma yields

$$dV(S,t) = \sigma S\frac{\partial V}{\partial S}dW_t + \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt.$$  \hspace{1cm} (2.4)

The first method is the so-called martingale method (or the risk neutral pricing principle), which states that the discounted asset price is a martingale, i.e.,

$$\mathbb{E}[e^{-rT}S_T|S_t] = e^{-rt}S_t, \ \forall \ T > t,$$  \hspace{1cm} (2.5)

and

$$\mathbb{E}[e^{-rT}V(S_T,T)|S_t] = e^{-rt}V(S_t,t), \ \forall \ T > t.$$  \hspace{1cm} (2.6)

If one applies (2.5), it is not difficult to find that $\mu$ in (2.1) should be replaced by $r$. Equation (2.6) further implies

$$\mathbb{E}[d(e^{-rt}V(S_t,t))|S_t] = 0, \ \forall \ t > 0.$$

(2.7)
and thus we can obtain
\[
\mathbb{E}[\exp(-rT)V(S_t, t) + e^{-rT}dV(S_t, t)|S_t] = 0
\]
\[
\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.8}
\]

The other method is the hedging method, where we construct a self-financing portfolio consisting of a bond and \(-\Delta\) shares of the underlying asset. In this case, the value of this portfolio is
\[
\Pi = V - \Delta S, \tag{2.9}
\]
from which we have
\[
d\Pi = dV - \Delta dS. \tag{2.10}
\]

Again, using Itô's Lemma leads to
\[
d\Pi = \sigma S \frac{\partial V}{\partial S} dW + \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right) dt - \Delta(\mu S dt + \sigma S dW)
\]
\[
= \sigma S \left(\frac{\partial V}{\partial S} - \Delta\right) dW + \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - \Delta \mu S\right) dt. \tag{2.11}
\]

To make the portfolio a risk-free asset, any stochastic term should be eliminated. Therefore, setting
\[
\Delta = \frac{\partial V}{\partial S}, \tag{2.12}
\]
yields
\[
d\Pi = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial S} \mu S\right) dt
\]
\[
= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.13}
\]

On the other hand, since \(\Pi\) is a risk-free asset, we have
\[
d\Pi = r\Pi dt = r(V - S \frac{\partial V}{\partial S}) dt. \tag{2.14}
\]

Then, we obtain
\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} = r(V - S \frac{\partial V}{\partial S})
\]
\[
\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.15}
\]

Actually, there is a famous theorem, the Feynman-Kac theorem, that establishes the relationship between the SDE of the underlying asset price and the PDE of the financial derivative price. The content of this theorem is provided below.
Theorem 2.1.1 (Feynman-Kac theorem) If the underlying asset price $S_t$ satisfies the SDE
\[ dS_t = \mu(S_t)S_t dt + \sigma(S_t)S_t dW_t, \] (2.16)
and the financial derivative price $V(S,t)$ solves the PDE
\[
\begin{aligned}
& \frac{\partial V}{\partial t}(S,t) + \mu(S,t)\frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2}(S,t) - u(S,t)V(S,t) + f(S,t) = 0, \\
& V(S,T) = \phi(S),
\end{aligned}
\] (2.17)
for all $S \geq 0$ and $t \in [0,T]$, the solution of $V(S,t)$ can be written as a conditional expectation
\[ V(S,t) = \mathbb{E}_Q^Q[Z_T | S_t = S]. \] (2.18)

Remark: The known functions, $\mu(S,t)$, $\sigma(S,t)$ and $u(S,t)$, should satisfy the following conditions, respectively.
\[
\begin{aligned}
\mu(S,t) : \mathbb{P}[\int_0^\infty |\mu(S,\tau)|d\tau] &= 1, \forall 0 \leq t \leq \infty, \\
\sigma(S,t) : \mathbb{P}[\int_0^\infty |\sigma^2(S,\tau)|d\tau] &= 1, \forall 0 \leq t \leq \infty, \\
u(S,t) : \mathbb{R} \times [0,T] \to \mathbb{R}.
\end{aligned}
\] (2.19-2.21)

2.1.2 Boundary conditions

In this subsection, the boundaries conditions as well as the terminal condition are set up to close the pricing PDE system. Firstly, the terminal condition can be easily given with the payoff of the convertible bond:
\[ V(S,T) = \max\{nS,Z\}, \] (2.22)
where $n$ is the conversion ratio, which is the amount of stocks the holder can obtain when he/she converts the bond, and $Z$ is the face value of the bond. When the price of the stock, $S$, approaches zero, it is almost impossible for the holder to convert the convertible bond into stocks with such a low price. Thus, in this case, the holder will choose to hold the bond until the expiry and receive the face value, implying that the bond price is actually the discounted face value, i.e.,
\[ V(0,t) = Ze^{-r(T-t)}. \] (2.23)

Other boundary conditions should be considered separately, as they can be different and should be determined according to their style, the European-style or the
American-style. For the European-style convertible bond, the second boundary condition should be specified when $S$ approaches infinity, and its value is equal to that of the bond if it was converted into the stocks right now. This is because if the underlying asset price is high enough, the convertible bond will definitely be converted at expiry, which means that the holder will be receiving $n$ shares of stocks. In this case, we have

$$
\lim_{S \to \infty} V(S,t) = nS.
$$

(2.24)

In contrast, when an American-style convertible bond is taken into consideration, there will be an unknown optimal conversion boundary, $S_f(t)$, that needs to be considered, as a result of the holder being entitled to the right to convert the bond before the expiry time. In particular, when the underlying asset price is higher than the optimal conversion boundary, the bond should be converted immediately, otherwise the holder is willing to wait until the time to expiry to receive the face value, since in this case the value of the obtained stocks after early conversion is less than the value of the contract. Thus, we should have

$$
V(S_f(t),t) = n S_f(t),
$$

(2.25)

which is accompanied by a smooth pasting condition

$$
\frac{\partial V}{\partial S}(S_f(t),t) = n.
$$

(2.26)

In this case, the PDE systems for the vanilla European-style and American-style convertible bond prices under the B-S model can be written as

$$
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV &= 0, \\
V(S,T) &= \max\{nS, Z\}, \\
V(0,t) &= Ze^{-r(T-t)}, \\
\lim_{S \to \infty} V(S,t) &= nS,
\end{align*}
$$

(2.27)

and

$$
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV &= 0, \\
V(S,T) &= \max\{nS, Z\}, \\
V(0,t) &= Ze^{-r(T-t)}, \\
V(S_f(t),t) &= nS_f(t), \\
\frac{\partial V}{\partial S}(S_f(t),t) &= n,
\end{align*}
$$

(2.28)

respectively.
2.2 Stochastic volatility or/and interest rate model

Since the B-S model is too simple to capture the main characteristics exhibited by real market data, various modifications have been proposed, among which stochastic volatility and stochastic interest rate models have received a lot of attention. In particular, a list of popular stochastic interest rate models\[^{22}\] is presented in Table 2.1.

<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merton</td>
<td>( dr = \alpha dt + \sigma dW )</td>
</tr>
<tr>
<td>Vasicek</td>
<td>( dr = (\alpha + \beta r)dt + \sigma dW )</td>
</tr>
<tr>
<td>CIR SR</td>
<td>( dr = (\alpha + \beta r)dt + \sigma r^{1/2}dW )</td>
</tr>
<tr>
<td>Dothan</td>
<td>( dr = \sigma rdW )</td>
</tr>
<tr>
<td>GBM</td>
<td>( dr = \beta r dt + \sigma r dW )</td>
</tr>
<tr>
<td>CIR VR</td>
<td>( dr = \sigma r^{3/2}dW )</td>
</tr>
<tr>
<td>CEV</td>
<td>( dr = \beta r dt + \sigma r^2 dW )</td>
</tr>
</tbody>
</table>

In summary, all of the models in this table can be written as: \( dr = \kappa (\eta + r) dt + \sigma r^\alpha dW \); choosing different values for these parameters, \( \kappa, \eta \) and \( \alpha \), would yield different models. Similarly, popular stochastic volatility models can also be generally expressed as \( dv = a (\beta + v) dt + \theta v^3 dW \) (here, for illustration purposes, we deliberately choose different parameters for volatility and interest rate processes). Furthermore, combing these two classes will give rise to hybrid stochastic volatility and interest rate models. Having the knowledge of the SDE under stochastic volatility or/and interest rate models, we are now ready to derive the PDE for the bond prices under these models. For simplicity, we will pick one model in each class to show the details on how to derive the PDE.

2.2.1 PDE for the Heston model

Under the Heston model, the underlying asset price is assumed to follow the dynamic

\[
    dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t,
\]

where \( \mu \) is the drift term and \( W_t \) is a standard Brownian motion. \( v_t \) represents stochastic volatility, satisfying

\[
    dv_t = \theta (\omega - v_t) dt + \xi \sqrt{v_t} dB_t,
\]

where \( \theta, \omega \) and \( \xi \) are the mean reversion speed, the long-term mean and the volatility of the volatility, respectively. \( B_t \) is also a standard Brownian motion, and it is correlated with \( W_t \) with the correlation \( \rho \). To derive the PDE, the hedging method
is used, involving the construction of a self-financing portfolio. However, it should be noted that unlike the B-S model, where only one random variable is involved, there are two stochastic sources in this model. Thus, the self-financing portfolio, \( \Pi \), should not only contain the bond, \( V(S,v,t) \), and \(-\Delta_1 \) shares of the underlying assets, it should also have \(-\Delta_2 \) shares of another bond \( U(S,v,t) \). In other words,

\[
\Pi = V - \Delta_1 S - \Delta_2 U. \tag{2.31}
\]

According to Itô’s lemma,

\[
d\Pi = dV - \Delta_1 dS - \Delta_2 dU = \\
\frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv + \frac{\partial V}{\partial t} dt + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} d^2 S + \frac{\partial^2 V}{\partial v^2} d^2 v + 2 \frac{\partial^2 V}{\partial v \partial S} dv dS \right) - \Delta_1 dS - \Delta_2 \left[ \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial t} dt + \frac{1}{2} \left( \frac{\partial^2 U}{\partial S^2} d^2 S + \frac{\partial^2 U}{\partial v^2} d^2 v + 2 \frac{\partial^2 U}{\partial v \partial S} dv dS \right) \right] \\
= \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} - \Delta_1 \right) dS + \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U}{\partial v} \right) dv + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t} \right) dt \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) d^2 S + \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U}{\partial v^2} \right) d^2 v + \frac{\partial^2 V}{\partial v \partial S} dv dS \\
= \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} - \Delta_1 \right) \left( \mu S dt + \sqrt{\rho} \sqrt{S} dW_t \right) + \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U}{\partial v} \right) \left( \theta (\omega - \nu) dt + \xi \sqrt{v} dB_t \right) \\
+ \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2 dt + \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U}{\partial v^2} \right) \xi^2 v dt + \frac{\partial^2 V}{\partial v \partial S} \rho \xi v S dt \\
= \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} - \Delta_1 \right) \mu S + \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U}{\partial v} \right) \theta (\omega - \nu) + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t} \right) \\
+ \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U}{\partial v^2} \right) \xi^2 v + \frac{\partial^2 V}{\partial v \partial S} \rho \xi v S dt \\
+ \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} - \Delta_1 \sqrt{v} S dW_t + \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U}{\partial v} \right) \xi \sqrt{v} dB_t. \tag{2.32}
\]

Applying the strategy of dynamic hedging, the term \( dW_t \) and \( dB_t \) should be eliminated, which implies that we need to set

\[
\Delta_2 = \frac{\partial V}{\partial v} / \frac{\partial U}{\partial v}, \tag{2.33}
\]

\[
\Delta_1 = \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S}. \tag{2.34}
\]

On the other hand, since this portfolio is risk-free, it should satisfy the following risk-free condition

\[
d\Pi = r \Pi dt = r(V - \Delta_1 S - \Delta_2 U) dt. \tag{2.35}
\]
These eventually yield

\[ r(V - \Delta_1 S - \Delta_2 U) = \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} - \Delta_1 \right) \mu S + \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U}{\partial v} \right) \theta (\omega - v) + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t} \right) \]

\[ + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) v S^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U}{\partial v^2} \right) \xi^2 v + \left( \frac{\partial^2 V}{\partial v \partial S} - \Delta_2 \frac{\partial^2 U}{\partial v \partial S} \right) \rho \xi v S \]

\[ \Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} + \theta (\omega - v) \frac{\partial V}{\partial v} + \rho \xi v S \frac{\partial^2 V}{\partial v \partial S} - r V \]

\[ = \Delta_2 \left( \frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + r S \frac{\partial U}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 U}{\partial v^2} + \theta (\omega - v) \frac{\partial U}{\partial v} + \rho \xi v S \frac{\partial^2 U}{\partial v \partial S} - r U \right) \]

\[ \Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} + \theta (\omega - v) \frac{\partial V}{\partial v} + \rho \xi v S \frac{\partial^2 V}{\partial v \partial S} - r V \]

\[ = \left( \frac{\partial U}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + r S \frac{\partial U}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 U}{\partial v^2} + \theta (\omega - v) \frac{\partial U}{\partial v} + \rho \xi v S \frac{\partial^2 U}{\partial v \partial S} - r U \right) \frac{\partial U}{\partial v}. \]

(2.36)

Clearly, the left hand side of the above equation only involves the function \( V \), while the right hand side contains \( U \) only. Thus, both sides of the above equation should only depend on the variables \( S, v \) and \( t \), i.e.,

\[ \frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} + \theta (\omega - v) \frac{\partial V}{\partial v} + \rho \xi v S \frac{\partial^2 V}{\partial v \partial S} - r V = \lambda(S,v,t). \]

(2.37)

As a result, we obtain

\[ \frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} + \theta (\omega - \lambda) \frac{\partial V}{\partial v} + \rho \xi v S \frac{\partial^2 V}{\partial v \partial S} - r V = 0. \]

(2.38)

By now, the PDE for pricing the convertible bond under the Heston model has been derived, which involves an arbitrary function \( \lambda(S,v,t) \), which is consistent with the argument that a market with stochastic volatility is incomplete and there exists different risk-neutral measures. Actually, it is the market price of the volatility risk, and it can be selected according to different financially meaningful arguments. For simplicity, \( \lambda(S,v,t) \equiv 0 \) is a common choice.
2.2.2 PDE for the CIR interest rate model

We now consider a well-known stochastic interest rate model, the CIR model, under which the dynamic of the underlying asset price can still be formulated as

\[ dS_t = (r_t - D_0)S_t dt + \sigma S_t dW_t, \]  

(2.39)

where \( D_0 \) is the continuous dividend yield, \( \sigma \) is the constant volatility of the underlying asset, and \( W_t \) is a standard Brownian motion. However, the interest rate is no longer a constant, but a random variable, following the CIR process

\[ dr_t = \kappa(\eta - r_t)dt + \zeta \sqrt{r_t} dB_t, \]

(2.40)

where \( \kappa, \eta \) and \( \zeta \) are the mean reversion speed, the long-term mean and the volatility of the interest rate, respectively. The correlation between \( B_t \), another standard Brownian motion, and \( W_t \) is \( \rho \). Similar to the previous subsection, we also construct a self-financing portfolio, consisting of the bond, \( V(S, r, t) \), \( -\Delta_1 \) shares of the underlying assets and \( -\Delta_2 \) shares of another bond \( U(S, r, t) \), leading to

\[ \Pi = V - \Delta_1 S - \Delta_2 U. \]

(2.41)

Using the Ito’s lemma, we can obtain

\begin{align*}
\text{d}\Pi &= dV - \Delta_1 dS - \Delta_2 dU \\
&= \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \left( \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial S \partial r} dS dr \right) \right) \\
&\quad - \Delta_1 \left[ \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial r} dr + \frac{1}{2} \left( \frac{\partial^2 U}{\partial S^2} dS^2 + \frac{\partial^2 U}{\partial r^2} dr^2 + \frac{\partial^2 U}{\partial S \partial r} dS dr \right) \right] \\
&\quad + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) dS^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U}{\partial r^2} \right) dr^2 + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U}{\partial S \partial r} \right) dS dr \\
&\quad + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t} \right) dt + \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} \right) dS + \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U}{\partial r} \right) dr + \frac{\partial V}{\partial S} dS \left( \kappa(\eta - r) dt + \zeta \sqrt{r} dB \right) \\
&\quad + \frac{\partial V}{\partial r} dr \left( \kappa(\eta - r) dt + \zeta \sqrt{r} dB \right) + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) \sigma^2 S^2 dt + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U}{\partial r^2} \right) \zeta^2 r^2 dt + \frac{1}{2} \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U}{\partial S \partial r} \right) \rho \sigma \zeta S \sqrt{r} dt \\
&\quad + \left( \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S} \right) \sigma S dW + \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U}{\partial r} \right) \zeta \sqrt{r} dB. \\
\end{align*}

(2.42)
The strategy of dynamic hedging requires that the stochastic term \(dW_t\) and \(dB_t\) should be removed, as a result of which we need
\[
\Delta_2 = \frac{\partial V}{\partial r} / \frac{\partial r}{\partial t},
\]
\[
\Delta_1 = \frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S}.
\]
Furthermore, by making use of the fact that the self-financing portfolio is risk free, we can obtain
\[
d\Pi = r\Pi dt = r(V - \Delta_1 S - \Delta_2 U) dt.
\]
Thus,
\[
r(V - \Delta_1 S - \Delta_2 U) = (\frac{\partial V}{\partial S} - \Delta_1 - \Delta_2 \frac{\partial U}{\partial S})(r - D_0) + \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t} \kappa(\eta - r) + \frac{\partial V}{\partial r} - \frac{\partial U}{\partial r}
\]
\[
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) \sigma^2 S^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U}{\partial r^2} \right) \xi^2 r + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U}{\partial S \partial r} \right) \rho \sigma \xi \sqrt{r}
\]
\[
\Rightarrow r[V - \Delta_2 U - (\frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U}{\partial S}) S] = \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U}{\partial r} \kappa(\eta - r) + \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U}{\partial t}
\]
\[
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} - \Delta_2 \frac{\partial^2 U}{\partial S^2} \right) \sigma^2 S^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U}{\partial r^2} \right) \xi^2 r + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U}{\partial S \partial r} \right) \rho \sigma \xi \sqrt{r}
\]
\[
\Rightarrow rV - rS \frac{\partial V}{\partial S} - \kappa(\eta - r) \frac{\partial V}{\partial r} - \frac{\partial V}{\partial t} \frac{\partial U}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} - \rho \sigma \xi \sqrt{r} \frac{\partial^2 V}{\partial S \partial r}
\]
\[
= \Delta_2 [rU - rS \frac{\partial U}{\partial S} - \kappa(\eta - r) \frac{\partial U}{\partial r} - \frac{\partial U}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - \frac{1}{2} \xi^2 r \frac{\partial^2 U}{\partial r^2} - \rho \sigma \xi \sqrt{r} \frac{\partial^2 U}{\partial S \partial r}]
\]
\[
\Rightarrow \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + \kappa(\eta - r) \frac{\partial V}{\partial r} + \rho \sigma \xi \sqrt{r} \frac{\partial^2 V}{\partial S \partial r} - rV \right] \frac{\partial V}{\partial r}
\]
\[
= \left[ \frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} + \frac{1}{2} \xi^2 r \frac{\partial^2 U}{\partial r^2} + \kappa(\eta - r) \frac{\partial U}{\partial r} + \rho \sigma \xi \sqrt{r} \frac{\partial^2 U}{\partial S \partial r} - rU \right] \frac{\partial U}{\partial r}.
\]

It is clearly shown that the left hand side and the right hand side of the above equation are only dependent on \(V\) and \(U\), respectively. Therefore, both sides of the above equation should be equal to a certain function \(\lambda(S,r,t)\) with the variables \(S\), \(r\) and \(t\), which implies
\[
\left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + \kappa(\eta - r) \frac{\partial V}{\partial r} + \rho \sigma \xi \sqrt{r} \frac{\partial^2 V}{\partial S \partial r} - rV \right] \frac{\partial V}{\partial r} = \lambda(S,r,t).
\]

\[
\left[ \frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} + \frac{1}{2} \xi^2 r \frac{\partial^2 U}{\partial r^2} + \kappa(\eta - r) \frac{\partial U}{\partial r} + \rho \sigma \xi \sqrt{r} \frac{\partial^2 U}{\partial S \partial r} - rU \right] \frac{\partial U}{\partial r} = \lambda(S,r,t).
\]
Thus, we can finally arrive at
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + [\kappa (\eta - r) - \lambda] \frac{\partial V}{\partial r} + \rho \sigma \xi \sqrt{r} \frac{\partial^2 V}{\partial S \partial r} - rV = 0.
\]
(2.48)

Similar to the Heston model, we still need to choose a function for \(\lambda(S, r, t)\), which is expected as the market here is also incomplete.

### 2.2.3 PDE for the hybrid stochastic volatility and interest rate model

In this subsection, we consider a particular model obtained by combining the Heston model and the CIR model together. If the underlying asset price, stochastic interest rate and stochastic volatility are denoted by \(S_t\), \(r_t\) and \(v_t\), respectively, this hybrid model can be specified as

\[
dS_t = (r_t - D_0)S_t dt + \sqrt{v_t}S_t dW_t,
\]
(2.49)

\[
\text{d}r_t = \kappa (\eta - r_t) dt + \zeta \sqrt{r_t} dB_1^1,
\]
(2.50)

\[
\text{d}v_t = \theta (\omega - v_t) dt + \xi \sqrt{v_t} dB_2^2,
\]
(2.51)

where \(D_0\) is the continuous dividend yield, \(\kappa\) and \(\theta\) are the mean reversion speed of the interest rate and that of the volatility, respectively, \(\eta\) and \(\omega\) are the long term mean of the interest rate and the corresponding one of the volatility, respectively, and \(\zeta\) and \(\xi\) are the volatility of the interest rate and that of the volatility, respectively. \(W_t, B_1^1\) and \(B_2^2\) are all standard Brownian motions, and the correlations between each two are

\[
\text{Cor}(W_t, B_1^1) = \rho_1, \quad \text{Cor}(W_t, B_2^2) = \rho_2, \quad \text{Cor}(B_1^1, B_2^2) = \rho_3.
\]
(2.52)

We now construct a self-financing portfolio whose value can be expressed as

\[
\Pi = V - \Delta_1 S - \Delta_2 U_1 - \Delta_3 U_2,
\]
(2.53)

where \(V(S, v, r, t)\) represents the target bond price, and \(U_1(S, v, r, t)\) and \(U_2(S, v, r, t)\) are the prices of another two bonds, the introduction of which are due to the fact that there are three stochastic variables in this model. Using It\(\hat{o}\)'s lemma, we obtain

\[
d\Pi = dV - \Delta_1 dS - \Delta_2 dU_1 - \Delta_3 dU_2
\]

\[
= \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial v} dv + \frac{\partial V}{\partial t} dt
\]
\[
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial^2 V}{\partial r^2} dr^2 + \frac{\partial^2 V}{\partial v^2} dv^2 + 2 \frac{\partial^2 V}{\partial S \partial r} dS dr + 2 \frac{\partial^2 V}{\partial S \partial v} dS dv + 2 \frac{\partial^2 V}{\partial r \partial v} dr dv \right)
\]
\[
- \Delta_1 dS - \Delta_2 \frac{\partial U_1}{\partial S} dS + \frac{\partial U_1}{\partial r} dr + \frac{\partial U_1}{\partial v} dv + \frac{\partial U_1}{\partial t} dt.
\]
\[
+ \frac{1}{2} \left( \frac{\partial^2 U_1}{\partial S^2} d^2 S + \frac{\partial^2 U_1}{\partial r^2} d^2 r + \frac{\partial^2 U_1}{\partial v^2} d^2 v + 2 \frac{\partial^2 U_1}{\partial S \partial r} dSdr + 2 \frac{\partial^2 U_1}{\partial S \partial v} dSdv + 2 \frac{\partial^2 U_1}{\partial r \partial v} drdv \right) \\
- \Delta S \left[ \frac{\partial U_2}{\partial s} dS + \frac{\partial U_2}{\partial r} dr + \frac{\partial U_2}{\partial v} dv + \frac{\partial U_2}{\partial t} dt \right] \\
+ \frac{1}{2} \left( \frac{\partial^2 U_2}{\partial S^2} d^2 S + \frac{\partial^2 U_2}{\partial r^2} d^2 r + \frac{\partial^2 U_2}{\partial v^2} d^2 v + 2 \frac{\partial^2 U_2}{\partial S \partial r} dSdr + 2 \frac{\partial^2 U_2}{\partial S \partial v} dSdv + 2 \frac{\partial^2 U_2}{\partial r \partial v} drdv \right) \\
= \left( \frac{\partial V}{\partial s} - \Delta_2 \frac{\partial U_1}{\partial s} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dS + \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dr \\
+ \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dv + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U_1}{\partial t} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dt \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial s^2} - \Delta_2 \frac{\partial^2 U_1}{\partial s^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) d^2 S + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U_1}{\partial r^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) d^2 r \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U_1}{\partial v^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) d^2 v + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial r} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial r} \right) dSdr \\
+ \left( \frac{\partial^2 V}{\partial S \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial v} \right) dSdv + \left( \frac{\partial^2 V}{\partial r \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial v} \right) drdv \\
= \left( \frac{\partial V}{\partial s} - \Delta_2 \frac{\partial U_1}{\partial s} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \left[(r - D_0) dt + \sqrt{\gamma} S dW \right] \\
+ \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \left[ \kappa (\eta - r) dt + \sqrt{\gamma} dB^1 \right] \\
+ \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \left[ \theta (\omega - v) dt + \sqrt{\gamma} dB^2 \right] + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U_1}{\partial t} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dt \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial s^2} - \Delta_2 \frac{\partial^2 U_1}{\partial s^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) S^2 dt + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U_1}{\partial r^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) \xi^2 r dt \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U_1}{\partial v^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) \xi^2 v dt + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial r} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial r} \right) \rho_1 \xi \sqrt{\gamma} S dt \\
+ \left( \frac{\partial^2 V}{\partial S \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial v} \right) \rho_2 \xi v S dt + \left( \frac{\partial^2 V}{\partial r \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial v} \right) \rho_3 \xi \sqrt{\gamma} v dt \\
= \left[ \left( \frac{\partial V}{\partial s} - \Delta_2 \frac{\partial U_1}{\partial s} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \left(r - D_0) \right) S + \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \kappa (\eta - r) \right] \\
+ \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \theta (\omega - v) + \left( \frac{\partial V}{\partial t} - \Delta_2 \frac{\partial U_1}{\partial t} - \Delta_3 \frac{\partial U_2}{\partial S} \right) dt \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial s^2} - \Delta_2 \frac{\partial^2 U_1}{\partial s^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) S^2 + \frac{1}{2} \left( \frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U_1}{\partial r^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) \xi^2 r \\
+ \frac{1}{2} \left( \frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U_1}{\partial v^2} - \Delta_3 \frac{\partial^2 U_2}{\partial S^2} \right) \xi^2 v + \left( \frac{\partial^2 V}{\partial S \partial r} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial r} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial r} \right) \rho_1 \xi \sqrt{\gamma} S \\
+ \left( \frac{\partial^2 V}{\partial S \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial S \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial S \partial v} \right) \rho_2 \xi v S + \left( \frac{\partial^2 V}{\partial r \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial v} \right) \rho_3 \xi \sqrt{\gamma} v dt \\
+ \left( \frac{\partial V}{\partial s} - \Delta_2 \frac{\partial U_1}{\partial s} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \sqrt{\gamma} S dt + \left( \frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \xi \sqrt{\gamma} dB^1 \\
+ \left( \frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial S} \right) \xi \sqrt{\gamma} dB^2. \right] 
\right) (2.54)
Again, we need to eliminate the stochastic terms, $dW_t$, $dB^1_t$ and $dB^2_t$, to make the portfolio risk free, i.e.,

$$\frac{\partial V}{\partial S} - \Delta_1 - \Delta_2 \frac{\partial U_1}{\partial S} - \Delta_3 \frac{\partial U_2}{\partial S} = 0,$$  \hspace{1cm} (2.55)

$$\frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial r} = 0,$$  \hspace{1cm} (2.56)

$$\frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial v} = 0.$$  \hspace{1cm} (2.57)

Furthermore, since we have already made the self-financing portfolio risk free, we also have

$$d\Pi = r\Pi dt = r(V - \Delta_1 S - \Delta_2 U_1 - \Delta_3 U_2)dt.$$  \hspace{1cm} (2.58)

Thus,

$$\begin{align*}
    r(V - \Delta_1 S - \Delta_2 U_1 - \Delta_3 U_2) & = (\frac{\partial V}{\partial S} - \Delta_1 - \Delta_2 \frac{\partial U_1}{\partial S} - \Delta_3 \frac{\partial U_2}{\partial S})(r - D_0)S \\
    & + (\frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial r})k(\eta - r) + (\frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial v})\theta(\omega - v) \\
    & + (\frac{\partial V}{\partial \rho} - \Delta_2 \frac{\partial U_1}{\partial \rho} - \Delta_3 \frac{\partial U_2}{\partial \rho}) + \frac{1}{2} (\frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U_1}{\partial r^2} - \Delta_3 \frac{\partial^2 U_2}{\partial r^2})\xi^2 r \\
    & + \frac{1}{2} (\frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U_1}{\partial v^2} - \Delta_3 \frac{\partial^2 U_2}{\partial v^2})\xi^2 v \\
    & + (\frac{\partial^2 V}{\partial \rho^2} - \Delta_2 \frac{\partial^2 U_1}{\partial \rho^2} - \Delta_3 \frac{\partial^2 U_2}{\partial \rho^2})\rho_1 \xi \sqrt{r} S & + (\frac{\partial^2 V}{\partial r \partial \rho} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial \rho} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial \rho})\rho_2 \xi v S \\
    & + (\frac{\partial^2 V}{\partial r \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial v})\rho_3 \xi \sqrt{r} v \\
    \Rightarrow & \quad r[V - (\frac{\partial V}{\partial S} - \Delta_2 \frac{\partial U_1}{\partial S} - \Delta_3 \frac{\partial U_2}{\partial S})S - \Delta_2 U_1 - \Delta_3 U_2] = (\frac{\partial V}{\partial r} - \Delta_2 \frac{\partial U_1}{\partial r} - \Delta_3 \frac{\partial U_2}{\partial r})k(\eta - r) \\
    & + (\frac{\partial V}{\partial v} - \Delta_2 \frac{\partial U_1}{\partial v} - \Delta_3 \frac{\partial U_2}{\partial v})\theta(\omega - v) + (\frac{\partial V}{\partial \rho} - \Delta_2 \frac{\partial U_1}{\partial \rho} - \Delta_3 \frac{\partial U_2}{\partial \rho}) \\
    & + \frac{1}{2} (\frac{\partial^2 V}{\partial r^2} - \Delta_2 \frac{\partial^2 U_1}{\partial r^2} - \Delta_3 \frac{\partial^2 U_2}{\partial r^2})\xi^2 r \\
    & + \frac{1}{2} (\frac{\partial^2 V}{\partial v^2} - \Delta_2 \frac{\partial^2 U_1}{\partial v^2} - \Delta_3 \frac{\partial^2 U_2}{\partial v^2})\xi^2 v \\
    & + (\frac{\partial^2 V}{\partial \rho^2} - \Delta_2 \frac{\partial^2 U_1}{\partial \rho^2} - \Delta_3 \frac{\partial^2 U_2}{\partial \rho^2})\rho_1 \xi \sqrt{r} S \\
    & + (\frac{\partial^2 V}{\partial r \partial \rho} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial \rho} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial \rho})\rho_2 \xi v S \\
    & + (\frac{\partial^2 V}{\partial r \partial v} - \Delta_2 \frac{\partial^2 U_1}{\partial r \partial v} - \Delta_3 \frac{\partial^2 U_2}{\partial r \partial v})\rho_3 \xi \sqrt{r} v \\
    \Rightarrow & \quad \frac{\partial V}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S} \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + (\frac{\partial V}{\partial \rho} - \Delta_2 \frac{\partial U_1}{\partial \rho} - \Delta_3 \frac{\partial U_2}{\partial \rho}) \theta(\omega - v) \\
    & + \rho_1 \xi \sqrt{r} S \frac{\partial^2 V}{\partial S \partial \rho} + \rho_2 \xi v S \frac{\partial^2 V}{\partial \rho \partial v} + \rho_3 \xi \sqrt{r} v \frac{\partial^2 V}{\partial r \partial v} - r V \\
    & - \Delta_2 \frac{\partial U_1}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 U_1}{\partial S^2} + \frac{\partial U_1}{\partial S} \frac{\partial^2 U_1}{\partial S \partial r} + \frac{1}{2} \xi^2 r \frac{\partial^2 U_1}{\partial r^2} + k(\eta - r) \frac{\partial U_1}{\partial r} + \frac{1}{2} \xi^2 v \frac{\partial^2 U_1}{\partial v^2} + \theta(\omega - v) \frac{\partial U_1}{\partial v} \\
    & + \rho_1 \xi \sqrt{r} S \frac{\partial^2 U_1}{\partial S \partial \rho} + \rho_2 \xi v S \frac{\partial^2 U_1}{\partial \rho \partial v} + \rho_3 \xi \sqrt{r} v \frac{\partial^2 U_1}{\partial r \partial v} - r U_1]\end{align*}
for arbitrary functions

With the utilization of

interest rate are given, while those with respect to the underlying asset price and

dimensional cases

2.2.4 Boundary conditions along the direction of the volatility and

In this section, some boundary conditions in the direction of the volatility and

interest rate model.

which is the PDE for the bond price under the hybrid stochastic volatility and

interest rate model.

for arbitrary functions \( \lambda_1(S,v,r,t) \) and \( \lambda_2(S,v,r,t) \). Thus, we can finally arrive at

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interest rate model.

2.2.4 Boundary conditions along the direction of the volatility and

interest rate

In this section, some boundary conditions in the direction of the volatility and

interest rate are given, while those with respect to the underlying asset price and
time are omitted since these are the same as what have been specified for the B-S
model in the previous section.
One of the choices for the boundary conditions in the volatility direction can be given as
\[
\lim_{v \to 0} U(S,v,t) = \max\{nS, Ze^{r(T-t)}\},
\]
\[
\lim_{v \to \infty} \frac{\partial U}{\partial v}(S,v,t) = 0,
\]
and the boundary conditions in the interest rate direction can be selected as
\[
\lim_{r \to 0} U(S,r,t) = U_{BS}(S,t)|_{r=0},
\]
\[
\lim_{r \to \infty} U(S,r,t) = 0.
\]
The reason for such kind of choices is elaborated in Chapter 5, and is thus omitted here.

Before we end this section, it should also be pointed out that sometimes a certain boundary condition is not necessary to close a PDE system, and we introduce the famous Fichera’s result \cite{44, 90} below.

Fichera’s result

Consider the linear second-order differential equation
\[
- \sum_{i,j=1}^{m} a_{ij}(x)u_{x_i x_j} - \sum_{i=1}^{m} b_i(x)u_{x_i} + c(x)u = f(x),
\]
where \(x = (x_1, x_2, \ldots, x_m) \in \Omega \subset \mathbb{R}^m\), \(a_{ij} = a_{ji}\), \(i, j = 1, 2, \ldots, m\), \(\forall x\). We assume the equation is the non-negative type in \(\Omega\), i.e.,
\[
\sum_{i,j=1}^{m} a_{ij}(x)\xi_i \xi_j \geq 0, \forall x \in \Omega, \xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m,
\]
and denote the unit outer-normal direction of \(\partial \Omega\) by \(n = (n_1, n_2, \ldots, n_m)\). In this case, the Fichera’s function can be defined as
\[
B(x) = \sum_{i=1}^{m} [b_i(x) - \sum_{j=1}^{m} \frac{\partial}{\partial x_j} a_{ij}(x)] n_i.
\]
If the whole boundary domain is separated into \(\partial \Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\), and for each domain, we have
\[
\sum_{i,j=1}^{m} a_{ij}(x)n_in_j > 0, \ x \in \Gamma_3,
\]
\begin{equation}
\sum_{i,j=1}^{m} a_{ij}(x)n_in_j = 0, \quad x \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2,
\end{equation}

with

\begin{align*}
B(x) &= 0, \quad x \in \Gamma_0, \\
B(x) &< 0, \quad x \in \Gamma_1, \\
B(x) &> 0, \quad x \in \Gamma_2,
\end{align*}

then, we only need the boundary conditions on \( \Gamma_2 \) and \( \Gamma_3 \) to close the PDE system, while the boundary conditions on \( \Gamma_0 \) and \( \Gamma_1 \) are not necessary.

It should be remarked here that whether a certain boundary condition is needed or not is dependent on the choices of parameter values, and one always needs to look into the case once parameters have been determined.

2.3 Numerical methods

It should be pointed out that it is often very difficult to price even a simple financial contract, especially when the adopted model captures the main characteristics of the underlying asset price, such as incorporating stochastic volatility and/or stochastic interest rate. Therefore, numerical methods must be resorted to in most cases, and several basic numerical approaches are illustrated below.

2.3.1 Monte-Carlo method

The Monte-Carlo method is a classical but useful method to price financial derivatives. The main idea of it is to generate a set of sample paths satisfying the given SDE, and all the computation/approximation is dependent the generated sample paths. The main advantages of the Monte-Carlo method are its generality, relative ease of use, and flexibility, and it is useful in pricing many complex financial derivatives, especially when the lattice and PDE framework cannot be applied. However, it also suffers from one main drawback that it is very time intensive if one wants to ensure that the approximation error is small. In the following, a simple approach is to be illustrated to generate \( N \) sample paths for the B-S model

\begin{equation}
\frac{dS_t}{S_t} = \mu S_t dt + \sigma S_t dW_t,
\end{equation}

when \( t \in [0,T] \).

We first need to uniformly discretize the domain [0,T] into \( 0 = t_0 < t_1 < t_2 < \cdots < t_J = T \) with \( dt = T/J \) and \( t_j = j*dt, \quad j = 0, 1, \ldots, J \). We start by rewriting the B-S
model as
\[ S_{t_{j+1}} = S_{t_j} + \mu S_{t_j} dt + \sigma S_{t_j} dW_{t_j}. \]  \( (2.79) \)

According to the property of the Brownian motion, \( dW_{t_j} = W_{t_{j+1}} - W_{t_j}, j = 0, 1, ..., J - 1 \) are independent of each other, following a normal distribution with 0 and \( dt \) as the mean and variance, respectively. Therefore, for the \( n \)-th sample path where \( n = 1, 2, ..., N \), we only need to independently generate \( J \) standard normally distributed numbers, \( M_j, j = 0, 2, ..., J - 1 \), so that for each \( j = 0, 2, ..., J - 1 \), we can compute
\[ S_{n_{t_{j+1}}} = S_{n_{t_j}} + S_{n_{t_j}} \mu * dt + S_{n_{t_j}} \sigma * sqrt(dt) * M_j, \]  \( (2.80) \)
to yield a complete sample path of the underlying price.

Once we obtain all the sample paths, the task left is straightforward; for each path, we compute the derivative price, which is conditional upon the information of the underlying asset, according to the payoff function, and the target price is just the average of all these obtained conditional prices.

### 2.3.2 Finite difference method

Another popular approach that is often applied in derivative pricing is the finite difference method, which is mainly used to solve differential equations numerically. There are in fact different kinds of the finite difference method, which depend on the choices of differences defined below.

**Definition 2.3.1 (Finite differences)** For the first-order derivative function, \( df/ds \), there are mainly three types of differences:

(i) Forward difference: \( \frac{df}{ds} = \frac{f_{n+1} - f_n}{\Delta s} \);
(ii) Backward difference: \( \frac{df}{ds} = \frac{f_n - f_{n-1}}{\Delta s} \);
(iii) Central difference: \( \frac{df}{ds} = \frac{f_{n+1} - f_{n-1}}{2\Delta s} \).

On the other hand, for the second-order derivative function, \( d^2f/ds^2 \), the most popular operator is the half-central difference, \( \frac{d^2f}{ds^2} = \frac{f_{n+1} - 2f_n + f_{n-1}}{(\Delta s)^2} \), which is also named as the second-order central difference.

Here, an example using a vanilla heat PDE, which is a degenerated equation of the considered problems in this thesis, is presented to show how to establish three different kinds of the finite difference method to solve this PDE.

**Example 2.3.1** Consider the following PDE
\[ \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \]  \( (2.81) \)
with the initial condition
\[ U(x,0) = f(x), \quad (2.82) \]
and the boundary conditions
\[ U(0,t) = 0 = U(1,t), \quad (2.83) \]
where \( x \) and \( t \) are both defined on \([0,1]\). To apply the finite difference method to solve this PDE system, we first need to divide the time domain and the space domain as
\[ [t_0,t_1],[t_1,t_2],\cdots,[t_{N-1},t_N], \text{ with } t_i = i \cdot dt, \]
\[ [x_0,x_1],[x_1,x_2],\cdots,[x_{L-1},x_L], \text{ with } x_j = j \cdot dx, \]
where \( dt = 1/N \) and \( dx = 1/L \). If we denote the value \( U_i^j \) as the numerical approximation of \( U(x_j,t_i) \), the explicit method is defined below.

Definition 2.3.2 (Explicit method) Using the forward difference scheme and the second-order central difference scheme to replace the time derivative and the second-order space derivative, respectively, at point \((t_i,x_j)\), yields:
\[ \frac{U_{i+1}^j - U_i^j}{dt} = \frac{U_{i+1}^{j+1} - 2U_i^j + U_{i-1}^j}{(dx)^2}, \quad (2.84) \]
a rearrangement of which leads to
\[ U_{i+1}^j = U_i^j + dt\left[\frac{U_{i+1}^{j+1} - 2U_i^j + U_{i-1}^j}{(dx)^2}\right] \]
\[ \Rightarrow U_{i+1}^j = (1 - 2\lambda)U_i^j + \lambda U_{i+1}^{j+1} + \lambda U_{i-1}^{j-1}, \quad (2.85) \]
where \( \lambda = dt/(dx)^2 \). Clearly, once we know the function values at time \( t_i \), the corresponding values at time \( t_{i+1} \) can be calculated directly.

Remark: Although the Explicit method is very easy to implement, it is not always stable and a stability condition for this case is \( 0 < \lambda \leq 1/2 \).

Definition 2.3.3 (Implicit method) Using the backward difference scheme and the second-order central difference scheme to replace the time derivative and the second-order space derivative, respectively, at point \((t_i,x_j)\), yields:
\[ \frac{U_{i+1}^j - U_i^j}{dt} = \frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j-1}}{(dx)^2}, \quad (2.86) \]
which can be simplified as

\[ U_{i+1}^j - dt \left( \frac{U_{i+1}^{j+1} - 2U_{i+1}^j + U_{i+1}^{j-1}}{(dx)^2} \right) = U_i^j \]

\[ \Rightarrow (1 + 2\lambda)U_{i+1}^j - \lambda U_{i+1}^{j+1} - \lambda U_{i+1}^{j-1} = U_i^j, \tag{2.87} \]

where \( \lambda = dt/(dx)^2 \).

Remark: The Implicit method has overcome the stability problem of the Explicit method.

Definition 2.3.4 (Crank-Nicolson method) A summation of the Explicit and Implicit method, or adding Equation (2.84) and (2.86), yields

\[ \frac{U_{i+1}^j - U_i^j}{dt} = \frac{1}{2} \left[ \frac{U_{i+1}^{j+1} - 2U_{i+1}^j + U_{i+1}^{j-1}}{(dx)^2} \right] + \frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{(dx)^2} \],

(2.88)

from which it is easy to obtain

\[ 2U_{i+1}^j - dt \left[ \frac{U_{i+1}^{j+1} - 2U_{i+1}^j + U_{i+1}^{j-1}}{(dx)^2} \right] = 2U_i^j + dt \left[ \frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{(dx)^2} \right] \]

\[ \Rightarrow (1 + 2\lambda)U_{i+1}^j - \lambda U_{i+1}^{j+1} - \lambda U_{i+1}^{j-1} = (1 - 2\lambda)U_i^j + \lambda U_i^{j+1} + \lambda U_i^{j-1}, \tag{2.89} \]

where \( \lambda = dt/(dx)^2 \).

Remark: This method is also unconditionally stable, In fact, we can use \( p \ast (2.84) + q \ast (2.86) \) to obtain different schemes, as long as \( p, q \in (0, 1) \) and \( p + q = 1 \).

2.3.3 Binomial tree pricing method

In this subsection, a simple but useful numerical approach to price financial derivatives is introduced, and this is applicable when the model of the underlying asset is discrete.

As an example, we consider a single period where the underlying price starts at \( S_0 \). At the next time instant, \( dt \), we assume that the underlying price can only become either \( uS_0 \) or \( dS_0 \) with probabilities \( P_u \) and \( P_d \), respectively, where \( u > d \), \( P_u, P_d \in [0, 1] \) and \( P_u + P_d = 1 \). Clearly, if we respectively denote \( V_u \) and \( V_d \) as the payoff corresponding to the two cases, and assume \( P_u \) and \( P_d \) are given, then the price of the derivative at the current time, \( V_0 \), can be directly computed as

\[ V_0 = e^{-rdt}(P_uV_u + P_dV_d), \tag{2.90} \]

using the risk neutral pricing principle. However, \( P_u \) and \( P_d \) are usually unknown, and an alternative approach is needed to find the derivative price.
We now construct a portfolio consisting of the risk-free asset valued at $\Phi$, and $\Delta$ shares of the underlying asset, which implies that the initial value of this portfolio is

$$\Pi_0 = \Delta S_0 + \Phi,$$

and its possible future value at $dt$ will be either

$$\Pi_u = \Delta u S_0 + \Phi e^{rdt} \quad \text{or} \quad \Pi_d = \Delta d S_0 + \Phi e^{rdt}.$$

If we try to use this portfolio to replicate the payoff of the target derivative, i.e.,

$$\Delta u S_0 + \Phi e^{rdt} = V_u,$$

$$\Delta d S_0 + \Phi e^{rdt} = V_d,$$

we can easily obtain

$$\Delta = \frac{V_u - V_d}{S_0 (u - d)},$$

$$\Phi = e^{-rdt} \frac{u V_d - d V_u}{u - d}.$$

In this case, in order to avoid arbitrage opportunities, the initial value of this portfolio must be equal to $V_0$, yielding

$$V_0 = \Delta S_0 + \Phi = \frac{V_u - V_d}{u - d} + e^{-rdt} \frac{u V_d - d V_u}{u - d} = e^{-rdt} \left( \frac{e^{rdt} - d}{u - d} V_u + \frac{u - e^{rdt}}{u - d} V_d \right).$$

From this, it is not difficult to deduce that

$$P_u = \frac{e^{rdt} - d}{u - d}, \quad P_d = \frac{u - e^{rdt}}{u - d}.$$

For the easiness of understanding, we have also provided a figure below to illustrate the main idea of the binomial tree method.

Although the binomial method is a numerical approximation approach, the derived result can be treated as the true value of this financial derivative if the number of the time steps is large enough. This is especially useful when American-style derivatives are taken into consideration, as for such kind of derivatives, it is usually impossible to find analytical solutions, and we would always need a benchmark to check the accuracy of a certain approach.
2.3.4 Predictor-corrector method

A predictor-corrector method is adopted in this thesis to solve the pricing PDE of the convertible bonds. In fact, it refers to a class of algorithms, designed to solve ODEs. Although it can give rise to different algorithms depending on the different forms of ODEs and different numerical schemes used, the main idea behind it is actually the same, and represents a two-step solution process as

Step 1: (Predictor) Given the function-values and derivative-values at a preceding set of points, extrapolation is used to obtain the value of the target function at a subsequent new point. For this step, the numerical scheme should be an explicit one.

Step 2: (Corrector) Refine the prediction obtained in Step 1 by using another method to interpolate the unknown function’s value at the same subsequent point. For this step, the numerical scheme should be an implicit one.

In the following, some simple and classical examples are given to describe this method more clearly.

Example 2.3.2 Consider the ODE

\[ y' = f(x,y), \quad y(x_0) = y_0. \tag{2.99} \]

The step size here is denoted as \( dx \) such that \( x_i = x_0 + i \cdot dt \), and the value of the function at each point, \( y(x_i) \), is represented by \( y_i \).

Firstly, a classical algorithm is considered. For the predictor step, if \( y_i \) is known, applying the Euler scheme yields

\[ \tilde{y}_{i+1} = y_i + f(x_i, y_i)dx. \tag{2.100} \]

Once we obtain the predicted value, the corrector step implements the trapezoidal
rule such that the prediction can be refined as
\[ y_{i+1} = y_i + \frac{1}{2} (f(x_i, y_i) + f(x_{i+1}, \tilde{y}_{i+1}))dx, \]  
(2.101)
which is the corrected value of the function at \( x_{i+1} \). It should be pointed out that this is the so-called Predict-Evaluate-Correct-Evaluate (PECE) mode, where we always update the function values \( f \) according to the update of the value \( y \). However, it is also possible to evaluate the function \( f \) only once per step by using the method of the Predict-Evaluate-Correct (PEC) mode:
\[ \tilde{y}_{i+1} = y_i + f(x_i, \tilde{y}_i)dx, \]  
(2.102)
\[ y_{i+1} = y_i + \frac{1}{2} (f(x_i, y_i) + f(x_{i+1}, \tilde{y}_{i+1}))dx. \]  
(2.103)
In addition, the corrector step can be repeated in the hope that this achieves an even better approximation to the true solution. For example, if the corrector method is run twice, this yields the PECECE mode:
\[ \tilde{y}_{i+1} = y_i + f(x_i, y_i)dx, \]  
(2.104)
\[ \hat{y}_{i+1} = y_i + \frac{1}{2} (f(x_i, y_i) + f(x_{i+1}, \tilde{y}_{i+1}))dx, \]  
(2.105)
\[ y_{i+1} = y_i + \frac{1}{2} (f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1}))dx. \]  
(2.106)
Remark: When this method is applied to solve PDEs, one should be very careful as we need to fix other variables when we are working on one particular variable.

2.3.5 Alternating direction implicit method
The Alternating Direction Implicit (ADI) method is a very useful method to solve the parabolic equations on rectangular domains. Let us consider a standard form of the parabolic equations
\[ U_t = b_1 U_{xx} + b_2 U_{yy}, \]  
(2.107)
defined on a rectangular domain. Let \( A_1 \) and \( A_2 \) be two linear operators defined as
\[ A_1 U = b_1 U_{xx}, \]  
(2.108)
\[ A_2 U = b_2 U_{yy}. \]  
(2.109)
Then, the problem can be rewritten as
\[ U_t = A_1 U + A_2 U. \]  
(2.110)
The ADI method is able to transform the initial two-dimensional problem into a set of simple one-dimensional ones. In the following, two different schemes are introduced.

The Peaceman-Rachford Method

Using the central-time finite difference scheme at \( t = (n + 1/2)dt \), Equation (2.110) becomes

\[
\frac{U^{n+1} - U^n}{dt} = \frac{1}{2}(A_1U^{n+1} + A_1U^n) + \frac{1}{2}(A_2U^{n+1} + A_2U^n),
\]

which can be rewritten as

\[
(\mathbb{I} - \frac{dt}{2}A_1 - \frac{dt}{2}A_2)U^{n+1} = (\mathbb{I} + \frac{dt}{2}A_1 + \frac{dt}{2}A_2)U^n.
\]

To solve this equation, a term \( dt^2A_1A_2U^{n+1}/4 \) is added to both sides of the above equation, i.e.,

\[
(\mathbb{I} - \frac{dt}{2}A_1 - \frac{dt}{2}A_2)U^{n+1} + \frac{dt^2}{4}A_1A_2U^{n+1} = (\mathbb{I} + \frac{dt}{2}A_1 + \frac{dt}{2}A_2)U^n + \frac{dt^2}{4}A_1A_2U^{n+1}.
\]

A simple rearrangement leads to

\[
(\mathbb{I} - \frac{dt}{2}A_1)(\mathbb{I} - \frac{dt}{2}A_2)U^{n+1} = (\mathbb{I} + \frac{dt}{2}A_1)(\mathbb{I} + \frac{dt}{2}A_2)U^n + \frac{dt^2}{4}A_1A_2(U^{n+1} - U^n).
\]

Since \( U^{n+1} = U^n + O(dt) \), we can further obtain

\[
(\mathbb{I} - \frac{dt}{2}A_1)(\mathbb{I} - \frac{dt}{2}A_2)U^{n+1} = (\mathbb{I} + \frac{dt}{2}A_1)(\mathbb{I} + \frac{dt}{2}A_2)U^n + O(dt^3).
\]

If \( A_{1h} \) and \( A_{2h} \) are the second-order approximations of \( A_1 \) and \( A_2 \), respectively, the above equation can be expressed as

\[
(\mathbb{I} - \frac{dt}{2}A_{1h})(\mathbb{I} - \frac{dt}{2}A_{2h})U^{n+1} = (\mathbb{I} + \frac{dt}{2}A_{1h})(\mathbb{I} + \frac{dt}{2}A_{2h})U^n + O(dt^3) + O(dth^2).
\]

According to this formulation, the ADI scheme can be finally derived as

\[
(\mathbb{I} - \frac{dt}{2}A_{1h})(\mathbb{I} - \frac{dt}{2}A_{2h})v^{n+1} = (\mathbb{I} + \frac{dt}{2}A_{1h})(\mathbb{I} + \frac{dt}{2}A_{2h})v^n.
\]

To solve the above equation, Peaceman & Rachford\[^{[91]}\] designed a two-step process
as

\[
(\mathbb{I} - \frac{dt}{2} A_{1h}) v^{n+1/2} = (\mathbb{I} + \frac{dt}{2} A_{2h}) v^n, \quad (2.118)
\]

\[
(\mathbb{I} - \frac{dt}{2} A_{2h}) v^{n+1} = (\mathbb{I} + \frac{dt}{2} A_{1h}) v^{n+1/2}, \quad (2.119)
\]

which is the original form of the ADI method. One can clearly see that this particular ADI method is constructed with two steps, each dealing with a one-dimensional problem.

The Douglas-Rachord Method

Another well-known ADI scheme that is widely adopted in derivative pricing is the Douglas-Rachord method\[^{38}\], since its accuracy is first-order in time and second-order in space. We still start with Equation (2.110), but apply the backward-time central-space scheme so that

\[
(\mathbb{I} - dt A_1 - dt A_2) U^{n+1} = U^n. \quad (2.120)
\]

Adding a term \(dt^2 A_1 A_2 U^{n+1}\) on both sides yields

\[
(\mathbb{I} - dt A_1 - dt A_2 + dt^2 A_1 A_2) U^{n+1} = U^n + dt^2 A_1 A_2 U^{n+1}
\]

\[
\Rightarrow (\mathbb{I} - dt A_1 - dt A_2 + dt^2 A_1 A_2) U^{n+1} = U^n + dt^2 A_1 A_2 U^{n+1} - dt^2 A_1 A_2 U^n + dt^2 A_1 A_2 U^n \\
\Rightarrow (\mathbb{I} - dt A_1 - dt A_2 + dt^2 A_1 A_2) U^{n+1} = U^n + dt^2 A_1 A_2 U^n + dt^2 A_1 A_2 (U^{n+1} - U^n),
\]

\[
(2.121)
\]

which implies

\[
(\mathbb{I} - dt A_1 - dt A_2 + dt^2 A_1 A_2) U^{n+1} = U^n + dt^2 A_1 A_2 U^n + O(dt^3). \quad (2.122)
\]

Omitting the term \(O(dt^3)\), the scheme we can obtain is

\[
(\mathbb{I} - dt A_1 - dt A_2 + dt^2 A_1 A_2) v^{n+1} = (\mathbb{I} + dt^2 A_1 A_2) v^n. \quad (2.123)
\]

for which the Douglas-Rachord method is

\[
(\mathbb{I} - dt A_{1h}) v^{n+1/2} = (\mathbb{I} + dt A_{2h}) v^n, \quad (2.124)
\]

\[
(\mathbb{I} - dt A_{2h}) v^{n+1} = v^{n+1/2} - dt A_{2h} v^n. \quad (2.125)
\]
2.4 Integral equation method and Fourier transform

A main disadvantage of purely numerical approaches mentioned in the previous section is that errors are always introduced at a very early stage of computation, which could sometimes severely affect the accuracy of the obtained results. One possible way to overcome such a disadvantage is to use semi-analytical approaches, in which analytical analysis is performed until a point beyond which numerical calculations must be resorted to. Belonging to this category, the integral equation method is one of the most popular approaches that have wide applications in derivative pricing. A key step in realizing the integral equation approach is to derive an integral equation representation for the target derivative price, involving the utilization of several useful techniques. In particular, the Fourier transform is one well-known method that can be applied to derive the integral equation representations\(^a\), and its definition as well as that of the Fourier inversion transform are presented below.

Definition 2.4.1 (Fourier Transform) The Fourier transform of a smooth function \(U(x)\) is defined as

\[
\mathcal{F}\{U(x)\} = \int_{-\infty}^{\infty} U(x) e^{i\omega x} \, dx, \tag{2.126}
\]

for any real number \(\omega\), and it is often denoted as \(\hat{U}(\omega)\).

Definition 2.4.2 (Fourier Inversion Transform) The Fourier inversion transform of the function \(\hat{U}(\omega)\) in the Fourier space is defined as

\[
U(x) = \mathcal{F}^{-1}\{\hat{U}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\omega) e^{-i\omega x} \, d\omega. \tag{2.127}
\]

The Fourier transform possesses some useful properties that are often used in the process of applying this particular transform, and these are illustrated in the following proposition and theorem.

Proposition 2.4.1 (Linearity) If

\[
P(x) = aU(x) + bV(x), \tag{2.128}
\]

then

\[
\hat{P}(\omega) = a\hat{U}(\omega) + b\hat{V}(\omega), \tag{2.129}
\]

for any complex numbers \(a\) and \(b\).

Theorem 2.4.1 (Convolution theorem) If

\[
P(x) = (U * V)(x) = \int_{-\infty}^{\infty} U(u) V(x-u) \, du, \tag{2.130}
\]

\(^a\)Fourier transform also has wide applications in the area of financial mathematics.
where * is the convolution operator, then

\[ \hat{P}(\omega) = \hat{U}(\omega) \cdot \hat{V}(\omega). \]  

(2.131)

It also means that if

\[ \hat{P}(\omega) = \hat{U}(\omega) \cdot \hat{V}(\omega), \]  

(2.132)

then

\[ P(x) = \mathcal{F}^{-1}\{\hat{P}(\omega)\} = \mathcal{F}^{-1}\{\hat{U}(\omega) \cdot \hat{V}(\omega)\} = (U \ast V)(x). \]  

(2.133)

Now, a simple example is presented to explain how to apply the Fourier transform to solve PDE systems.

Example 2.4.1 Consider a classical heat equation

\[ \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \]  

(2.134)

with the initial condition

\[ U(x,0) = f(x), \]  

(2.135)

and the boundary conditions

\[ U(-\infty,t) = 0, \]  

(2.136)

\[ U(\infty,t) = 0, \]  

(2.137)

where the domains of \( x \) and \( t \) are \((-\infty, \infty)\) and \([0, \infty)\), respectively.

To solve the target system, applying the Fourier transform with respect to \( x \) on Equation (2.134) yields

\[ \mathcal{F}\{\frac{\partial U}{\partial t}\} = \mathcal{F}\{\frac{\partial^2 U}{\partial x^2}\}. \]  

(2.138)

The left hand side of the equation can be computed through

\[ \mathcal{F}\{\frac{\partial U}{\partial t}\} = \int_{-\infty}^{\infty} \frac{\partial U(x,t)}{\partial t} e^{i\omega x} dx \]  

\[ = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} U(x,t) e^{i\omega x} dx \]  

\[ = \frac{\partial \hat{U}}{\partial t}(\omega,t), \]  

(2.139)
and the right hand side can be calculated as

$$\mathcal{F}\left\{ \frac{\partial^2 U}{\partial x^2} \right\} = \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2}(x,t)e^{i\omega x}dx$$

$$= \frac{\partial U}{\partial x}(x,t)e^{i\omega x}|_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} \frac{\partial U}{\partial x}(x,t)e^{i\omega x}dx$$

$$= -i\omega \int_{-\infty}^{\infty} \frac{\partial U}{\partial x}(x,t)e^{i\omega x}dx$$

$$= -i\omega U(x,t)e^{i\omega x}|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} U(x,t)e^{i\omega x}dx$$

$$= -\omega^2 \hat{U}(\omega,t). \quad (2.140)$$

A combination of the two equations leads to

$$\begin{cases} 
\frac{\partial \hat{U}}{\partial t}(\omega,t) + \omega^2 \hat{U}(\omega,t) = 0 \\
\hat{U}(\omega,0) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx.
\end{cases} \quad (2.141)$$

Clearly, this is a first-order linear ODE with an initial condition, the solution to which can be easily formulated as

$$\hat{U}(\omega,t) = \hat{U}(\omega,0)e^{-\omega^2 t}. \quad (2.142)$$

By now, the solution to Equation (2.134) has been derived in the Fourier space, and to obtain the solution in the original space, the Fourier inversion transform needs to be applied, which yields

$$U(x,t) = \mathcal{F}^{-1}\{\hat{U}(\omega,t)\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\omega,t)e^{-i\omega x}d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\omega,0)e^{-\omega^2 t}e^{-i\omega x}d\omega. \quad (2.143)$$

Using the Convolution theorem, we further obtain

$$U(x,t) = U(x,0) \ast \mathcal{F}^{-1}\{e^{-\omega^2 t}\}$$

$$= f(x) \ast \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t}e^{-i\omega x}d\omega$$

$$= f(x) \ast \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 t-i\omega x}d\omega$$

$$= f(x) \ast \frac{1}{\pi} \sqrt{\frac{\pi}{t}}e^{-\frac{x^2}{4t}}$$

$$= f(x) \ast \sqrt{\frac{1}{4\pi t}}e^{-\frac{x^2}{\pi t}}.$$
\[
= \sqrt{\frac{1}{4\pi t}} \int_{-\infty}^{\infty} f(u) \cdot e^{-\frac{(x-u)^2}{4t}} \, du,
\]

which is the desired result.
Chapter 3

Pricing puttable convertible bonds with integral equation approaches

3.1 Introduction

A convertible bond (CB) is one of the widely-used hybrid financial instruments. It gives the holder the right to convert a bond into a predetermined number of stocks at any time during the life of the bond, or to hold the bond until maturity to receive the principal payment. Such a conversion right gives the holder the possibility to gain a maximum benefit. But, this particular conversion feature has made the valuation problem more complicated because the optimal conversion boundary needs to be determined as part of the solution of the problem.

The theoretical framework for pricing CBs under the Black-Scholes model was initially proposed by Ingersoll\textsuperscript{[62]} and Brennan & Schwartz\textsuperscript{[12]}. They priced a convertible bond by using contingent claims, in which they took the firm value as the underlying variable. However, the model is not practical since the firm value is not observable in market. In 1986, McConnel & Schwartz\textsuperscript{[84]} proposed a single-factor pricing model for a zero-coupon convertible bond, using stock price as the underlying variable.

Since then, various approaches have been proposed to price convertible bonds. Analytical solutions are only available for CBs with very simply exercises clauses. For example, Nyborg\textsuperscript{[88]} obtained a closed-form solution for a simple convertible bond, which can only be converted at maturity, while Zhu\textsuperscript{[106]} presented a closed-form analytical solution for a convertible bond, which can be converted at any time on or before maturity, using the homotopy analysis method. Recently, Chan & Zhu\textsuperscript{[23]} provided an approximate solution for the price of a convertible bond under the regime-switching model.

On the other hand, numerical approaches are resorted to when CBs with more
complex exercise clauses need to be priced. Among them, the finite element approach\cite{6}, the finite difference approach\cite{98} and the finite volume approach\cite{112} have been adopted by various authors. In terms of integral equation formulations for pricing CBs, Zhu & Zhang\cite{111} used a decomposition approach to obtain an integral equation formulation for pricing a vanilla convertible bond without any additional feature such as the puttabiliy discussed in this chapter.

Apart from the Black-Scholes model, there are other models having been adopted for the evaluation of CBs. For example, Brennan & Schwartz\cite{14} proposed a stochastic interest rate model to price convertible bonds, taking the value of the issuing firm as the underlying state variable. Carayannopoulos\cite{17} priced convertible bonds with a different stochastic interest rate model (the so-called CIR model (Cox-Ingersoll-Ross)\cite{29}), while David & Lischka\cite{34} adopted the Vasicek’s model\cite{100}. All these models are based on an assumption that CBs are usually designed for a long time period, during which interest rate itself may be subject to changes. However, such an addition of stochastic nature of interest rate would not be necessary if one only needs to price a CB with short time to expiry. It is certainly not necessary if one aims to develop numerical approaches as their first step. Furthermore, Hung & Wang\cite{59} used the binomial tree model to value the convertible bond, taking the risk of interest rate change as well as the default risk of the issuer into consideration, while Chambers & Lu\cite{21} further extended Hung & Wang’s work by allowing correlations among those two stochastic processes.

In addition to model complexity contributing to the pricing of CBs, various added additional rights to either or both the bond issuer and/or the bond holder, may also make the pricing problem more complicated, which demands better numerical solution approaches. For example, call and put features can be added to convertible bonds to form the so-called callable convertible bonds and puttable convertible bonds\cite{2}, respectively. A callable convertible bond is a bond in which the issuer has the right to call (repurchase) the bond from the investor for a predetermined call price within a predetermined callable period. The call feature in a convertible bond is in favor to the issuer, as if the underlying price increases significantly beyond the call price, the issuer can call back the bond. As a result, a callable convertible bond should be worth less than that of a vanilla convertible bond. A puttable convertible bond, on the other hand, allows the holder to sell the bond back to the issuer, prior to maturity, at a price that is specified at the time that the bond is issued. This price is commonly referred to as the put price\cite{80}, which is also called the strike or exercise price\cite{84}. Obviously, the put feature benefits the holder of the bond, and hence, a puttable convertible bond trades at a higher price than that of a vanilla convertible bond.

The pricing problem of callable convertible bonds has been studied for many
years. For example, Brennan & Schwartz\cite{12} explained in theory how to price such contracts, and provided numerical solutions in their later article\cite{13}, Bernini\cite{7} used a binomial tree method to obtain their numerical solution. It is interesting to note that Kifer\cite{67} presented a new derivative security called game options, similar to the callable convertible bond, which was used by Yagi & Sawaki\cite{103} to study callable convertible bonds. There are also many references on puttable convertible bonds in the literature. For example, Nyborg\cite{88} presented the boundary condition of puttable CBs, and checked if the boundary condition is reasonable and correct, while Lvov et al.\cite{80} obtained the numerical solution by using Monte Carlo simulations. However, there has not been any integral equation formulation for puttable convertible bonds, which forms the base of the current research.

In this chapter, we present two integral equation formulations to analyze a puttable convertible bond under the Black-Scholes model. It should be pointed out that although it is more practical to adopt a stochastic interest rate for convertible bond pricing, we assume a constant interest rate in our formulation. This is because it is more feasible to start with a simpler model when introducing a new solution approach to an already complicated problem with two free boundaries. There are two partial differential equation (PDE) systems governing the price of a puttable convertible bond, as the lifetime of a puttable CB is divided into two intervals by the time when the face value of the bond discounted by the time to expiry equals the predetermined put price. From this critical time, only convertible bond boundary conditions need to be considered since the price of a puttable CB is always greater than the put price during this time period there is no financial incentive to exercise the put feature. Thus, the PDE system for this part should be the same as that for the vanilla CB presented in\cite{106}. On the other hand, from the beginning of the contract until the critical time, the minimal price of puttable CB would be floored below by the put price, forming a second free boundary. Financially, the bound price is bounded below is because the holder would otherwise sell the bond back to the issuer at the put price with the warranted puttability. As a result, the puttable CB can no longer be treated as a vanilla CB and another PDE system is needed with two free boundary conditions associated with the conversion and put feature, respectively.

In order to obtain the first integral equation formulation, we apply the method of incomplete Fourier transform\cite{26} to both of the two PDE systems. However, the resulting integral equations possess a discontinuity at both of two free boundaries and they contain the first-order derivatives of the unknown free boundaries. These problems could lead to computational difficulties when the numerical results are calculated. To overcome the problems, we derive a second integral equation representation from the first integral representation.
The chapter is organized as follows. In Section 2, the PDE systems governing the price of a puttable CB are established to reflect all the unique features associated with conversion and puttability at any time prior to expiry. In Section 3, the first form of integral equation is derived by using the incomplete Fourier transform, which serves as a base to obtain another integral equation representation. In Section 4, we compared our results with the known benchmarks such as the convergent results obtained with the binomial tree method. Numerical examples are presented in Section 5, followed by some concluding remarks given in the last section.

3.2 The model

In this section, we will establish the PDE systems to price a puttable convertible bond.

Let $S$ be an underlying asset price and we assume that its dynamics follows the stochastic differential equation:

$$dS = (r - D_0)Sdt + \sigma SdW_t,$$  \hspace{1cm} (3.1)

where $W_t$ is a Brownian motion, $\sigma$ is the volatility of the underlying asset, $r$ is the risk-free interest rate, and $D_0$ is the rate of continuous dividend.

Now, consider a puttable convertible bond of maturity $T$, with face value $Z$, conversion ratio $n$ and put price $M$. Let the time to expiry be $\tau = T - t$, there exists a critical value of $\tau = \tau_M$ when the minimum value of the puttable CB (face value discounted by time to expiry) equals the put price, that is, $Ze^{-r\tau_M} = M$, or $\tau_M = -\frac{1}{r} \log \frac{M}{Z}$.

Let $V_1(S, \tau)$ be the value of the puttable CB in the interval $\tau \in [0, \tau_M]$. The price of the CB is always greater than the put price in this time interval, and thus the optimal put exercise price is always equal to zero. As a result, there is no difference between the price of a vanilla CB and that of a puttable CB in this time interval, and it should satisfy the following PDE system (cf. Zhu$^{[106]}$).

$$-\frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0)S \frac{\partial V_1}{\partial S} - rV_1 = 0,$$

$$V_1(S, 0) = \max\{nS, Z\},$$

$$V_1(S_c(\tau), \tau) = nS_c(\tau),$$

$$\frac{\partial V_1}{\partial S}(S_c(\tau), \tau) = n,$$

$$V_1(0, \tau) = Ze^{-r\tau},$$ \hspace{1cm} (3.2)

where $S_c(\tau)$ is the optimal conversion boundary, $S \in [0, S_c(\tau)]$ and $\tau \in [0, \tau_M]$.

In the interval $\tau \in [\tau_M, T]$, the price of bond should not fall below the put price, as
the holder would otherwise sell the bond back to the issuer at the predetermined put price. Therefore, when $\tau > \tau_M$, the value of the bond is bounded below by the put price, which is an important feature of puttable CBs. In fact there exist an optimal put price $S_p(\tau)$ associated with the puttable, as well as an optimal conversion price $S_c(\tau)$ in this time interval. The price of the puttable CB, $V_2(S, \tau)$, satisfies the following PDE system:

$$
\begin{align*}
- \frac{\partial V_2}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - D_0)S \frac{\partial V_2}{\partial S} - rV_2 &= 0, \\
V_2(S_c(\tau), \tau) &= nS_c(\tau), \\
\frac{\partial V_2}{\partial S}(S_c(\tau), \tau) &= n, \\
V_2(S_p(\tau), \tau) &= M, \\
\frac{\partial V_2}{\partial S}(S_p(\tau), \tau) &= 0, \\
V_2(S, \tau^+_M) &= V_1(S, \tau^-_M),
\end{align*}
$$

(3.3)

The value of the puttable convertible bond for the lifetime $\tau \in [0, T]$ can be found by solving the two PDE systems (3.2) and (3.3). We start the solution process of the systems by making the following variable transforms

$$
x = \log(S), \quad v_1(x, \tau) = V_1(S, \tau), \quad v_2(x, \tau) = V_2(S, \tau).
$$

The PDE systems (3.2) and (3.3) become

$$
\begin{align*}
- \frac{\partial v_1}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial v_1}{\partial x} - rv_1 &= 0, \\
v_1(x, 0) &= \max\{ne^x, Z\}, \\
v_1(\ln(S_c(\tau)), \tau) &= nS_c(\tau), \\
\frac{\partial v_1}{\partial x}(\ln(S_c(\tau)), \tau) &= nS_c(\tau), \\
v_1(-\infty, \tau) &= Ze^{-r\tau},
\end{align*}
$$

(3.4)

with the domain $x \in (-\infty, \ln(S_c(\tau))]$ and $\tau \in [0, \tau_M]$, and

$$
\begin{align*}
- \frac{\partial v_2}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 v_2}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial v_2}{\partial x} - rv_2 &= 0, \\
v_2(\ln(S_c(\tau)), \tau) &= nS_c(\tau), \\
\frac{\partial v_2}{\partial x}(\ln(S_c(\tau)), \tau) &= nS_c(\tau), \\
v_2(\ln(S_p(\tau)), \tau) &= M, \\
\frac{\partial v_2}{\partial x}(\ln(S_p(\tau)), \tau) &= 0, \\
v_2(x, \tau^+_M) &= v_1(x, \tau^-_M),
\end{align*}
$$

(3.5)
with the domain of $x$ and $\tau$ being $[\ln(S_p(\tau)), \ln(S_c(\tau))]$ and $[\tau_M, T]$, respectively. By now, we have derived two dimensionless PDE systems. In next section, the solution techniques to obtain integral equation formulations for Systems (3.4) and (3.5) will be discussed.

### 3.3 Integral equation formulations of puttable convertible bond

In this section, two forms of integral equations will be presented for pricing a puttable convertible bond. One is obtained by applying the so-called incomplete Fourier transform to the PDE systems directly, and the second one is a further extension of the first one, in order to avoid some potential numerical problems.

#### 3.3.1 First integral equation formulation of puttable convertible bond

In this subsection, we use the incomplete Fourier transform method to derive an integral equation representation to price a puttable convertible bond. The incomplete Fourier transform is adopted as a result of the presence of free boundaries, which have limited the domain of $x$ to a semi-infinite domain, rather than an infinite domain from $-\infty$ to $\infty$, on which the classical Fourier transform can be applied\cite{26}. Before applying the incomplete Fourier transform to System (3.4), it should be noted that the boundary condition at infinity is non-zero, which can cause problems. Therefore, a simple transform

$$U(x, \tau) = v_1(x, \tau) - Ze^{-r\tau}, \quad (3.6)$$

is introduced, so that System (3.4) can be rewritten as

$$
\begin{cases}
\frac{\partial U}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial U}{\partial x} - rU = 0, \\
U(x, 0) = \max\{ne^x - Z, 0\}, \\
U(\ln(S_c(\tau)), \tau) = nS_c(\tau) - Ze^{-r\tau}, \\
\frac{\partial U}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\
U(-\infty, \tau) = 0.
\end{cases} \quad (3.7)
$$

Now, define the following incomplete Fourier transform

$$\mathcal{F}\{U(x, \tau)\} = \int_{-\infty}^{\ln(S_c(\tau))} U(x, \tau)e^{i\omega x}dx \triangleq \hat{U}(\omega, \tau), \quad (3.8)$$
and by applying (3.8) to System (3.7), we can obtain the following ordinary differential equation (ODE) system

\[
\begin{align*}
\frac{\partial \hat{U}}{\partial \tau}(\omega, \tau) + B(\omega)\hat{U}(\omega, \tau) &= f(\omega, \tau), \\
\hat{U}(\omega, 0) &= \int_{-\infty}^{\ln(S_c(0))} \max\{ne^x - Z, 0\} e^{i\omega x} dx,
\end{align*}
\tag{3.9}
\]

where

\[
B(\omega) = \frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2)i\omega + \sigma^2,
\]

\[
f(\omega, \tau) = (nS_c(\tau) - Ze^{-r\tau}) e^{i\omega \ln(S_c(\tau))} \left[ \frac{S_c'(\tau)}{S_c(\tau)} \right] + r - D_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 i\omega + \frac{1}{2} \sigma^2 nS_c(\tau) e^{i\omega \ln(S_c(\tau))}.\]

System (3.9) is a non-homogeneous first-order linear ODE system with an initial condition. The solution of this system is as follows

\[
\hat{U}(\omega, \tau) = \hat{U}(\omega, 0) e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi) e^{-B(\omega)(\tau - \xi)} d\xi.
\tag{3.10}
\]

By now, the integral equation formulation in the Fourier space has been derived. In order to obtain the formulation in the original space, we define the following incomplete Fourier inversion transform

\[
\mathcal{F}^{-1}\{\hat{U}(\omega, \tau)\} = \int_{-\infty}^{\infty} \hat{U}(\omega, \tau) e^{-i\omega x} d\omega.
\tag{3.11}
\]

The incomplete Fourier inversion transform appears to be the same as the classical one, but there is a difference in the domain of \(x\), in our case, the domain of \(x\) is replaced by \((-\infty, \ln(S_c(\tau)))\]. Applying this new definition to (3.10) and after some tedious algebraic manipulations (see Appendix A.1), we obtain

\[
U(x, \tau) = \int_{-\infty}^{\ln(S_c(0))} \frac{e^{-r\tau}}{\sigma \sqrt{2\pi \tau}} e^{-\frac{\left[(r-D_0-\frac{1}{2}\sigma^2)\tau+x-u^2\right]}{2\sigma^2 \tau}} \cdot \max\{ne^u - Z, 0\} du
+ \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma \sqrt{2\pi (\tau-\xi)}} e^{-\frac{\left[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+1\right]}{2\sigma^2 (\tau-\xi)}} \cdot \{nS_c(\xi) - Ze^{-r\xi}\} \frac{S_c'(\xi)}{S_c(\xi)} \left[ \frac{1}{2} (r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau - \xi}) + \frac{1}{2} \sigma^2 S_c(\xi) \right] d\xi.
\tag{3.12}
\]

Rewriting the integral equation using the original parameters, we derive an integral equation formulation of System (3.2) as follows

\[
V_t(S, \tau) = \int_{-\infty}^{\ln(S(0))} \frac{e^{-r\tau}}{\sigma \sqrt{2\pi \tau}} e^{-\frac{\left[(r-D_0-\frac{1}{2}\sigma^2)\tau+\ln(S)-u^2\right]}{2\sigma^2 \tau}} \cdot \max\{ne^u - Z, 0\} du
\]
Finally, we arrive at another definition of the incomplete Fourier transform as

\[
\hat{v}_2(\omega, \tau) = \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx \triangleq \hat{v}_2(\omega, \tau).
\]  

(3.14)

Substituting Equation (3.14) into System (3.5) yields the following ODE system

\[
\begin{align*}
\frac{\partial \hat{v}_2}{\partial \tau}(\omega, \tau) + B(\omega)\hat{v}_2(\omega, \tau) &= g(\omega, \tau), \\
\hat{v}_2(\omega, \tau_M) &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} v_1(x, \tau_M) e^{i\omega x} dx,
\end{align*}
\]  

(3.15)

where

\[
g(\omega, \tau) = nS_c(\tau) e^{i\omega \ln(S_c(\tau))} \left[ \frac{S'_c(\tau)}{S_c(\tau)} - \frac{1}{2} \sigma^2 i\omega + r - D_0 \right] - Me^{i\omega \ln(S_p(\tau))} \left[ \frac{S'_p(\tau)}{S_p(\tau)} - \frac{1}{2} \sigma^2 i\omega + r - D_0 - \frac{1}{2} \sigma^2 \right].
\]  

(3.16)

System (3.15) is again a non-homogeneous first-order linear ODE system, the solution of which can be derived as

\[
\hat{v}_2(\omega, \tau) = \hat{v}_2(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} + \int_0^{\tau - \tau_M} g(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\xi.
\]  

(3.17)

To obtain the integral equation formulation in the original space, we apply the Fourier inversion transform to Equation (3.17) (the details are left in Appendix A.2). Finally, we arrive at

\[
V_2(S, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) e^{-r(\tau - \tau_M)} \frac{e^{-(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} du \\
+ \int_0^{\tau - \tau_M} nS_c(\tau_M + \xi) e^{-r(\tau - \tau_M - \xi)} \frac{e^{-(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-D_0 u(\tau - \tau_M)} e^{-\frac{(\tau - \tau_M - \xi)}{\sigma^2(\tau - \tau_M)}} d\xi.
\]
\[ nS_c(\tau) = \int_{-\infty}^{\ln(S_c(0))} e^{-\frac{\tau}{2\sigma^2}} \frac{e^{-r\tau} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - u]^2}{2\sigma^2\tau}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ + \int_{0}^{\tau} e^{-r(\tau - \xi)} \cdot \frac{e^{-\frac{\tau}{2\sigma^2}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - \ln(S_c(\xi)) + \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ - \int_{0}^{\tau - \tau_M} \frac{e^{-r(\tau - \xi)}}{\sigma \sqrt{2\pi}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - \ln(S_c(\xi)) + \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right) \cdot \max\{ne^u - Z, 0\} du \]

Equation (3.13) and Equation (3.18) could be used, for \( \tau \in [0, \tau_M] \) and \( \tau \in [\tau_M, T] \), respectively, to determine the value of puttable CBs. However, both (3.13) and (3.18) involve the optimal conversion price and the optimal put price, \( S_c(\tau) \) and \( S_p(\tau) \), which still remain unknown. Fortunately, we can derive three integral equations for the boundary using the free boundaries conditions

\[ nS_c(\tau) = \int_{-\infty}^{\ln(S_c(0))} e^{-\frac{\tau}{2\sigma^2}} \frac{e^{-r\tau} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 (\pi - \ln(S_c(\tau))) - u]^2}{2\sigma^2\tau}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ + \int_{0}^{\tau} e^{-r(\tau - \xi)} \cdot \frac{e^{-\frac{\tau}{2\sigma^2}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 (\pi - \ln(S_c(\tau)) + \ln(S_c(\xi)) + \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ - \int_{0}^{\tau - \tau_M} \frac{e^{-r(\tau - \xi)}}{\sigma \sqrt{2\pi}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 (\pi - \ln(S_c(\tau)) - \ln(S_c(\xi)) - \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right) \cdot \max\{ne^u - Z, 0\} du \]

\[ \frac{M}{2} = \int_{-\infty}^{\ln(S_c(\tau_M))} e^{-\frac{\tau}{2\sigma^2}} \frac{e^{-r\tau} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - u]^2}{2\sigma^2\tau}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ + \int_{0}^{\tau} e^{-r(\tau - \xi)} \cdot \frac{e^{-\frac{\tau}{2\sigma^2}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - \ln(S_c(\xi)) + \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right)}{\sigma \sqrt{2\pi}} \cdot \max\{ne^u - Z, 0\} du \]

\[ - \int_{0}^{\tau - \tau_M} \frac{e^{-r(\tau - \xi)}}{\sigma \sqrt{2\pi}} \cdot \exp\left(\frac{[r - D_0 - \frac{1}{2}\sigma^2 T + \ln(S_c(\tau)) - \ln(S_c(\xi)) - \ln(S_c(\tau))]^2}{2\sigma^2(\tau - \xi)}\right) \cdot \max\{ne^u - Z, 0\} du \]
\[ \left( \frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} \right) + \frac{1}{2} \left[ r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S_p(\tau))}{\tau - \tau_M - \xi} \right] d\xi \]  

(3.21)

It should be noted that there is a factor of $1/2$ on the left hand side of Equations (3.19), (3.20) and (3.21), which arises by performing the incomplete Fourier transform. Actually, it can be viewed as the complete Fourier transform of a discontinuous function, and thus the corresponding Fourier inversion converges to the midpoint of the discontinuity\(^{[36]}\). Sometimes such discontinuity can lead to problems when numerical experiments are conducted. An even worse problem is that both of these two integral equation formulations contain the first-order derivative of the optimal exercise prices which can lead to large numerical errors due to the infinite slope associated with these derivative functions at expiry for the optimal conversion price and at threshold value of the time to expiry for the optimal put price. To overcome these shortfalls, we propose another integral equation formulation in the next subsection.

### 3.3.2 Second integral equation formulation for puttable convertible bond

As pointed out in the previous subsection, the integral representations (3.13) and (3.18) and the integral equations (3.19), (3.20) and (3.21) are not ideal to be used for computing the value of a puttable convertible bond and its optimal boundaries, since they all contain first-order derivatives of the optimal exercise prices. So we derive the second integral representation as an extension from the first one. While we shall leave the details of the derivation in Appendix A.3 and Appendix A.4, the main results are summarized here

\[ V_1(S, \tau) = \int_0^\tau nSD_0 e^{-D_0(\tau - \xi)} \mathcal{N} \left( \frac{r - D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau - \xi}} \right) d\xi \]

\[ + nSe^{-D_0 \tau} \mathcal{N} \left( \frac{r - D_0 + \frac{1}{2} \sigma^2 \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}} \right) \]

\[ - Ze^{-r \tau} \mathcal{N} \left( \frac{r - D_0 - \frac{1}{2} \sigma^2 \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}} \right) + Ze^{-r \tau}, \]

(3.22)

and

\[ V_2(S, \tau) = \int_{\ln(S_c(\tau_M))}^{\ln(S(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{|r - D_0 - \frac{1}{2} \sigma^2 (\tau - \tau_M) + \ln(S) - u|^2}{2\sigma^2(\tau - \tau_M)}} du \]

\[ + \int_0^{\tau - \tau_M} nD_0 e^{-D_0(\tau - \tau_M - \xi)} \]
To determine the price of a puttable convertible bond, \( V_1(S, \tau) \) and \( V_2(S, \tau) \), the two free boundaries, \( S_p \) and \( S_c \), need to be computed first from the following three integral equations constructed from substituting (3.22) and (3.23) into the boundaries conditions of Systems (3.2) and (3.3):

\[
\begin{align*}
\text{(3.23)}
\end{align*}
\]

\[
\begin{align*}
nS_c(\tau) &= \int_0^\tau nS_c(\tau)D_0e^{-D_0(\tau-\xi)}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2(\tau-\xi)+\ln(S_c(\tau))-\ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right)d\xi \\
&+ nS_c(\tau)e^{-D_0\tau}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2\tau+\ln(S_c(\tau))-\ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \\
&- Ze^{-r\tau}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2\tau+\ln(S_c(\tau))-\ln(S_c(0))}{\sigma\sqrt{\tau}}\right) + Ze^{-r\tau}, \\
\end{align*}
\]

\[
\begin{align*}
nS_c(\tau) &= \int_0^{\ln(S_c(\tauM))} V_1(e^u, \tauM) \frac{e^{-r(\tau-\tauM)}}{\sigma\sqrt{2\pi(\tau-\tauM)}} e^{-\frac{(r-D_0-1/2\sigma^2(\tau-\tauM)+\ln(S_c(\tau))-\mu)^2}{2\sigma^2(\tau-\tauM)}} du \\
&+ \int_0^{\tau-\tauM} nD_0S_c(\tau)e^{-D_0(\tau-\tauM-\xi)}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2(\tau-\tauM-\xi)+\ln(S_c(\tau))-\ln(S_c(\tauM+\xi))}{\sigma\sqrt{\tau-\tauM-\xi}}\right)d\xi \\
&- \int_0^{\tau-\tauM} rMe^{-r(\tau-\tauM-\xi)}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2(\tau-\tauM-\xi)+\ln(S_c(\tau))-\ln(S_c(\tauM+\xi))}{\sigma\sqrt{\tau-\tauM-\xi}}\right)d\xi \\
&+ nS_c(\tau)e^{-D_0(\tau-\tauM)}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2(\tau-\tauM)+\ln(S_c(\tau))-\ln(S_c(\tauM))}{\sigma\sqrt{\tau-\tauM}}\right) \\
&- Me^{-r(\tau-\tauM)}\mathcal{N}\left(\frac{r-D_0+1/2\sigma^2(\tau-\tauM)+\ln(S_p(\tau))-\ln(S_c(\tauM))}{\sigma\sqrt{\tau-\tauM}}\right) + M, \\
\end{align*}
\]

\[
\begin{align*}
M &= \int_0^{\ln(S_c(\tauM))} V_1(e^u, \tauM) \frac{e^{-r(\tau-\tauM)}}{\sigma\sqrt{2\pi(\tau-\tauM)}} e^{-\frac{(r-D_0-1/2\sigma^2(\tau-\tauM)+\ln(S_p(\tau))-\mu)^2}{2\sigma^2(\tau-\tauM)}} du \\
&+ \int_0^{\tau-\tauM} nD_0S_p(\tau)e^{-D_0(\tau-\tauM-\xi)}
\end{align*}
\]
and thus we obtain a set of non-linear algebraic equations for the free boundaries are known, we only need to numerically integrate (3.24) and (3.26) would give rise to the optimal boundaries, which can be plugged into the integral representations (3.22) and (3.23) to calculate the bond price. However, the integral equations are highly non-linear that a numerical method is needed to obtain their solutions. Therefore, in the next section, the numerical implementation of the solution procedure for pricing a puttable convertible bond will be presented.

3.4 The numerical implementation

In the following, we will provide an outline of our numerical scheme and its validation.

The major task in obtaining the numerical solutions of the integral equations is to find the values of free boundaries from Equations (3.24), (3.25) and (3.26). Once the free boundaries are known, we only need to numerically integrate (3.22) and (3.23) to obtain the bond prices. Our solution procedure for the free boundaries is as follows:

First, Equation (3.24) is used to obtain the values of the function $S_c(\tau)$ when $\tau \in [0, \tau_M]$. In this process, we discretize uniformly the time interval

$$0 = s_1 < s_2 < \cdots < s_N = \tau_M \quad \text{where} \quad s_i = (i - 1) \times \tau_M/(N - 1),$$

and thus we obtain a set of non-linear algebraic equations for $S_c(s_i)$ (denoted by $S_c^{(i)}$) for $i = 1, 2, \ldots, N$,

\[
\begin{align*}
    nS_c^{(i)} &= \sum_{k=1}^{i-1} nS_c^{(i)} D_0 e^{-D_0(s_i-s_k)} N\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(s_i-s_k)+\ln(S_c^{(i)})-\ln(S_c^{(k)})}{\sigma\sqrt{s_i-s_k}}\right)(s_{k+1}-s_k) \\
    &= nS_c^{(i)} D_0 e^{-D_0(s_i-s_1)} N\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(s_i-s_1)+\ln(S_c^{(i)})-\ln(S_c^{(1)})}{\sigma\sqrt{s_i-s_1}}\right) \frac{s_2-s_1}{2}
\end{align*}
\]
\[
\begin{align*}
+ nS_c^{(i)}D_0 \cdot \frac{s_i - s_{i-1}}{4} + nS_c^{(i)}e^{-D_0s_i} N\left( \frac{r - D_0 + \frac{1}{2}\sigma^2 s_i + \ln(S_c^{(i)}) - \ln(S_c^{(1)})}{\sigma \sqrt{s_i}} \right) \\
- Ze^{-r s_i} N\left( \frac{r - D_0 - \frac{1}{2}\sigma^2 s_i + \ln(S_c^{(i)}) - \ln(S_c^{(1)})}{\sigma \sqrt{s_i}} \right) + Ze^{-r s_i}.
\end{align*}
\] (3.27)

Since the terminal value of the free boundary, \( S_c^{(1)} = \frac{Z}{n} \), is known, we can calculate \( S_c^{(i)} \) for \( i = 2, 3, ..., N \) recursively with a MATLAB built-in root finding function (\texttt{fsolve}). It should be noted that the integral term here and those in other places are replaced by summations using a standard quadrature rule, the trapezoidal rule. The above procedure is similar to the one used in \(^{68}\).

Equations (3.25) and (3.26) are discretized and solved simultaneously to obtain the values of functions \( S_c(\tau) \) and \( S_p(\tau) \) in the interval \( \tau \in [\tau_M, T] \) by using the same method mentioned above. Instead of providing lengthy discretized equations, here we give brief outlines only. The time interval \( \tau \in [\tau_M, T] \) is again divided uniformly into \( L - 1 \) time intervals: \([h_1, h_2], [h_2, h_3] \cdots [h_{L-1}, h_L]\), where \( h_1 = \tau_M \) and \( h_L = T \). The discretized free boundaries \( S_c(h_i) \) and \( S_p(h_i) \) are denoted by \( S_c^{(i)} \) and \( S_p^{(i)} \), respectively, for \( i = 1, 2, ..., L \). Since the value of the function \( S_p(\tau) \) at \( \tau = \tau_M \) is equal to 0, we have \( S_p^{(1)} = 0 \). In addition, \( S_c^{(1)} \) should be the same as that of \( S_c^{(N)} \) obtained in \([0, \tau_M]\). Knowing \( S_c^{(1)} \) and \( S_p^{(1)} \), we calculate \( S_c^{(i)} \) and \( S_p^{(i)} \) for \( i = 2, 3, ..., L \), recursively using another MATLAB built-in root finding function (\texttt{lsqnonlin}) as this is a two dimensional problem.

Once the values of the functions \( S_c(\tau) \) and \( S_p(\tau) \) are obtained, the value of the bond can be straightforwardly computed through Equations (3.22) and (3.23).

We are now ready to validate our numerical scheme. Unless otherwise stated, parameters used are listed below (the same parameter setting will be used in the next section):

- Face value \( Z = 100 \),
- Conversion ratio \( n = 1 \),
- Maturity \( T = 1 \) (year),
- Risk-free annual interest rate \( r = 0.1 \),
- Rate of continuous dividend payment \( D_0 = 0.07 \),
- Volatility \( \sigma = 0.4 \),
- The put price \( M = 95 \).

Under these parameters, the critical value of time to expiry \( \tau_M = 0.5129 \) (year).
We choose to use the results calculated from the binomial tree method as the benchmark to validate our numerical scheme. Prior to benchmarking the numerical results obtained from the integral equation approach against the benchmark, numerical experiments are conducted in order to make sure that the benchmark itself obtained from the binomial tree method displays convergency. This is indeed verified as our numerical test results show that the convergence of the binomial tree method match with those reported in the literature, i.e., convergence has been established, as shown in Table 3.1, when the length of time interval is reduced to $\frac{1}{10000}$\footnote{Following a similar procedure presented by Goswami & Saini\cite{46}, it is not difficult to show that the computational complexity of our integral equation approach is $O(N^2)$, which is the same as that of the binomial tree method. This theoretical result is indeed consistent with what is displayed here.}. Since the binomial tree method will always converge according to\footnote{Following a similar procedure presented by Goswami & Saini\cite{46}, it is not difficult to show that the computational complexity of our integral equation approach is $O(N^2)$, which is the same as that of the binomial tree method. This theoretical result is indeed consistent with what is displayed here.}, it can be used as a benchmark, and unless otherwise stated, 1,000 time steps will be used to produce the binomial-tree results in the remaining numerical experiments for comparison purposes, so that the number of time steps used in both methods match with each other.

Table 3.1: Convergency test of the Binomial tree method

<table>
<thead>
<tr>
<th>Puttable Convertible bond price at $t = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>110</td>
</tr>
<tr>
<td>120</td>
</tr>
<tr>
<td>130</td>
</tr>
</tbody>
</table>

Now, we are ready to carry out numerical experiments to benchmark the accuracy and efficiency of our integral equation approach against the binomial tree method. From Table 3.2, one can see that all of the results from the integral equation approach at different $N$ (the number of time intervals) agree very well with the benchmark results with maximum relative error within the order of $10^{-4}$. It is observed, from the CPU time listed in Table 3.2, that the integral equation approach is slightly more efficient than the binomial tree method\footnote{Following a similar procedure presented by Goswami & Saini\cite{46}, it is not difficult to show that the computational complexity of our integral equation approach is $O(N^2)$, which is the same as that of the binomial tree method. This theoretical result is indeed consistent with what is displayed here.}. In addition, it should be noted that the time consumed in the integral equation approach includes the computation of the free boundaries, whereas the much longer time spent in the binomial tree method is only limited to producing the bond price, which makes the computational speed of the integral equation approach even more impressive. To further illustrate the accuracy of the integral equation method, the optimal boundaries obtained by the integral equation method are compared with those obtained by the binomial tree method in Table 3.3. It should be remarked that optimal boundaries produced by
the binomial tree method with 1,000 time steps are not accurate enough, and thus the number of time steps is further increased to 10,000. In this case, the results from the two approaches agree very well with the maximum relative error being less than 0.5%. This again shows the superiority of our integral equation method. Overall, the benchmark tests clearly demonstrated the accuracy and efficiency of our integral equation method.

Table 3.2: Accuracy and efficiency test of IE method

<table>
<thead>
<tr>
<th>S</th>
<th>Benchmark N=1,000</th>
<th>IE N=1,000</th>
<th>IE N=2,000</th>
<th>IE N=3,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>107.6894</td>
<td>107.6895</td>
<td>107.6873</td>
<td>107.6866</td>
</tr>
<tr>
<td>105</td>
<td>110.9297</td>
<td>110.9307</td>
<td>110.9291</td>
<td>110.9286</td>
</tr>
<tr>
<td>110</td>
<td>114.4379</td>
<td>114.4381</td>
<td>114.4367</td>
<td>114.4363</td>
</tr>
<tr>
<td>115</td>
<td>118.1927</td>
<td>118.1923</td>
<td>118.1911</td>
<td>118.1907</td>
</tr>
<tr>
<td>120</td>
<td>122.1746</td>
<td>122.1756</td>
<td>122.1745</td>
<td>122.1741</td>
</tr>
</tbody>
</table>
max. relative error | - | 8.80 × 10⁻⁵ | 1.95 × 10⁻⁵ | 2.60 × 10⁻⁵ |
Time (second) | 13.8836 | 10.4098 | 21.8943 | 38.6281 |

Table 3.3: Accuracy and efficiency test of IE method

Optimal boundaries

<table>
<thead>
<tr>
<th>τ</th>
<th>$S_c$ Benchmark N=1,000</th>
<th>$S_c$ Benchmark N=10,000</th>
<th>$S_c$ IE N=2,000</th>
<th>$S_p$ Benchmark N=1,000</th>
<th>$S_p$ Benchmark N=10,000</th>
<th>$S_p$ IE N=2,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1021</td>
<td>123.5048</td>
<td>123.9848</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.2560</td>
<td>132.5256</td>
<td>133.0856</td>
<td>133.3356</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.4098</td>
<td>137.4900</td>
<td>138.0200</td>
<td>138.2800</td>
<td>61.0431</td>
<td>63.4205</td>
<td>63.3805</td>
</tr>
<tr>
<td>0.5612</td>
<td>140.5231</td>
<td>141.0831</td>
<td>141.3731</td>
<td>58.4331</td>
<td>63.4205</td>
<td>63.8305</td>
</tr>
<tr>
<td>0.7073</td>
<td>143.4547</td>
<td>143.0447</td>
<td>143.3247</td>
<td>65.6605</td>
<td>63.4205</td>
<td>67.2047</td>
</tr>
<tr>
<td>0.8534</td>
<td>143.7388</td>
<td>144.3288</td>
<td>144.6088</td>
<td>66.6848</td>
<td>66.6847</td>
<td>67.2047</td>
</tr>
</tbody>
</table>
max. RE | $6.11 \times 10^{-3}$ | $2.06 \times 10^{-3}$ | -                | $4.34 \times 10^{-2}$ | $7.80 \times 10^{-3}$ | -                |

In the following section, the number of time intervals in solving our integral equations is set to be 2000 to achieve a balance between accuracy and efficiency. In addition, all of our calculations in this chapter are done on a PC with the following specifications: Intel(R) Xeon(R), CPU E5-1640 v4 @3.60GHz 3.60 GHz, and 32.0 GB of RAM.
3.5 Examples and discussions

In this section, numerical examples are provided to illustrate various properties of puttable convertible bonds, and the difference between vanilla and puttable CBs is also demonstrated.

Figure 3.1 shows both the optimal conversion price and the optimal put price with respect to the time to expiry. It can be seen that both the optimal conversion price and the optimal put price are the monotonically increasing functions of the time to expiry, $\tau = T - t$. And as the conversion ratio becomes larger, the optimal exercise curves become flatter. Naturally, both the optimal conversion price and the optimal put price vary inversely with the conversion ratio, and the optimal conversion prices at expiry are the strike price divided by the conversion ratio. In fact, these properties are the same as those of the vanilla convertible bond. For puttable CBs, it should be observed that there is only one free boundary, the optimal conversion boundary, during $[0, \tau_M]$, since the value of the optimal put boundary is equal to zero in this time interval. When the time to expiry is greater than the critical value of the time to expiry, $\tau_M$, the optimal put boundary “appears” due to the existence of the “put” feature. It is observed that when time to expiry is closer to zero, the optimal conversion price decreases quickly to the value of the strike price divided by the conversion ratio, and that as time to expiry approaches $\tau_M$, the optimal put price.
drops rapidly to zero. The large slope of \( S_c(\tau) \) and that of \( S_p(\tau) \) near \( \tau = 0 \) and \( \tau = \tau_M \), respectively, are similar to the behavior of the optimal exercise price near expiry.\(^{[42]}\).

Depicted in Figure 3.2 are the price curves of the puttable CB versus the underlying asset value, \( S \), at times \( t = 0.0000, t = 0.2435, t = 0.4871, t = 0.7435 \). We observe that the slope of the price curves is zero when the underlying asset is worthless, increasing slowly at first at lower underlying asset price, and eventually all curves become tangent to the payoff line. This observation indicates that the first-order partial derivative of the bond price with respect to the underlying asset price is between 0 and the conversion ratio \( n \), that is \( 0 \leq \frac{\partial V}{\partial S} \leq n \). It can be seen that the bond price remains almost unchanged when the underlying asset price is low, the greater the time, the higher the bond price. However, when the underlying asset price increases to a certain extend, a completely different phenomenon can be observed: the bond price becomes lower as time increases.

Figure 3.3 displays the value of a puttable CB and its vanilla counterpart. It can be seen that the value of the puttable convertible bond is higher than that of the vanilla one. This certainly makes sense since the holder of a puttable convertible bond has an additional right to sell the bond back to the issuer, and thus the holder should be expected to pay an extra amount as a “premium”. It is interesting to observe that such a premium decreases as the underlying asset price becomes
higher, and when the price of the underlying asset is very high, this premium is almost equal to zero. In other words, the price of the puttable convertible bond and that of the vanilla counterpart are almost equivalent to each other when the price of the underlying asset is very high. This can be easily explained since when the price of the underlying asset is high, there is no financial incentive for the holder to sell the bond back to the issuer. In this case, the puttable convertible bond can almost be replaced by the vanilla convertible bond. In contrast, the puttable convertible bond is worth more when the price of the underlying asset is low.

Figures 3.4 and 3.5 show the effects of the volatility on the bond price as well as its optimal conversion price and optimal put price. In particular, exhibited in Figure 3.4 are the bond prices corresponding to three different volatility values. When the stock price is very low, the bond prices are insensitive to the variation of volatility, since when the stock price is low, the bond price remains almost unchanged and are equal when $S$ is zero. Moreover, the value of the puttable CB is a monotonically increasing function of volatility. This is reasonable because when the volatility becomes larger, there is a higher risk, which will lead to a higher price. On the other hand, from Figure 3.5, it is easy to note that both the optimal conversion price and the optimal put price are the increasing functions of the time to expiry. Another interesting phenomenon is that a higher volatility will lead to a higher optimal conversion price while it will lead to a less optimal put price.
In Figures 3.6 and 3.7, we show how the price of a puttable convertible bond and both its optimal conversion price and optimal put price change with the risk-free
interest rate. Comparing the bond price as well as its two free boundaries shown in this figure with those shown in Figures 3.4 and 3.5, respectively, one can observe that
the risk-free interest rate has quite a different influence than volatility does. If we increase the risk-free interest rate, the bond price will decrease, this can be easily explained since when the risk-free interest rate is higher, it gives more incentive for investors to leave their money in a risk-free environment than buying a risky bond, resulting a lower CB price as displayed in Figure 3.6. On the other hand, in Figure 3.7 opposite trends for the two sets of free boundaries are observed, the optimal conversion price is a decreasing function of the risk-free interest rate while the optimal put price is an increasing function.

3.6 Conclusion

In this chapter, the pricing problem of a puttable convertible bond on a single underlying asset with constant dividend is considered, two integral equation formulations are presented for the first time. The integral equations are solved numerically to obtain the two free boundaries, and the bond price is then calculated from the integral representations in their respective domains. Numerical examples are provided to show some interesting properties of puttable convertible bonds, subject to different values of the volatility and the interest rate.

It should be remarked here that the current approach can be extended to solving the pricing problem for callable convertible bonds, in which case there exists at most one free boundary.
Chapter 4

Pricing callable-puttable convertible bonds with an integral equation approach

4.1 Introduction

Convertible bonds (CBs) are widely used financial instruments, which are different from bonds and stocks. However, CBs can be treated as a combination of bonds and options, since they possess the essential characteristics of these two. A CB gives its holders a right to convert the bond into a predetermined number of underlying stocks either only at the expiry (the so-called European-style) or during the entire life of the bond (the so-called American-style). Although such a right enables the holders to benefit from both the security of a bond as well as a possible higher return through a more risky underlying asset such as stocks, it also results in a much more complex pricing problem, especially for those of American-style since they are allowed to be converted at any time.

Various models have been used to price CBs. A simple choice was the Black-Scholes model\(^6\). Ingersoll\(^{62}\) and Brennan & Schwartz\(^{12}\) were the first to work on the problem under this model. In their approach, the firm value was utilized as the underlying asset. However, firm values are not observable in real markets and thus their approach has some drawbacks in practice as far as model calibration is concerned. Later on, McConnel & Schwartz\(^{84}\) proposed to adopt the stock price instead of the firm value as the underlying variable to price CBs.

Since then, research activities in the area of pricing CBs intensified. Among a large number of papers published in the past 30 years, numerical approaches, such as the finite element method\(^6\), the finite difference method\(^{98}\) and the finite volume method\(^{112}\), are often adopted. However, two main drawbacks, i.e. the accuracy problem and the time-consuming feature that exist in most of the numerical methods, prompted researchers to seek analytical solution approaches for their simplicity.
and analytical elegance, though they are quite often restricted to some relatively simple cases. For example, a closed-form solution for a simple CB, which can only be converted at maturity, was obtained by Nyborg[88], while Zhu[106] presented an analytical solution in the form of a Taylor series expansion for the simplest American style CB without any other clauses being added, using the Homotopy Analysis Method[74].

As one of the most popularly used financial derivatives in financial practice for firms to raise needed capital, CBs today, stemming from the very basic original concept, have many variations with some quite involved terms, clauses and conditions. Among them, callable CBs and puttable CBs are two kinds of the most popular CBs[2]. The former is a bond that allows the issuer to call (repurchase) the bond from the holder for a predetermined call price, which is used to protect the issuer against the risk of the underlying running unreasonably much higher than initially expected. When the underlying asset price increases beyond a preset critical value that is related to the conversion ratio and the call price, the issuer can call back the bond at the call price. Therefore, the price of the callable CB should be less than that of the vanilla counterpart, as a result of the holder’s potential return is capped from the above. On the other hand, puttability permits the holder to sell the bond back to the issuer at a predetermined put price. Obviously, the put feature benefits the holder of the bond, and thus a puttable CB is traded at a higher price than that of its vanilla counterpart.

Regarding solving the pricing problem of callable CBs, there are many reference materials. While Brennan & Schwartz[12] explained in theory on how to price the callable CB, and provided solutions using the finite difference method in their later article[13], Bernini[7] used the binomial tree method to obtain the solution. Yagi & Sawaki[103] priced the callable CBs with the utilization of the game options defined by Kifer[67]. On the other hand, there are also a few references on pricing puttable CBs in the literature. For instance, Nyborg[88] presented the boundary condition of puttable CBs, while Lvov et al.[80] obtained the numerical solution by using Monte Carlo simulations.

In this chapter, two types of CBs mentioned above are combined together to form a new type of CBs, called callable-puttable CBs, which should be considered on behalf of both the issuer and the holder. An integral equation formulation is presented to price callable-puttable CBs under the Black-Scholes model with the method of incomplete Fourier transform[26] and the Green’s function[39]. One may argue that it is more practical to adopt stochastic interest rate models[14,29,100] for pricing CBs, as CBs are usually designed for a long time period, during which the interest rate itself may be subject to changes. However, we still assume a constant interest rate in this study, since the pricing exercise is already very complicated
even under this simple model, resulting from the tangled presence of callability, puttability, as well as conversion, which have led to possible co-existence of two moving boundaries at the same time, depending on the values of the call price, the put price and the conversion ratio.

If a callable-puttable CB needs to be priced at a time sufficiently far away from the expiry, only the moving boundary associated with the puttability needs to be dealt with. For this situation, the partial differential equation (PDE) system governing the price of a callable-puttable CB is presented. When the pricing time is closer to expiry beyond a critical value, it is then possible to have two distinct cases. While the two moving boundaries associated with conversion and puttability co-exist in one case, they may both disappear in another with callability remaining to be the only issue that needs to be dealt with. The former case can be solved through one of the PDE systems presented in [109], while the PDE system for the latter case can be built without the presence of any free boundaries. Furthermore, there exists another critical value, beyond which the callable-puttable CB can be treated as the vanilla counterpart, solving which requires the utilization of the PDE system presented in [106]. In summary, the pricing problem for our issue should be designed with three different scenarios, and in each case, there are three or two PDE systems governing the price of a callable-puttable CB.

This chapter is organized as follows. In Section 2, the pricing problem is divided into three cases, and the PDE systems governing the price of a callable-puttable CB are established for each case, and also the form of integral equation is derived. In Section 3, we compared our results with the known benchmark. Numerical examples are presented in Section 4, followed by some concluding remarks given in the last section.

4.2 Models and results

In this section, the PDE systems are established to price callable-puttable CBs under the Black-Scholes model, and the integral equation formulations are obtained by solving these systems. As mentioned above, the pricing problem should be divided into three scenarios according to the relationship between the values of the call price and that of the put price. In particular, due to the fact that two additional rights are actually added into the vanilla CB, there will be two critical moments in the callable-puttable CB, corresponding to the time instances when the callability disappears and the same instance when the puttability disappears. The different order of the arrival of these two moments makes the pricing problem for the bond quite different, and three scenarios distinguished by which moment arrives earlier are thus considered.
Before we discuss the difference between three scenarios of the callable-puttable CB, the two similarities among these three should be pointed out first. One is that both of the puttability and callability are possible at the beginning of the bond, otherwise there are no financial incentive to exercise both of the two features during the lifetime of the bond. The reason is that maximum and minimum values of the bond are a decreasing and an increasing function of the time, respectively, which makes it impossible for the bond value to reach either the value of the call price or the value of the put price if initially the maximal and minimal bond value is lower and higher than the value of the call price and the value of the put price, respectively. Another one is that it is not an optimal choice to call or put the bond when the time is sufficiently close to expiry, since during this time period the maximal bond value is smaller than the value of the call price, $K$, and the minimal bond value is larger than the value of the put price, $M$. Therefore, the PDE systems corresponding to these two time intervals are the same for three scenarios. On the other hand, one should also be noted that the two critical moments can separate the time zone into two or three parts, and this means that the difference between these three cases is only the middle part. In the following, these three cases are discussed one by one.

In Case 1, when the value of the call price is sufficiently small, the callability disappears later since a small value of the call price makes it harder for the maximal price of the bond to drop down below the value of the call price compared with the case that the minimal value of the bond hits the value of the put price. Considering the property of a callable CB, the moment when the callability disappears is also the time when the value of the optimal conversion boundary gets to the value of the call price divided by the value of the conversion ratio, $\frac{K}{n}$, where $n$ is the conversion ratio. Thus, for this case, the first part actually consists of the time to expiry period when the value of the optimal put boundary is equal to zero and the value of the optimal conversion boundary is less than a certain value, which implies that the PDE system for this part is actually as same as that for the vanilla CB. The second part of this case represents the time to expiry period when callability is available while there is no sense to exercise puttability, which clearly shows that the value of the optimal put boundary is equal to zero and the PDE system for this part is the same as that for the callable CB. At last, the third part of Case 1 is the real callable-puttable CB part, in which both the callability and the puttability are possible and the value of the optimal put boundary is no longer zero.

In Case 2, when the value of the call price is sufficiently large, the moment when the minimal value of the bond hits the value of the put price comes later. Similar
to Case 1, the first part still models the time to expiry period when the callability and the puttability are not active, and the third part denotes the situation when the callability and the puttability both exist. However, being different from Case 1, the callability, which is possible in the second part of Case 1, is no longer meaningful, while both the puttability and conversion come into effect in the current situation.

Case 3 is actually a special case that the two moments arrive at the same time, and thus there are only two parts with the first one being equivalent to the vanilla CB and the last one being the callable-puttable one.

Having been aware of the similarity and difference among these three cases, the valuation of callable-puttable conversion bonds under the three cases will be discussed in next three subsections, respectively.

4.2.1 Case 1

In this subsection, the pricing problem of callable-puttable CBs for Case 1 will be discussed, in which the moment that the optimal conversion price reaches \( \frac{K}{n} \) later than the moment that the minimum value of the bond gets to the value of the put price. Firstly, let \( S_t \) be an underlying asset price and we assume that its dynamics follows the stochastic differential equation (the same assumption will be used in the next two cases):

\[
dS_t = (r - D_0)S_t dt + \sigma S_t dW_t,
\]

(4.1)

where \( W_t \) is a Brownian motion, \( \sigma \) is the volatility of the underlying asset, \( r \) is the risk-free interest rate, and \( D_0 \) is the continuous dividend rate.

Let \( V_1(S, \tau) \) be the value of the callable-puttable CB for Case 1, with the time to expiry, \( \tau = T - t \). Then, when the time to expiry is small enough, the value of the optimal put boundary is always equal to zero since it is not optimal for the holder to sell the bond back to the issuer, and at the same time, the value of the optimal conversion boundary does not reach the value \( \frac{K}{n} \), which means the issuer will not choose to call back the bond. During this time interval, the callable-puttable CB can be treated as a vanilla one. Therefore, \( V_1(S, \tau) \) in Part 1 should satisfy the following PDE system (c.f.\cite{[106]}):

\[
\left\{ \begin{array}{ll}
- \frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0)S \frac{\partial V_1}{\partial S} - rV_1 = 0, \\
V_1(S, 0) = \min\{K, \max\{nS, Z\}\}, \\
V_1(S_c(\tau), \tau) = nS_c(\tau), \\
\frac{\partial V_1}{\partial S}(S_c(\tau), \tau) = n, \\
V_1(0, \tau) = Ze^{-r\tau},
\end{array} \right.
\]

(4.2)
with the domain of $S$ and $\tau$ being $[0, S_c(\tau)]$ and $[0, \tau_K]$, respectively. Here, $Z$ is the face value of the bond, $S_c(\tau)$ is the value of the optimal conversion boundary and $\tau_K$ is the moment that the value of the optimal conversion boundary reaches $\frac{K}{n}$, which is the maximum value of the optimal conversion boundary for a callable-puttable CB.

With the PDE system for Part 1 of Case 1 being established, we can proceed to Part 2. According to the property of the callable CB, the value of the optimal conversion boundary should be constant and be the same as the value that the issuer calls the bond back divided to the value of the conversion ratio. Also, the value of the optimal put boundary is equal to zero in this interval, since the minimal value of the bond is not bounded below by the value of the put price during this time zone. So, the PDE system can be built as follows:

$$
\begin{align*}
- \frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0) S \frac{\partial V_1}{\partial S} - rV_1 &= 0, \\
V_1(S, \tau_K) &= V_1(S, \tau_K^-), \\
V_1(\frac{K}{n}, \tau) &= K, \\
V_1(0, \tau) &= Ze^{-r\tau},
\end{align*}
$$

(4.3)

with the domain of $S$ and $\tau$ being $[0, \frac{K}{n}]$ and $[\tau_K, \tau_M]$, respectively, where $\tau_M$ is the moment that the minimal value of the bond hits the value of the put price, i.e. the value of the vanilla CB at $S = 0$ is bounded below by $M$, since the value of the bond at any certain time is an increasing function with the value of underlying asset. Therefore, $\tau_M$ determined by the following equation: $Ze^{-r\tau_M} = M$, i.e. $\tau_M = -\frac{1}{r} \log \frac{M}{Z}$. Another fact that should be noted is that the value of two boundaries in this PDE system are all constants.

On the other hand, for the third time interval of Case 1, the value of the optimal conversion boundary is still a constant due to the existence of callability, while the value of the optimal put boundary will change with time to expiry since it is incentive to exercise puttability. Thus, the PDE system for Part 3 of Case 1 can be set up as:

$$
\begin{align*}
- \frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0) S \frac{\partial V_1}{\partial S} - rV_1 &= 0, \\
V_1(S, \tau_M^+) &= V_1(S, \tau_M^-), \\
V_1(\frac{K}{n}, \tau) &= K, \\
V_1(S_p(\tau), \tau) &= M, \\
\frac{\partial V_1}{\partial S} (S_p(\tau), \tau) &= 0,
\end{align*}
$$

(4.4)

with the domain of $S$ and $\tau$ being $[S_p(\tau), \frac{K}{n}]$ and $[\tau_M, T]$, respectively. Here, $S_p(\tau)$
is the value of the optimal put boundary. By now, the PDE systems for pricing callable-puttable CBs of Case 1 have been established, and the integral equation formulation will be obtained to determine the bond value, starting from Part 1.

The integral equation representation for Part 1 of Case 1

For this part, only one free boundary should be considered when we derive the integral equation formula. According to \[26\], the incomplete Fourier transform can be used to solve this problem, before applying which a classical transform needs to be applied to the PDE system (4.2) first, so that a dimensionless PDE system can be obtained. Let

\[ x = \ln(S), \quad v_1(x, \tau) = V_1(S, \tau), \]

and then the target PDE system can be simplified to

\[
\begin{cases}
-\frac{\partial v_1}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial v_1}{\partial x} - rv_1 = 0, \\
v_1(x, 0) = \min\{K, \max\{ne^x, Z\}\}, \\
v_1(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\
\frac{\partial v_1}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\
v_1(-\infty, \tau) = Ze^{-r\tau},
\end{cases}
\]

with the domain of \(x\) and \(\tau\) being \([-\infty, \ln(S_c(\tau))]\) and \([0, \tau_K]\), respectively. It needs to be pointed out that the so-called incomplete Fourier transform can not be applied to System (4.5) directly, since the boundary condition at infinity is non-zero. Thus, another transform:

\[U_1(x, \tau) = v_1(x, \tau) - Ze^{-r\tau},\]

is used, and then the PDE system (4.5) can be rewritten as:

\[
\begin{cases}
-\frac{\partial U_1}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 U_1}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial U_1}{\partial x} - rU_1 = 0, \\
U_1(x, 0) = \min\{K - Z, \max\{ne^x - Z, 0\}\}, \\
U_1(\ln(S_c(\tau)), \tau) = nS_c(\tau) - Ze^{-r\tau}, \\
\frac{\partial U_1}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\
U_1(-\infty, \tau) = 0.
\end{cases}
\]

Considering the definition of the domain, the incomplete Fourier transform for this part can be defined as follows:

\[
\mathcal{F}\{U_1(x, \tau)\} = \int_{-\infty}^{\ln(S_c(\tau))} U_1(x, \tau) \cdot e^{i\omega x} dx \triangleq \hat{U}_1(\omega, \tau).
\]
Applying the incomplete Fourier transform (4.7) to System (4.6) yields the following ordinary differential equation (ODE) system (the details are presented in Appendix B.1)

\[
\begin{align*}
\frac{\partial \hat{U}_1}{\partial \tau}(\omega, \tau) + B(\omega)\hat{U}_1(\omega, \tau) &= f(\omega, \tau), \\
\hat{U}_1(\omega, 0) &= \int_{\ln(S_c(0))}^{\infty} \min\{K - Z, \max\{ne^x - Z, 0\}\} \cdot e^{i\omega x} dx,
\end{align*}
\]

where

\[
B(\omega) = \frac{1}{2}\sigma^2 \omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r,
\]

\[
f(\omega, \tau) = (nS_c(\tau) - Ze^{-r\tau})e^{i\omega \ln(S_c(\tau))}
\]

\[
\left[\frac{S'_c(\tau)}{S_c(\tau)} + (r - D_0 - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 i\omega\right] + \frac{1}{2}\sigma^2 nS_c(\tau)e^{i\omega \ln(S_c(\tau))}.
\]

Clearly, System (4.8) is a non-homogeneous first-order linear ODE system with an initial condition, using the general solution for which can lead to the integral equation formulation of the bond value in the Fourier space

\[
\hat{U}_1(\omega, \tau) = \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} d\xi.
\]

But, it is better to derive the formulation in the original space than leaving it in the Fourier space, since numerically inverting Fourier transform consumes a lot of time. Therefore, the incomplete Fourier inversion transform is defined as

\[
U_1(x, \tau) = \mathcal{F}^{-1}\{\hat{U}_1(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, \tau) e^{-i\omega x} d\omega,
\]

needs to be applied to Equation (4.11).

It should be noted that the definition of the incomplete Fourier inversion transform is as same as the standard counterpart. In fact, according to\(^{[26]}\), although the definition of the incomplete Fourier transform can be quite different from the standard one, the inversion transforms for these two are the same except the domain of the definition. Therefore, we will use the same definition for Fourier inversion transform of different incomplete Fourier transforms in the following.

By applying Fourier inversion transform (4.12) on Equation (4.11), we obtain

\[
U_1(x, \tau) = \int_{-\infty}^{\ln(S_c(0))} \min\{K - Z, \max\{ne^x - Z, 0\}\} \cdot e^{-r\tau} \cdot e^{\frac{[(r-D_0-\frac{1}{2}\sigma^2)i\omega+u^2]}{2\sigma^2 \tau}} du
\]

\[
+ \int_0^\tau e^{-r(\tau - \xi)} \cdot \frac{e^{\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau - \xi)+i\omega \ln(S_c(\xi))]^2}{2\sigma^2 (\tau - \xi)}}}{\sqrt{2\pi \sigma^2 (\tau - \xi)}} \cdot \{nS_c(\xi) - Ze^{-r\xi}\} d\xi.
\]
\[
\frac{S_c' (\xi)}{S_c (\xi)} + \frac{1}{2} (r - D_0 - \frac{1}{2} \sigma^2 + \ln(S_c (\xi)) - x) + \frac{1}{2} \sigma^2 n S_c (\xi) d\xi. \tag{4.13}
\]

Then, the derivation of which is put in Appendix B.2, the integral representation in the original space can be expressed with the initial parameters as

\[
V_1(S, \tau) = \int_{-\infty}^{\ln(S_c(0))} \min \{K - Z, \max \{ne^n - Z, 0\} \} \cdot e^{-r\tau} \cdot e^{-\frac{|(r - D_0 - \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c(0))|^2}{2\sigma^2\tau}} du + \int_{0}^{\tau} \int_{0}^{\tau} \sqrt{2\pi\sigma^2} e^{-\frac{|(r - D_0 - \frac{1}{2} \sigma^2) (\tau - \xi) + \ln(S_c (\xi)) - \ln(S_c (\xi))|^2}{2\sigma^2(\tau - \xi)}} \cdot \{nS_c (\xi) - Ze^{-r \xi}\}
\]

\[
\cdot \frac{S_c' (\xi)}{S_c (\xi)} + \frac{1}{2} (r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_c (\xi)) - \ln(S_c (\xi))}{\tau - \xi}) + \frac{1}{2} \sigma^2 n S_c (\xi) d\xi + Ze^{-r\tau}. \tag{4.14}
\]

By now, we have presented the integral representation for Part 1 of Case 1. However, it should be noted that the integral equation formulation obtained by the incomplete Fourier transform is not perfect to price a bond\cite{26}. Therefore, a better one can be derived as

\[
V_1(S, \tau) = D_0 S \int_{0}^{\tau} ne^{-D_0 (\tau - \xi)} \cdot \frac{(r - D_0 + \frac{1}{2} \sigma^2) (\tau - \xi) - \ln(S_c (\tau)) + \ln(S)}{\sigma \sqrt{\tau - \xi}} d\xi + nS e^{-D_0\tau} \cdot \frac{(r - D_0 + \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c (0))}{\sigma \sqrt{\tau}} - Ze^{-r\tau} \cdot \frac{(r - D_0 + \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c (0))}{\sigma \sqrt{\tau}} + Ze^{-r\tau}, \tag{4.15}
\]

after some complex computations presented in Appendix B.3. Clearly, the integral representation just derived for pricing a callable-puttable CB of Part 1 for Case 1 is in form of an unknown function, \( S_c (\tau) \), and the method to obtain the value of the unknown function is displayed at the end of this subsection, after the forms of integral equation to PDE systems of Part 2 and Part 3 are obtained.

The integral equation representation for Part 2 of Case 1

It is interesting to note that there is no free boundary in the PDE system (4.3), and thus its analytical solution can be directly derived using the Green’s function

\[
V_1(S, \tau) = \frac{1}{\sqrt{2\pi\sigma^2(\tau - \tau_K)}} e^{-\frac{|(r - D_0 + \frac{1}{2} \sigma^2)^2 + 2\sigma^2(\tau - \tau_K)|}{2\sigma^2(\tau - \tau_K)}} \cdot \int_{-\infty}^{\ln(S)} \exp(-\frac{\ln(S) + y}{2\sigma^2(\tau - \tau_K)}) - \exp(-\frac{\ln(S) + 2\ln(\frac{K}{n}) + y}{2\sigma^2(\tau - \tau_K)})] \cdot \exp(-\frac{r - D_0 + \frac{1}{2} \sigma^2}{\sigma^2} (-\ln(S) + y)) \cdot (V_1(e^y, \tau_K) - Ze^{-r\tau_K}) dy
\]
+ \int_0^\tau \frac{-\ln(S) + \ln(K_n)}{\sqrt{2\pi\sigma^2(\tau - \tau_K - \xi)}} \cdot \exp(-\frac{[-\ln(S) + \ln(K_n)]^2}{2\sigma^2(\tau - \tau_K - \xi)}) \cdot (K - Ze^{-r(\tau_K + \xi)}) \cdot \exp(-\frac{r - D_0 - \frac{1}{2}\sigma^2}{\sigma^2}(\ln(K_n) - \ln(S)) + \frac{(r - D_0 - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}{2\sigma^2}((\xi + \tau_K - \tau)d\xi + Ze^{-r\tau},}

\text{(4.16)}

the complicated solution process of which is illustrated in Appendix B.4. With integral representations of bond values for the first two parts in Case 1 being obtained, the left task is to work out the third one, details of which are shown below.

The Integral representation for Part 3 of Case 1

The PDE system actually contains one free boundary, and thus the incomplete Fourier transform can again be used to obtain the integral equation representation. To transform the PDE system to a dimensionless one, a classical transform is applied first:

\[ x = \ln(S), \quad v_1(x, \tau) = V_1(S, \tau), \]

which leads to

\[ \begin{cases} 
- \frac{\partial v_1}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r - D_0 - \frac{1}{2}\sigma^2) \frac{\partial v_1}{\partial x} - rv_1 = 0, \\
\displaystyle \frac{K}{n} \quad (\xi + \tau_K - \tau) \quad d\xi \\

v_1(\ln(K_n), \tau) = K, \\
v_1(\ln(S_p(\tau)), \tau) = M, \\
\frac{\partial v_1}{\partial \xi}(\ln(S_p(\tau)), \tau) = 0,
\end{cases} \quad (4.17) \]

with the domain of \( x \) and \( \tau \) being \([\ln(S_p(\tau)), \ln(K_n)]\) and \([\tau_M, T]\), respectively. It should be emphasized that the domain of \( x \) for this part is different from that for Part 1 of Case 1, implying that another definition of the incomplete Fourier transform will be introduced:

\[ \mathcal{F}\{v_1(x, \tau)\} = \int_{\ln(S_p(\tau))}^{\ln(K_n)} v_1(x, \tau)e^{i\omega x} dx \triangleq \hat{v}_1(\omega, \tau). \quad (4.18) \]

When we try to apply the incomplete Fourier transform just defined (4.18) to System (4.17) directly, there exists an obstacle that the first-order derivative of the CB price at \( S = \frac{K}{n} \) is unknown. Therefore, we introduce a time-dependent function

\[ \frac{\partial V_1}{\partial S}(\frac{K}{n}, \tau) = A(\tau), \quad (4.19) \]
which is determined simultaneously with the unknown free boundaries later. Thus, System (4.17) should be rewritten with one more boundary condition

\[
\begin{align*}
\frac{\partial v_1}{\partial \tau} &+ \frac{1}{2} \sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r-D_0 - \frac{1}{2} \sigma^2) \frac{\partial v_1}{\partial x} - rv_1 = 0, \\
v_1(x, \tau_M) &= v_1(x, \tau_M), \\
v_1(\ln(K/n), \tau) &= K, \\
\frac{\partial v_1}{\partial x}(\ln(K/n), \tau) &= \frac{K}{n} A(\tau), \\
v_1(\ln(S_p(\tau)), \tau) &= M, \\
\frac{\partial v_1}{\partial x}(\ln(S_p(\tau)), \tau) &= 0.
\end{align*}
\]

(4.20)

In this case, formally applying the incomplete Fourier transform to the above PDE system, gives the following ODE system

\[
\begin{align*}
\frac{\partial \hat{v}_1}{\partial \tau}(\omega, \tau) + B(\omega) \hat{v}_1(\omega, \tau) &= f_1(\omega, \tau) - f_2(\omega, \tau), \\
\hat{v}_1(\omega, \tau_M) &= \int_{\ln(S_p(\tau_M))}^{\ln(K/n)} v_1(x, \tau_M) e^{i\omega x} dx,
\end{align*}
\]

where

\[
\begin{align*}
f_1(\omega, \tau) &= Ke^{i\omega \ln(K/n)} \cdot \left[ \frac{1}{2} \sigma^2 \frac{A(\tau)}{n} - \frac{1}{2} \sigma^2 i\omega + (r-D_0 - \frac{1}{2} \sigma^2) \right], \\
f_2(\omega, \tau) &= Me^{i\omega \ln(S_p(\tau))} \cdot \left[ \frac{S_p'(\tau)}{S_p(\tau)} - \frac{1}{2} \sigma^2 i\omega + (r-D_0 - \frac{1}{2} \sigma^2) \right],
\end{align*}
\]

and $B(\omega)$ is as the same definition as before. We refer interested readers to Appendix B.5 for derivation details.

One can easily find that this is again a non-homogeneous first-order linear ODE system, and the solution of it can be derived as

\[
\begin{align*}
\hat{v}_1(\omega, \tau) &= e^{-B(\omega)(\tau-\tau_M)} \cdot \hat{v}_1(\omega, \tau_M) \\
&+ \int_0^{\tau-\tau_M} f_1(\omega, \xi + \tau_M) e^{-B(\omega)(\tau-\tau_M-\xi)} d\xi \\
&- \int_0^{\tau-\tau_M} f_2(\omega, \xi + \tau_M) e^{-B(\omega)(\tau-\tau_M-\xi)} d\xi.
\end{align*}
\]

(4.24)

which is the solution in the Fourier space. Upon applying the Fourier inversion transform (the procedures are in Appendix B.6), the integral representation in the original space with the original parameters can be obtained

\[
V_1(S, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(K/n)} V_1(e^n, \tau_M) \cdot \frac{e^{-r(\tau-\tau_M)}}{\sqrt{2\pi \sigma^2(\tau-\tau_M)}} \cdot \frac{1}{u} e^{-\frac{|(r-D_0 - \frac{1}{2} \sigma^2)(\tau-\tau_M)+\ln(S)-u|^2}{2\sigma^2(\tau-\tau_M)}} du
\]
A CB of Case 1 have been derived, with three unknown functions, an alternative one can be derived after some complex computations boundary and the higher requirement for the smoothness of the free boundary. Fourier transform may have some problems, such as the accuracy problem in the boundaries conditions.

By now, the integral equation representations for pricing the callable-puttable (\( S, \tau \)), Fortunately, we can derive three integral equations for them using the free conditions:

\[
V_1(S, \tau) = \int_{\ln(S_p(\tau_M + \xi))}^{\ln(\frac{S}{n})} e^{-\frac{(r-D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}} \, \sigma A(\tau_M + \xi) \, du 
\]

\[
+ \int_0^{\tau - \tau_M} Ke^{-r(\tau - \tau_M - \xi)} \frac{e^{-\frac{(r-D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}}}{2\pi\sigma^2(\tau - \tau_M - \xi)} \, d\xi 
\]

\[- \int_0^{\tau - \tau_M} M e^{-r(\tau - \tau_M - \xi)} \frac{\sigma A(\tau_M + \xi)}{2n\sqrt{\tau - \tau_M - \xi}} \, d\xi 
\]

As mentioned above, the integral representation obtained by the incomplete Fourier transform may have some problems, such as the accuracy problem in the boundary and the higher requirement for the smoothness of the free boundary functions. As a result, it is not perfect for us to use this formula to price a bond, and an alternative one can be derived after some complex computations.

\[
\frac{\frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2}(r-D_0 - 1 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi})}{2} \, d\xi \] (4.25)

Again, the details are left in Appendix B.7.

By now, the integral equation representations for pricing the callable-puttable CB of Case 1 have been derived, with three unknown functions, \( S_c(\tau) \), \( S_p(\tau) \) and \( A(\tau) \). Fortunately, we can derive three integral equations for them using the free boundaries conditions.

\[
nS_c(\tau) = nS_c(\tau)e^{-D_0\tau}\mathcal{N}\left(\frac{(r-D_0 + \frac{1}{2}\sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \]
\[ - Ze^{-r_\tau} N\left(\frac{r-D_0 - \frac{1}{2} \sigma^2}{\sigma \sqrt{\tau}} + \ln(S_c(\tau)) - \ln(S_c(0))\right) \]
\[ + D_0 S_c(\tau) \int_0^\tau ne^{-D_0 (\tau-\xi)} N\left(\frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S_c(\tau))}{\sigma \sqrt{\tau - \xi}}\right) d\xi \]
\[ + Ze^{-r_\tau}, \]  
\[ K = \int_{\ln(S_p(\tau_M)))}^{\ln(S_p(\tau_M)))} V_1(e^{\mu}, \tau_M) \cdot e^{-r(\tau - \tau_M)} \cdot \frac{\sqrt{2\pi \sigma^2(\tau - \tau_M)}}{\sigma \sqrt{\tau - \tau_M}} e^{\frac{[r-D_0 - \frac{1}{2} \sigma^2(\tau - \tau_M) + \ln(S_p(\xi))]}{2\sigma^2(\tau - \tau_M)}} d\xi \]
\[ + \int_0^{\tau - \tau_M} Ke^{-r(\tau - \tau_M - \xi)} \cdot A\left(\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}\right) d\xi \]
\[ - Ke^{-r(\tau - \tau_M)} \cdot A\left(-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M)}{\sigma \sqrt{\tau - \tau_M}}\right) \]
\[ - Me^{-r(\tau - \tau_M)} \cdot A\left(-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M}}\right), \]
\[ (4.27) \]
\[ M = \int_{\ln(S_p(\tau_M)))}^{\ln(S_p(\tau_M)))} V_1(e^{\mu}, \tau_M) \cdot e^{-r(\tau - \tau_M)} \cdot \frac{\sqrt{2\pi \sigma^2(\tau - \tau_M)}}{\sigma \sqrt{\tau - \tau_M}} e^{\frac{[r-D_0 - \frac{1}{2} \sigma^2(\tau - \tau_M) + \ln(S_p(\xi))]}{2\sigma^2(\tau - \tau_M)}} d\xi \]
\[ + \int_0^{\tau - \tau_M} Ke^{-r(\tau - \tau_M - \xi)} \cdot A\left(\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}\right) d\xi \]
\[ - Ke^{-r(\tau - \tau_M)} \cdot A\left(-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M)}{\sigma \sqrt{\tau - \tau_M}}\right) \]
\[ - Me^{-r(\tau - \tau_M)} \cdot A\left(-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M}}\right), \]
\[ (4.28) \]
Therefore, using these three integral equations, the value of three unknown functions can be obtained, after which the price of the bond can be presented through the integral equation formulation directly.

With the valuation problem of a callable-puttable CB for Case 1 being successfully
solved, it is time to consider Case 2, which will be discussed in the next subsection.

### 4.2.2 Case 2

In this subsection, the PDE systems for the second case of callable-puttable CBs are established. For this case, we assume that the moment when the value of the optimal conversion boundary reaches \( \frac{K}{n} \) appears earlier than the moment when the minimum value of the bond gets to the value of the put price. Similar to Subsection 4.2.1, the PDE systems also be set up with respect to the time to expiry. When the time is sufficiently close to the expiry, it is not optimal for the holder to sell the bond back to the issuer, so the value of the optimal put boundary is always equal to zero in this interval until the time to expiry reaches a certain value, \( \tau_M \). Furthermore, since it is close to expiry, the value of the optimal conversion boundary is unable to reach \( \frac{K}{n} \), and thus the boundary condition for the optimal convertible boundary is same to the vanilla CB. As a result, the PDE system for this part is as same as that for the vanilla CB.

If the value of the callable-puttable CB for Case 2 is assumed \( V_2(S, \tau) \), then the PDE system for the first part of Case 2 is

\[
-\frac{\partial V_2}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - D_0)S \frac{\partial V_2}{\partial S} - rV_2 = 0, \\
V_2(S, 0) = \min\{K, \max\{nS, Z\}\}, \\
V_2(S_c(\tau), \tau) = nS_c(\tau), \\
\frac{\partial V_2}{\partial S}(S_c(\tau), \tau) = n, \\
V_2(0, \tau) = Ze^{-r\tau},
\]

with the domain of \( S \) and \( \tau \) being \([0, S_c(\tau)]\) and \([0, \tau_M]\), respectively.

For Part 2 of Case 2, the terminal time of this part will be the moment that the value of the optimal conversion boundary reaches \( \frac{K}{n} \). Thus, during this time zone, the issuer is not willing to call the bond back either, which leads to the optimal conversion boundary condition for this part being the same as that for Part 1, while the value of the optimal put boundary is longer equal to zero. This can give rise to
the PDE system for this time zone

\[
\begin{aligned}
&-\frac{\partial V_2}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r-D_0)S \frac{\partial V_2}{\partial S} - rV_2 = 0, \\
&V_2(S, \tau_M) = V_2(S, \tau^+_M), \\
&V_2(S_c(\tau), \tau) = nS_c(\tau), \\
&\frac{\partial V_2}{\partial S}(S_c(\tau), \tau) = n, \\
&V_2(S_p(\tau), \tau) = M, \\
&\frac{\partial V_2}{\partial S}(S_p(\tau), \tau) = 0,
\end{aligned}
\] (4.31)

with the domain of \( S \) and \( \tau \) being \([S_p(\tau), S_c(\tau)]\) and \([\tau_M, \tau_K]\), respectively.

Now, the PDE system for the third time interval of Case 2 is set up. In fact, it should be noted that the situation of two free boundaries in this time interval is as the same as that in Part 3 of Case 1. Therefore, the PDE system can be built directly by making use of the corresponding counterpart in Case 1

\[
\begin{aligned}
&-\frac{\partial V_2}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r-D_0)S \frac{\partial V_2}{\partial S} - rV_2 = 0, \\
&V_2(S, \tau^n) = V_2(S, \tau^+_K), \\
&V_2(K, \tau) = K, \\
&V_2(S_p(\tau), \tau) = M, \\
&\frac{\partial V_2}{\partial S}(S_p(\tau), \tau) = 0,
\end{aligned}
\] (4.32)

with the domain of \( S \) and \( \tau \) being \([S_p(\tau), K]\) and \([\tau_K, T]\), respectively.

Obviously, the PDE systems for pricing callable-puttable CBs of Case 2 have been set up as (4.30), (4.31) and (4.32). It is interesting to observe that the PDE system for Part 1 of this case is almost as same as that of Case 1, and the same phenomenon happens for Part 3. As a result, the integral equation representations in Case 1 for these two parts can be applied to this case directly.

The integral equation representation for Part 1 of Case 2

Since there is no different between this part and Part 1 of Case 1, the integral equation representation can be written as

\[
V_2(S, \tau) = \text{nSe}^{-D_0\tau} \mathcal{N}\left(\frac{(r-D_0 + \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right)
- \text{Ze}^{-r\tau} \mathcal{N}\left(\frac{(r-D_0 - \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right)
\]
\[ + D_0 S \int_0^\tau ne^{-D_0(\tau-\xi)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau-\xi) + \ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau - \xi}} \right) d\xi \]

\[ + Ze^{-r\tau}. \] (4.33)

The integral equation representation for Part 2 of Case 2

With a careful observation, it is not difficult to find that this part is actually equivalent to the second part of the puttable CB, derived in\(^{[109]}\). Thus, the corresponding results there can be directly utilized so that the integral equation representation can be expressed

\[
V_2(S, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_2(e^{u'}, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sqrt{2\pi\sigma^2(\tau-\tau_M)}} d\xi + \int_0^{\tau-\tau_M} nD_0 Se^{-D_0(\tau-\tau_M-\xi)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau-\tau_M-\xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi
\]

\[ - \int_0^{\tau-\tau_M} rMe^{-r(\tau-\tau_M-\xi)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau-\tau_M-\xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi \]

\[ + nSe^{-D_0(\tau-\tau_M)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau-\tau_M) + \ln(S) - \ln(S_c(\tau_M))}{\sigma \sqrt{\tau - \tau_M}} \right)
\]

\[ - Me^{-r(\tau-\tau_M)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau-\tau_M) + \ln(S) - \ln(S_p(\tau_M))}{\sigma \sqrt{\tau - \tau_M}} \right). \] (4.34)

The integral equation representation for Part 3 of Case 2

The PDE system for this part shows that it is almost as the same as the Part 3 of Case 1, except the domain of the time interval. Hence, the formulation of the integral equation formulation can be presented as

\[
V_2(S, \tau) = \int_{\ln(S_c(\tau_K))}^{\ln(S_c(\tau_K+\xi))} V_2(e^{u'}, \tau_K) \frac{e^{-r(\tau-\tau_K)}}{\sqrt{2\pi\sigma^2(\tau-\tau_K)}} d\xi + \int_0^{\tau-\tau_K} Ke^{-r(\tau-\tau_K-\xi)} e^{-\frac{|(r-D_0 - \frac{1}{2} \sigma^2)(\tau-\tau_K-\xi) + \ln(S) - \ln(S_c(\tau_K + \xi))|^2}{2\sigma^2(\tau-\tau_K-\xi)}} \frac{\sigma A(\tau_K+\xi) + \xi}{2\sqrt{\tau - \tau_K - \xi}} d\xi
\]

\[ - \int_0^{\tau-\tau_K} rMe^{-r(\tau-\tau_K-\xi)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau-\tau_K-\xi) + \ln(S) - \ln(S_c(\tau_K + \xi))}{\sigma \sqrt{\tau - \tau_K - \xi}} \right) d\xi
\]

\[ - \int_0^{\tau-\tau_K} rKe^{-r(\tau-\tau_K-\xi)} \]
\[ nS_c(\tau) = nS_c(\tau)e^{-D_0\tau}\mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) 
- Z_0^{-\tau}\mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi))}{\sigma\sqrt{\tau - \xi}}\right) 
+ D_0S_c(\tau)\int_0^\tau ne^{-D_0(\tau - \xi)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right)d\xi 
- Z_0^{-\tau}, \tag{4.36} \]

\[ M = \int_{\ln(S_p(\tau_M))}^{\ln(S_p(\tau_M))} V_2(e^{\mu}, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sqrt{2\pi\sigma^2(\tau - \tau_M)}} e^{-\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(S_p(\tau_M))}{2\sigma^2(\tau - \tau_M)}} du 
+ \int_{\tau - \tau_M}^{\tau - \tau_M} nD_0S_p(\tau)e^{-D_0(\tau - \tau_M - \xi)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right)d\xi 
- Me^{-r(\tau - \tau_M)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(S_p(\tau_M))}{\sigma\sqrt{\tau - \tau_M}}\right), \tag{4.37} \]
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\[ \begin{align*}
&- \int_0^{\tau-\tau_M} rMe^{-r(\tau-\tau_M-\xi)} \\
&\qquad \cdot \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&\quad + nS_p(\sigma) e^{-D_0(\tau-\tau_M)} \mathcal{N}\left( \frac{(r-D_0 + \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(S_c(\tau_M))}{\sigma \sqrt{\tau - \tau_M}} \right) \\
&\quad - Me^{-r(\tau-\tau_M)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(S_p(\tau_M))}{\sigma \sqrt{\tau - \tau_M}} \right), \quad (4.38)
\end{align*} \]

\[ \begin{align*}
K &= \int_{\ln(K/\pi)}^{\ln(S_p(\tau_k))} V_2(e^u, \tau_k) \frac{e^{-r(\tau-\tau_k)}}{\sqrt{2\pi}} e^{\frac{|(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi) + [\ln(K/\pi) - \ln(S_p(\tau_k + \xi))]|^2}{2\sigma^2(\tau - \tau_k)}} du \\
&\quad + \int_0^{\tau-\tau_k} rKe^{-r(\tau-\tau_k-\xi)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi)}{\sigma \sqrt{\tau - \tau_k - \xi}} \right) d\xi \\
&\quad - \int_0^{\tau-\tau_k} rKe^{-r(\tau-\tau_k-\xi)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi) + (\ln(K/\pi) - \ln(S_p(\tau_k + \xi)))]}{\sigma \sqrt{\tau - \tau_k - \xi}} \right) d\xi \\
&\quad - Ke^{-r(\tau-\tau_k)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi)}{\sigma \sqrt{\tau - \tau_k}} \right) \\
&\quad - Me^{-r(\tau-\tau_k)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k) + \ln(S_p(\tau)) - \ln(S_p(\tau_k))}{\sigma \sqrt{\tau - \tau_k}} \right), \quad (4.39)
\end{align*} \]

\[ \begin{align*}
M &= \int_{\ln(K/\pi)}^{\ln(S_p(\tau_k))} V_2(e^u, \tau_k) \frac{e^{-r(\tau-\tau_k)}}{\sqrt{2\pi}} e^{\frac{|(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi) + [\ln(K/\pi) - \ln(S_p(\tau_k + \xi))]|^2}{2\sigma^2(\tau - \tau_k)}} du \\
&\quad + \int_0^{\tau-\tau_k} rKe^{-r(\tau-\tau_k-\xi)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi) + [\ln(K/\pi) - \ln(S_p(\tau_k + \xi))]}{\sigma \sqrt{\tau - \tau_k - \xi}} \right) d\xi \\
&\quad - \int_0^{\tau-\tau_k} rKe^{-r(\tau-\tau_k-\xi)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k - \xi) + (\ln(K/\pi) - \ln(S_p(\tau_k + \xi)))]}{\sigma \sqrt{\tau - \tau_k - \xi}} \right) d\xi \\
&\quad - Ke^{-r(\tau-\tau_k)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k) + \ln(S_p(\tau)) - \ln(K/\pi)}{\sigma \sqrt{\tau - \tau_k}} \right) \\
&\quad - Me^{-r(\tau-\tau_k)} \mathcal{N}\left( \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_k) + \ln(S_p(\tau)) - \ln(S_p(\tau_k))}{\sigma \sqrt{\tau - \tau_k}} \right). \quad (4.40)
\end{align*} \]

Hence, these five integral equations can be utilized to derive the value of the
unknown functions, which are contained in the formulation of the pricing for the callable-puttable CB, and the bond values can be subsequently obtained, once the value of these unknown functions are obtained. One may get confused about the current results as there are only three unknown functions, whereas five equations governing these functions are derived. But, it is actually reasonable since we have three parts, and we are trying to find the value of the unknown functions on these parts separately. In particular, there is only one unknown term in Part 1, and we use Equation (4.36) to solve it, while Equation (4.37) - (4.38) and (4.39) - (4.40) are used to derive the two unknown terms of Part 2 and Part 3, respectively.

Being clear about the PDE systems to value the callable-puttable CB for Case 1 and Case 2 and the corresponding solutions, we can now proceed to the remaining case, which is a special one, in the next subsection.

4.2.3 Case 3

In this subsection, the PDE systems are built for pricing the callable-puttable CB of Case 3. This is in fact a special case, and is actually the easiest one among these three, since the moment that the value of the optimal conversion boundary reaches \( \frac{K}{n} \) and the moment that the minimum value of the bond hits the value of the put price arrive at the same time, i.e. \( \tau_K = \tau_M \). Therefore, there are only two time intervals, one of which is near the expiration, being the same as Part 1 of Case 1 and Case 2, where the value of the optimal put boundary is always equal to zero. Let the value of the callable-puttable CB for this case be \( V_3(S, \tau) \), and the PDE system for this part can be written as:

\[
\begin{align*}
-\frac{\partial V_3}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_3}{\partial S^2} + (r - D_0)S \frac{\partial V_3}{\partial S} - rV_3 &= 0, \\
V_3(S, 0) &= \min\{K, \max\{nS, Z\}\}, \\
V_3(S_c(\tau), \tau) &= nS_c(\tau), \\
\frac{\partial V_3}{\partial S}(S_c(\tau), \tau) &= n, \\
V_3(0, \tau) &= Ze^{-r\tau},
\end{align*}
\]

with the domain of \( S \) and \( \tau \) being \([0, S_c(\tau)]\) and \([0, \tau_M]\), respectively.

The other part is the one that the value of the optimal conversion boundary is a constant while the value of the optimal put boundary is not equal to zero, which is the same as the third part of the last two cases. Therefore, the PDE system for this
time zone is built as follows:

\[
\begin{align*}
- \frac{\partial V_3}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_3}{\partial S^2} + (r - D_0) S \frac{\partial V_3}{\partial S} - r V_3 &= 0, \\
V_3(S, \tau^n_M) &= V_3(S, \tau^-_M), \\
V_3(\frac{K}{n}, \tau) &= K, \\
V_3(S_p(\tau), \tau) &= M, \\
\frac{\partial V_3}{\partial S}(S_p(\tau), \tau) &= 0,
\end{align*}
\]

(4.42)

with the domain of \( S \) and \( \tau \) being \([S_p(\tau), \frac{K}{n}]\) and \([\tau_M, \tau]\), respectively.

With the PDE systems being built up, the integral equation representations can be given below.

The integral equation representation for Part 1 of Case 3

By comparing the PDE system of Part 1 among these three case, we can write the integral equation formulation

\[
V_3(S, \tau) = nS e^{-D_0 \tau} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right)
- Z e^{-r \tau} \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2} \sigma^2) \tau + \ln(S) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right)
+ D_0 S \int_0^{\tau} n e^{-D_0 (\tau - \xi)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2} \sigma^2) (\tau - \xi) + \ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau - \xi}}\right) d\xi
+ Z e^{-r \tau}.
\]

(4.43)

The integral equation representation for Part 2 of Case 3

As mentioned above, the PDE system for this part is as the same as Part 3 of Case 1 and Case 2. Following the results above, the integral equation formulation for this part is

\[
V_3(S, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_3(e^u, \tau_M) e^{-r(\tau - \tau_M)} \frac{\sigma A(\tau_M + \xi)}{2 \pi \sqrt{\sigma^2 \tau - \tau_M - \xi^2}} e^{\frac{[r - D_0 - \frac{1}{2} \sigma^2] (\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}} du
+ \int_0^{\tau - \tau_M} Ke^{-r(\tau - \tau_M - \xi)} e^{\frac{[r - D_0 - \frac{1}{2} \sigma^2] (\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}} \cdot \frac{\sigma A(\tau_M + \xi)}{2 \pi \sqrt{\sigma^2 \tau - \tau_M - \xi}} d\xi
- \int_0^{\tau - \tau_M} r Me^{-r(\tau - \tau_M - \xi)} e^{\frac{[r - D_0 - \frac{1}{2} \sigma^2] (\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}} d\xi
- \int_0^{\tau - \tau_M} r Ke^{-r(\tau - \tau_M - \xi)} e^{\frac{[r - D_0 - \frac{1}{2} \sigma^2] (\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}} d\xi.
\]
\[ \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S)}{\sigma \sqrt{\tau - \tau_M - \xi}}\right) \]

\[ - K e^{-r(\tau - \tau_M)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma \sqrt{\tau - \tau_M}}\right) \]

\[ - M e^{-r(\tau - \tau_M)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma \sqrt{\tau - \tau_M}}\right). \] (4.44)

Combining Equations (4.43) and (4.44), we have successfully derived the integral equation representations for the third case of the callable-puttable CB in form of three unknown functions, \( S_c(\tau), S_p(\tau) \) and \( A(\tau) \). Thus, the boundary conditions can again be used to determine the value of these unknown functions.

\[ nS_c(\tau) = nS_c(\tau) e^{-D_0(\tau - \xi)} \mathcal{N}\left(-\frac{(r - D_0 + \frac{1}{2} \sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right) \]

\[ - Ze^{-r\tau} \mathcal{N}\left(-\frac{(r - D_0 + \frac{1}{2} \sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma \sqrt{\tau}}\right) \]

\[ + D_0 S_c(\tau) \int_{0}^{\tau} ne^{-D_0(\tau - \xi)} \mathcal{N}\left(-\frac{(r - D_0 + \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi))}{\sigma \sqrt{\tau - \xi}}\right) d\xi \]

\[ + Ze^{-r\tau}, \] (4.45)

\[ K = \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_3(e^{u}, \tau_M) e^{-r(\tau - \tau_M)} \frac{e^{-\frac{\ln(\xi)}{2\sigma^2}(\tau - \tau_M)}}{\sqrt{2\pi \sigma^2(\tau - \tau_M)}} du \]

\[ + \int_{0}^{\tau - \tau_M} K e^{-r(\tau - \tau_M - \xi)} \frac{e^{-\frac{|(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi)|^2}{2\sigma^2(\tau - \tau_M - \xi)}}}{\sqrt{2\pi}} A(\tau_M + \xi) \frac{\sigma A(\tau_M + \xi)}{2n \sqrt{\tau - \tau_M - \xi}} d\xi \]

\[ - \int_{0}^{\tau - \tau_M} r M e^{-r(\tau - \tau_M - \xi)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}}\right) d\xi \]

\[ - \int_{0}^{\tau - \tau_M} r K e^{-r(\tau - \tau_M - \xi)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi)}{\sigma \sqrt{\tau - \tau_M}}\right) d\xi \]

\[ - K e^{-r(\tau - \tau_M)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M)}{\sigma \sqrt{\tau - \tau_M}}\right) \]

\[ - M e^{-r(\tau - \tau_M)} \mathcal{N}\left(-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau_M))}{\sigma \sqrt{\tau - \tau_M}}\right), \] (4.46)

\[ M = \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_3(e^{u}, \tau_M) e^{-r(\tau - \tau_M)} \frac{e^{-\frac{\ln(\xi)}{2\sigma^2}(\tau - \tau_M)}}{\sqrt{2\pi \sigma^2(\tau - \tau_M)}} du \]

\[ + \int_{0}^{\tau - \tau_M} K e^{-r(\tau - \tau_M - \xi)} \frac{e^{-\frac{|(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi)|^2}{2\sigma^2(\tau - \tau_M - \xi)}}}{\sqrt{2\pi}} \frac{\sigma A(\tau_M + \xi)}{2n \sqrt{\tau - \tau_M - \xi}} d\xi \]

\[ - \int_{0}^{\tau - \tau_M} r M e^{-r(\tau - \tau_M - \xi)} \]
\begin{align}
\mathcal{N}(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_p(\tau_M + \xi)) \bigg) d\xi
- \int_0^{\tau - \tau_M} r e^{-r(\tau - \tau_M - \xi)}
\mathcal{N}(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(K) \bigg) d\xi
- K e^{-r(\tau - \tau_M)} \mathcal{N}(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(K) \bigg) \frac{\sigma \sqrt{\tau - \tau_M - \xi}}{\sigma \sqrt{\tau - \tau_M - \xi}}
- M e^{-r(\tau - \tau_M)} \mathcal{N}(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - \ln(S_p(\tau_M)) \bigg) \frac{\sigma \sqrt{\tau - \tau_M - \xi}}{\sigma \sqrt{\tau - \tau_M}}.
\end{align} 

(4.47)

Afterwards, the integral equation representations for pricing the callable-puttable CB can be utilized to get the bond values by direct substitution of the derived function values.

By now, the integral equation representations for pricing the callable-puttable CB have been presented. However, it should be noted that, for all these three cases, the integral equations are all nonlinear, and thus a numerical method needs to be utilized to obtain the numerical solution, the details of which are provided in the next two sections.

4.3 The numerical implementation

In this section, the numerical scheme adopted to solve the integral equations shown in the previous section will be introduced. It should be noted that although different integral equations corresponding to different scenarios need to be solved, the method used for numerical implementation of each one is very similar. Therefore, we will only introduce the scheme for solving Part 2 of Case 3 as an example; the others can be similarly derived.

The main procedure involved in finding numerical solutions through our approach is to solve the coupled Equation (4.47) and Equation (4.47) to obtain the value of two unknown functions, the optimal put boundary, \( S_p(\tau) \), and the unknown function, \( A(\tau) \), in Equation (4.44). Once these two functions are determined, the bond price can be calculated directly using Equation (4.44). In the following, the process in determining the two unknown functions is illustrated.

Firstly, the time interval \([\tau_M, T]\) is separated into several uniform time steps:

\[ [s_1, s_2], [s_2, s_3] \cdots [s_{L-1}, s_L] \]

where \( s_1 = \tau_M \) and \( s_L = T \),

and the discretized unknown functions \( S_p(s_i) \) and \( A(s_i) \) are denoted by \( S_p(i) \) and \( A(i) \), respectively, for \( i = 1, 2, \ldots, L \). And thus we obtain a set of non-linear algebraic
equations for \( S_p^{(i)} \) and \( A^{(i)} \) with \( i = 1, 2, \ldots, L \), as

\[
K = \int_{\ln(S_p^{(i)})}^{\ln(S_p^{(i)})} V_3(e^{\mu t}, \tau_M) \frac{e^{-r(s_i-s_1)}}{\sqrt{2\pi}\sigma^2(s_i-s_1)} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(s_i-s_1)+\ln(s_i^{(i)})-u]^2}{2\sigma^2(s_i-s_1)}} \, du
\]

\[
+ \sum_{k=1}^{i-1} Ke^{-r(s_i-s_k)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_k}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) (s_{k+1} - s_k)
\]

\[
- \sum_{k=1}^{i-1} rMe^{-r(s_i-s_k)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_k}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot \frac{(s_2-s_1)}{2}
\]

\[
rM \frac{s_2-s_1}{2} - rK \frac{s_2-s_1}{4}
\]

\[
- \sum_{k=1}^{i-1} rKe^{-r(s_i-s_k)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_k}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot (s_{k+1} - s_k)
\]

\[
rKe^{-r(s_i-s_1)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_1}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot \frac{(s_2-s_1)}{2}
\]

\[
- Ke^{-r(s_i-s_1)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_1}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right)
\]

\[
M = \int_{\ln(s^{(i)})}^{\ln(s^{(i)})} V_3(e^\mu t, \tau_M) \frac{e^{-r(s_i-s_1)}}{\sqrt{2\pi}\sigma^2(s_i-s_1)} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(s_i-s_1)+\ln(s_i^{(i)})-u]^2}{2\sigma^2(s_i-s_1)}} \, du
\]

\[
+ \sum_{k=1}^{i-1} Ke^{-r(s_i-s_k)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_k}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot \frac{(s_2-s_1)}{2}
\]

\[
rM \frac{s_2-s_1}{4}
\]

\[
- \sum_{k=1}^{i-1} rKe^{-r(s_i-s_k)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_k}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot (s_{k+1} - s_k)
\]

\[
rKe^{-r(s_i-s_1)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_1}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right) \cdot \frac{(s_2-s_1)}{2}
\]

\[
- Ke^{-r(s_i-s_1)} \cdot \mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma\sqrt{s_i-s_1}} \cdot \ln(s^{(i)}) - \ln(s^{(i)})\right)
\]
\[ - M e^{-r(s_i-s_1)} A_i \left( \frac{(r-D_0 - \frac{1}{2}\sigma^2)(s_i-s_1) + \ln(S_p^{(i)}) - \ln(S_p^{(1)})}{\sigma \sqrt{s_i-s_1}} \right). \] (4.49)

It should be pointed out that the value of the optimal put boundary at \( i = 1 \) is known, because \( S_p^{(1)} \) can be obtained from the solution for the time period \( \tau \in [0, \tau_M] \), and \( A^{(1)} \) can be figured out from its definition, i.e. \( A^{(1)} = n \). Thus, we can calculate \( S_p^{(i)} \) and \( A^{(i)} \) for \( i = 2, 3, ..., L \), simultaneously, with a MATLAB built-in root finding function (lsqnonlin). Once the values of the functions \( S_p(\tau) \) and \( A(\tau) \) are determined, the value of the bond can be found directly through Equation (4.44).

Before we proceed to studying the properties of callable-puttable CBs, it is necessary to validate the designed numerical scheme by assessing its accuracy and efficiency. Unless otherwise stated, parameters listed below are also used in the next section.

- Face value \( Z = 100 \),
- Conversion ratio \( n = 1 \),
- Time to expiration \( T = 1 \) (year),
- Risk-free annual interest rate \( r = 0.1 \),
- Rate of continuous dividend payment \( D_0 = 0.07 \),
- Volatility \( \sigma = 0.4 \),
- The put price \( M = 95 \),
- The call price for Case 1 \( K_1 = 135 \),
- The call price for Case 2 \( K_2 = 145 \).

Under these parameters, the threshold value of time to expiry, \( \tau_M \), is 0.5129 (year), and the value of the call price for Case 3, \( K_3 \), is 140.5114.

In \cite{109}, the convergence of the binomial tree method has already been shown, and it has also pointed out that the solution of the binomial tree method with 1,000 time-steps can be used as the benchmark. Thus, we omit the details here, and directly compare the accuracy and efficiency between the benchmark and our integral equation approach. Table 4.1 shows that all of the results obtained by the integral equation approach with the different value of \( N \) (the number of time intervals) match very well with the benchmark results with the maximum relative error being in the order of \( 10^{-2} \). On the other hand, the CPU time consumed by the integral equation approach is much less than that cost by the binomial tree method. It should be remarked that the time consumed in the integral equation approach has
included the computation of computing the value of two free boundaries, while the binomial tree method only yields the bond price within the listed time. Therefore, the benchmark test clearly demonstrates the accuracy and efficiency of our integral equation method.

### Table 4.1: Accuracy and efficiency test of IE method

<table>
<thead>
<tr>
<th>Callable-puttable CB price at $t=0$</th>
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<tbody>
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<td>100</td>
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<td>110</td>
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<tr>
<td>120</td>
</tr>
<tr>
<td>130</td>
</tr>
<tr>
<td>max. relative error</td>
</tr>
<tr>
<td>Time (second)</td>
</tr>
</tbody>
</table>

In the following section, the number of time intervals in solving our integral equations is set to be 2000 to achieve a balance between accuracy and efficiency. In addition, all of our calculations in this chapter is done on a PC with the following specifications: Intel(R) Xeon(R), CPU E5-1640 v4 @3.60GHz 3.60 GHz, and 32.0 GB of RAM.

### 4.4 Numerical results and discussions

In this section, the price of the callable-puttable CB will be presented with the value of its free boundaries, i.e., the optimal conversion boundary and the optimal put boundary. Various properties will be discussed based on the numerical experimental results.

Depicted in Figure 4.1 are the values of the optimal exercise boundaries with time to expiry for three cases. It is obvious that no matter in which case, the value of
Figure 4.1: The value of the optimal exercise boundaries for three different values of the conversion ratios.
two free boundaries, including the optimal conversion boundary and the optimal put boundary, are the increasing functions with the time to expiry, $\tau = T - t$. Also, the value of the optimal conversion boundary is constant when the time to expiry is greater than $\tau_K$, since in this situation, the call feature is switched on with the fixed boundary, $\frac{K}{n}$. In addition, the value of the optimal put boundary during the time to expiry interval $[0, \tau_M]$ is always equal to zero for all the three cases, while it arises when the time to expiry is greater than another critical value of the time to expiry, $\tau_M$, with the put feature being meaningful. Another property that can be seen in all the three figures is that a higher value of the conversion ratio leads to lower values of the free boundaries. This is because a higher value of the conversion ratio means a larger number of the underlying assets the holder can get when they convert the bond, and thus the value of the optimal conversion boundary should be lower. On the other hand, the main difference in the three figures result from the magnitude of the two critical points, $\tau_K$ and $\tau_M$. In particular, the value of $\tau_K$ is lower than that of $\tau_M$ in Figure 4.1(a), while an apposite phenomenon can be observed in Figure 4.1(b), with Figure 4.1(c) presenting a special case in which the value of $\tau_K$ is equal to that of $\tau_M$.

![Figure 4.2: The optimal exercise prices for three cases](image)

In order to compare these three scenarios clearly, we merge three sub-figures in Figure 4.1 into a single figure (the value of the conversion ratio is equal to one) to form Figure 4.2, with $\tau_{K1}$, $\tau_{K2}$ and $\tau_{K3}$ being used to represent $\tau_K$ in Case 1, Case 2
and Case 3, respectively. Firstly, it should be noticed that the gap between $\tau_{K2}$ and $\tau_{K3}$ is larger than that between $\tau_{K1}$ and $\tau_{K3}$, with the average value of the call price for Case 1 and Case 2 being almost equal to the value of the call price for Case 3. This is reasonable since the value of the optimal conversion boundary climbs fast when the time to expiry is close to zero, while it becomes flatter and flatter with the increase of the time to expiry. Therefore, it takes the value of the conversion boundary needs more time to increase from the value of the call price for Case 3 to the value of the call price for Case 2 compared with the case when it increases from the value of the call price for Case 1 to the value of the call price for Case 3. If we look at these optimal conversion boundaries curves, what can be noticed first is that when the time to expiry is less than $\tau_{K1}$, all three equal to each other, and when the time to expiry is close to zero, the value of the optimal conversion boundary decreases quickly to the face value divided by the value of the conversion ratio. Moreover, during the time zone $[\tau_{K1}, \tau_{K3}]$, the values of the optimal conversion boundary for Case 2 and Case 3 still equal to each other, increasing with the time to expiry, while that for Case 1 remains a constant. When the time to expiry stays within $[\tau_{K3}, \tau_{K2}]$, the value of the two optimal conversion boundaries of Case 1 and Case 3 become constants, and if we further increase the time to expiry such that it becomes larger than $\tau_{K2}$, the values of the optimal conversion boundary for all of three cases become constants. Overall, the value of the conversion boundary for Case 2 is obviously the highest while that of Case 1 is the lowest, since the value of the conversion boundaries for each case are all increasing functions with the time to expiry before the critical values, $\tau_{Ki}$, respectively. On the other hand, if we return to the optimal put boundaries, it is clear that they equal to zero during the time zone $[0, \tau_{M}]$, while they no longer take the value of zero when the time to expiry is greater than $\tau_{M}$. Although the value of three put boundaries in the time interval $[\tau_{M}, \tau_{K2}]$ almost equal to each other, the value of the optimal put boundary for Case 1 and Case 2 is actually the highest and lowest one, respectively. In contract, when the time to expiry is greater than $\tau_{K2}$, there is a surge in the optimal put boundary for Case 2, making it the highest among the three cases, which means that a higher value of the call price leads to a greater value of the put boundary. It is also interesting to find that the slope of $S_c(\tau)$ near $\tau = 0$ and that of $S_p(\tau)$ near $\tau = \tau_{M}$ are both very large, which is similar to American-style call options.

Depicted in Figure 4.3 is the bond price with the underlying asset price at different moments for three cases. Obviously, the price of the callable-puttable CB is always an increasing function with the underlying asset price, and all of the price curves are smoothly tangent to the payoff curve, which means the slope of the price curves at the optimal conversion boundary (or the optimal call boundary) is equal to the value of the conversion ratio. Also, when the underlying asset price is very low, the
Figure 4.3: The bond price for different time moments.
increase in the underlying asset price will not lead to a significant change in the bond price, and the slope of the price curve will become zero when the underlying asset price is zero. In this situation, a greater time to expiry will result in a lower value of the bond, while a completely opposite phenomenon can be observed when the underlying asset price increases to a certain extend.

![Figure 4.4: The bond prices of three cases.](image)

On the other hand, Figure 4.4 is a combination of all sub-figures in Figure 4.3, aiming at making comparison of the bond prices in three cases at the same moment, $\tau = T$. The three prices are very similar to each other when the underlying price is small, since the boundary conditions for the optimal put boundary of these three cases are the same. With the increase of the underlying asset price, it is interesting to find that the bond prices corresponding to Case 1 and Case 3 are still almost equal to each other, while there is a gap between these two prices and the bond price of Case 2. This can be explained by the fact that when the time to expiry is large enough, there exists a situation that the holder of Case 2 will still choose to sell the bond back, while the holder of Case 1 or Case 3 will keep the bond since the value of the optimal put boundary of Case 2 is much larger than that of Case 1 and Case 3, as shown in Figure 4.2. Thus, the bond price corresponding to Case 2 is still equal to the value of the put price, while that for Case 1 or Case 3 is higher than the value of the put price. When we further increase the underlying asset price, these three prices again become close to each other. In order to demonstrate the
difference among the three cases, a zoom-in chart is embedded in this figure, which clearly shows that the bond price for Case 1 is the highest while that for Case 2 is the lowest.

![Figure 4.5: The prices of CBs, PCBs and CPCBs.](image)

It is also interesting to compare the prices of vanilla and puttable convertible bonds considered in Chapter 3 and those of callable-puttable convertible bonds to show the difference between these three types. For the illustration purposes, we use Case 2 as an example, and the results are presented in Figure 4.5. One can easily observe from this figure that being consistent with the results in Chapter 3, the price of a puttable convertible bond is always higher than that of the corresponding vanilla convertible bond, since puttable convertible bonds give an additional right to the holder to sell the bond back to the issuer, when the stock price falls down to a certain level, potentially protecting the benefit of the holder. It is also interesting to notice that the price of a callable-puttable convertible bond is lower than both prices of puttable and vanilla convertible bonds when the underlying price is beyond a certain level. This is because when the underlying price is large enough, the possibility for the holder to sell the bond back to the issuer becomes very low, while the call feature has enabled the issuer to call the bond back, which means that the contract of the callable-puttable convertible bond is more favorable to the issuer.

Figure 4.6 shows the effects of the volatility on the bond price and its optimal
(a) The price of the bond for three different volatilities.

(b) Optimal exercise prices for three different volatilities.

Figure 4.6: Comparison by three different volatilities.
exercise boundaries\(^\text{b}\). From Figure 4.6(a), it can be noticed that the value of the callable-puttable CB is a monotonically increasing function of volatility. It is reasonable since when the value of the volatility becomes larger, there is a higher risk, leading to a higher price. When we turn to Figure 4.6(b), it can be found that a higher value of the volatility will lead to a larger value of the optimal conversion boundary while it will lead to a lower value of the optimal put boundary. The rationale behind this phenomenon is that the increase in the value of the volatility will contribute to an increase of the bond price for the same underlying price and time to expiry, and if we increase the level of the volatility, the bond price at the conversion boundary corresponding to lower volatility is no longer equal to the value of the optimal conversion boundary times the value of the conversion ratio, but higher than it. In other words, the bond holder will not be willing to convert unless the underlying price reaches a higher level. Due to a similar reason, the bond price at the put boundary correspond to the lower level of the volatility is higher than the value of the put price, and in this case the holder will not sell it back to issuer unless the underlying price drops to a further point.

In Figure 4.7, we show how the price of a callable-puttable CB and both its optimal conversion price and optimal put price change with the risk-free interest rate. Comparing the bond price as well as the values of its two free boundaries shown in this figure with those shown in Figure 4.6, one can observe that the risk-free interest rate has quite a different influence than volatility. In specific, Figure 4.7(a) displays if we increase the value of the risk-free interest rate, the bond price will decrease, which can be easily explained since when the risk-free interest rate is higher, investors are more willing to leave their money in a risk-free bank account than buying a risky bond, resulting in a lower CB price. When the two sets of free boundaries are taken in to consideration, opposite trends are also shown in Figure 4.7(b); the optimal conversion price and the optimal put price are a decreasing and an increasing function of the risk-free interest rate, respectively. The main explanation for this can be analogous to that for the volatility case. Taking the conversion boundary as an example. If we decrease the value of the risk free interest rate, the bond price increases for the same underlying price and time to expiry, and thus the bond price at the conversion boundary corresponding to higher interest rate is higher than the value of the conversion boundary times the value of the conversion ratio, making the holder to keep the bond until the underlying price reaches a higher level.

\(^{\text{b}}\)We will only use Case 3 as example for illustration, since there is no essential difference among the three cases, as far as the influence of the parameters on the bond prices.
(a) The price of the bond for three different interest rates.

(b) Optimal exercise prices for three different interest rates.

Figure 4.7: Comparison by three different interest rates.
4.5 Conclusion

In this chapter, the integral equation formulations for the valuation of the callable-puttable conversion bond on a single underlying asset with a constant dividend are derived with the incomplete Fourier transform method and the Green’s function, and the numerical implementation of the formulations is also discussed to provide some guidance for practical application. The accuracy and efficiency of the newly derived integral equation formulations are demonstrated through numerical comparison with the binomial tree method, and the quantitative impact of different parameters on the bond price as well as it free boundaries are also studied.
Chapter 5

Pricing resettable convertible bonds with an integral equation approach

5.1 Introduction

A vanilla bond is one of the widely-used instruments, through which the issuer borrows money from the holder with a preset time period and a preset interest rate. When other clauses are added into the vanilla bond, some particular bonds arise. One of the most popular variations of vanilla bonds is the convertible bonds (CBs), which allows the holder to convert the bond into the preset number of the underlying assets, during the lifetime of the bond or only at maturity. In contrast to vanilla CBs, we can attain other types of CBs, if additional clauses are incorporated. For example, a callable convertible bond (CCB) enables the issuer to call back the bond when the underlying price is large enough, while a puttable convertible bond (PCB) gives a right to the holder so that he/she can sell the bond back to the issuer when the underlying price is small enough. A resettable convertible bond (RCB), as another common CBs, contains a clause that when the underlying price drops or increases to the preset reset price, the conversion ratio will be reseted. Due to the existence of these additional clauses, the pricing problems of the corresponding bonds become more complicated, although they are more flexible and useful in real markets.

To price these financial instruments, many theoretical frameworks are proposed by the researchers. One of the most popular models is the Black-Scholes (B-S) model proposed by Black & Scholes\textsuperscript{[9]}, in 1973, with the underlying price being assumed to follow a geometric Brownian motion (GBM). Under this particular model, Ingersoll\textsuperscript{[62]} and Brennan & Schwartz\textsuperscript{[12]} took the firm value as the underlying variable to price CBs. However, the firm value is very difficult to obtain in real markets, and thus McConnel & Schwartz\textsuperscript{[84]} improved this method by using the stock price as
the underlying variable. Since then, various approaches have been proposed to price CBs. While a closed-form solution was obtained for the European-style CBs, which can only be converted at maturity, in \[88\], Zhu \[106\] presented an analytical solution under the B-S model using the homotopy analysis method for the American-style CBs, which can be converted at any time during their lifetime. Moreover, Tsiveriotis & Fernandes \[99\] priced the cash-only CBs by applying the finite difference method on the coupled B-S equations, and Ohtake et al. \[89\] presented the definitions of the call and reset clause. Recently, Zhu et al. \[109\] derived an integral equation representation to price a PCB with the utilization of the incomplete Fourier transform.

Although the B-S model is widely used in today’s financial market, it is usually not adequate to model the underlying price, especially for the long-term period contracts, since the volatility and the risk free interest rate are always not constant. Therefore, adding more random variables into the B-S model become the first choice, including the stochastic interest rate models and the stochastic volatility models. Belonging to the former category, the Merton model \[85\], CEV model \[27,30\], Vasicek model \[100\], Dothan model \[37\], Brennan-Schwartz model \[14\], CIR-VR model \[28\], GBM model \[83\] and CIR model \[29\] are widely adopted, while the Hull-White model \[55\] and the Heston model \[52\] are two of the most popular ones included in the latter category. Although the models with the additional random variables may provide better fit to the real market data, the corresponding pricing problem would become much more complicated, and thus the numerical methods often have to be resorted to in finding the solution. For example, the finite difference approach \[105\], the finite element approach \[6\], the finite volume approach \[112\], the binomial tree method \[21,59\] and the Monte Carlo simulation method \[3,80\] have already been adopted to price CBs under these complex models.

In this chapter, a resettable convertible bond is considered, and the reset clause studied here is that the conversion ratio will be adjusted upwards once the underlying price drops below the preset reset price. One may be interested in the difference between the vanilla CBs and the RCBs, and why we need RCBs. In the situation of vanilla CBs, the holder will not convert the bond until the maturity to obtain the face value if the underlying price is not large enough, which causes a heavy burden of cash flows for the issuer. However, when this particular reset clause is added into the vanilla CB, this bond may also be converted when the underlying price is relatively small. Moreover, one should also notice that the value of a RCB is higher than that of a corresponding CB since the reset clause brings benefit to the holder, which means that it is advantageous for the issuer to release the RCBs instead of the CBs.

In fact, pricing resettable convertible bonds has not been investigated until very recently, and there are only some numerical solutions derived to price RCBs \[43,70\],
while no partial different equation (PDE) system has been set up for the value of RCBs. In the following, we will work under the B-S model by assuming that the volatility of the underlying asset and the risk free interest rate are constant, and establish a closed PDE system for the bond price. It should be pointed out that although it is more appropriate to model the underlying price under the stochastic volatility model and/or the stochastic interest rate model, we still use the B-S model in this study. This is because it is more feasible to start with a simpler model when introducing a new solution approach to an already complicated problem with the free boundary, given that very few results have been presented on how to price RCBs.

Once the PDE system is successfully built up, a natural question is how to find its solution. It is observed that the newly established system contains an optimal conversion price, which needs to be solved with the bond price simultaneously. To deal with this kind of problem, one of the most efficient methods is the incomplete Fourier transform technique, which has been utilized to derive the value of the American option and that of the PCB in\cite{26} and\cite{109}, respectively. Therefore, the incomplete Fourier transform is adapted in our study, based on which, we obtain an integral equation representation for the bond price, involving the unknown optimal conversion price. The optimal conversion price can then be found through numerical solving the nonlinear equation we obtain, after which the value of the resettable convertible bond can be derived directly.

The main contribution of this paper can be summarized from two aspects. First of all, a closed PDE system for the pricing of RCBs under the B-S model is successfully established for the first time, based on which an integral equation representation for the prices of RCBs is derived, which is shown to be accurate and efficient from numerical experiments. Secondly, we clearly articulate, through a rigorous theoretical proof of a proposition, a unique feature of RCB’s price: it is not always a monotonically increasing function of the underlying asset price, which may appear to be incomprehensible for classic convertible bonds. Such a theoretical proof is also supplemented by some numerical examples to further illustrate this quite amazing phenomenon.

It should be pointed out that our approach can possibly be extended for the pricing of RCBs under other models, such as stochastic volatility or interest rate models, and jump-diffusion models, as the essential feature of the moving boundary is still the same. However, for any stochastic volatility or stochastic interest rate model, the one-dimensional free boundary curve in the B-S model will become a two-dimensional surface, and this will certainly make the target problem more complex. On the other hand, when jump-diffusion models are taken into consideration, there will be an additional integral component in the PDE, leading to a partial integro-
differential equation. In this case, the ordinary differential equation (ODE) obtained after applying the Fourier transform would be different from the one derived under the B-S model. A challenge then is to seek the analytical solution to the new ODE as well as to analytically invert the Fourier transform as what we did here.

It should also be noted that the main focus of this paper is to develop an efficient method for the pricing of RCBs. Of course, it is interesting to apply our newly proposed approach to real market data. However, resettable CBs are mainly over-the-counter derivative products and thus collecting their market data is never as easy as acquiring data of exchange-traded derivative products. Without access to the needed data, it is impossible for us to conduct an empirical study, and thus we choose to tackle the first part of this complicated problem (i.e., the theoretical part of the problem) first and report our methodology in this chapter.

This chapter is organized as follows: In Section 2, the PDE system for the bond price is set up under the Black-Scholes model, and then the incomplete Fourier transform is applied on the system to derive the integral equation representation, as shown in Section 3. Numerical schemes and several interesting results are displayed with a set of graphs presented in Section 4. At last, some conclusion remarks are given in Section 5.

5.2 Model set up

In this section, the PDE system to price a resettable convertible bonds is set up. We begin by assuming the dynamic of the underlying asset price, $S$, satisfies the following stochastic different equation:

$$dS = (r - D_0)Sdt + \sigma SdW_t,$$  (5.1)

where $r$, $D_0$, and $\sigma$ are the risk-free interest rate, the continuous dividend rate, and the volatility of the underlying asset, respectively, and $W_t$ is a standard Brownian motion. Then, if the value of a resettable convertible bond is denoted by $V(S,t)$, it should satisfy the following PDE under the Black-Scholes model, by using the Feynman-Kac formula

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV = 0.$$  (5.2)

To close the PDE system, we need to give the terminal condition and boundary conditions. With the maturity of the bond, $T$, the face value, $Z$, the initial conversion ratio, $n_1$ and the reset conversion ratio, $n_2$, the payoff of the RCBs, or the terminal
condition can be represented as

\[ V(S, T) = \max\{n_1 S, Z\}, \quad (5.3) \]

which is the same as that of the vanilla convertible bonds with the initial conversion ratio, since the bond will not be reset at the maturity. It should be remarked that the value of the reset conversion ratio should be higher than that of the initial one, as part of the requirement according to the reset clause we set up here.

In addition, the boundary conditions at the optimal conversion boundary should also be in the same form as those of the vanilla CBs, involving the initial conversion ratio only, although there are two conversion ratios in our case. This can be explained by the fact that when the underlying price is large enough, it is almost impossible for the underlying price to drop below the reset price. In other words, the reset clause will not be exercised in this case, and thus the RCB can be treated as the vanilla one with the initial conversion ratio. Therefore, the optimal conversion boundary conditions can be written as

\[ V(S_c(t), t) = n_1 S_c(t), \quad (5.4) \]

\[ \frac{\partial V}{\partial S}(S_c(t), t) = n_1, \quad (5.5) \]

where \( S_c(t) \) is the optimal conversion boundary. It should be pointed out that the pricing PDE system has not been closed yet, and one more boundary condition is needed, which is the bond price at the reset price, \( S_r \). Since the bond price should be a continuous function of the underlying price, we need to investigate the case when the underlying price touches the reset price. In fact, if the underlying price touches the reset price from above, the bond should be reset immediately and automatically, after which the RCB becomes the vanilla CB with the reset conversion ratio. One can easily deduce from this that the bond price at the reset price should be defined as

\[ F(t) = \begin{cases} 
  n_2 \cdot S_r, & S_{c_2}(t) \leq S_r, \\
  V_2(S_r, t), & S_{c_2}(t) > S_r, 
\end{cases} \quad (5.6) \]

where \( V_2(S_r, t) \) is the value of the CB with the reset conversion ratio, and \( S_{c_2}(t) \) is its corresponding optimal conversion boundary. It should be pointed out that once the bond has been reset, \( S_{c_2}(t) \) becomes the optimal exercise price of the RCB. In
summary, the closed PDE system can be formulated as follows

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV &= 0, \\
V(S, T) &= \max\{n_1 S, Z\}, \\
V(S_r, t) &= F(t), \\
V(S_c(t), t) &= n_1 \cdot S_c(t), \\
\frac{\partial V}{\partial S}(S_c(t), t) &= n_1.
\end{aligned}
\]  

(5.7)

It should also be noted that if the reset price equals to 0, i.e. \(S_r = 0\), the RCBs are actually equivalent to the vanilla CBs with the initial conversion ratio. In the following, we would like to only discuss the case when the resettable right is meaningful (otherwise the RCBs reduce to the vanilla CBs with the reset conversion ratio), i.e. \(S_{c_1}(T) > S_r\), where \(S_{c_1}(t)\) is the optimal conversion boundary for the vanilla convertible bonds with the initial conversion ratio, in which case the domain of \(t\) and that of \(S\) for the PDE system are \([0, T]\) and \([S_r, S_c(t)]\), respectively.

5.3 Integral equation representation

In this section, the integral equation representation to price RCB is obtained by using the incomplete Fourier transform method, which is in fact one of the standard methods for pricing the financial derivatives. Now, we start with the following two simple transforms

\[
\tau = T - t, \quad x = \ln(S),
\]  

(5.8)

with which the initial PDE system (5.7) is transformed into

\[
\begin{aligned}
-\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial V}{\partial x} - rV &= 0, \\
V(x, 0) &= \max\{n_1 e^x, Z\}, \\
V(\ln(S_r), \tau) &= F(\tau), \\
V(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau), \\
\frac{\partial V}{\partial x}(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau),
\end{aligned}
\]  

(5.9)

with the domain of \(x\) and \(\tau\) being \([\ln(S_r), \ln(S_c(\tau))]\) and \([0, T]\), respectively.

If we define the incomplete Fourier transform for our issue as

\[
\mathcal{F}\{V(x, \tau)\} = \int_{\ln(S_r)}^{\ln(S_c(\tau))} V(x, \tau) \cdot e^{i\omega x} dx \triangleq \hat{V}(\omega, \tau),
\]  

(5.10)

and then when we apply this incomplete Fourier transform on the PDE system (5.9),
a problem immediately arises that the value of the first-order derivative, $\frac{\partial V}{\partial x}(x, \tau)$, at $x = \ln(S_T)$ is needed in the computation, which is not available in the system. Therefore, we add one more boundary condition, $\frac{\partial V}{\partial S}(S_r, t) = G(t)$, into the original PDE system (5.7) so that the PDE system (5.9) becomes

$$
\begin{aligned}
-\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial V}{\partial x} - rV &= 0, \\
V(x, 0) &= \max\{n_1 e^x, Z\}, \\
V(\ln(S_r), \tau) &= F(\tau), \\
\frac{\partial V}{\partial x}(\ln(S_r), \tau) &= S_r \cdot G(\tau), \\
V(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau), \\
\frac{\partial V}{\partial x}(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau).
\end{aligned}
$$

(5.11)

Applying the incomplete Fourier transform (5.10) on the new PDE system (5.11) yields the following ordinary different equation (ODE) system

$$
\begin{aligned}
\frac{\partial \hat{V}}{\partial \tau}(\omega, \tau) + B(\omega)\hat{V}(\omega, \tau) &= F_1(\omega, \tau) - F_2(\omega, \tau), \\
\hat{V}(\omega, 0) &= \int_{\ln(S_r)} e^{\ln(S_c(\xi))} \max\{n_1 e^x, Z\} \cdot e^{i\omega x} dx,
\end{aligned}
$$

(5.12)

where

$$
\begin{aligned}
B(\omega) &= \frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2)i\omega + r, \\
F_1(\omega, \tau) &= \left[ \frac{S_r'(\tau)}{S_c(\tau)} - \frac{1}{2} \sigma^2 i\omega + (r - D_0) \right] \cdot n_1 S_c(\tau)e^{i\omega \ln(S_c(\tau))}, \\
F_2(\omega, \tau) &= \left[ \frac{1}{2} \sigma^2 G(\tau)S_r - \frac{1}{2} \sigma^2 i\omega F(\tau) + (r - D_0 - \frac{1}{2} \sigma^2)F(\tau) \right] \cdot e^{i\omega \ln(S_r)},
\end{aligned}
$$

(5.13) \hspace{1cm} (5.14) \hspace{1cm} (5.15)

with the derivation process being left in Appendix C.1. It should be noted that this is a non-homogeneous first-order linear ODE with an initial condition, thus the solution to which can be easily derived as follows

$$
\hat{V}(\omega, \tau) = \hat{V}(\omega, 0) \cdot e^{-B(\omega)\tau} + \int_0^\tau \left[ F_1(\omega, \xi) - F_2(\omega, \xi) \right] \cdot e^{-B(\omega)(\tau - \xi)} d\xi.
$$

(5.16)

Clearly, the integral equation formulation for the RCB price in the Fourier space has been presented. However, it should be pointed out that it costs a lot of time to use numerical methods to conduct the Fourier inversion transform, and it is desired to find the solution in the original space. Fortunately, after some complex computation, with details left in Appendix C.2, we have successfully derived an analytical expression of the solution in the original space by applying the Fourier
inversion transform on Equation (5.16), with its formulation presented as

\[
V(S, \tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(S_0)}^{\ln(S)} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S) - u]^2}{2\sigma^2\tau}} \cdot \max\{n_1 e^u, Z\} du \\
+ \int_0^\tau n_1 S_c(\tau) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi\sigma^2(\tau - \xi)}} \cdot e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S) + \ln(S_c(\xi))]}{2(\tau - \xi)}} d\xi \\
- \int_0^\tau \frac{1}{2}\sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi\sigma^2(\tau - \xi)}} \cdot e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S) - \ln(S_r)]^2}{2\sigma^2(\tau - \xi)}} d\xi \\
- \int_0^\tau F(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi\sigma^2(\tau - \xi)}} \cdot e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S) + \ln(S_r)]}{2(\tau - \xi)}} d\xi. 
\]

(5.17)

It should be noted that this integral equation representation contains the unknown functions, \(S_c(\tau)\) and \(G(\tau)\), and the numerical method has to be used to obtain their values, since it is impossible to obtain their explicit expressions. Moreover, this integral equation representation also contains the first-derivative of the unknown function, \(S'_c(\tau)\), which leads to the requirement of the higher smoothness of the interpolation function used in the numerical solution procedure. Therefore, a new representation is obtained based on the formation (5.17), using the integration by parts, and it has the form of

\[
V(S, \tau) = \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(S_0)}^{\ln(S)} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S) - u]^2}{2\sigma^2\tau}} \cdot \max\{n_1 e^u, Z\} du \\
+n_1 S_c e^{-D_0\tau} \cdot \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)\tau + \ln(S) - \ln(S_c(0))}{\sqrt{\sigma^2\tau}}\right) \\
+ \int_0^\tau D_0 n_1 S_c e^{-D_0(\tau - \xi)} \cdot \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S) - \ln(S_c(\xi))}{\sqrt{\sigma^2(\tau - \xi)}}\right) d\xi \\
- \int_0^\tau \frac{1}{2}\sigma^2 G(\xi) S_r e^{-D_0\tau} \cdot \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S) - \ln(S_r)}{\sqrt{\sigma^2(\tau - \xi)}}\right) d\xi \\
- \int_0^\tau F(\xi) e^{-D_0(\tau - \xi)} \cdot \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S) + \ln(S_r)}{\sqrt{\sigma^2(\tau - \xi)}}\right) d\xi. 
\]

(5.18)

We refer interested readers to Appendix C.3 for the technical details. By now, the final integral equation representation has been obtained. However, this formulation can not be directly used since it involves the unknown functions, \(S_c(\tau)\) and \(G(\tau)\),
implying that two equations are needed to obtain the values of these two unknown functions. If we recall two of the boundary conditions\(^a\) here

\[
V(S_r, \tau) = \frac{1}{2} F(\tau), \quad \text{(5.19)}
\]

\[
V(S_c(\tau), \tau) = n_1 \cdot S_c(\tau), \quad \text{(5.20)}
\]

substituting these two boundary conditions into representation (5.18) can lead to the following two integral equations

\[
\frac{1}{2} F(\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi \sigma^2 \tau}} \int_{\ln(S(0))}^{\ln(S_r(\tau))} e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 \tau + \ln(S_r(\tau)) - u)^2}{2\sigma^2 \tau}} \cdot \max\{n_1 e^u, Z\} du 
+ n_1 S_r \cdot e^{-D_0 \tau} \cdot \mathcal{N}'( \frac{(r-D_0 + \frac{1}{2} \sigma^2 \tau + \ln(S_r(\tau)) - \ln(S_c(0)))}{\sqrt{\sigma^2 \tau}} )
+ \int_0^\tau D_0 n_1 S_r \cdot e^{-D_0 (\tau - \xi)} \cdot \mathcal{N}'( \frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_r) - \ln(S_c(\xi))}{\sqrt{\sigma^2 (\tau - \xi)}} ) d\xi 
- \int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2 (\tau - \xi)}} \cdot e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_r) - \ln(S(\tau)))^2}{2\sigma^2 (\tau - \xi)}} d\xi 
- \int_0^\tau F(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2 (\tau - \xi)}} \cdot e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_r) - \ln(S(\tau)))^2}{2\sigma^2 (\tau - \xi)}} d\xi 
\cdot \frac{(r-D_0 - \frac{1}{2} \sigma^2 (\tau - \xi) - \ln(S_r) + \ln(S_c))}{2(\tau - \xi)}, \quad \text{(5.21)}
\]

and

\[
n_1 \cdot S_c(\tau) = \frac{e^{-r\tau}}{\sqrt{2\pi \sigma^2 \tau}} \int_{\ln(S(0))}^{\ln(S_c(\tau))} e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 \tau + \ln(S_c(\tau)) - u)^2}{2\sigma^2 \tau}} \cdot \max\{n_1 e^u, Z\} du 
+ n_1 S_c(\tau) \cdot e^{-D_0 \tau} \cdot \mathcal{N}'( \frac{(r-D_0 + \frac{1}{2} \sigma^2 \tau + \ln(S_c(\tau)) - \ln(S_c(0)))}{\sqrt{\sigma^2 \tau}} )
+ \int_0^\tau D_0 n_1 S_c(\tau) \cdot e^{-D_0 (\tau - \xi)} \cdot \mathcal{N}'( \frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi))}{\sqrt{\sigma^2 (\tau - \xi)}} ) d\xi 
- \int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_c \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2 (\tau - \xi)}} \cdot e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi)))^2}{2\sigma^2 (\tau - \xi)}} d\xi 
- \int_0^\tau F(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2 (\tau - \xi)}} \cdot e^{-\frac{(r-D_0 + \frac{1}{2} \sigma^2 (\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi)))^2}{2\sigma^2 (\tau - \xi)}} d\xi 
\cdot \frac{(r-D_0 - \frac{1}{2} \sigma^2 (\tau - \xi) - \ln(S_c(\tau)) + \ln(S_c))}{2(\tau - \xi)}, \quad \text{(5.22)}
\]

\(^a\)The factor of 1/2 contained in one of the boundary conditions is a result of applying the incomplete Fourier transform when deriving the integral equation representation.
functions, contained in the integral equation representation (5.18), can be obtained, and then the value of the RCBs can be computed directly. In the next section, the numerical scheme will be designed to obtain the value of the optimal conversion price, based on which, the bond price can be derived. It should be pointed out that numerically deriving the optimal conversion boundary requires the knowledge of the optimal conversion price at expiry, and thus its value should be determined first, which is provided in the following proposition.

Proposition 5.3.1 The optimal conversion price at expiry is \( S_c(\tau) |_{\tau = 0} = Z/n_1 \).

The details of the proof are left in Appendix C.4.

Before we present numerical examples in the next section, it needs to be pointed out that because of the additional reset clause, the bond price is not always a monotonically increasing function of the underlying price, which is impossible for other types of convertible bonds. To address this unique property, we use a particular example presented in the following proposition.

Proposition 5.3.2 (Non-monotonicity) The price of the resettable convertible bond is not a monotonically increasing function of the underlying price when the time to expiry is sufficiently small, when \( S_c(\tau) |_{\tau = 0} = Z/n_1 = S_r \).

The detailed proof is left in Appendix C.5. It should be remarked that the property shown by Proposition 5.3.2 also holds when \( S_c(\tau) |_{\tau = 0} - S_r \) is not very large.

5.4 Numerical schemes and the results

In this section, we will provided the numerical scheme to obtain the value of the unknown functions, \( S_c \) and \( G \), numerically. After that, some results will be displayed to illustrate the properties of the RCBs. Before we present the scheme, the time interval, \([0, T]\), should be discretized uniformly as: \( 0 = \tau_1 < \tau_2 < \cdots < \tau_N < \tau_{N+1} = T \), with \( \tau_n = (n - 1) \cdot T / N \). In this case, the discretized unknown functions \( S_c(\tau_n) \) and \( G(\tau_n) \) are denoted as \( S_c^{(n)} \) and \( G^{(n)} \), respectively, and at the same time, the known function \( F(\tau_n) \) is denoted as \( F^{(n)} \). Therefore, the numerical scheme is constructed by two sets of the coupled non-linear algebraic equations:

\[
\frac{1}{2} F^{(n)} = e^{-r\tau_n} \int_{\ln(S_r)}^{\ln(S_c^{(1)})} e^{- \frac{(r - D_0 + \frac{1}{2} \sigma^2)\tau_n + \ln(S_r) - \ln(S_c^{(1)})}{2\sigma^2\tau_n}} \cdot \max\{n_1 e^u, Z\} du \\
+ n_1 S_r \cdot e^{-D_0\tau_n} \cdot \mathcal{N} \left( \frac{(r - D_0 + \frac{1}{2} \sigma^2)\tau_n + \ln(S_r) - \ln(S_c^{(1)})}{\sqrt{\sigma^2\tau_n}} \right) \\
+ \sum_{i=1}^{n-1} D_0 n_1 S_r \cdot e^{-D_0(\tau_n - \tau_i)} \cdot \mathcal{N} \left( \frac{(r - D_0 + \frac{1}{2} \sigma^2)(\tau_n - \tau_i) + \ln(S_r) - \ln(S_c^{(i)})}{\sqrt{\sigma^2(\tau_n - \tau_i)}} \right) \cdot \frac{T}{N}
\]
\[ - \frac{1}{2} D_0 n_1 S_r \cdot e^{-D_0 \tau_n} \cdot \mathcal{N} \left( \frac{r - D_0 + \frac{1}{2} \sigma^2 \tau_n}{\sqrt{\sigma^2 \tau_n}}, T \right) \]

\[ - \sum_{i=1}^{n-1} \frac{1}{2} \sigma^2 G^{(i)} S_r \cdot e^{-r(\tau_n - \tau_i)} \cdot \frac{e^{-\frac{[(r-D_0) \frac{1}{2} \sigma^2 (\tau_n - \tau_i)]^2}{2 \sigma^2 (\tau_n - \tau_i)}}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{T}{N} \]

\[ - \sum_{i=1}^{n-1} F(i) \cdot \frac{e^{-r(\tau_n - \tau_i)}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{e^{-\frac{[(r-D_0) \frac{1}{2} \sigma^2 (\tau_n - \tau_i)]^2}{2 \sigma^2 (\tau_n - \tau_i)}}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{(r - D_0 + \frac{1}{2} \sigma^2)(\tau_n - \tau_i) - \ln(S^{(i)}_c) + \ln(S_r)}{2(\tau_n - \tau_i)} \cdot \frac{T}{N}, \tag{5.23} \]

and

\[ n_1 \cdot S^{(n)}_c = \frac{e^{-r \tau_n}}{\sqrt{2 \pi \sigma^2 \tau_n}} \int_{\ln(S_r)}^{\ln(S^{(1)}_c)} e^{\frac{[(r-D_0) \frac{1}{2} \sigma^2 (\tau_n - \tau_i)]^2}{2 \sigma^2 (\tau_n - \tau_i)}} \cdot \max\{n_1 e^u, Z\} \, du \]

\[ + n_1 S^{(n)}_c \cdot e^{-D_0 \tau_n} \cdot \mathcal{N} \left( \frac{r - D_0 + \frac{1}{2} \sigma^2 \tau_n + \ln(S^{(n)}_c) - \ln(S^{(1)}_c)}{\sqrt{\sigma^2 \tau_n}}, T \right) \]

\[ + \sum_{i=1}^{n-1} D_0 n_1 S^{(i)}_c \cdot e^{-D_0 (\tau_n - \tau_i)} \cdot \mathcal{N} \left( \frac{(r - D_0 + \frac{1}{2} \sigma^2) \tau_n + \ln(S^{(i)}_c) - \ln(S^{(1)}_c)}{\sqrt{\sigma^2 \tau_n}}, T \right) \]

\[ - \frac{1}{2} D_0 n_1 S^{(n)}_c \cdot e^{-D_0 \tau_n} \cdot \mathcal{N} \left( \frac{(r - D_0 + \frac{1}{2} \sigma^2) \tau_n + \ln(S^{(n)}_c) - \ln(S^{(1)}_c)}{\sqrt{\sigma^2 \tau_n}}, T \right) \]

\[ + \frac{1}{4} D_0 n_1 S^{(n)}_c \cdot T \]

\[ - \sum_{i=1}^{n-1} \frac{1}{2} \sigma^2 G^{(i)} S_r \cdot e^{-r(\tau_n - \tau_i)} \cdot \frac{e^{-\frac{[(r-D_0) \frac{1}{2} \sigma^2 (\tau_n - \tau_i)]^2}{2 \sigma^2 (\tau_n - \tau_i)}}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{T}{N} \]

\[ - \sum_{i=1}^{n-1} F(i) \cdot \frac{e^{-r(\tau_n - \tau_i)}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{e^{-\frac{[(r-D_0) \frac{1}{2} \sigma^2 (\tau_n - \tau_i)]^2}{2 \sigma^2 (\tau_n - \tau_i)}}}{\sqrt{2 \pi \sigma^2 (\tau_n - \tau_i)}} \cdot \frac{(r - D_0 + \frac{1}{2} \sigma^2)(\tau_n - \tau_i) - \ln(S^{(i)}_c) + \ln(S_r)}{2(\tau_n - \tau_i)} \cdot \frac{T}{N}. \tag{5.24} \]

Since the terminal value of the optimal conversion boundary, \( S^{(1)}_c = Z/n_1 \), is known, we can use Equation (5.23) to calculate the value of the unknown function, \( G^{(1)} \), with \( n = 2 \), and then Equation (5.24) can be utilized to calculate the value of \( S^{(2)}_c \) with \( n = 2 \). With the values of the unknown functions computed in the previous steps, Equation (5.23) and Equation (5.24) can be used to find the value of unknown function, \( G \), and the value of the optimal conversion boundary, \( S_c \), in the following steps, until \( n = N + 1 \).

With the numerical scheme being established, we are now able to present numerical solutions. Unless otherwise stated, the parameters used in our study are listed below:

- Face value \( Z = 100 \),
- Reset price \( S_r = 100 \),
• Maturity $T = 1$ (year),
• Risk-free annual interest rate $r = 0.1$,
• Rate of continuous dividend payment $D_0 = 0.07$,
• Volatility $\sigma = 0.4$,
• Initial conversion ratio $n_1 = 1$,
• Reset conversion ratio $n_2 = 1.2$.

It should be noted that all of our calculations in this paper are done using Matlab R2017a on a PC with the following specifications: Intel(R) Core(TM), i7-4790 CPU@3.60GHz 3.60 GHz, and 16.0 GB of RAM.

To validate our numerical scheme, the results obtained through the Monte-Carlo simulation are chosen as a benchmark, the implementation of which is through the Least Square Monte-Carlo (LSMC) approach proposed by Longstaff & Schwartz. The main idea of LSMC is to assume that there is a finite number of possible exercise dates, and the holder of the bond needs to determine whether the bond should be exercised or not at each discrete exercise time. Such a decision is made by comparing the immediate exercise value, which is the amount that the holder can obtain if the bond is exercised now, and the continuation value, which is defined as the amount that the holder can receive if the bond is exercised at a future time. Therefore, it is vital to estimate the continuation value, which can be achieved through the least squares regression with the cross-sectional information coming from the Monte-Carlo simulation. The comparison between the estimated continuation value and the immediate exercise value will then determine the optimal stopping rule. This procedure is conducted backward in time as we know the payoff function, and the resulted cash flow, if discounted back to the current time, would yield the bond price. It should be pointed out that we can almost apply the same procedure as illustrated above to price RCBs using LSMC, and the only exception is that one should always determine the conversion ratio according to the simulated underlying price in each simulation path before any calculation is carried out, due to the existence of “reset” feature. With 5,000 time steps and 200,000 sampling paths, we are now ready to present the comparison results.

Table 5.1 clearly demonstrates the accuracy of our scheme, with the maximum relative error for these two methods being less than 0.5%. However, one should notice that the Monte-Carlo simulation is very time-intensive, while the integral equation method is computationally efficient. In particular, the average CPU time consumed by the Monte-Carlo simulations to calculate the RCB price is 100.0745 seconds, while it only takes 0.0173 seconds to produce one price with the integral
equation approach, which demonstrates the superiority of the integral equation approach, as far as the computational efficiency is concerned.

Table 5.1: MC method vs IE method

<table>
<thead>
<tr>
<th>RCB price at $t = 0$</th>
<th>S=120</th>
<th>S=130</th>
<th>S=140</th>
<th>S=150</th>
<th>S=160</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>130.2665</td>
<td>136.4142</td>
<td>143.7459</td>
<td>151.8093</td>
<td>160.6040</td>
<td>100.0745</td>
</tr>
<tr>
<td>IE</td>
<td>130.8426</td>
<td>136.9983</td>
<td>144.1337</td>
<td>152.1121</td>
<td>160.8153</td>
<td>0.0173</td>
</tr>
<tr>
<td>Relative error</td>
<td>$4.42 \times 10^{-3}$</td>
<td>$4.28 \times 10^{-3}$</td>
<td>$2.70 \times 10^{-3}$</td>
<td>$1.99 \times 10^{-3}$</td>
<td>$1.32 \times 10^{-3}$</td>
<td>-</td>
</tr>
</tbody>
</table>

Figure 5.1: The optimal conversion price of RCB and also two vanilla CBs.

In Figure 5.1, we show the value of the optimal conversion boundary of the RCB and that of the vanilla convertible bonds with two conversion ratios. What we can see first is that all three optimal conversion prices are the monotonically increasing functions of the time to expiry, with the optimal conversion price of the RCB being the highest one. This is reasonable since the holder of the RCB has one more benefit that the conversion ratio can be reset to a higher value when the underlying price is low enough, implying that the optimal conversion price of the RCB should be higher than that of the CB with initial conversion ratio. Moreover, when the time to expiry is equal to zero, or at maturity, the optimal conversion price of the RCB is as same as that of the CB with initial conversion ratio, and both of which are
equal to \( Z/n_1 \), because the holder of both contracts face the same choice between receiving the face value \( Z \) and getting \( n_1 \) shares of stocks.

Figures 5.2-5.5 show the price of the RCB with respect to the underlying price at three time moments, \( \tau = 0.05 \), \( \tau = 0.50 \) and \( \tau = 0.95 \), respectively. One of the most noticeable phenomena is the non-monotonicity of the bond price displayed in Figure 5.2, which was discussed in theory in Proposition 5.3.2 already. Financially, this is a natural consequence of the introduction of the resettable clause; in this particular case forcing the bond price being bounded below, as a result of bond holders would benefit from the bond being reset to a higher conversion ratio when the underlying price is sufficiently small. On the other hand, our observations show that such a non-monotonic phenomenon never occurs when it is still quite far away from expiry (see Figure 5.3 and Figure 5.4). Financially, the fact that the non-monotonicity can only be observed when the time is sufficiently close to expiry, with a sufficiently large reset conversion ratio as discussed in Proposition 5.3.2, is because the optimal conversion price is a decreasing function of time and when the expiry time is approached, the bond price corresponding to this price, \( V(S_c(\tau), \tau) \), becomes smaller than the bond price corresponding to the reset, \( V(S_r, \tau) \). Thus, unless the bond holder chooses not to convert, then he/she should be prepared to accept the price non-monotonicity between \( S_r \) and \( S_c(\tau) \).

In addition, it should also be pointed out that all these three figures have shown
that the value of the resettable convertible bond is lower than that of the convertible bond with the reset conversion ratio, while is higher than that of the convertible
bond with the initial conversion ratio. This is reasonable because the price of the convertible bond is an increasing function of the conversion ratio, according to the properties of the convertible bond in [106], implying that the price of the bond with the reset conversion ratio is higher than that of the bond with the initial conversion ratio. With this in mind, the value of the resettable convertible bond is higher than that of the convertible bond with the initial conversion ratio because it is bounded below due to the existence of the reset clause, while it is lower than that of the convertible bond with the reset conversion ratio because it will only become a more valuable contract when the underlying price is small enough.

To further demonstrate the properties of the bond price, Figure 5.5 combines the three resettable convertible bond price curves together. From this figure, one can clearly observe that all three price curves are tangent to the payoff line corresponding to the initial conversion ratio, instead of that with the reset conversion ratio. Moreover, the value of the optimal conversion boundary is a monotonically increasing function of the time to expiry, which is consistent with the property shown in Figure 5.1.

Figure 5.6 and Figure 5.7 are used to show the impact of different reset prices on the bond price and that on the optimal conversion price, respectively. When we focus on Figure 5.6, what can be observed is that a higher value of the reset price leads to a higher value of the resettable convertible bonds. This can be explained...
from the property mentioned above that the price of the RCB should lay between the convertible bond prices with the two conversion ratios, leading to the case that a
larger reset price yields a higher value of the RCB at the reset price. Take the RCB prices at the highest reset price, $S_r = 115$, as an example to further illustrate this. When the underlying price equals to the reset price, the price of the RCB with the reset price being to 115 is actually the price of the convertible bond with the reset conversion ratio, while the prices of another two RCBs should be lower than it since they have not been reseted yet. Moreover, it is also clear in this figure that a higher value of the reset price leads to a higher value of the optimal conversion boundary, which is also shown in Figure 5.7. It is simply because the higher value of the reset price implies a higher value of the resettable convertible bond, and thus the holder will naturally hold the bond unless the underlying price increases to a higher level.

![Figure 5.8: The bond price of RCB.](image)

Depicted in Figure 5.8 and Figure 5.9 are the changes of the bond price and the optimal conversion price with three different values of the reset conversion ratios, respectively. In specific, Figure 5.8 clearly shows that the bond price appears to be increasing with respect to the reset conversion ratio. The rational behind this phenomenon is that the price of the vanilla convertible bond is an increasing function of the conversion ratio, as discussed above, and the increase in the reset conversion ratio actually raises the lower bound of the RCB price. This also explains the phenomenon in Figure 5.9 that a higher value of the bond leads to a higher optimal conversion price.
5.5 Conclusion

In this chapter, the pricing problem of a resettable convertible bond is considered. A PDE system for the bond price is built up, and an integral equation formulation for the bond price is obtained with the use of the incomplete Fourier transform, involving the optimal conversion price as an unknown function to be solved. After the establishment of an appropriate numerical scheme, the value of the optimal conversion boundary is obtained, with which the bond price can be directly calculated from the integral equation representation, and some numerical examples are provided to show the properties of the resettable convertible bond price as well as the optimal conversion price.
Chapter 6

Pricing convertible bonds under stochastic volatility or interest rate

6.1 Introduction

A bond is an instrument that can be treated as the issuer borrowing money from the holders for a pre-specified period. If a clause is added to the contract so that the holders can choose to convert the bond into a predetermined number of stocks or not, it then becomes a convertible bond (CB). While it provides a great incentives to a bond holder to invest in CBs rather than a conventional bond, this additional right of the holder indeed makes its pricing problem much more complicated, since the bond price and the optimal conversion price\(^a\) should be determined simultaneously.

In 1973, Black & Scholes\(^9\) proposed to model the underlying price with a geometric Brownian motion (GBM) for the option pricing problem, and shortly after, Ingersoll\(^62\) and Brennan & Schwartz\(^12\) considered the valuation problem of CBs with the firm value being taken as underlying variable following this particular Black-Scholes (B-S) model. This approach was improved by McConnel & Schwartz\(^84\) by replacing the firm value with the stock price as the underlying variable since the firm value can not be directly observed in real markets. Since then, various approaches have been proposed to price CBs. For example, Nyborg\(^88\) obtained a closed-form solution under the B-S model for a simple convertible bond, which can only be converted at maturity, while Zhu\(^106\) presented an analytical solution under the same model for a convertible bond, which can be converted at any time on or before maturity, using the homotopy analysis method.

If the issuer or the holder of a CB is entitled with some additional rights, different kinds of CBs will be formulated, such as callable CBs, puttable CBs, resettable CBs and so on, making the corresponding pricing problem even more complex. Thus,

\(^a\)The optimal conversion price is referred to as the critical stock price beyond which the holder will choose to convert the bond into stocks.
numerical methods must be adopted in most cases. For example, Tsiveriotis & Fernandes\cite{99} priced the cash-only CBs by applying the finite difference method on the coupled B-S equations. Ohtake et al.\cite{89} presented the definitions of the call and reset clause, depending on which resettable CBs were considered by Kimura & Shinohara\cite{70} with the Monte Carlo method. Recently, Zhu et al.\cite{109} derived an integral equation formulation for pricing the puttable convertible bond.

Of course, the B-S model is usually not adequate to model the underlying price, and one of the most popular approaches is to introduce additional random variables into the B-S model, which can mainly be divided into two categories, i.e., stochastic interest rate models and stochastic volatility models. Examples in the former category include the Merton model\cite{85}, CEV model\cite{27,30}, Vasicek model\cite{100}, Dothan model\cite{37}, Brennan-Schwartz model\cite{14}, CIR-VR model\cite{28}, GBM model\cite{83} and CIR model\cite{29}, while the Heston model\cite{52} and Hull-White model\cite{55} are very popular among many others included in the latter category. Unfortunately, the additional random variables make the pricing problem much more complicated, and thus the numerical methods are often resorted to in these cases. In particular, the finite difference approach\cite{105}, the finite element approach\cite{6}, the finite volume approach\cite{112}, the binomial tree method\cite{21,59} and the Monte Carlo simulation method\cite{3,80} have already been adopted to price the convertible bonds under these complex models.

Another popular numerical approach is the predictor-corrector scheme\cite{16,93}. It is a method to solve the ordinary differential equation (ODE) with two steps; a prediction step computing the value of the function at a preceding set of points to obtain the value of this function at a subsequent point, and then a correction step refining the value of the unknown function at the same subsequent point using a suitable approach. In other words, it is a method with suitable association of an implicit scheme and an explicit scheme. In fact, this method has already been applied to solve the pricing problem of the security instruments, even though the governing equations for these pricing problems are all partial differential equations (PDEs). A typical example is provided in\cite{110}, where Zhu & Zhang chose a suitable combination of a prediction scheme and a correction scheme to obtain a new scheme for evaluating American options under the B-S model. On the other hand, the Alternating Direction Implicit (ADI) method is a very useful technique to solve the PDEs on the rectangular domains\cite{32,33,58,91}, especially for the parabolic ones as for the other two cases the problem can become quite complex\cite{96}. Fortunately, the equations governing the prices of financial derivatives under most existing models, including the B-S model and the stochastic volatility/interest-rate models, are all parabolic differential equations, and thus the ADI method is ideal to be utilized for these pricing problems\cite{48,60}.

In this chapter, we adopt a particular predictor-corrector scheme, constructed by
the two methods mentioned above being combined together with the ADI scheme used as the correction step. It was proposed by Zhu & Chen\cite{107} in solving the pricing problem of American puts with stochastic volatility, which was utilized by Chen et al\cite{25} for the pricing of the stock loan with stochastic interest rate. In order to determine the price of CBs, we firstly establish two PDE systems for the price of CBs under a stochastic volatility and a stochastic interest rate model, respectively. Since the moving boundary exists in both of the two systems, Landau transform\cite{71} is used to transform the free boundary problem into a fixed one, at the cost of the original linear PDE becoming a nonlinear one, after which the predictor-corrector scheme is adopted for each time step to convert the nonlinear PDE into two linearized difference equations associated with the prediction and correction phase, respectively. For the prediction step, an explicit Euler scheme is used to predict the value of the optimal conversion boundary, and at the correction step, the value of the bond is then determined through the ADI scheme, based on which the correction of the optimal conversion boundary is obtained. Another contribution of this paper is the proposition of the boundary conditions along the volatility and interest rate direction, which contribute to the development of the closed PDE system for pricing CBs under stochastic volatility and interest rate models.

The chapter is organized as follows. In Section 2, the pricing problem for CBs under a stochastic volatility model is considered, presenting the numerical scheme as well as the numerical results we obtain. In Section 3, how to price CBs under a stochastic interest rate model is illustrated. Concluding remarks are given in the last section.

6.2 Pricing convertible bonds with stochastic volatility

In this section, the pricing problem of convertible bonds when the volatility is made to be another random variable is discussed. We will use the Heston model as an example to illustrate this since the processes in solving the pricing problem under different stochastic volatility models are very similar and the Heston model is one of the most popular models.

In the following, the PDE system governing the price of the CBs is firstly set up and then how to obtain the predictor-corrector scheme with the ADI method to value the CBs are illustrated, after which the accuracy of the proposed method is demonstrated through numerical experiments and the properties of the CBs with stochastic volatility are also studied.
6.2.1 The PDE system under the Heston model

To build the PDE system for pricing a CB, the dynamics of the adopted model should be specified first. Let $S_t$ be the underlying asset price, and then its dynamic under a risk-neutral measure is assumed to satisfy the following stochastic different equation (SDE):

$$dS_t = (r - D_0)S_t dt + \sqrt{v_t} S_t dW_1,$$

(6.1)

where $r$ is the risk-free interest rate and $D_0$ is the continuous dividend rate. $W_1$ is a standard Brownian motion, and $v_t$ is the stochastic volatility, which is governed by the following SDE:

$$dv_t = \kappa(\eta - v_t)dt + \sigma \sqrt{v_t} dW_2,$$

(6.2)

with $\kappa$ denoting the rate of relaxation to this mean, $\eta$ representing the long time mean of $v_t$, and $\sigma$ being the volatility of volatility. $W_2$ is also a standard Brownian motion, being correlated $W_1$ with $\rho \in [-1, 1]$. If the value of the bond is denoted by $U(S, v, t)$, its governing PDE can be written as follows:

$$\frac{1}{2} v S^2 \frac{\partial^2 U}{\partial S^2} + \sigma \rho v S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 U}{\partial v^2} + (r - D_0)S \frac{\partial U}{\partial S} + \kappa(\eta - v) \frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0.$$

(6.3)

The terminal condition for Equation (6.3) is actually the payoff function of CBs

$$U(S, v, T) = \max\{C_RS, Z\},$$

(6.4)

where $C_R$ is the conversion ratio and $Z$ is the face value of the bond. The boundary conditions in the direction of the underlying asset price $S$ are given as

$$U(0, v, t) = Ze^{-r(T-t)},$$

(6.5)

$$U(S_f(v, t), v, t) = C_R \cdot S_f(v, t),$$

(6.6)

$$\frac{\partial U}{\partial S}(S_f(v, t), v, t) = C_R,$$

(6.7)

where $S_f$ is the optimal conversion price. It should be noted that all of the conditions mentioned above are very similar to that under the B-S model, and the main difference between the B-S model and the stochastic volatility model is that the bond price and the optimal conversion price are both the functions of the volatility for the stochastic volatility model. Therefore, the boundary conditions for $v$ are needed to close the PDE system. In this study, the boundary conditions are chosen as

$$\lim_{v \to 0} U(S, v, t) = \max\{C_RS, Ze^{-r(T-t)}\},$$

(6.8)
\begin{equation}
\lim_{v \to \infty} \frac{\partial U}{\partial v}(S,v,t) = 0.
\end{equation}

We would like to explain a bit on how we choose the boundary conditions in the direction of \(v\). On one hand, for the boundary condition at \(v = 0\), it needs to be pointed out that this boundary condition is not necessary if the Fichera function \([44]\) along \(v = 0\) satisfies \(\kappa \eta - \frac{\sigma^2}{2} \geq 0\), while the boundary condition at \(v = 0\) is needed to close the system when \(\kappa \eta < \frac{\sigma^2}{2}\). Moreover, the solution of SDE (6.1) when \(v = 0\) can be approximated as, \(S = e^{(r-D_0)t}S_0\), there is virtually no risk with the underlying asset. This demonstrates that if \(C_RS > Ze^{-r(T-t)}\), there is no sense to hold the bond, and it should be exercised immediately, implying that the bond price at this situation is \(C_RS\). In contrast, if \(C_RS \leq Ze^{-r(T-t)}\), the bond should be held until the expiry and its value should be \(Ze^{-r(T-t)}\) instead. Therefore, both cases show that the boundary condition at \(v = 0\) is \(\lim_{v \to 0} U(S,v,t) = \max\{C_RS, Ze^{-r(T-t)}\}\). On the other hand, when the volatility approaches infinity, the bond price should be independent of the volatility change, otherwise, the bond price will reach infinity, since the bond price is an increasing function with respect to the volatility.

In summary, the PDE system for pricing a CB under the Heston model can be established as

\begin{equation}
\begin{aligned}
\frac{1}{2} \sigma v^2 \frac{\partial^2 U}{\partial S^2} + \sigma v \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial v^2} + (r-D_0)S \frac{\partial U}{\partial S} \\
+ \kappa(\eta - v) \frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0,
\end{aligned}
\end{equation}

for \(S \in [0,S_f(v,t)]\), \(v \in [0,\infty]\) and \(t \in [0,T]\). In the following, we are going to present the details on how to apply the predictor-corrector method with ADI scheme on the PDE system governing the value of CBs.

### 6.2.2 Discretize the PDE system

In this subsection, the PDE system is discretized with some rules. Before discretization, it should be noted that one of the boundaries of System (6.10) in the direction of \(S\) is not fixed, which poses an obstacle in applying the predictor-corrector method. Therefore, a classical transform, Landau transform \([71]\), i.e. \(x = \ln\left(\frac{S}{S_f}\right)\), should be
adopted to this PDE system to solve this issue, and at the same time the value of the optimal conversion boundary is now a part of the solution. Moreover, a simple transform, $\tau = T - t$, is also applied to the PDE system to change the terminal condition problem to an initial counterpart. In addition, another transform, $V(x,v,\tau) = U(x,v,\tau) - Ze^{-r\tau}$, is also made here so that the PDE system (6.10) can be rewritten as

$$
\begin{align*}
\mathcal{L} V(x,v,\tau) &= 0, \\
V(x,v,0) &= \max\{C_f \cdot S_f(v,0) \cdot e^x - Z, 0\}, \\
\lim_{x \to \infty} V(x,v,\tau) &= 0, \\
V(0,v,\tau) &= C_f \cdot S_f(v,\tau) - Ze^{-r\tau}, \\
\frac{\partial V}{\partial x}(0,v,\tau) &= C_f \cdot S_f(v,\tau), \\
\lim_{v \to 0} V(x,v,\tau) &= \max\{C_f \cdot S_f(0,\tau) \cdot e^x - Ze^{-r\tau}, 0\}, \\
\lim_{v \to \infty} \frac{\partial V}{\partial v}(x,v,\tau) &= 0,
\end{align*}
$$

(6.11)

with $x \in (-\infty,0]$, $v \in [0,\infty)$ and $\tau \in [0,T]$, and

$$
\mathcal{L} = \left[ \frac{v}{2} + \frac{1}{2} \sigma^2 v \right] \left( \frac{\partial S_f}{\partial v} \right)^2 - \rho \sigma v \frac{\partial S_f}{\partial v} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma^2 v \frac{\partial^2}{\partial v^2} + (\rho \sigma v - \frac{\sigma^2 v \partial S_f}{S_f} \frac{\partial}{\partial x} v \frac{\partial}{\partial v}) \frac{\partial^2}{\partial x \partial v}
\right.
$$

$$
+ \left[-\frac{v}{2} + \frac{1}{2} \sigma^2 v \right] \left( \frac{\partial S_f}{\partial v} \right)^2 - \frac{1}{2} \sigma^2 v \frac{\partial^2 \lambda}{\partial v^2} + r - D_0 - \kappa(\eta - v) \frac{1}{2} \frac{\partial S_f}{\partial v} + \frac{1}{2} \frac{\partial S_f}{\partial \tau} \frac{\partial}{\partial \tau}
+ \kappa(\eta - v) \frac{\partial}{\partial v} - r - \frac{\partial}{\partial \tau}.
$$

(6.12)

To further simplify the notation of $\mathcal{L}$, some new notations are defined here

$$
\xi = \frac{1}{S_f} \frac{\partial S_f}{\partial v}, \quad \beta = \frac{1}{S_f} \frac{\partial^2 S_f}{\partial v^2}, \quad \lambda = \frac{1}{S_f} \frac{\partial S_f}{\partial \tau},
$$

(6.13)

with which $\mathcal{L}$ can be represented by

$$
\mathcal{L} = a(v) \frac{\partial^2}{\partial x^2} + b(v) \frac{\partial^2}{\partial v^2} + c(v) \frac{\partial^2}{\partial x \partial v} + \left[ d(v) + \lambda \right] \frac{\partial}{\partial x} + e(v) \frac{\partial}{\partial v} - r - \frac{\partial}{\partial \tau},
$$

(6.14)

where

$$
a(v) = \frac{1}{2} v + \frac{1}{2} \sigma^2 v \xi^2 - \rho \sigma v \xi, \\
b(v) = \frac{1}{2} \sigma^2 v, \\
c(v) = \rho \sigma v - \sigma^2 v \xi, \\
d(v) = \frac{1}{2} v + \frac{1}{2} \sigma^2 v - \frac{1}{2} \sigma^2 v \beta + r - D_0 - \kappa(\eta - v) \xi,
$$

(6.15) - (6.18)
Before our approach can be applied to obtain the numerical solution, System (6.11) should be discretized first. Specifically, the semi-infinite domain should be firstly truncated into a finite one as follows

\[
\{(x, v, \tau) | x \in [x_{\min}, 0], v \in [0, v_{\max}], \tau \in [0, T]\},
\]

and the values of \(x_{\min}\) and \(v_{\max}\) will be chosen when we present the numerical results. Then, the finite domain will be separated into \(N_\tau\) uniform grids in the direction of \(\tau\), \(N_x\) uniform grids in the direction of \(x\) and \(N_v\) uniform grids in the direction of \(v\). Thus, we have

\[
\Delta \tau = \frac{T}{N_\tau}, \quad \Delta x = \frac{-x_{\min}}{N_x}, \quad \Delta v = \frac{v_{\max}}{N_v},
\]

and

\[
\tau_n = n\Delta \tau, \quad \text{with} \quad \tau_0 = 0 \text{ and } \tau_{N_\tau} = T,
\]

\[
x_i = x_{\min} + i\Delta x, \quad \text{with} \quad x_0 = x_{\min} \text{ and } x_{N_x} = 0,
\]

\[
v_j = j\Delta v, \quad \text{with} \quad v_0 = 0 \text{ and } v_{N_v} = v_{\max}.
\]

In this case, the value of the unknown functions at a grid point, \(V(x_i, v_j, \tau_n)\) and \(S_f(v_j, \tau_n)\), are denoted as \(V^{(n)}_{i,j}\) and \(S^{(n)}_f(j)\), respectively, for \(i = 0, 1, \cdots, N_x\), \(j = 0, 1, \cdots, N_v\) and \(n = 0, 1, \cdots, N_\tau\).

In order to apply our numerical method, we now classify the entire domain into two parts, with the first one being the interior of the domain

\[
\mathcal{D} = \{(x_i, v_j) | i = 1, \cdots, N_x - 1, j = 1, \cdots, N_v - 1\},
\]

and another one representing the boundaries. For each grid point in \(\mathcal{D}\), the standard central difference scheme and the second-order half-central difference scheme are used to approximate the first-order derivative (including the cross-derivative) and the second-order derivative, respectively. Thus, all the derivatives belonging to \(\mathcal{D}\) are discretized as following:

\[
\frac{\partial V^{(n)}_{i,j}}{\partial x} = \frac{V^{(n)}_{i+1,j} - V^{(n)}_{i-1,j}}{2\Delta x},
\]

\[
\frac{\partial V^{(n)}_{i,j}}{\partial v} = \frac{V^{(n)}_{i,j+1} - V^{(n)}_{i,j-1}}{2\Delta v},
\]

\[
\frac{\partial^2 V^{(n)}_{i,j}}{\partial x^2} = \frac{V^{(n)}_{i+1,j} - 2V^{(n)}_{i,j} + V^{(n)}_{i-1,j}}{(\Delta x)^2},
\]
\[
\frac{\partial^2 V_{i,j}^{(n)}}{\partial v^2} = \frac{V_{i,j}^{(n)} - 2V_{i,j}^{(n)} + V_{i,j-1}^{(n)}}{(\Delta v)^2}, \quad (6.29)
\]

\[
\frac{\partial^2 V_{i,j}^{(n)}}{\partial x \partial v} = \frac{V_{i+1,j+1}^{(n)} - V_{i+1,j-1}^{(n)} - V_{i-1,j+1}^{(n)} + V_{i-1,j-1}^{(n)}}{4\Delta v\Delta x}, \quad (6.30)
\]

For the boundary part, it is easy to deal with the Dirichlet boundary conditions, while it is quite difficult to approximate the Neumann boundary condition. This is because in general, the first-order derivative at \(x = 0\) should be presented

\[
\partial_x V_{N_1,j}^{(n)} = \frac{V_{N_1,j+1}^{(n)} - V_{N_1,j-1}^{(n)}}{2\Delta x}, \quad (6.31)
\]

whereas it is impossible to obtain the value of \(V_{N_1,j+1}^{(n)}\). Therefore, we have to use an alternative approach, the so-called one-sided difference, instead of the central one. It is a form of extrapolation that determines the value of the unknown function on the boundary in terms of its values at the interior grid points. With the use of the Taylor series, we can obtain the following equations:

\[
V_{N_1-1,j}^{(n)} = V_{N_1,j}^{(n)} - \Delta x \frac{\partial V_{N_1,j}^{(n)}}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V_{N_1,j}^{(n)}}{\partial x^2} + o((\Delta x)^3), \quad (6.32)
\]

\[
V_{N_1-2,j}^{(n)} = V_{N_1,j}^{(n)} - 2\Delta x \frac{\partial V_{N_1,j}^{(n)}}{\partial x} + \frac{1}{2}(2\Delta x)^2 \frac{\partial^2 V_{N_1,j}^{(n)}}{\partial x^2} + o((\Delta x)^3). \quad (6.33)
\]

By eliminating \(\frac{\partial^2 V_{N_1,j}^{(n)}}{\partial x^2}\), we can further obtain

\[
\frac{\partial V_{N_1,j}^{(n)}}{\partial x} = \frac{3V_{N_1,j}^{(n)} + V_{N_1-2,j}^{(n)} - 4V_{N_1-1,j}^{(n)}}{2\Delta x} + o((\Delta x)^3), \quad (6.34)
\]

where the value of \(\frac{\partial V_{N_1,j}^{(n)}}{\partial x}\) is expressed in the form of the values for \(V_{N_1-2,j}^{(n)}, V_{N_1-1,j}^{(n)}\), and the unknown boundary value \(V_{N_1,j}^{(n)}\) approximately. In summary, the finite difference equation (FDE) system written on a grid point for System (6.11) can be
specified as

\[
\begin{align*}
\frac{\partial v^{(n)}_{i,j}}{\partial t} &= a_j \frac{\partial^2 v^{(n)}_{i,j}}{\partial x^2} + b_j \frac{\partial v^{(n)}_{i,j}}{\partial x} + c_j \frac{\partial^2 v^{(n)}_{i,j}}{\partial x^2} + [d_j + \lambda_j] \frac{\partial v^{(n)}_{i,j}}{\partial x} + e_j \frac{\partial v^{(n)}_{i,j}}{\partial x} - r V^{(n)}_{i,j}, \\
V_{i,j}^{(0)} &= \max\{C_R \cdot S_f^{(0)}(j)e^{v_i} - Z, 0\}, \\
V_{0,j}^{(n)} &= 0, \\
V_{N_i,j}^{(n)} &= C_R \cdot S_f^{(n)}(j) - Z e^{-r \tau}, \\
\frac{3 V_{N_i,j}^{(n)} + V_{N_i-2,j-1}^{(n)} - 4 V_{N_i-1,j}^{(n)}}{2 \Delta x} &= C_R \cdot S_f^{(n)}(j), \\
V_{i,0}^{(n)} &= \max\{C_R \cdot S_f^{(n)}(0)e^{v_i} - Z e^{-r \tau}, 0\}, \\
V_{i,N_v}^{(n)} &= 0,
\end{align*}
\]

where

\[
\begin{align*}
a_j &= \frac{1}{2} v_j + \frac{1}{2} \sigma^2 v_j \xi_j^2 - \rho \sigma v_j \xi_j, \\
b_j &= \frac{1}{2} \sigma^2 v_j, \\
c_j &= \rho \sigma v_j - \sigma^2 v_j \xi_j, \\
d_j &= -\frac{1}{2} v_j + \frac{1}{2} \sigma^2 v_j \xi_j - \frac{1}{2} \sigma^2 v_j \beta_j + r - D_0 - \kappa (\eta - v_j) \xi_j, \\
e_j &= \kappa (\eta - v_j),
\end{align*}
\]

with

\[
\begin{align*}
\xi_j &= \frac{1}{S_f(j)} \frac{S_f(j+1) - S_f(j-1)}{2 \Delta v}, \\
\beta_j &= \frac{1}{S_f(j)} \frac{S_f(j+1) - 2S_f(j) + S_f(j-1)}{(\Delta v)^2}, \\
\lambda_j &= \frac{1}{S_f(j)} \frac{\partial S_f(j)}{\partial \tau}.
\end{align*}
\]

It should be remarked here that the boundary condition in the FDE system is changed from the Neumann one to the Dirichlet counterpart when \(v \to \infty\). This can be explained by nothing that the initial Neumann boundary condition implies that the value of the bond at \(v \to \infty\) should be a constant (independent on \(v\)), and such a constant should be equal to 0 since the holder will not choose to convert the bond as the market is too volatile and the bond price in this case should equal to \(Ze^{-r \tau}\), or in other words, \(\lim_{v \to \infty} V(x,v,\tau) = 0\). Another thing should also be noted that the time derivatives have not been discretized by now, and the explicit Euler scheme and the implicit Euler scheme are applied to the time derivative \(\frac{\partial V^{(n)}_{i,j}}{\partial \tau}\) and \(\frac{\partial S_f^{(n)}(j)}{\partial \tau}\), respectively, in the process of applying the predictor-corrector scheme, the details of which are illustrated in the next two subsections.
6.2.3 Numerical scheme for the prediction step

In this subsection, the numerical scheme for predicting the value of the optimal conversion boundary will be presented. Firstly, we should recall the discretized formulation of the boundary conditions at \( x = 0 \)

\[
V^{(n)}_{N_x,j} = C_R \cdot S^{(n)}_f (j) - Z e^{-r \tau_n}, \quad (6.44)
\]

\[
C_R \cdot S^{(n)}_f (j) = \frac{3V^{(n)}_{N_x,j} + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j}}{2\Delta x}, \quad (6.45)
\]

Then, we can obtain

\[
C_R \cdot S^{(n)}_f (j) = \frac{3(C_R \cdot S^{(n)}_f (j) - Z e^{-r \tau_n}) + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j}}{2\Delta x} \\
\Rightarrow 2\Delta x \cdot C_R \cdot S^{(n)}_f (j) = 3(C_R \cdot S^{(n)}_f (j) - Z e^{-r \tau_n}) + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j} \\
\Rightarrow (3 - 2\Delta x) \cdot C_R \cdot S^{(n)}_f (j) = 3Ze^{-r \tau_n} + 4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j} \\
\Rightarrow S^{(n)}_f (j) = \frac{3Ze^{-r \tau_n} + 4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j}}{(3 - 2\Delta x) \cdot C_R}. \quad (6.46)
\]

If we assume the values of the bond and its optimal conversion boundary at the \( n \)th time step are known, the optimal conversion boundary at the \((n+1)\)th time step can be determined by using the formulation above

\[
S^{(n+1)}_f (j) = \frac{3Ze^{-r \tau_{n+1}} + 4V^{(n+1)}_{N_x-1,j} - V^{(n+1)}_{N_x-2,j}}{(3 - 2\Delta x) \cdot C_R}. \quad (6.47)
\]

With this newly obtained expression, if the explicit Euler scheme and the implicit Euler scheme are applied to the time derivative \( \frac{\partial V^{(n)}_{i,j}}{\partial \tau} \) and \( \frac{\partial S^{(n)}_f (j)}{\partial \tau} \), respectively, we can obtain

\[
V^{(n+1)}_{i,j} = V^{(n)}_{i,j} + \Delta \tau \left\{ a_i \frac{\partial^2 V^{(n)}_{i,j}}{\partial x^2} + b_i \frac{\partial^2 V^{(n)}_{i,j}}{\partial v^2} + c_i \frac{\partial^2 V^{(n)}_{i,j}}{\partial x \partial v} + \left[ d_j + \frac{S^{(n+1)}_f (j) - S^{(n)}_f (j)}{\Delta \tau S^{(n)}_f (j)} \right] \frac{\partial V^{(n)}_{i,j}}{\partial x} \\
+ e_j \frac{\partial V^{(n)}_{i,j}}{\partial v} \right\} - r \Delta \tau V^{(n)}_{i,j}, \quad i = N_x - 2, N_x - 1. \quad (6.48)
\]

After some complex computation, the predicted value of the optimal conversion
boundary at \((n+1)\)th time step, \(\tilde{S}_f^{(n+1)}(j)\), can be presented

\[
\tilde{S}_f^{(n+1)}(j) = \frac{3Ze^{-r\tau_{n+1}} + \mathcal{F}[4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j}]}{(3 - 2\Delta x)CR - \frac{\delta_x(4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j})}{S_f^{(n)}(j)}},
\]

(6.49)

with

\[
\mathcal{F} = I + \Delta \tau [a_j \delta_{\Delta x} + b_j \delta_{\Delta v} + c_j \delta_{\Delta v} + (d_j - \frac{1}{\Delta \tau}) \delta_{\Delta x} + e_j \delta_{\Delta v}] - r \Delta \tau.
\]

(6.50)

Here, all the derivatives are replaced by the corresponding \(\delta^*_\). Therefore, the predicted value of the corresponding bond price, \(\tilde{V}^{(n+1)}_{N_x,j}\), can be calculated as

\[
\tilde{V}^{(n+1)}_{N_x,j} = CR \cdot \tilde{S}_f^{(n+1)}(j) - Ze^{-r\tau_{n+1}}.
\]

(6.51)

In summary, the value of the optimal conversion boundary and the bond price on this boundary can be predicted by Equation (6.50) and Equation (6.51), respectively, with which the algorithm of ADI method is used to obtain all the values at the \((n+1)\)th time step, \(V^{(n+1)}\), in the next subsection. Afterwards, we can correct the values of \(S_f^{(n+1)}\) and \(V^{(n+1)}_{N_x}\) by using the newly obtained \(V^{(n+1)}_{N_x-2}\) and \(V^{(n+1)}_{N_x-1}\).

### 6.2.4 Numerical scheme for the correction step

In this subsection, ADI method is utilized for the correction step. As mentioned above, this method is a very useful technique to solve PDEs on rectangular domains, especially for the parabolic ones. Moreover, what we choose in this study is the Douglas-Rachford (D-R) method, whose accuracy is of first-order in time and second-order in space. Now, the FDE should be rewritten, so that the ADI method can be applied

\[
(\mathbb{I} - \phi A_1)(\mathbb{I} - \phi A_2)V^{(n+1)} = [\mathbb{I} + A_0 + (1 - \phi)A_1 + A_2]V^{(n)} - (\mathbb{I} - \phi A_1)\phi A_2 V^{(n)},
\]

(6.52)

where \(\phi \in [0, 1]\). The procedures to obtain all operators, \(A_0, A_1\) and \(A_2\), are left in Appendix D.1. For the D-R method, there are two steps that need to be conducted before we can obtain the final scheme. Firstly, we should calculate the intermediate value, \(Y\), with the following equation

\[
(\mathbb{I} - \phi A_1)Y = [\mathbb{I} + A_0 + (1 - \phi)A_1 + A_2]V^{(n)},
\]

(6.53)

where we fix the \(v\) direction. Then, the above equation can be simplified as

\[
AY_j = P_j + Bx_j,
\]

(6.54)
with the details of $A$, $Y_j$, $P_j$ and $B x_j$ being presented in Appendix D.2. It should be noted that matrix $A$ is a tridiagonal, and thus the Thomas algorithm\cite{96} can be applied to improve the speed and the accuracy of our method. It should also be pointed out that the value $Y$ is actually obtained by the loop of $j$ and each $Y_j$ is a vector with $(N_x + 1) \times 1$. Once the value of $Y$ is known, we can then compute $V^{(n+1)}$ from the following equation

$$ (1 - \phi A_2) V^{(n+1)} + \phi A_2 V^{(n)} = Y, \quad (6.55) $$

with the $x$ direction being fixed. This equation can be further represented as

$$ C V_i^{(n+1)} = Q_i + B v_i, \quad (6.56) $$

and the details of $C$, $V_i^{(n+1)}$, $Q_i$ and $B v_i$ are left in Appendix D.3. Similarly to Equation (6.54), Matrix $C$ is also a tridiagonal, and the Thomas algorithm is utilized here again. For this equation, the loop of $i$ is used to obtain the value of $V^{(n+1)}$, leading to our desired result, since it can be easily proved that solving Equation (6.54) and Equation (6.56) means solving the initial Equation (6.52). Clearly, with these two steps illustrated in this subsection, the algorithm to derive the corrector scheme has already been designed.

In order to numerically implement our scheme in the next subsection, we need to make it clear how to calculate the boundary value of the intermediate value $Y$, which is presented as

$$ Y_0 = 0, \quad (6.57) $$

$$ Y_{N_x} = (1 - \phi A_2) \tilde{V}_N^{(n+1)} + \phi A_2 V_N^{(n)} $$

$$ = (1 - \phi A_2) (C_R \cdot \tilde{S}_{f}^{(n+1)} - Z e^{-r t_n + 1}) + \phi A_2 V_N^{(n)}, \quad (6.58) $$

with the use of the predicted value of $\tilde{S}_{f}^{(n+1)}$. Clearly, this has completed the establishment of the predictor-corrector scheme with the ADI method for pricing CBs if we combine the predictor scheme presented in the previous subsection and the corrector scheme shown in this subsection, and we are now ready to conduct numerical experiments to study the properties of CBs under the stochastic volatility model, the details of which are shown in the next subsection.

### 6.2.5 Numerical examples

In this subsection, the accuracy of our method is tested first, and then the numerical results are provided to illustrate several properties of the convertible bond under a stochastic volatility model. Unless otherwise stated, the parameters used are listed
below:

- Face value $Z = 10$,
- Conversion ratio $C_R = 1$,
- Maturity $T = 1$ (year),
- Risk-free annual interest rate $r = 0.1$,
- Rate of continuous dividend payment $D_0 = 0.07$,
- Reversion rate $\kappa = 1.5$,
- Reversion level $\eta = 0.16$,
- Volatility of the volatility $\sigma = 0.4$,
- Correlation factor $\rho = 0.1$.

Before we present the numerical results, it is necessary to determine the domain that we are going to operate on. Although $x$ can take any value being less than 0, we need to truncate the semi-infinite domain into a finite one so as to implement our numerical scheme. As mentioned in\cite{102} that it suffices to set the minimum of $x$ to be $-\ln 5$, the domain of our model is assumed as

\[
\{(x,v,t) | x \in [-\ln 5, 0] \times [0, 1] \times [0, 1]\}. \tag{6.59}
\]

Here, setting $v_{\text{max}} = 1$ is sufficient and reasonable since the value of the volatility is usually very small. After the domain is uniformly discretized, with the step size in the direction of $x$, $v$ and $t$ being 101, 201 and 5001, respectively, the numerical results are presented in the following. It should also be pointed out that all of our calculations in this paper are done using Matlab R2017a on a PC with the following specifications: Intel(R) Core(TM), i7-4790 CPU@3.60GHz 3.60 GHz, and 16.0 GB of RAM.

To validate our numerical scheme, a degenerate case is considered as the benchmark, where the volatility is a fixed value instead of being stochastic, and the values of the optimal conversion boundary calculated with our method and those derived through the integral equation approach\cite{109} are displayed in both Table 6.1 and Figure 6.1. In particular, Table 6.1 shows the prices of the optimal conversion boundary at the current time, $t = 0$, and one can easily observe that with the increase in the number of grid points, our results converge to the benchmark, as the relative error between the two prices are decreasing. If we turn to Figure 6.1, it is clear that both values match very well with each other, demonstrating the accuracy of our
Table 6.1: Convergence test

<table>
<thead>
<tr>
<th>Volatility value</th>
<th>( (N_x, N_y, N_t) )</th>
<th>ADI</th>
<th>IE</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v = 0.1 )</td>
<td>( (25, 50, 1250) )</td>
<td>12.8669</td>
<td></td>
<td>( 5.97 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (50, 100, 2500) )</td>
<td>12.8110</td>
<td>12.7905</td>
<td>( 1.61 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (100, 200, 5000) )</td>
<td>12.7969</td>
<td></td>
<td>( 5.00 \times 10^{-4} )</td>
</tr>
<tr>
<td>( v = 0.2 )</td>
<td>( (25, 50, 1250) )</td>
<td>15.7500</td>
<td></td>
<td>( 6.17 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (50, 100, 2500) )</td>
<td>15.6796</td>
<td>15.6533</td>
<td>( 1.67 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (100, 200, 5000) )</td>
<td>15.6618</td>
<td></td>
<td>( 5.38 \times 10^{-4} )</td>
</tr>
<tr>
<td>( v = 0.4 )</td>
<td>( (25, 50, 1250) )</td>
<td>21.2244</td>
<td></td>
<td>( 8.19 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (50, 100, 2500) )</td>
<td>21.1017</td>
<td>21.0519</td>
<td>( 2.36 \times 10^{-3} )</td>
</tr>
<tr>
<td></td>
<td>( (100, 200, 5000) )</td>
<td>21.0709</td>
<td></td>
<td>( 9.00 \times 10^{-4} )</td>
</tr>
</tbody>
</table>

Figure 6.1: The comparison of the optimal conversion boundary obtained by our method and that from the integral equation method\(^{[109]}\), at \( v = 0.1 \).

method for this case. Of course, we still need to check whether our approach works for the general case when the stochastic volatility is incorporated. Thus, the values of CBs obtained from our method are also compared with those generated through the Monte Carlo method, the results of which are presented in Figure 6.2. It can be easily noticed that both prices are point-wisely close to each other, which certainly reflects that our method is reliable. On the other hand, it should be point out that the average CPU time consumed by the Monte Carlo method, with 100 time steps and 500,000 simple paths, is 10.3828 seconds, while it only takes \( 7.9 \times 10^{-7} \) seconds to produce one price with the method introduced by this paper. Such a great differ-
ence certainly indicates that the proposed predictor-corrector scheme is much more efficient.

With the confidence in our approach, we are now studying the properties of the optimal conversion boundary as well as the bond prices when the stochastic volatility is considered. Depicted in Figure 6.3 are the values of the optimal conversion boundary with respect to the volatility and the time to expiry. In order to show clearly the effects of the volatility and the time to expiry on the optimal conversion price, two figures, Figure 6.4 and Figure 6.5, are displayed by fixing one direction. An interesting phenomenon that can be observed in Figure 6.4 is that the optimal conversion boundary is not a monotonic increasing function of the time to expiry; it increases with the time to expiry when the time to expiry is small, and it will show a downward trend once the time to expiry is large enough. This is consistent with the theoretical result\cite{104} that the value of the perpetual optimal conversion boundary equals to zero. From the financial point of view, it can be understood from the extreme case that when the time to expiry approaches infinity, the current value of the face value is almost zero, implying that it is meaningless to continue to hold the CB and the investor should convert it into stocks immediately. On the other hand, when we turn to Figure 6.5, it should be noted that the optimal conversion boundary is always a monotonic increasing function of the volatility, no matter the
value of the time to expiry, which is reasonable since a higher volatility means a higher risk, leading to a higher premium of the CB.
What is shown in Figure 6.6 is the change of the bond price with respect to the volatility and the underlying price, when the CB has not been converted into
stocks. Clearly, no matter what the value of the underlying asset is, the bond price is a monotonic increasing function of the volatility as a larger volatility always implies a higher risk. Moreover, a higher value of the underlying asset leads to the larger slope of the bond price with respect to the volatility. In other words, the bond price is also an increasing function of the underlying asset, when the volatility is fixed, which is also clearly presented in Figure 6.7. This is financially meaningful since when the underlying asset price increases, there will be a higher probability for the holder to convert the bond, leading to the higher value of the bond. Another phenomenon that can be noticed here is that increasing the value of the volatility is actually increasing the value of the optimal conversion boundary, which confirms the result presented in Figure 6.3.

6.3 Pricing convertible bonds with stochastic interest rate

In this section, we study the pricing problem of the convertible bond with a stochastic interest rate model (CIR model). Given the fact that the PDE system in this section is very similar to that in the last section and the same predictor-corrector scheme will also be utilized here, the details on some tedious but very similar computational processes are thus omitted.
6.3.1 The PDE system and its numerical scheme

We now begin by assuming that the stochastic interest rate satisfies the following SDE

\[ dr = \kappa(\eta - r)dt + \xi \sqrt{r}dW_3, \quad (6.60) \]

where \( \kappa, \eta \) and \( \xi \) are the mean reversion speed, the long term mean and the volatility of the interest rate, respectively, while \( W_3 \) is another standard Brownian motion. In addition, the dynamics of the underlying asset price is assumed as

\[ dS_t = (r - D_0)S_t dt + \sigma S_t dW_1^t, \quad (6.61) \]

which is the same as Equation (6.1) except that the constant volatility of the underlying asset is denoted as \( \sigma \). The correlation between \( W_1 \) and \( W_3 \) is also represented by \( \rho \), which can vary within \([-1,1]\). In this case, if the bond price is denoted as \( U(S,r,t) \), its governing PDE system can be set up

\[
\begin{aligned}
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + \sigma \rho \xi \sqrt{r} S \frac{\partial^2 U}{\partial S \partial r} + \frac{1}{2} \xi^2 r \frac{\partial^2 U}{\partial r^2} + (r - D_0) S \frac{\partial U}{\partial S} &+ \kappa(\eta - r) \frac{\partial U}{\partial r} - r U + \frac{\partial U}{\partial t} = 0, \\
U(S,r,T) &= \max\{C_R \cdot S, Z\}, \\
U(0,r,t) &= \mathbb{E}[Ze^{-\int_t^T r(s)ds} | r_t] \triangleq Z \cdot F(r,t), \\
U(S_c(r,t),r,t) &= C_R \cdot S_c(r,t), \\
\frac{\partial U}{\partial S}(S_c(r,t),r,t) &= C_R, \\
\lim_{r \to 0} U(S,r,t) &= U_{BS}(S,t), \\
\lim_{r \to \infty} U(S,r,t) &= 0,
\end{aligned}
\]

where \( S_c(r,t) \) and \( U_{BS}(S,t) \) are the optimal conversion boundary and the convertible bond price under the Black-Scholes model with \( r = 0 \), respectively, and \( F(r,t) \) satisfies the following PDE system

\[
\begin{aligned}
\frac{\partial F}{\partial t}(r,t) + \kappa(\eta - r) \frac{\partial F}{\partial r}(r,t) + \frac{1}{2} \xi^2 r \frac{\partial^2 F}{\partial r^2}(r,t) - r F(r,t) &= 0, \\
F(r,T) &= 1.
\end{aligned}
\]

The solution to this PDE system can be found as

\[ F(r,t) = e^{A(t) - B(t) r}, \quad (6.64) \]
where

\[
A(t) = -\kappa \eta \left\{ \frac{4}{(m - \kappa)(m + \kappa)} \ln\left[ \frac{2m + (m + \kappa)(e^{m(T-t)} - 1)}{2m} \right] + \frac{2}{\kappa - m}(T-t) \right\},
\]

\[
B(t) = \frac{2(e^{\sqrt{m(T-t)}} - 1)}{2m + (\kappa + m)(e^{m(T-t)} - 1)},
\]

with \( m = \sqrt{\kappa + 2\xi^2} \), the derivation of which can be found in \[^{50}\]. Before we proceed further, it is necessary for us to explain the boundary conditions we gave in the system in the direction of \( r \). On one hand, for the boundary condition at \( r = 0 \), it needs to be pointed out that this boundary condition is not necessary if the Fichera function \[^{44}\] along \( r = 0 \) satisfies \( \kappa \eta - \xi^2 \geq 0 \), while a boundary condition at \( r = 0 \) is needed to close the system when \( \kappa \eta < \xi^2 \). If a boundary condition is needed, we adopt the bond price under the Black-Scholes model with \( r = 0 \) as an approximation, which is based on an assumption that when \( O(\kappa \eta \frac{\partial U}{\partial r}) \) is much smaller than the order of the other terms in (6.62) when \( r = 0 \), the resulting equation degenerates to the Black-Scholes equation. On the other hand, when the value of the risk-free interest rate goes to infinity, the best way to achieve the best return for an investor is leave the money in a risk-free bank account, which implies that no one would invest in a convertible bond, resulting in the bond value being equal to zero.

Now, applying the following transform

\[
V(S, r, t) = U(S, r, t) - \mathbb{E}[Ze^{-\int_0^T r(s)ds} | r_t] = U(S, r, t) - Z \cdot F(r, t),
\]

(6.67)

to the above PDE system yields

\[
\begin{align*}
\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} &+ \sigma \rho \xi \sqrt{r} S \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2} \xi^2 r \frac{\partial^2 V}{\partial r^2} + (r - D_0)S \frac{\partial V}{\partial S} \\
&\quad+ \kappa(\eta - r) \frac{\partial V}{\partial r} - rV + \frac{\partial V}{\partial t} = 0,
\end{align*}
\]

\[
\begin{align*}
V(S, r, T) &= \max\{C_R \cdot S - Z, 0\}, \\
V(0, r, t) &= 0, \\
V(S_c(r, t), r, t) &= C_R \cdot S_c(r, t) - Z \cdot F(r, t), \\
\frac{\partial V}{\partial S}(S_c(r, t), r, t) &= C_R, \\
\lim_{r \to 0} V(S, r, t) &= U_{BS}(S, t) - Z \cdot F(0, t), \\
\lim_{r \to \infty} V(S, r, t) &= 0.
\end{align*}
\]

(6.68)

Then, in order to transform the target PDE system to a dimensionless one and also transform the free boundary problem into a fixed boundary one to facilitate the
numerical computation, we make the following transformation

$$
\tau = T - t, \quad x = \ln \left( \frac{S}{S_c} \right), \quad (6.69)
$$

so that we have

$$
\begin{cases}
\mathcal{L}[V] = 0, \\
V(x, r, 0) = \max \{C_R \cdot S_c(r, 0) \cdot e^x - Z, 0\}, \\
\lim_{x \to -\infty} V(x, r, \tau) = 0, \\
V(0, r, \tau) = C_R \cdot S_c(r, \tau) - Z \cdot F(r, \tau), \\
\frac{\partial V}{\partial x}(0, r, \tau) = C_R \cdot S_c(r, \tau), \\
\lim_{r \to 0} V(x, r, \tau) = 0, \\
\lim_{r \to \infty} V(x, r, \tau) = 0,
\end{cases} \quad (6.70)
$$

where

$$
\mathcal{L} = a(r) \frac{\partial^2}{\partial x^2} + b(r) \frac{\partial^2}{\partial r^2} + c(r) \frac{\partial^2}{\partial x \partial r} + [d(r) + \lambda] \frac{\partial}{\partial x} + e(r) \frac{\partial}{\partial r} - \frac{\partial}{\partial \tau}, \quad (6.71)
$$

with

$$
a(r) = \frac{1}{2} \sigma^2 + \frac{1}{2} \xi^2 r \zeta^2 - \sigma \rho \xi \sqrt{r} \zeta, \quad (6.72)
$$

$$
b(r) = \frac{1}{2} \xi^2 r, \quad (6.73)
$$

$$
c(r) = \sigma \rho \xi \sqrt{r} - \xi^2 r \zeta, \quad (6.74)
$$

$$
d(r) = -\frac{1}{2} \sigma^2 + \frac{1}{2} \xi^2 r \zeta^2 - \frac{1}{2} \xi^2 r \beta + r - D_0 - \kappa(\eta - r) \zeta, \quad (6.75)
$$

$$
e(r) = \kappa(\eta - r). \quad (6.76)
$$

Here, we denote

$$
\zeta = \frac{1}{S_c} \cdot \frac{\partial S_c}{\partial r}, \quad \beta = \frac{1}{S_c} \cdot \frac{\partial^2 S_c}{\partial r^2}, \quad \lambda = \frac{1}{S_c} \cdot \frac{\partial S_c}{\partial \tau}. \quad (6.77)
$$

Since this system is very similar to the last one, the FDE system is directly provided
below without presenting all the details

\[
\begin{align*}
\frac{\partial V_{i,j}^{(n)}}{\partial \tau} &= a_j \frac{\partial^2 V_{i,j}^{(n)}}{\partial x^2} + b_j \frac{\partial^2 V_{i,j}^{(n)}}{\partial x \partial r} + c_j \frac{\partial^2 V_{i,j}^{(n)}}{\partial r^2} + [d_j + \lambda_j] \frac{\partial V_{i,j}^{(n)}}{\partial x} + e_j \frac{\partial V_{i,j}^{(n)}}{\partial r} - r_j V_{i,j}^{(n)}, \\
V_{i,j}^{(0)} &= \max\{C_R \cdot S_c^{(0)}(j)e^{\kappa_i} - Z, 0\}, \\
V_{0,j}^{(n)} &= 0, \\
V_{N_x,j}^{(n)} &= C_R \cdot S_c^{(n)}(j) - Z \cdot F^{(n)}(j), \\
\frac{3V_{N_x,j}^{(n)} + V_{N_x-2,j}^{(n)} - 4V_{N_x-1,j}^{(n)}}{2\Delta x} &= C_R \cdot S_c^{(n)}(j), \\
V_{i,0}^{(n)} &= U_{BS}(S_c^{(n)}(0) \cdot e^{\kappa_i}, \tau_n) - Z \cdot F^{(n)}(0), \\
V_{i,N_r}^{(n)} &= 0,
\end{align*}
\]

(6.78)

with the divided finite domain being \{(x_i, r_j, \tau_n) | x_i = x_{\min} + i\Delta x, \ for \ i = 0, \cdots, N_x; \ r_j = j\Delta r, \ for \ j = 0, \cdots, N_r; \ \tau_n = n\Delta \tau, \ for \ n = 0, \cdots, N_\tau\}. Here, we have \(\Delta \tau = -\frac{x_{\min}}{N_x},\)

\[
\Delta r = \frac{r_{\max}}{N_r} \quad \text{and} \quad \Delta \tau = \frac{T}{N_\tau},
\]

and the parameters are

\[
\begin{align*}
a_j &= \frac{1}{2} \sigma^2 + \frac{1}{2} \xi_2^2 \eta_2 - \sigma \rho \sqrt{\eta} j, \\
b_j &= \frac{1}{2} \xi_2 r_j, \\
c_j &= \sigma \rho \sqrt{\eta} j - \xi_2 r_j, \\
d_j &= -\frac{1}{2} \sigma^2 + \frac{1}{2} \xi_2^2 \eta_2 + \frac{1}{2} \xi_2 r_j \beta + r_j - D_0 - \kappa(\eta - r_j) \xi_j, \\
e_j &= \kappa(\eta - r_j),
\end{align*}
\]

(6.79)

(6.80)

(6.81)

(6.82)

(6.83)

with

\[
\begin{align*}
\zeta_j &= \frac{1}{S_c(j)} \cdot \frac{S_c(j+1) - S_c(j-1)}{2\Delta r}, \\
\beta_j &= \frac{1}{S_c(j)} \cdot \frac{S_c(j+1) - 2S_c(j) + S_c(j-1)}{(\Delta r)^2}, \\
\lambda_j &= \frac{1}{S_c(j)} \cdot \frac{\partial S_c(j)}{\partial \tau}.
\end{align*}
\]

(6.84)

(6.85)

(6.86)

In order to numerically solve the FDE system, we are now ready to set up the predictor-corrector scheme again with two steps. We will first briefly discuss how the predictor scheme can be established. By using the boundary conditions at \(x = 0\)

\[
\begin{align*}
C_R \cdot S_c^{(n)}(j) &= \frac{3V_{N_x,j}^{(n)} + V_{N_x-2,j}^{(n)} - 4V_{N_x-1,j}^{(n)}}{2\Delta x}, \\
V_{N_x,j}^{(n)} &= C_R \cdot S_c^{(n)}(j) - Z \cdot F^{(n)}(j).
\end{align*}
\]

(6.77)

(6.78)
we can further obtain
\[
C_R \cdot S_c^{(n)}(j) = \frac{3V^{(n)}_{N_x,j} + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j}}{2\Delta x}
\]
\[
\Rightarrow C_R \cdot S_c^{(n)}(j) = \frac{3(C_R \cdot S_c^{(n)}(j) - Z \cdot F^{(n)}(j)) + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j}}{2\Delta x}
\]
\[
\Rightarrow 2\Delta x \cdot C_R \cdot S_c^{(n)}(j) = 3(C_R \cdot S_c^{(n)}(j) - Z \cdot F^{(n)}(j)) + V^{(n)}_{N_x-2,j} - 4V^{(n)}_{N_x-1,j}
\]
\[
\Rightarrow (3 - 2\Delta x) \cdot C_R \cdot S_c^{(n)}(j) = 3Z \cdot F^{(n)}(j) - V^{(n)}_{N_x-2,j} + 4V^{(n)}_{N_x-1,j}
\]
\[
\Rightarrow S_c^{(n)}(j) = \frac{3Z \cdot F^{(n)}(j) - V^{(n)}_{N_x-2,j} + 4V^{(n)}_{N_x-1,j}}{(3 - 2\Delta x) \cdot C_R}.
\]

(6.89)

Therefore, if we assume the value of the bond and that of the optimal conversion boundary at \(n\)th time step are known, then the optimal conversion price at \((n+1)\)th time step can be expressed
\[
S_c^{(n+1)}(j) = \frac{3Z \cdot F^{(n+1)}(j) + 4V^{(n+1)}_{N_x-1,j} - V^{(n+1)}_{N_x-2,j}}{(3 - 2\Delta x) \cdot C_R}.
\]

(6.90)

With the explicit Euler scheme and implicit Euler scheme being applied to the time derivative \(\frac{\partial V^{(n)}_{i,j}}{\partial \tau}\) and \(\frac{\partial S_c^{(n)}(j)}{\partial \tau}\), respectively, the following formulation can be obtained
\[
V^{(n+1)}_{i,j} = V^{(n)}_{i,j} + \Delta \tau \left\{ a_j \frac{\partial^2 V^{(n)}_{i,j}}{\partial x^2} + b_j \frac{\partial^2 V^{(n)}_{i,j}}{\partial r^2} + c_j \frac{\partial^2 V^{(n)}_{i,j}}{\partial x \partial r} + [d_j + \frac{S_c^{(n+1)}(j) - S_c^{(n)}(j)}{\Delta \tau \cdot S_c^{(n)}(j)}] \frac{\partial V^{(n)}_{i,j}}{\partial x} \\
+ e_j \frac{\partial V^{(n)}_{i,j}}{\partial r} \right\} - r_j \Delta \tau V^{(n)}_{i,j}, \quad i = N_x - 2, N_x - 1.
\]

(6.91)

As a result, the representation of the predicted optimal conversion price at \((n+1)\)th time step, \(\tilde{S}_c^{(n+1)}(j)\), should be
\[
\tilde{S}_c^{(n+1)}(j) = \frac{3Z \cdot F^{(n+1)}(j) + \mathcal{F}[4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j}]}{(3 - 2\Delta x)C_R - \frac{\delta(4V^{(n)}_{N_x-1,j} - V^{(n)}_{N_x-2,j})}{S_c^{(n)}(j)}},
\]

(6.92)

where
\[
\mathcal{F} = 1 + \Delta \tau \left\{ a_j \delta_{xx} + b_j \delta_{rr} + c_j \delta_{x r} + (d_j - \frac{1}{\Delta \tau}) \delta_x + e_j \delta_r \right\} - r_j \Delta \tau.
\]

(6.93)

Here, all the derivatives are again replaced by the corresponding \(\delta_x\). Hence, the
predicted convertible bond price at \( x = 0 \), \( \tilde{V}_{N_x,j}^{(n+1)} \), is

\[
\tilde{V}_{N_x,j}^{(n+1)} = C_R \cdot \tilde{S}_c^{(n+1)}(j) - Z \cdot F^{(n+1)}(j). \tag{6.94}
\]

By now, the numerical scheme for the prediction step, where the optimal conversion price is predicted via Equation (6.93), and the bond price at \( x = 0 \) is predicted through Equation (6.94), have been obtained, and the next step is to correct the values of \( S_c^{(n+1)} \) and \( V_{N_x}^{(n+1)} \) by using the obtained \( V_{N_x-2}^{(n+1)} \) and \( V_{N_x-1}^{(n+1)} \) at the prediction step using the ADI technique.

We are now again presenting the numerical scheme for the correction step directly, while omitting the details for the derivation. In order to apply the ADI method, the FDE also needs to be reformulated as

\[
(I - \phi A_1)(I - \phi A_2)V^{(n+1)} = [I + A_0 + (1 - \phi)A_1 + A_2]V^{(n)} - (I - \phi A_1)\phi A_2 V^{(n)}, \tag{6.95}
\]

where the definitions of all operators, \( A_0, A_1 \) and \( A_2 \), are left in Appendix D.4. For the adopted D-R method, two steps should be taken into consideration. Firstly, we should fix the \( r \) direction, and an intermediate value, \( Y \), satisfies the following equation

\[
(I - \phi A_1)Y = [I + A_0 + (1 - \phi)A_1 + A_2]V^{(n)}, \tag{6.96}
\]

should be determined. Simplifying this equation leads to

\[
A Y_j = P_j + B x_j, \tag{6.97}
\]

with the definitions of \( A, Y_j, P_j \) and \( B x_j \) left in Appendix D.5, and solving the set of equations here will give the value of \( Y \). Once the value of \( Y \) is known, the value of \( V^{(n+1)} \) can be obtained from the following equation

\[
(I - \phi A_2)V^{(n+1)} + \phi A_2 V^{(n)} = Y, \tag{6.98}
\]

with the \( x \) direction being fixed. To deal with the above equation, it is again transformed into a set of equations

\[
\mathbb{C} V_i^{(n+1)} = Q_i + \mathbb{B} r_i, \tag{6.99}
\]

with the details of \( \mathbb{C}, V_i^{(n+1)}, Q_i \) and \( \mathbb{B} r_i \) left in Appendix D.6. Of course, it is also easy to show that solving Equation (6.97) and Equation (6.99) means solving the initial Equation (6.95). In order to numerically implement our scheme, we need to make it clear how to calculate the boundary value of the intermediate value \( Y \), which
is presented as

\[
Y_0 = 0, \quad \text{ (6.100)}
\]

\[
Y_{N_t} = (I - \phi A_2)\tilde{V}_{N_t}^{(n+1)} + \phi A_2 V_{N_t}^{(n)}
= (I - \phi A_2)(C_R \cdot \tilde{S}_c^{(n+1)} - Z \cdot F^{(n+1)}) + \phi A_2 V_{N_t}^{(n)}, \quad \text{ (6.101)}
\]

with the use of the predicted value of \(\tilde{S}_c^{(n+1)}\). With the numerical scheme being established for the PDE system governing the bond price under the stochastic interest rate model, we are now ready to conduct numerical experiments, the details of which are presented in the following subsection.

6.3.2 Numerical examples

In this subsection, the numerical examples are presented with the same values of the corresponding parameters used in the last section. The only exception is that the constant volatility, \(\sigma\), is assumed to be 0.4 in this section. It should be pointed out that in this section, we will do not check the accuracy of our method, since the algorithm of the stochastic interest rate model is almost same as that of the stochastic volatility model, which has been confirmed as accurate in the last section.

\[\text{Figure 6.8: The optimal conversion price.}\]

Figure 6.8 displays that the value of the optimal conversion boundary with respect
to the time to expiry and the interest rate. To clearly demonstrate the properties of the optimal conversion boundary, two figures are presented by fixing one direction.
Firstly, a similar phenomenon as shown in the case of stochastic volatility can be observed in Figure 6.9 that the optimal convertible price is not an increasing function of the time to expiry; the value of the optimal conversion boundary increases with the time to expiry initially, before it decreases. The main explanation for this is also the fact that the optimal conversion boundary of a perpetual CB is zero, as discussed in the case of stochastic volatility. When we look at Figure 6.10, we can find that the value of the optimal conversion boundary is actually a decreasing function of the interest rate, no matter what the lifetime of the bond is. This is also financially reasonable since the higher the interest rate, the lower the present value of the face value will be, implying that the holder will choose to convert the bond at a smaller underlying price.

In Figure 6.11, the effects of the interest rate and the underlying price on the bond price with a certain time to expiry are demonstrated, and we again only consider the case when the CB has not been converted. When the interest rate is taken into consideration, it is not difficult to find that the price of the CB is a monotonic decreasing function with respect to the interest rate, which is reasonable since a higher value of the interest rate means that it is more incentive for the holder to leave their money in a risk-free environment than buying a risky bond, leading to a lower bond value. When we turn to the underlying price, the effect of which is displayed in Figure 6.12, it can be observed that the price of the convertible bond
is actually an increasing function of the underlying price, which can be understood with a similar reason as provided for the case of stochastic volatility.

6.4 Conclusion

In this paper, the pricing problems of convertible bonds with a stochastic volatility model and a stochastic interest rate model are considered, respectively. An efficient numerical scheme, the predictor-corrector scheme, is established for these two cases. Being able to provide the entire optimal conversion boundary as part of the solution procedure, this new approach requires no embedded iterations at all. Finally, numerical experiments are also carried out to show the reliability of our approach, and different properties of the convertible bond price as well as the optimal convertible boundary are investigated.
Chapter 7

Summary and Conclusion

In this thesis, we consider the pricing of various types of American-style convertible bonds under different models, and both semi-analytical and numerical approaches are applied. In particular, integral equation approaches are applied to evaluate puttable convertible bonds, callable-puttable convertible bonds and resettable convertible bonds under the B-S model. We have also discussed the pricing problem of vanilla convertible bonds when stochastic volatility or stochastic interest rate is incorporated into the B-S model, utilizing a predictor-corrector scheme.

Firstly, two integral equation formulations for the puttable convertible bond prices under the B-S model are presented in Chapter 3. The bond prices as well as optimal conversion and put boundaries can be numerically derived by solving the obtained integral equations. This approach is shown to be superior to the binomial tree method, as far as the accuracy and efficiency is concerned. Numerical results also confirm that the price of a puttable convertible bond is higher than that of the corresponding vanilla one, which is a result of the holder being entitled to the right to sell the bond back to the issuer, potentially protecting the benefit of the holder.

Chapter 4 moves a step further to price callable-puttable convertible bonds, which combine the call and put feature together, under the B-S model. Being different from vanilla convertible bonds, three different cases, depending on the parameters of the target contract, are discussed, and the corresponding PDE systems for these cases are established. By using the incomplete Fourier transform method and Green's function, the integral equation formulations for the target bond prices corresponding to the three cases are all presented, and the details of their numerical implementation are further discussed. The newly derived formulations are shown to be very accurate, while they are much more efficient than the binomial tree method.

In Chapter 5, the reset feature is embedded into vanilla convertible bonds to formulate resettable convertible bonds. To value these contracts under the B-S model, a closed PDE system is established, and applying the incomplete Fourier transform leads to an integral equation formulation for the bond price. This new formulation
involves an unknown optimal conversion boundary, which is solved after an appropriate numerical scheme is designed. It is also shown through a rigorous theoretical proof that the price of a resettable convertible bond is not always a monotonically increasing function of the underlying asset price, and this quite amazing phenomenon is further illustrated through some numerical examples.

Lastly, stochastic volatility and interest rate are respectively introduced into the B-S model for the valuation of vanilla convertible bonds in Chapter 6. We establish a new predictor-corrector scheme, which involves no embedded iterations, so that the entire optimal conversion boundary can be simultaneously determined, together with the bond prices. The reliability of this approach is tested through the comparison with different existing approaches, and it is also shown that our approach is far more efficient than the Monte-Carlo method. Interestingly, it is demonstrated through numerical experiments that the optimal conversion boundary is not always an increasing function of the time to expiry; rather it will show a downward trend with respect to the time to expiry when the time to expiry reaches a critical value.
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Appendix A

Appendix for Chapter 3

A.1 Appendix A.1

In this appendix, we will present the detail of the computation process from System (3.7) to Equation (3.12). Under the definition of (3.8), we can transform System (3.7) to System (3.9), and here are several important processes. Since the incomplete Fourier transform operator is linear, we can obtain the following equation:

\[-\mathcal{F}\{\frac{\partial U}{\partial \tau}\} + \frac{1}{2}\sigma^2 \mathcal{F}\{\frac{\partial^2 U}{\partial x^2}\} + (r - D_0 - \frac{1}{2})\mathcal{F}\{\frac{\partial U}{\partial x}\} - r \mathcal{F}\{U\} = 0. \tag{A.1}\]

Now, we compute every term of Equation (A.1):

\[\mathcal{F}\{\frac{\partial U}{\partial \tau}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U}{\partial \tau}(x, \tau)e^{i\omega x}dx = \frac{\partial}{\partial \tau}\left[\int_{-\infty}^{\ln(S_c(\tau))} U(x, \tau)e^{i\omega x}dx\right] - \frac{S_c'(\tau)}{S_c(\tau)}U(\ln(S_c(\tau)), \tau)e^{i\omega \ln(S_c(\tau))}\]

\[\mathcal{F}\{\frac{\partial U}{\partial x}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U}{\partial x}(x, \tau)e^{i\omega x}dx = \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x}dU(x, \tau) = e^{i\omega \ln(S_c(\tau))}\left[nS_c(\tau) - Ze^{-r\tau}\right] - i\omega \hat{U}(\omega, \tau) \tag{A.2}\]

\[\mathcal{F}\{\frac{\partial^2 U}{\partial x^2}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial^2 U}{\partial x^2}(x, \tau)e^{i\omega x}dx = \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x}d\frac{\partial U}{\partial x}(x, \omega) = e^{i\omega \ln(S_c(\tau))}\left[nS_c(\tau) - Ze^{-r\tau}\right] - i\omega \hat{U}(\omega, \tau) \tag{A.3}\]
\[ = nS_c(\tau)e^{i\omega \ln(S_c(\tau))} - i\omega [nS_c(\tau) - Ze^{-r\tau}] - i\omega \hat{U}(\omega, \tau) \]
\[ = (1 - i\omega)nS_c(\tau)e^{i\omega \ln(S_c(\tau))} + i\omega Ze^{-r\tau}e^{i\omega \ln(S_c(\tau))} - \omega^2 \mathcal{F}(\omega, \tau). \quad (A.4) \]

Apply these three results to Equation (A.1), we can obtain the following equation:
\[
\frac{\partial \hat{U}}{\partial \tau}(\omega, \tau) + \left[ \frac{1}{2}\sigma^2 \omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r \right] \hat{U}(\omega, \tau) = (nS_c(\tau) - Ze^{-r\tau})e^{i\omega \ln(S_c(\tau))}\left[ \frac{S_c'(\tau)}{S_c(\tau)} + (r - D_0 - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 i\omega \right] + \frac{1}{2}\sigma^2 e^{i\omega \ln(S_c(\tau))}nS_c(\tau). \quad (A.5) \]

Therefore, System (3.9) can be derived directly. Using the technique of the solution of ODE system, we can write the solution of System (3.9)
\[
\hat{U}(\omega, \tau) = \hat{U}(\omega, 0)e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi)e^{-B(\omega)(\tau - \xi)}d\xi. \quad (A.6) \]

Now, the integral equation formulation in the Fourier space has been presented. To obtain the integral equation formulation in the original space, the Fourier Inversion transform should be applied to Equation (A.6). And then we obtain
\[
U(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, \tau)d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0)e^{-B(\omega)\tau}d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_0^\tau f(\omega, \xi)e^{-B(\omega)(\tau - \xi)}d\xi d\omega \Delta I_1 + I_2. \quad (A.7) \]

Now, we compute \( I_1 \) and \( I_2 \) respectively, and \( I_1 \) first.
\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0)e^{-B(\omega)\tau}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) \cdot e^{-\left[ \frac{1}{2}\sigma^2 \omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r \right] \tau}d\omega \Delta \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) \cdot G(\omega, \tau)d\omega. \quad (A.8) \]

where \( G(\omega, \tau) = e^{-\left[ \frac{1}{2}\sigma^2 \omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r \right] \tau}. \)

In order to use the Convolution theorem \(^{[11]}\) to obtain the value of \( I_1 \), we should obtain the Fourier Inversion transform of \( G(\omega, \tau) \) first. Define
\[
g(x, \tau) \triangleq \mathcal{F}^{-1}\{G(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} G(\omega, \tau)d\omega
\]
Therefore, $I_1$ can be expressed

$$I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda x} \Phi(\omega, 0) \cdot G(\omega, \tau) d\omega$$

$$= U(x, 0) \ast g(x, \tau)$$

$$= \int_{-\infty}^{\ln(S_c(0))} \max\{ne^u - Z, 0\} e^{-\lambda u} e^{-\frac{|(r-D_0 - \frac{1}{2}\sigma^2)(\tau-x)^2|}{2\sigma^2}} du. \quad (A.10)$$

Then, we compute $I_2$

$$I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda x} \int_{\infty}^{\infty} f(\omega, \xi) e^{B(\omega)(\tau-\xi)} d\xi d\omega$$

$$= \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda x} f(\omega, \xi) e^{B(\omega)(\tau-\xi)} d\omega d\xi$$

$$= \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda x - \frac{1}{2}\sigma^2\omega^2 + (r-D_0 - \frac{1}{2}\sigma^2)i\omega + r(\tau-\xi)} e^{i\omega\ln(S_c(\xi))}$$

$$\cdot \left\{ (nS_c(\xi) - Ze^{-r\xi}) \frac{S'_c(\xi)}{S_c(\xi)} + (r - D_0 - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 i\omega \right\} d\omega d\xi$$

$$= \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda x - \frac{1}{2}\sigma^2\omega^2 + (r-D_0 - \frac{1}{2}\sigma^2)i\omega + r(\tau-\xi)} e^{i\omega\ln(S_c(\xi))} \cdot \left\{ f_1(\xi) - f_2(\xi) \omega \right\} d\omega d\xi, \quad (A.11)$$

where

$$f_1(\xi) = (nS_c(\xi) - Ze^{-r\xi}) \frac{S'_c(\xi)}{S_c(\xi)} + (r - D_0 - \frac{1}{2}\sigma^2)$$

$$f_2(\xi) = \frac{1}{2}\sigma^2 i(nS_c(\xi) - Ze^{-r\xi}).$$

Now, we compute $I_2$:

$$I_2 = \int_{0}^{\tau} \frac{e^{-\lambda (\tau-\xi)}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\xi)\omega^2 - |(r-D_0 - \frac{1}{2}\sigma^2)(\tau-x) + x\ln(S_c(\xi))| i\omega} \cdot \left\{ f_1(\xi) - f_2(\xi) \omega \right\} d\omega d\xi$$

$$= \int_{0}^{\tau} \frac{e^{-\lambda (\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)\sigma}} \cdot e^{-\frac{|(r-D_0 - \frac{1}{2}\sigma^2)(\tau-\xi) + x\ln(S_c(\xi))|^2}{2\sigma^2(\tau-\xi)}} \cdot \left\{ (nS_c(\xi) - Ze^{-r\xi}) \right. $$

$$\cdot \left\{ \frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2} (r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau - \xi}) \right\} + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi, \quad (A.12)$$
where the following equation is used

\[
\int_{-\infty}^{\infty} e^{-\rho \omega^2 -q \omega^q} d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{\omega^2}{2p}}.
\]

Combining \(I_1\) and \(I_2\), \(U(x, \tau)\) can be expressed as

\[
U(x, \tau) = \int_{-\infty}^{\ln(S_c(0))} \max\{ne^{\nu} - Z, 0\} \frac{e^{-r\tau}}{\sqrt{2\pi r}} e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau - x)}{2\alpha^2}} du + \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau - \xi) + x - \ln(S_c(\xi))}{2\alpha^2(\tau - \xi)}} \cdot \left( nS_c(\xi) - Ze^{-r\xi} \right) \frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau - \xi}) + \frac{1}{2}n\sigma^2S_c(\xi) d\xi, \quad (A.13)
\]

which gives Equation (3.12).

### A.2 Appendix A.2

In this appendix, we will present the detail of the computation process from System (3.5) to Equation (3.18). First, we apply the incomplete Fourier transform (3.14) to System (3.5). Since incomplete Fourier transform operator is linear, we can obtain the following equation:

\[
-\mathcal{F}\left\{ \frac{\partial v_2}{\partial \tau} \right\} + \frac{1}{2}\sigma^2 \mathcal{F}\left\{ \frac{\partial^2 v_2}{\partial x^2} \right\} + (r - D_0 - \frac{1}{2}\sigma^2)\mathcal{F}\left\{ \frac{\partial v_2}{\partial x} \right\} - r\mathcal{F}\{v_2\} = 0. \quad (A.14)
\]

Now, we compute every term of Equation (A.14):

\[
\mathcal{F}\left\{ \frac{\partial v_2}{\partial \tau} (x, \tau) \right\} = \int_{\ln(S_c(\tau))}^{\ln(S_p(\tau))} \frac{\partial v_2}{\partial \tau} (x, \tau) e^{i\omega x} dx
\]

\[
= \frac{\partial}{\partial \tau} \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx - \frac{S'_c(\tau)}{S_c(\tau)} nS_c(\tau) e^{i\omega \ln(S_c(\tau))} + \frac{S'_p(\tau)}{S_p(\tau)} Me^{i\omega \ln(S_p(\tau))}, \quad (A.15)
\]

\[
\mathcal{F}\left\{ \frac{\partial v_2}{\partial x} (x, \tau) \right\} = \int_{\ln(S_c(\tau))}^{\ln(S_p(\tau))} \frac{\partial v_2}{\partial x} (x, \tau) e^{i\omega x} dx
\]

\[
= \int_{\ln(S_c(\tau))}^{\ln(S_p(\tau))} e^{i\omega x} dv_2(x, \tau)
\]

\[
= \frac{\partial v_2}{\partial x} (x, \tau) e^{i\omega x} \ln(S_c(\tau)) - i\omega \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx
\]

\[
= e^{i\omega \ln(S_c(\tau))} nS_c(\tau) - e^{i\omega \ln(S_p(\tau))} M - i\omega \hat{v}_2(\omega, \tau), \quad (A.16)
\]

\[
\mathcal{F}\left\{ \frac{\partial^2 v_2}{\partial x^2} (x, \tau) \right\} = \int_{\ln(S_c(\tau))}^{\ln(S_p(\tau))} \frac{\partial^2 v_2}{\partial x^2} (x, \tau) e^{i\omega x} dx
\]
We compute \( v_2(x, \tau) \) first.

\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) e^{-\frac{1}{2} \omega^2 + (r-D_0 - \frac{1}{2} \sigma^2) i\omega + r}(\tau - \tau_M) d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) \cdot G(\omega, \tau) d\omega, \tag{A.21}
\]
where \( G(\omega, \tau) = e^{-\frac{1}{2} \sigma^2 \omega^2 + (r-D_0 - \frac{1}{2} \sigma^2) i \omega + r} \). To use the Convolution theorem here, we need to obtain the inverse Fourier transform of \( G(\omega, \tau) \). Define

\[
g(x, \tau) \triangleq G(\omega, \tau) = \mathcal{F}^{-1}\{ G(\omega, \tau) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-\frac{1}{2} \sigma^2 \omega^2 + (r-D_0 - \frac{1}{2} \sigma^2) i \omega + r} e^{-\frac{i}{2} \sigma^2 (\tau - \tau_M) \omega + i \omega + r} d\omega = \frac{1}{\sqrt{2\pi(\tau - \tau_M) \sigma}} e^{-\frac{1}{2} \sigma^2 (\tau - \tau_M)^2}.
\]

Then

\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \chi_1(\omega, \tau_M) G(\omega, \tau) d\omega = v_1(x, \tau) * g(x, \tau) = \int_{\ln(S_c(\tau_M))}^{\ln(S_p(\tau_M))} v_1(u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sqrt{2\pi(\tau - \tau_M) \sigma}} e^{-\frac{1}{2} \sigma^2 (\tau - \tau_M)^2} du.
\]

Now, we compute \( I_2 \)

\[
I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_{0}^{\tau - \tau_M} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\xi d\omega = \int_{0}^{\tau - \tau_M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\omega d\xi = \int_{0}^{\tau - \tau_M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - \frac{1}{2} \sigma^2 \omega^2 + (r-D_0 - \frac{1}{2} \sigma^2) i \omega + r} \omega d\omega d\xi \cdot \{ nS_c(\tau_M + \xi) e^{i\omega \ln(S_c(\tau_M + \xi))} \} [ \frac{S_c'((\tau_M + \xi))}{S_c(\tau_M + \xi)} - \frac{1}{2} \sigma^2 i \omega + r - D_0 ] - Me^{i\omega \ln(S_p(\tau_M + \xi))} \frac{S_p'((\tau_M + \xi))}{S_p(\tau_M + \xi)} - \frac{1}{2} \sigma^2 i \omega + r - D_0 - \frac{1}{2} \sigma^2 \} d\omega d\xi
\]

\[
\triangleq \int_{0}^{\tau - \tau_M} nS_c(\tau_M + \xi) e^{-r(\tau - \tau_M - \xi)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sigma^2 (\tau - \tau_M - \xi)^2} - (r-D_0 - \frac{1}{2} \sigma^2) (\tau - \tau_M - \xi) + x - \ln(S_c(\tau_M + \xi)) i \omega \cdot \{ f_1(\xi) - f_2(\xi) \omega \} d\omega d\xi
\]

\[
- \int_{0}^{\tau - \tau_M} Me^{-r(\tau - \tau_M - \xi)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sigma^2 (\tau - \tau_M - \xi)^2} - (r-D_0 - \frac{1}{2} \sigma^2) (\tau - \tau_M - \xi) + x - \ln(S_p(\tau_M + \xi)) i \omega \cdot \{ f_3(\xi) - f_4(\xi) \omega \} d\omega d\xi,
\]

(A.24)
where

\[
\begin{align*}
  f_1(\xi) &= \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0, \\
  f_2(\xi) &= \frac{1}{2} \sigma_i^2, \\
  f_3(\xi) &= \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + r - D_0 - \frac{1}{2} \sigma_i^2, \\
  f_4(\xi) &= \frac{1}{2} \sigma_i^2.
\end{align*}
\]

So, \( I_2 \) can be computed as

\[
\begin{align*}
  I_2 &= \int_0^{\tau_M} \frac{nS_c(\tau_M + \xi)}{2\pi} e^{-r(\tau - \tau_M - \xi)} \\
  &\quad \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \omega^2 \cdot \left| \frac{1}{2} \sigma_i^2 \right| \omega \cdot f_1(\xi) - f_2(\xi) \omega \right) d\omega d\xi \\
  &\quad - \int_0^{\tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{2\pi} \cdot e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| \omega \cdot f_3(\xi) - f_4(\xi) \omega \right) d\omega d\xi \\
  &= \int_0^{\tau_M} \frac{nS_c(\tau_M + \xi)}{2\pi(\tau - \tau_M - \xi) \sigma} \cdot e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| \omega \cdot f_1(\xi) - f_2(\xi) \omega \right) d\omega d\xi \\
  &\quad - \int_0^{\tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{2\pi(\tau - \tau_M - \xi) \sigma} \cdot e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| \omega \cdot f_3(\xi) - f_4(\xi) \omega \right) d\omega d\xi,
\end{align*}
\]

(A.25)

where the follow equation is used:

\[
\int_{-\infty}^{\infty} e^{-p\omega^2 + q\omega} \omega^\nu d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{q^2}{2p}}.
\]

Combining \( I_1 \) and \( I_2 \), the expression of \( v_2(x, \tau) \) is displayed as

\[
\begin{align*}
  v_2(x, \tau) &= \int_{-\infty}^{\infty} \left( \ln(S_c(\tau_M)) \right)^n V_1(u, \tau_M) e^{-r(\tau - \tau_M)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| (\tau - \tau_M + x - u) \right) d\omega \\
  &\quad + \int_0^{\tau_M} \frac{nS_c(\tau_M + \xi)}{2\pi(\tau - \tau_M - \xi) \sigma} \cdot e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| (\tau - \tau_M + x - \ln(S_c(\tau_M + \xi))) d\omega d\xi \\
  &\quad - \int_0^{\tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{2\pi(\tau - \tau_M - \xi) \sigma} \cdot e^{-\frac{1}{2} \sigma_i^2(\tau - \tau_M - \xi)} \cdot \left| \frac{1}{2} \sigma_i^2 \right| (\tau - \tau_M + x - \ln(S_p(\tau_M + \xi))) d\omega d\xi.
\end{align*}
\]
\[ \frac{S_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \{ r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - x}{\tau - \tau_M - \xi} \} \} d\xi. \]  

(A.26)

### A.3 Appendix A.3

Now, we present the detail of computing Equation (3.22) from Equation (3.13). First, we rewrite Equation (3.13) as follows

\[
V_1(S, \tau) = \int_0^\tau \frac{e^{-(r\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)\sigma}} e^{-\frac{(r - D_0 - \frac{1}{2} \sigma^2)\ln(S_c(\xi)) + \ln(S)}{2\sigma^2(\tau - \xi)}} \cdot \{ (nS_c(\xi) - Ze^{-r\xi}) \\
\cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2} \{ r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi} \} \right] + \frac{1}{2} n \sigma^2 S_c(\xi) \} d\xi + Ze^{-r\tau}. \]  

(A.27)

It should be noted that the first term of Equation (3.13) is missing. Actually, it is always zero, since \( S_c(0) = \frac{Z}{n} \). Now, we define

\[
h(S, \xi) \triangleq \frac{[(r - D_0 - \frac{1}{2} \sigma^2) (\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]^2}{2\sigma^2(\tau - \xi)} \]

\[
= \frac{1}{2(\tau - \xi)} \left[ \frac{(r - D_0 - \frac{1}{2} \sigma^2) (\tau - \xi) - \ln(S_c(\xi)) + \ln(S)}{\sigma} \right]^2
\]

\[
= \frac{1}{\sigma} \left[ \frac{(r - D_0 - \frac{1}{2} \sigma^2) \tau + \ln(S)}{\sigma} - \frac{(r - D_0 - \frac{1}{2} \sigma^2) \xi + \ln(S_c(\xi))}{\sigma} \right]^2
\]

\[
\triangleq \frac{1}{2(\tau - \xi)} \left[ y - P(\xi) \right]^2,
\]  

(A.28)

where \( y = \frac{(r - D_0 - \frac{1}{2} \sigma^2) \tau + \ln(S)}{\sigma} \) and \( P(\xi) = \frac{(r - D_0 - \frac{1}{2} \sigma^2) \xi + \ln(S_c(\xi))}{\sigma} \). It can be found that \( P'(\xi) = \frac{1}{\sigma} \left[ \frac{S'_c(\xi)}{S_c(\xi)} \right] + r - D_0 - \frac{1}{2} \sigma^2 \). Therefore, we apply it to Equation (A.27) and obtain

\[
V_1(S, \tau) = \int_0^\tau \frac{e^{-(r\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)\sigma}} e^{-\frac{(r - D_0 - \frac{1}{2} \sigma^2)\ln(S_c(\xi)) + \ln(S)}{2\sigma^2(\tau - \xi)}} \cdot \{ (nS_c(\xi) - Ze^{-r\xi}) \\
\cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2} \{ r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi} \} \right] + \frac{1}{2} n \sigma^2 S_c(\xi) \} d\xi + Ze^{-r\tau}
\]

\[
= \int_0^\tau \frac{e^{-(r\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)\sigma}} e^{-\frac{[y - P(\xi)]^2}{2(\tau - \xi)} \cdot \{ (nS_c(\xi) - Ze^{-r\xi}) \\
\cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} \{ r - D_0 - \frac{1}{2} \sigma^2 - \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi} \} \right] \right] + \frac{1}{2} n \sigma^2 S_c(\xi) \} d\xi + Ze^{-r\tau}
\]
\[ R_1(S, \tau) - ZR_2(S, \tau) + Ze^{-rT}, \quad (A.29) \]

where

\[ R_1(S, \tau) = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi(\tau - \xi)}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} \cdot nS_c(\xi) \left( \frac{\sigma}{2} + P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right) d\xi. \]

\[ R_2(S, \tau) = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi(\tau - \xi)}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} \cdot e^{-r\xi} [P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)}] d\xi. \]

Now, we compute \( R_1(S, \tau) \) and \( R_2(S, \tau) \), respectively.

\[ R_1(S, \tau) = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi(\tau - \xi)}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} \cdot nS_c(\xi) \left( \frac{\sigma}{2} + P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right) d\xi. \]

\[ = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} e^{\frac{(y - P(\xi))(\sigma + \sigma^2(\tau - \xi))}{2\tau - \xi} \cdot nS_c(\xi) \left( \frac{\sigma}{2} + P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right) d\xi. \]

\[ = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} \cdot nS_c(\xi) \left( \frac{\sigma(\tau - \xi) + 2P'(\xi)(\tau - \xi) - y + P(\xi)}{2(\tau - \xi)} \sqrt{\tau - \xi} \right) d\xi. \]

\[ = \int_0^\tau \frac{e^{-rT\tau}}{\sqrt{2\pi}} e^{-\frac{(y - P(\xi))^2}{2(\tau - \xi)}} \cdot nS_c(\xi) \left( \frac{y - \sigma(\tau - \xi) + 2P'(\xi)(\tau - \xi) - 2\sigma(\tau - \xi)}{2(\tau - \xi)} \sqrt{\tau - \xi} \right) d\xi. \]
\[
\begin{align*}
R_2(S, \tau) &= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot e^{-\frac{r(\xi)}{2(\tau-\xi)}} \cdot e^{-r(\xi)} [P'(\xi) - \frac{y - P(\xi)}{2(\tau-\xi)}] d\xi \\
&= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot \left[ \frac{2P'(\xi)(\tau-\xi) - y + P(\xi)}{2(\tau-\xi)} \right] d\xi \\
&= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot \left[ \frac{P'(\xi) \cdot 2(\tau-\xi) + (P(\xi)-y)}{2(\tau-\xi)} \right] d\xi \\
&= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot \left[ \frac{P'(\xi) \sqrt{\tau-\xi} + \frac{P(\xi)-y}{2\sqrt{\tau-\xi}}}{\sqrt{\tau-\xi}} \right] d\xi \\
&= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot \left[ \frac{\partial}{\partial \xi} \left( \frac{y-P(\xi)}{\sqrt{\tau-\xi}} \right) \right] d\xi \\
&= \int_0^\tau e^{-r(\tau-\xi)} \frac{e^{\frac{|P(\xi)-\sigma(\tau-\xi)(\xi)|}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)^2}} \cdot \left[ \frac{\partial}{\partial \xi} \left( \frac{y-P(\xi)}{\sqrt{\tau-\xi}} \right) \right] d\xi.
\end{align*}
\]
Thus, we substitute the expression of \(R_1(S, \tau)\) and \(R_2(S, \tau)\) into Equation (A.29).

Equation (3.22) can be obtained

\[
V_1(S, \tau) = R_1(S, \tau) - ZR_2(S, \tau) + Ze^{-rt} = \int_0^\tau -nSe^{-D_0(\tau-\xi)} \frac{\partial}{\partial \xi} N(y-P(\xi) + \sigma(\tau-\xi)) d\xi
\]

\[+ Z \int_0^\tau e^{-r(\tau-\xi)} \frac{\partial}{\partial \xi} N(y-P(\xi)) d\xi + Ze^{-rt} = \int_0^\tau -nSe^{-D_0(\tau-\xi)}dN(y-P(\xi) + \sigma(\tau-\xi)) d\xi + Ze^{-rt}
\]

\[= -nSe^{-D_0(\tau-\xi)}N(y-P(\xi) + \sigma(\tau-\xi)) \bigg|_{\xi=\tau}
\]
the same method mentioned in Appendix A.3. First, we rewrite Equation (3.21) from Equation (3.18), using the same method mentioned in Appendix A.3. First, we rewrite Equation (3.18) as follows

\[ +nSe^{-D_0(\tau-\xi)}\mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)|_{\xi=0} \]

\[ +\int_0^1 nD_0Se^{-D_0(\tau-\xi)}\mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi \]

\[ +Ze^{-r\tau}\mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)|_{\xi=\tau}-Ze^{-r\tau}\mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)|_{\xi=0} + Ze^{-r\tau} \]

\[ = -nS\mathcal{N}\left(\frac{\ln(S)-\ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right)|_{\xi=\tau} \]

\[ +nSe^{-D_0\tau}\mathcal{N}\left(\frac{r-D_0+\frac{1}{2}\sigma^2\tau-\ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \]

\[ +\int_0^1 nD_0Se^{-D_0(\tau-\xi)}\mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)d\xi \]

\[ +Ze^{-r\tau}\mathcal{N}\left(\frac{\ln(S)-\ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right)|_{\xi=\tau} \]

\[ = nSe^{-D_0\tau}\mathcal{N}\left(\frac{r-D_0+\frac{1}{2}\sigma^2\tau-\ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \]

\[ -Ze^{-r\tau}\mathcal{N}\left(\frac{r-D_0-\frac{1}{2}\sigma^2(\tau-\xi)-\ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}+\ln(S)\right) \]

\[ +\int_0^1 nD_0Se^{-D_0(\tau-\xi)} \]

\[ \cdot \mathcal{N}\left(\frac{r-D_0+\frac{1}{2}\sigma^2(\tau-\xi)-\ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}+\ln(S)\right)d\xi \]

\[ -1_{S=S_c(\tau)}(nS-Ze^{-r\tau}) + Ze^{-r\tau} \], \hspace{1cm} (A.32) \]

where

\[ 1_{S=S_c(\tau)} = \begin{cases} 
\frac{1}{2}, & S = S_c(\tau), \\
0, & S < S_c(\tau). 
\end{cases} \hspace{1cm} (A.33) \]

A.4 Appendix A.4

Now, we give the detail of computing Equation (3.23) from Equation (3.18), using the same method mentioned in Appendix A.3. First, we rewrite Equation (3.18) as follows

\[ V_2(S, \tau) = \int_{\ln(S_c(\tau_M))}^{\ln(S_c(\tau_M + \xi))} \frac{e^{r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{1}{2}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-\ln(S_c(\tau_M + \xi))}{2\sigma^2(\tau-\tau_M)}\right)^2 \hspace{1cm} (A.32) \]

\[ + \int_0^{\tau-\tau_M} nS_c(\tau_M + \xi) e^{-r(\tau-\tau_M-\xi)} \frac{1}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{1}{2}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M + \xi)))}{2\sigma^2(\tau-\tau_M-\xi)}\right)^2} \hspace{1cm} (A.33) \]

\[ + \int_0^{\tau-\tau_M} nS_c(\tau_M + \xi) e^{-r(\tau-\tau_M-\xi)} \frac{1}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{1}{2}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M + \xi)))}{2\sigma^2(\tau-\tau_M-\xi)}\right)^2} \hspace{1cm} (A.33) \]
\[
\frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[ r - D_0 + \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] d\xi \\
= \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{1}{2} \left[ (r - D_0) - \frac{1}{2} \sigma^2 \right] \left[ (\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi)) \right]^2}{2\sigma^2(\tau - \tau_M - \xi)} d\xi
\]

where

\[
R_1(S, \tau) = \int_0^{\tau - \tau_M} \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[ r - D_0 + \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] d\xi,
\]

\[
R_2(S, \tau) = \int_0^{\tau - \tau_M} \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[ r - D_0 + \frac{1}{2} \sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] d\xi.
\]

Now, we compute \(R_1(S, \tau)\) first. Define

\[
h_1(S, \xi) \triangleq \frac{\left[ (r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) \right]^2}{2\sigma^2(\tau - \tau_M - \xi)}
= \frac{1}{2(\tau - \tau_M - \xi)} \left[ (r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) \right]^2
= \frac{1}{2(\tau - \tau_M - \xi)} \left[ (r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S) - \frac{(r - D_0 - \frac{1}{2} \sigma^2)\xi + \ln(S_c(\tau_M + \xi))}{\sigma} \right]^2
= \frac{1}{2(\tau - \tau_M - \xi)} [y_1 - P_1(\xi)]^2,
\]

where

\[
y_1 = \frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma}\quad\text{and}\quad P_1(\xi) = \frac{(r - D_0 - \frac{1}{2} \sigma^2)\xi + \ln(S_c(\tau_M + \xi))}{\sigma}.
\]

It should be noted that \(P'_1(\xi) = \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{r - D_0 - \frac{1}{2} \sigma^2}{\sigma}\). Therefore, we can obtain

\[
R_1(S, \xi) = \int_0^{\tau - \tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{1}{2} \left[ (r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) \right]^2}{2\sigma^2(\tau - \tau_M - \xi)} d\xi
\]
\[\begin{align*}
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sigma\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_1 - P(\xi)]^2}{2(\tau - \tau_M - \xi)}} \, d\xi \\
&\cdot \left\{ \frac{S_c'(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0 + \frac{1}{2}\sigma^2 - \frac{1}{2} [r - D_0 + \frac{1}{2}\sigma^2 - \ln(S_c(\tau_M + \xi)) - \ln(S) - \frac{\sigma^2}{2}] \right\} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sigma\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_1 - P(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \frac{S_c'(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2 - \frac{1}{2} [r - D_0 - \frac{1}{2}\sigma^2 - \ln(S_c(\tau_M + \xi)) - \ln(S) - \frac{\sigma^2}{2}] \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_1 - P(\xi)]^2}{2(\tau - \tau_M - \xi)} + \frac{\sigma^2}{2}} \cdot [P_1'(\xi) - \frac{y_1 - P(\xi)}{2(\tau - \tau_M - \xi)} + \frac{\sigma}{2}] \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_1 - P(\xi)]^2}{2(\tau - \tau_M - \xi)} + \frac{\sigma^2}{2}} \cdot \frac{2P_1'(\xi)(\tau - \tau_M - \xi) - y_1 + P(\xi) + \sigma(\tau - \tau_M - \xi)}{2(\tau - \tau_M - \xi)} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)} + (r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) + \frac{\sigma^2}{2}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)^2}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)^2}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nS_c(\tau_M + \xi)e^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)^2}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nSe^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)^2}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi \\
&= \int_0^\tau \frac{nSe^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot \frac{\sigma(\tau - \tau_M - \xi)^2}{\sqrt{2(\tau - \tau_M - \xi)}} \, d\xi. \quad (A.36)
\end{align*}\]

Now, we compute \(R_2(S, \tau)\). Define

\[h_2(S, \xi) \triangleq \frac{\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\frac{2\sigma^2}{\sigma}(\tau - \tau_M - \xi)}}\]

\[\begin{align*}
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[ \frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\frac{2\sigma^2}{\sigma}(\tau - \tau_M - \xi)} \right]^2 \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[ \frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\frac{2\sigma^2}{\sigma}(\tau - \tau_M - \xi)} \right]^2 \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[ \frac{\sqrt{2 - \frac{\sigma^2}{\sigma}}(\tau - \tau_M - \xi) + \ln(S_c(\tau_M + \xi))}{\frac{2\sigma^2}{\sigma}(\tau - \tau_M - \xi)} \right]^2 \\
&\triangleq \frac{1}{2(\tau - \tau_M - \xi)} [\sqrt{2 - \frac{\sigma^2}{\sigma}}(\tau - \tau_M - \xi) + \ln(S_c(\tau_M + \xi))]^2, \quad (A.37)
\end{align*}\]
where \( y_2 = \frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma} \) and \( P_2(\xi) = \frac{(r - D_0 - \frac{1}{2} \sigma^2)\xi + \ln(S_p(\tau_M + \xi))}{\sigma} \).

It should be noted that \( P_2(\xi) = \frac{S_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + r - D_0 - \frac{1}{2} \sigma^2 \). Therefore, we can obtain

\[
R_2(S, \xi) = \int_0^{\tau - \tau_m} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{(y_2 - P_2(\xi))^2}{2(\tau - \tau_M - \xi)^2} \cdot \left\{ \frac{2P_2(\xi)}{2(\tau - \tau_M - \xi)} - \frac{1}{2(\tau - \tau_M - \xi)^2} \right\} d\xi
\]

\[
= \int_0^{\tau - \tau_m} - \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{(y_2 - P_2(\xi))^2}{2(\tau - \tau_M - \xi)^2} \cdot \left\{ \frac{\partial}{\partial \xi} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) \right\} d\xi.
\]

Now, we substitute the expression of \( R_1(S, \tau) \) and \( R_2(S, \tau) \) into \( V_2(S, \tau) \), we obtain

\[
V_2(S, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_1(e^{\mu}, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{2\sigma^2(\tau - \tau_M)}} du
\]

\[
+ R_1(S, \tau) - R_2(S, \tau)
\]

\[
= \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_1(e^{\mu}, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{2\sigma^2(\tau - \tau_M)}} du
\]

\[
- \int_0^{\tau - \tau_M} nSe^{-D_0(\tau - \tau_M - \xi)} \frac{\partial}{\partial \xi} \mathcal{N} \left( \frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi
\]

\[
+ \int_0^{\tau - \tau_M} Me^{-r(\tau - \tau_M - \xi)} \frac{\partial}{\partial \xi} \mathcal{N} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi
\]

\[
= \int_{\ln(S_p(\tau_M))}^{\ln(S)} V_1(e^{\mu}, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S)}{2\sigma^2(\tau - \tau_M)}} du
\]

\[
- \int_0^{\tau - \tau_M} nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi.
\]
\[
\begin{align*}
&+ \int_0^{\tau - \tau_M} Me^{-r(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) \\
&= \int_{\ln(S_c(\tau_M))}^{\ln(S_p(\tau_M))} V_1(e^\mu, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-(r-D_0) \frac{1}{2}\sigma^2(\tau - \tau_M + \ln(S) - \mu^2)} du \\
&\quad - nSe^{-D_0(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = \tau - \tau_M} \\
&\quad + nSe^{-D_0(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = 0} \\
&\quad + \int_0^{\tau - \tau_M} nD_0Se^{-D_0(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&\quad - Me^{-r(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = \tau - \tau_M} \\
&\quad - Me^{-r(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = 0} \\
&\quad - \int_0^{\tau - \tau_M} rMe^{-r(\tau - \tau_M - \xi)} \cdot d\mathcal{N} \left( \frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&= \int_{\ln(S_c(\tau_M))}^{\ln(S_p(\tau_M))} V_1(e^\mu, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-(r-D_0) \frac{1}{2}\sigma^2(\tau - \tau_M + \ln(S) - \mu^2)} du \\
&\quad - nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = \tau - \tau_M} \\
&\quad + nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = \tau - \tau_M} \\
&\quad - Me^{-r(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = \tau - \tau_M} \\
&\quad - Me^{-r(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) |_{\xi = 0} \\
&\quad + \int_0^{\tau - \tau_M} nD_0Se^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&\quad + nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&\quad - nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \mathcal{N} \left( \frac{\ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma \sqrt{\tau - \tau_M - \xi}} \right) d\xi
\end{align*}
\]
\[ N'(r - D_0 + \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) + 1 \]
\[ \sigma \sqrt{\tau - \tau_M - \xi} d\xi \]
\[ -Me^{-r(\tau - \tau_M)}N'(r - D_0 - \frac{1}{2} \sigma^2)(\tau - \tau_M) + \ln(S) - \ln(S_p(\tau_M)) \]
\[ \sigma \sqrt{\tau - \tau_M} \]
\[ + M \cdot 1_{S=S_p}(\tau) = \int_0^{\tau - \tau_M} rMe^{-r(\tau - \tau_M - \xi)} \]
\[ N'(r - D_0 + \frac{1}{2} \sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi)) \]
\[ \sigma \sqrt{\tau - \tau_M - \xi} d\xi, \quad (A.39) \]

where

\[ 1_{S=S_c}(\tau) = \begin{cases} 
\frac{1}{2} & S = S_c(\tau), \\
0 & S < S_c(\tau),
\end{cases} \quad (A.40) \]

\[ 1_{S=S_p}(\tau) = \begin{cases} 
\frac{1}{2} & S = S_p(\tau), \\
1 & S > S_p(\tau).
\end{cases} \quad (A.41) \]
Appendix B

Appendix for Chapter 4

B.1 Appendix B.1

In this appendix, we present the details of applying the incomplete Fourier transform to the PDE system (4.6). Firstly, it should be noted that the incomplete Fourier transform operate is a linear transform. Thus, when we apply it to the PDE, the following equation can be obtained:

\[-\mathcal{F}\{\frac{\partial U_1}{\partial \tau}\} + \frac{1}{2} \sigma^2 \mathcal{F}\{\frac{\partial^2 U_1}{\partial x^2}\} + (r - D_0 - \frac{1}{2} \sigma^2) \mathcal{F}\{\frac{\partial U_1}{\partial x}\} - r \mathcal{F}\{U_1\} = 0. \] (B.1)

Now, we compute every term of the Equation (B.1):

\[
\mathcal{F}\{\frac{\partial U_1}{\partial \tau}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U_1}{\partial \tau}(x, \tau) \cdot e^{i\omega x} dx
\]
\[
= \frac{\partial}{\partial \tau} \left[ \int_{-\infty}^{\ln(S_c(\tau))} U_1(x, \tau) \cdot e^{i\omega x} dx \right] - \frac{S'_c(\tau)}{S_c(\tau)} U_1(\ln(S_c(\tau)), \tau) e^{i\omega \ln(S_c(\tau))}
\]
\[
= \frac{\partial U_1}{\partial \tau}(\omega, \tau) - \frac{S'_c(\tau)}{S_c(\tau)} (nS_c(\tau) - Ze^{-r\tau}) e^{i\omega \ln(S_c(\tau))}, \quad \text{ (B.2)}
\]

\[
\mathcal{F}\{\frac{\partial U_1}{\partial x}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U_1}{\partial x}(x, \tau) \cdot e^{i\omega x} dx
\]
\[
= \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x} du_1(x, \tau)
\]
\[
= U_1(x, \tau) e^{i\omega x |\ln(S_c(\tau))|} - i\omega \int_{-\infty}^{\ln(S_c(\tau))} U_1(x, \tau) \cdot e^{i\omega x} dx
\]
\[
= (nS_c(\tau) - Ze^{-r\tau}) e^{i\omega \ln(S_c(\tau))} - i\omega \hat{U}_1(\omega, \tau), \quad \text{ (B.3)}
\]

\[
\mathcal{F}\{\frac{\partial^2 U_1}{\partial x^2}\} = \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial^2 U_1}{\partial x^2}(x, \tau) \cdot e^{i\omega x} dx
\]
\[
= \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x} \frac{\partial U_1}{\partial x}(x, \tau)
\]
\[
= \frac{\partial U_1}{\partial x}(x, \tau) e^{i\omega x |\ln(S_c(\tau))|} - i\omega \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U_1}{\partial x}(x, \tau) \cdot e^{i\omega x} dx
\]
Substituting these three results into Equation (B.1), the following equation can be obtained:

\[
\frac{\partial \hat{U}_1}{\partial \tau}(\omega, \tau) + \left[ \frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2) \omega + r \right] \hat{U}_1(\omega, \tau) = \left( nS_c(\tau) - Ze^{-rt} \right) e^{i\omega \ln(S_c(\tau))} \left( \frac{S_c'(\tau)}{S_c(\tau)} \right) + (r - D_0 - \frac{1}{2} \sigma^2) \frac{1}{2} \sigma^2 i\omega + \frac{1}{2} \sigma^2 e^{i\omega \ln(S_c(\tau))} nS_c(\tau).
\]

(B.5)

Then, System (4.8) is obtained.

B.2 Appendix B.2

In this appendix, the Fourier inversion transform is considered. Firstly, we rewrite Equation (4.11) as follows

\[
\hat{U}_1(\omega, \tau) = \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi) \cdot e^{-g(\omega)(\tau - \xi)} d\xi.
\]

(B.6)

Then, the Fourier inversion transform should be applied to this equation, and the following formulation can be obtained

\[
U_1(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} d\xi \cdot e^{-i\omega x} d\omega
\]

\[= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\tau f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} \cdot e^{-i\omega x} d\xi d\omega
\]

\[\triangleq I_1 + I_2,
\]

(B.7)

where \(I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega\) and \(I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^\tau f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} \cdot e^{-i\omega x} d\xi d\omega\).

Now, we calculate \(I_1\) and \(I_2\), respectively, and \(I_1\) first

\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega
\]

\[= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-[\frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2) i\omega + r] \tau} \cdot e^{-i\omega x} d\omega
\]

\[\triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-i\omega x} \cdot G(\omega, \tau) d\omega,
\]

(B.8)
Then, we compute

\[ I = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \cdot G(\omega, \tau) d\omega \]

Thus, \( I_1 \) can be rewritten

\[ I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_1(\omega, 0) \cdot e^{-i\omega x} \cdot G(\omega, \tau) d\omega = U_1(x, 0) \ast g(x, \tau) = \int_{-\infty}^{\ln(S(0))} \min\{K - Z, \max\{n\varepsilon - Z, 0\}\} \cdot e^{-\frac{r\tau}{\sqrt{2\pi\sigma^2}}} \cdot e^{-\frac{|\tau + x - \varepsilon|^2}{2\sigma^2}} d\tau. \]  

Then, we compute \( I_2 \)

\[ I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\tau} f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} \cdot e^{-i\omega x} d\xi d\omega \]

\[ = \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} \cdot e^{-i\omega x} d\omega d\xi \]

\[ = \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} \{(nS_c(\xi) - Ze^{-r\xi}) \cdot e^{i\omega \ln(S_c(\xi))} \cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 i\omega \right] + \frac{1}{2} \sigma^2 nS_c(\xi) \cdot e^{i\omega \ln(S_c(\xi))} \cdot e^{-B(\omega)(\tau - \xi)} \cdot e^{-i\omega x} d\omega d\xi \]

\[ = \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - \frac{1}{2}\sigma^2 x^2 + (r-D_0 - \frac{1}{2}\sigma^2 i\omega + r)(\tau - \xi) + i\omega \ln(S_c(\xi))} \cdot \{(nS_c(\xi) - Ze^{-r\xi}) \cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 i\omega \right] + \frac{1}{2} \sigma^2 nS_c(\xi) \} d\omega d\xi \]

\[ = \int_{0}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - \frac{1}{2}\sigma^2 x^2 + (r-D_0 - \frac{1}{2}\sigma^2 i\omega + r)(\tau - \xi) + i\omega \ln(S_c(\xi))} \cdot \{f_1(\xi) - f_2(\xi)\} d\omega d\xi, \]  

where

\[ f_1(\xi) = (nS_c(\xi) - Ze^{-r\xi}) \cdot \left[ \frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2} \sigma^2 \right] + \frac{1}{2} \sigma^2 nS_c(\xi), \]

\[ f_2(\xi) = \frac{1}{2} \sigma^2 (nS_c(\xi) - Ze^{-r\xi})i. \]
APPENDIX B. APPENDIX FOR CHAPTER 4

With the following equation used

\[ \int_{-\infty}^{\infty} e^{-p\omega^2 - q\omega} d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{q^2}{2}}} \]  

(B.14)

We can obtain

\[ I_2 = \int_0^{\tau} \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)\sigma^2}} \cdot e^{-\frac{[(r-D_0) - \frac{1}{2}\sigma^2(\tau - \xi) + \ln(S_c(\xi))]^2}{2\sigma^2(\tau - \xi)}} \cdot \left\{ (nS_c(\xi) - Ze^{-r\tau}) \right\} 
\cdot \left( \frac{S'_c(\xi)}{S_c(\xi)} \right) + \frac{1}{2} (r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - r}{\tau - \xi}) \right] + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi \]  

(B.15)

Combining \( I_1 \) and \( I_2 \), \( U_1 \) can be presented

\[ U_1(x, \tau) = \int_{-\infty}^{\ln(S_c(0))} \min\{K - Z, \max\{n\eta^\tau - Z, 0\}\} \frac{e^{-r\tau}}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{[(r-D_0) - \frac{1}{2}\sigma^2\tau + \ln(S_c(\xi))]^2}{2\sigma^2 \tau}} \cdot \left\{ (nS_c(\xi) - Ze^{-r\tau}) \right\} 
\cdot \left( \frac{S'_c(\xi)}{S_c(\xi)} \right) + \frac{1}{2} (r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - r}{\tau - \xi}) \right] + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi \]  

(B.16)

B.3 Appendix B.3

Now, we present the details of computing Equation (4.15) from Equation (4.14).

First, we rewrite Equation (4.14) as follows

\[ V_1(S, \tau) = \int_0^{\tau} \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)\sigma}} \cdot e^{-\frac{[(r-D_0) - \frac{1}{2}\sigma^2(\tau - \xi) + \ln(S_c(\xi))]^2}{2\sigma^2(\tau - \xi)}} \cdot \left\{ (nS_c(\xi) - Ze^{-r\tau}) \right\} 
\cdot \left( \frac{S'_c(\xi)}{S_c(\xi)} \right) + \frac{1}{2} (r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S_c(\xi))}{\tau - \xi}) \right] + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi + Ze^{-r\tau} \]  

(B.17)

It should be noted that the first term of Equation (4.14) is missing. Actually, it is always zero, due to \( S_c(0) = \frac{Z}{n} \). Now, we define

\[ h(S, \xi) \triangleq \frac{[(r-D_0) - \frac{1}{2}\sigma^2(\tau - \xi) - \ln(S_c(\xi)) + \ln(S_c(\xi))]}{2\sigma^2(\tau - \xi)} \]

\[ = \frac{1}{2(\tau - \xi)} \left[ (r-D_0) - \frac{1}{2}\sigma^2 \right] + \frac{\ln(S_c(\xi))}{\sigma} \]

\[ = \frac{1}{2(\tau - \xi)} \left[ (r-D_0) - \frac{1}{2}\sigma^2 \right] + \frac{\ln(S_c(\xi))}{\sigma} - \frac{(r-D_0) - \frac{1}{2}\sigma^2 \xi + \ln(S_c(\xi))}{\sigma} \]

\[ \triangleq \frac{1}{2(\tau - \xi)} [y - P(\xi)]^2 \]  

(B.18)
where $y = \frac{(r-D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S)}{\sigma}$ and $P(\xi) = \frac{(r-D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_c(\xi))}{\sigma}$. It can be found that $P'(\xi) = \frac{1}{\sigma} [S'_c(\xi) + r - D_0 - \frac{1}{2}\sigma^2]$. Therefore, we apply it to Equation (B.17) and obtain

$$V_1(S, \tau) = Z e^{-r\tau} + \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{2\sigma^2(\tau - \xi)} \cdot \left\{ n S_c(\xi) - Z e^{-r\xi} \right\} \right\} d\xi$$

$$= Z e^{-r\tau} + \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi$$

$$= Z e^{-r\tau} + \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi$$

$$= Z e^{-r\tau} + \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi$$

$$= \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi + Z e^{-r\tau}$$

$$= \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi + Z e^{-r\tau}$$

$$= \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi + Z e^{-r\tau}$$

$$\triangleq R_1(S, \tau) - Z R_2(S, \tau) + Ze^{-r\tau}, \quad (B.19)$$

where

$$R_1(S, \tau) = \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi \quad (B.20)$$

$$R_2(S, \tau) = \int_0^\tau \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi(\tau - \xi)}} \cdot \left\{ \frac{[r-D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]}{\sigma} + \frac{1}{2} n \sigma^2 S_c(\xi) \right\} d\xi. \quad (B.21)$$
Now, we compute $R_1(S, \tau)$ and $R_2(S, \tau)$, respectively.

\[
R_1(S, \tau) = \int_0^\tau e^{-r(t - \xi)} \frac{e^{-\frac{[y-P(\xi)]^2}{2(\tau - \xi)}}}{\sqrt{2\pi (\tau - \xi)}} \cdot nS_c(\xi) \frac{\sigma}{\tau} \cdot P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right] d\xi
\]

\[
= \int_0^\tau e^{-r(t - \xi)} \frac{e^{-\frac{[y-P(\xi)]^2}{2(\tau - \xi)}}}{\sqrt{2\pi (\tau - \xi)}} \cdot e^{(y-P(\xi))\sigma + \frac{\sigma^2}{2} \cdot \sigma \cdot nS_c(\xi) \frac{\sigma}{\tau} \cdot P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right] d\xi
\]

\[
= \int_0^\tau e^{-r(t - \xi) + (y-P(\xi))\sigma + \frac{\sigma^2}{2} \cdot \sigma \cdot nS_c(\xi) \frac{\sigma}{\tau} \cdot P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right] d\xi
\]

\[
= \int_0^\tau e^{-r(t - \xi) + (y-P(\xi))\sigma + \frac{\sigma^2}{2} \cdot \sigma \cdot nS_c(\xi) \frac{\sigma}{\tau} \cdot P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)} \right] d\xi
\]

\[
R_2(S, \tau) = \int_0^\tau e^{-r(t - \xi)} \frac{e^{-\frac{[y-P(\xi)]^2}{2(\tau - \xi)}}}{\sqrt{2\pi (\tau - \xi)}} \cdot e^{-r\xi} \cdot [P'(\xi) - \frac{y - P(\xi)}{2(\tau - \xi)}] d\xi
\]

\[
= \int_0^\tau e^{-r(t - \xi)} \frac{e^{-\frac{[y-P(\xi)]^2}{2(\tau - \xi)}}}{\sqrt{2\pi (\tau - \xi)}} \cdot \frac{\partial}{\partial \xi} \frac{y - P(\xi)}{\sqrt{\tau - \xi}} d\xi
\]

\[
= \int_0^\tau e^{-r(t - \xi)} \frac{e^{-\frac{[y-P(\xi)]^2}{2(\tau - \xi)}}}{\sqrt{2\pi (\tau - \xi)}} \cdot \frac{\partial}{\partial \xi} \frac{y - P(\xi)}{\sqrt{\tau - \xi}} d\xi
\]
Thus, we substitute the expression of $R_1(S, \tau)$ and $R_2(S, \tau)$ into Equation (B.19). Then, Equation (4.15) can be obtained

\[
V_1(S, \tau) = R_1(S, \tau) - ZR_2(S, \tau) + Z e^{-\tau}
\]

\[
= \int_0^\tau -nSe^{-D_0(\tau-\xi)} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi
\]

\[
+ Z \int_0^\tau e^{-\tau} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) d\xi + Ze^{-\tau}
\]

\[
= \int_0^\tau -nSe^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) + Z \int_0^\tau e^{-\tau} d\mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) + Ze^{-\tau}
\]

\[
= -nSe^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi)+\sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=\tau}
\]

\[
+ns\mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau} + Ze^{-\tau} \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau}
\]

\[
+Z e^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=\tau} - Ze^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=0} + Ze^{-\tau}
\]

\[
= -nS \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau} + Ze^{-\tau} \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau}
\]

\[
+Z e^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=\tau} - Ze^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=0} + Ze^{-\tau}
\]

\[
= -nS \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau} + Ze^{-\tau} \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau}
\]

\[
-1_{S=S_c(\tau)} (nS - Ze^{-\tau})
\]

\[
+ ns \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma \sqrt{\tau-\xi}}\right)_{\xi=\tau} + Ze^{-\tau}
\]

\[
+ Z e^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=\tau} - Ze^{-\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right)_{\xi=0} + Ze^{-\tau},
\]

(B.24)

where

\[
1_{S=S_c(\tau)} = \begin{cases} \frac{1}{2} & S = S_c(\tau), \\ 0 & S < S_c(\tau). \end{cases} \]

(B.25)
B.4 Appendix B.4

In this appendix, Green's function is used to obtain Equation (4.16) from System (4.3). Firstly, we rewrite the system here

\[
\begin{align*}
-\frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0) S \frac{\partial V_1}{\partial S} - rV_1 &= 0, \\
V_1(S, \tau^+) &= V_1(S, \tau^-), \\
V_1 \left( \frac{K}{n}, \tau \right) &= K, \\
V_1(0, \tau) &= Ze^{-r\tau}.
\end{align*}
\]  

(B.26)

A classical transform should be applied to this system to obtain a dimensionless PDE system, mentioned before

\[
x = \log(S), \quad v_1(x, \tau) = V_1(S, \tau),
\]

then, we obtain

\[
\begin{align*}
-\frac{\partial v_1}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial v_1}{\partial x} - rv_1 &= 0, \\
v_1(x, \tau^+) &= v_1(x, \tau^-), \\
v_1 \left( \ln \left( \frac{K}{n} \right), \tau \right) &= K, \\
v_1(-\infty, \tau) &= Ze^{-r\tau}.
\end{align*}
\]  

(B.27)

To apply the Green’s function method to the PDE system, we should make sure that the boundary condition at negative infinity is equal to zero. Thus, a simple transform can reach to it

\[
U(x, \tau) = v_1(x, \tau) - Ze^{-r\tau},
\]  

(B.28)

and then

\[
\begin{align*}
-\frac{\partial U}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial U}{\partial x} - rU &= 0, \\
U(x, \tau^+) &= U(x, \tau^-), \\
U \left( \ln \left( \frac{K}{n} \right), \tau \right) &= K - Ze^{-r\tau}, \\
U(-\infty, \tau) &= 0.
\end{align*}
\]  

(B.29)

Next step, this PDE system is transformed to a normal one using the transform

\[
W(x, \tau) = e^{\frac{r-D_0-\frac{1}{2}\sigma^2}{\sigma^2}x + \frac{(r-D_0-\frac{1}{2}\sigma^2)^2+2r\sigma^2}{2\sigma^4} \tau} \cdot U(x, \tau),
\]  

(B.30)
Then, we rewrite the form of integral equation with the original parameters

\[
\begin{aligned}
\frac{\partial W}{\partial \tau} &= \frac{1}{2} \sigma^2 \frac{\partial^2 W}{\partial x^2}, \\
W(x, \tau_K^n) &= W(x, \tau_K^n), \\
W(\ln(\frac{K}{n}), \tau) &= (K - Ze^{-r\tau})e^{-(r-D_0 - \frac{1}{2} \sigma^2 + 2r\sigma^2) + (r-D_0 - \frac{1}{2} \sigma^2 + 2r\sigma^2) + \ln(\frac{K}{n}) - (\ln(S) - \ln(n))}, \\
W(-\infty, \tau) &= 0.
\end{aligned}
\]  

(B.31)

By now, the Green’s function can be applied to the PDE system, and then the solution can be obtained

\[
W(x, \tau) = \frac{1}{\sqrt{2\pi\sigma^2(\tau - \tau_K)}} \int_0^\infty \left[ \exp\left(-\frac{[-x + \ln(\frac{K}{n}) - y]^2}{2\sigma^2(\tau - \tau_K)}\right) \right] \\
- \exp\left(-\frac{[-x + \ln(\frac{K}{n}) + y]^2}{2\sigma^2(\tau - \tau_K)}\right) \cdot W(\ln(\frac{K}{n}) - y, \tau_K)dy
\]

\[+ \int_0^{\tau - \tau_K} \frac{-x + \ln(\frac{K}{n})}{\sqrt{2\pi\sigma^2(\tau - \tau_K)}} \cdot \exp\left(-\frac{[-x + \ln(\frac{K}{n})]^2}{2\sigma^2(\tau - \tau_K - \xi)}\right) \cdot (K - Ze^{-r(\xi + \tau_K)}) \]

\[\cdot \exp\left(\frac{r - D_0 - \frac{1}{2} \sigma^2}{\sigma^2} \ln(\frac{K}{n}) + \frac{(r - D_0 - \frac{1}{2} \sigma^2)^2 + 2r\sigma^2}{2\sigma^2}(\xi + \tau_K)\right)d\xi.
\]

Then, we rewrite the form of integral equation with the original parameters

\[
V_1(S, \tau) = \frac{1}{\sqrt{2\pi\sigma^2(\tau - \tau_K)}} e^{-(r-D_0 - \frac{1}{2} \sigma^2 + 2r\sigma^2)(\tau - \tau_K)}
\]

\[
\cdot \int_{-\infty}^{\ln(\frac{K}{n})} \left[ \exp\left(-\frac{[-\ln(S) + y]^2}{2\sigma^2(\tau - \tau_K)}\right) - \exp\left(-\frac{[-\ln(S) + 2\ln(\frac{K}{n}) - y]^2}{2\sigma^2(\tau - \tau_K)}\right)\right]
\]

\[\cdot \exp\left(\frac{r - D_0 - \frac{1}{2} \sigma^2}{\sigma^2}(-\ln(S) + y)\right) \cdot (V_1(e^y, \tau_K) - Ze^{-r\tau_K})dy
\]

\[+ \int_0^{\tau - \tau_K} -\ln(S) + \ln(\frac{K}{n}) \cdot \exp\left(-\frac{[-\ln(S) + \ln(\frac{K}{n})]^2}{2\sigma^2(\tau - \tau_K - \xi)}\right) \cdot (K - Ze^{-r(\xi + \tau_K)})
\]

\[\cdot \exp\left(\frac{r - D_0 - \frac{1}{2} \sigma^2}{\sigma^2} \ln(\frac{K}{n}) + (r - D_0 - \frac{1}{2} \sigma^2)^2 + 2r\sigma^2(\xi + \tau_K - \tau)\right)d\xi
\]

\[+ Ze^{-r\tau},
\]  

(B.32)

it is Equation (4.16).

B.5 Appendix B.5

In this appendix, the details of applying the incomplete Fourier transform (4.18) to the PDE System (4.20) are displayed. Firstly, it should be noted that the incomplete Fourier transform operator is linear, and thus we can obtain the following equation:
Then, the ODE System (B.34) becomes

$$-\mathcal{F}\{ \frac{\partial v_1}{\partial \tau} \} + \frac{1}{2} \sigma^2 \mathcal{F}\{ \frac{\partial^2 v_1}{\partial x^2} \} + (r - D_0 - \frac{1}{2} \sigma^2) \mathcal{F}\{ \frac{\partial v_1}{\partial x} \} - r \mathcal{F}\{ v_1 \} = 0. \quad (B.33)$$

Now, we compute every term of Equation (B.33)

\[
\mathcal{F}\{ \frac{\partial v_1}{\partial \tau} (x, \tau) \} = \int_{\ln(S_P(\tau))}^{\ln(\xi)} \frac{\partial v_1}{\partial \tau}(x, \tau) \cdot e^{i\omega x} dx \\
= \frac{\partial}{\partial \tau} \left[ \int_{\ln(S_P(\tau))}^{\ln(\xi)} v_1(x, \tau) \cdot e^{i\omega x} dx \right] + \left( \frac{S_P'(\tau)}{S_P(\tau)} \right) v_1(\ln(S_P(\tau)), \tau) e^{i\omega \ln(S_P(\tau))} \\
= \frac{\partial \hat{v}_1}{\partial \tau}(\omega, \tau) + \frac{S_P'(\tau)}{S_P(\tau)} e^{i\omega \ln(S_P(\tau))}, \quad (B.34)
\]

\[
\mathcal{F}\{ \frac{\partial v_1}{\partial x} (x, \tau) \} = \int_{\ln(S_P(\tau))}^{\ln(\xi)} \frac{\partial v_1}{\partial x}(x, \tau) \cdot e^{i\omega x} dx \\
= \int_{\ln(S_P(\tau))}^{\ln(\xi)} e^{i\omega x} dv_1(x, \tau) \\
= v_1(x, \tau)e^{i\omega \ln(\xi)} - i\omega \int_{\ln(S_P(\tau))}^{\ln(\xi)} v_1(x, \tau) e^{i\omega x} dx \\
= Ke^{i\omega \ln(\xi)} - Me^{i\omega \ln(S_P(\tau))} - i\omega \hat{v}_1(\omega, \tau), \quad (B.35)
\]

\[
\mathcal{F}\{ \frac{\partial^2 v_1}{\partial x^2} (x, \tau) \} = \int_{\ln(S_P(\tau))}^{\ln(\xi)} \frac{\partial^2 v_1}{\partial x^2}(x, \tau) \cdot e^{i\omega x} dx \\
= \int_{\ln(S_P(\tau))}^{\ln(\xi)} \frac{\partial}{\partial x} \left( \frac{\partial v_1}{\partial x}(x, \tau) \cdot e^{i\omega x} dx \right) \\
= \frac{\partial v_1}{\partial x}(x, \tau)e^{i\omega \ln(\xi)} - i\omega \int_{\ln(S_P(\tau))}^{\ln(\xi)} \frac{\partial v_1}{\partial x}(x, \tau) \cdot e^{i\omega x} dx \\
= K \frac{A(\tau)}{n} e^{i\omega \ln(\xi)} - i\omega [Ke^{i\omega \ln(\xi)} - Me^{i\omega \ln(S_P(\tau))} - i\omega \hat{v}_1(\omega, \tau)] \\
= K \frac{A(\tau)}{n} e^{i\omega \ln(\xi)} - i\omega Ke^{i\omega \ln(\xi)} + i\omega Me^{i\omega \ln(S_P(\tau))} - \omega^2 \hat{v}_1(\omega, \tau). \quad (B.36)
\]

Substituting these three formulations into Equation (B.33), we can obtain

\[
\frac{\partial v_1}{\partial \tau}(\omega, \tau) + \left[ \frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2) i\omega + r \right] \hat{v}_1(\omega, \tau) \\
= Ke^{i\omega \ln(\xi)} \left[ \frac{1}{2} \sigma^2 A(\tau) - \frac{1}{2} \sigma^2 i\omega + (r - D_0 - \frac{1}{2} \sigma^2) \right] \\
- Me^{i\omega \ln(S_P(\tau))} \left[ \frac{S_P'(\tau)}{S_P(\tau)} - \frac{1}{2} \sigma^2 i\omega + (r - D_0 - \frac{1}{2} \sigma^2) \right]. \quad (B.37)
\]

Then, the ODE System (4.21) is obtained.
B.6 Appendix B.6

In this appendix, the Fourier inversion transform is applied to Equation (4.24), and Equation (4.25) will be obtained. Firstly, we rewrite Equation (4.24) here

\[ \hat{v}_1(\omega, \tau) = e^{-B(\omega)(\tau - \tau_M)} \cdot \hat{v}_1(\omega, \tau_M) + \int_0^{\tau - \tau_M} f_1(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \, d\xi \\
- \int_0^{\tau - \tau_M} f_2(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \, d\xi, \] (B.38)

where

\[ B(\omega) = \frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2) i\omega + r, \] (B.39)

\[ f_1(\omega, \tau) = Ke^{i\omega \ln(\frac{\xi}{\pi})} \cdot [\frac{1}{2} \sigma^2 A(\tau) - \frac{1}{2} \sigma^2 i\omega + (r - D_0 - \frac{1}{2} \sigma^2)], \] (B.40)

\[ f_2(\omega, \tau) = Me^{i\omega \ln(S_0(\tau))} \cdot [\frac{S'_p(\tau)}{S_p(\tau)} - \frac{1}{2} \sigma^2 i\omega + (r - D_0 - \frac{1}{2} \sigma^2)]. \] (B.41)

Then, the integral equation formation of \( v_1(x, \tau) \) can be presented

\[ v_1(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}_1(\omega, \tau)e^{-i\omega x} \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-B(\omega)(\tau - \tau_M)} \cdot \hat{v}_1(\omega, \tau_M) \cdot e^{-i\omega x} \, d\omega \\
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\tau - \tau_M} f_1(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \cdot e^{-i\omega x} \, d\xi \, d\omega \\
- \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\tau - \tau_M} f_2(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \cdot e^{-i\omega x} \, d\xi \, d\omega \\
= I_1 + I_2 - I_3, \] (B.42)

where \( I_1, I_2 \) and \( I_3 \) are the integral equation formulations, respectively. As follows, we compute these three one by one with \( I_1 \) first. It should be noted that the formula of \( I_1 \) in this appendix is very similar to the \( I_1 \) in the Appendix B. Therefore, we will only give the solution of it with no process

\[ I_1 = \int_{\ln(S_0(\tau_M))}^{\ln(\frac{\xi}{\pi})} v_1(u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sqrt{2\pi(\tau - \tau_M)}} \cdot e^{-\frac{|(r-\omega) - \frac{1}{2} \sigma^2 (\tau - \tau_M) + \xi|^2}{2\sigma^2(\tau - \tau_M)}} \, du. \] (B.43)

Now, we compute \( I_2 \)

\[ I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\tau - \tau_M} f_1(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \cdot e^{-i\omega x} \, d\xi \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\tau - \tau_M} f_1(\omega, \xi + \tau_M) \cdot e^{-B(\omega)(\tau - \tau_M - \xi)} \cdot e^{-i\omega x} \, d\xi \, d\omega. \]
\[
\begin{align*}
= & \int_{0}^{\tau-\tau_{M}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Ke}^{i\omega \ln(\xi \frac{2}{n})} \cdot e^{-\frac{1}{2} \sigma^{2} \omega^{2} + (r-D_{0} - \frac{1}{2} \sigma^{2}) i\omega + r(\tau-\tau_{M} - \xi)} \cdot e^{-i\omega x} \\
& \cdot \frac{1}{2} \sigma^{2} A(\tau) - \frac{1}{2} \sigma^{2} i\omega + (r-D_{0} - \frac{1}{2} \sigma^{2}) d\omega d\xi \\
= & \int_{0}^{\tau-\tau_{M}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega \ln(\xi \frac{2}{n}) - \frac{1}{2} \sigma^{2} \omega^{2} + (r-D_{0} - \frac{1}{2} \sigma^{2}) i\omega (\tau-\tau_{M} - \xi) - i\omega x} \\
& \cdot \frac{1}{2} \sigma^{2} A(\tau) - \frac{1}{2} \sigma^{2} i\omega + (r-D_{0} - \frac{1}{2} \sigma^{2}) d\omega d\xi \\
= & \int_{0}^{\tau-\tau_{M}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \sigma^{2} (\tau-\tau_{M} - \xi) \omega^{2} - (r-D_{0} - \frac{1}{2} \sigma^{2}) (\tau-\tau_{M} - \xi) + x - \ln(\xi \frac{2}{n})} i\omega \\
& \cdot [g_{1}(\xi) - g_{2}(\xi) \omega] d\omega d\xi, \quad (B.44)
\end{align*}
\]

where \( g_{1}(\xi) = \frac{1}{2} \sigma^{2} A(\tau) + (r-D_{0} - \frac{1}{2} \sigma^{2}) \) and \( g_{2}(\xi) = \frac{1}{2} \sigma^{2} i \). Thus, with the using of the conclusion \( \int_{-\infty}^{\infty} e^{-\rho \omega^{2} - q \omega} d\omega = (-1)^{n} \sqrt{\frac{\pi}{p}} \frac{\partial^{n} \omega^{n}}{\partial q^{n}} e^{\frac{q}{\rho}} \), we obtain

\[
I_{2} \quad = \quad \int_{0}^{\tau-\tau_{M}} \frac{\text{Ke}^{-r(\tau-\tau_{M} - \xi)}}{\sqrt{\pi(\tau-\tau_{M} - \xi)^{2}}} e^{-\frac{[(r-D_{0} - \frac{1}{2} \sigma^{2})(\tau-\tau_{M} - \xi) + x - \ln(\xi \frac{2}{n})]^{2}}{2\sigma^{2}(\tau-\tau_{M} - \xi)^{2}}} \\
& \quad \cdot \left( \frac{1}{2} \sigma^{2} A(\tau_{M} + \xi) \right) - \frac{1}{2} [r-D_{0} - \frac{1}{2} \sigma^{2} + \frac{\ln(\xi \frac{2}{n}) - x}{\tau-\tau_{M} - \xi}] d\xi. \quad (B.45)
\]

With the same method, we can obtain

\[
I_{3} \quad = \quad \int_{0}^{\tau-\tau_{M}} \frac{M e^{-r(\tau-\tau_{M} - \xi)}}{\sqrt{\pi(\tau-\tau_{M} - \xi)^{2}}} \cdot e^{-\frac{[(r-D_{0} - \frac{1}{2} \sigma^{2})(\tau-\tau_{M} - \xi) + x - \ln(S_{p}(\tau_{M} + \xi))]^{2}}{2\sigma^{2}(\tau-\tau_{M} - \xi)^{2}}} \\
& \quad \cdot \left( \frac{S_{p}(\tau_{M} + \xi)}{S_{p}(\tau_{M} + \xi)} + \frac{1}{2} [r-D_{0} - \frac{1}{2} \sigma^{2} + \frac{\ln(S_{p}(\tau_{M} + \xi)) - x}{\tau-\tau_{M} - \xi}] \right) d\xi. \quad (B.46)
\]

Combining these three formulae, and rewriting the integral equation formulation in the original parameters, then Equation (4.25) can be obtained.

### B.7 Appendix B.7

In this appendix, the details of how to obtain Equation (4.26) form Equation (4.25) can be presented. Firstly, we rewrite Equation (4.25)

\[
V_{1}(S, \tau) = \int_{\ln(S_{p}(\tau_{M}))}^{\ln(\xi \frac{2}{n})} V_{1}(e^{u}, \tau_{M}) \frac{e^{-r(\tau-\tau_{M})}}{\sqrt{2\pi \sigma^{2} (\tau-\tau_{M})}} e^{-\frac{[(r-D_{0} - \frac{1}{2} \sigma^{2})(\tau-\tau_{M} + \ln(S)) - u]^{2}}{2\sigma^{2}(\tau-\tau_{M})}} du \\
+ \int_{0}^{\tau-\tau_{M}} \frac{\text{Ke}^{-r(\tau-\tau_{M} - \xi)}}{\sqrt{2\pi \sigma^{2} (\tau-\tau_{M} - \xi)}} e^{-\frac{[(r-D_{0} - \frac{1}{2} \sigma^{2})(\tau-\tau_{M} - \xi) + \ln(S_{p}(\tau_{M} + \xi))]^{2}}{2\sigma^{2}(\tau-\tau_{M} - \xi)^{2}}} \\
& \quad \cdot \left( \frac{\frac{1}{2} \sigma^{2} A(\tau_{M} + \xi)}{n} + \frac{1}{2} [r-D_{0} - \frac{1}{2} \sigma^{2} + \frac{\ln(\xi \frac{2}{n}) - \ln(S)}{\tau-\tau_{M} - \xi}] \right) d\xi.
\]
\[ - \int_{0}^{\tau - \tau_{M}} \frac{Me^{-r(\tau - \tau_{M} - \xi)}}{\sqrt{2\pi\sigma^{2}(\tau - \tau_{M} - \xi)}} e^{\frac{[r - D_{0} - \frac{1}{2}\sigma^{2}(\tau - \tau_{M} - \xi) + \ln(S) - \ln(S_{p}(\tau_{M} + \xi))]^{2}}{2\sigma^{2}(\tau - \tau_{M} - \xi)}} \cdot \frac{S'_{p}(\tau_{M} + \xi)}{S_{p}(\tau_{M} + \xi)} + \frac{1}{2}(r - D_{0} - \frac{1}{2}\sigma^{2} + \frac{\ln(S_{p}(\tau_{M} + \xi)) - \ln(S)}{\tau - \tau_{M} - \xi}) \, d\xi. \]

(B.47)

It should be noted that there are only two items should be done the further computations, \(I_{2}\) and \(I_{3}\), where we use the same marks as in Appendix B.6. Thus, do \(I_{2}\) first

\[ I_{2} = \int_{0}^{\tau - \tau_{M}} \frac{Ke^{-r(\tau - \tau_{M} - \xi)}}{\sqrt{2\pi(\tau - \tau_{M} - \xi)\sigma^{2}}} \cdot e^{\frac{[r - D_{0} - \frac{1}{2}\sigma^{2}(\tau - \tau_{M} - \xi) + \ln(S) - \ln(S_{p}(\tau_{M} + \xi))]^{2}}{2\sigma^{2}(\tau - \tau_{M} - \xi)}} \cdot \left\{ \frac{1}{2}\sigma^{2}A(\tau_{M} + \xi) \right\} n + \frac{1}{2}[r - D_{0} - \frac{1}{2}\sigma^{2} + \frac{\ln(K_{n}) - \ln(S)}{\tau - \tau_{M} - \xi}] \, d\xi, \]  

(B.48)

and we define

\[ h_{2}(S, \xi) \triangleq \frac{[(r - D_{0} - \frac{1}{2}\sigma^{2})(\tau - \tau_{M} - \xi) + \ln(S) - \ln(K_{n})]}{2\sigma^{2}(\tau - \tau_{M} - \xi)} \]

\[ = \frac{1}{2(\tau - \tau_{M} - \xi)} \left[ \frac{(r - D_{0} - \frac{1}{2}\sigma^{2})(\tau - \tau_{M} - \xi) + \ln(S) - \ln(K_{n})}{\sigma} \right]^{2} \]

\[ = \frac{1}{2(\tau - \tau_{M} - \xi)} \left[ \frac{(r - D_{0} - \frac{1}{2}\sigma^{2})(\tau - \tau_{M} + \xi) + \ln(S) - \ln(K_{n})}{\sigma} \right]^{2} \]

\[ = \frac{1}{2(\tau - \tau_{M} - \xi)} [y - P(\xi)]^{2}, \]  

(B.49)

where \(y = \frac{(r - D_{0} - \frac{1}{2}\sigma^{2})(\tau - \tau_{M}) + \ln(S)}{\sigma} \) and \(P(\xi) = \frac{(r - D_{0} - \frac{1}{2}\sigma^{2})\xi + \ln(K_{n})}{\sigma}. \) It is interesting to find that \(P'(\xi) = \frac{r - D_{0} - \frac{1}{2}\sigma^{2}}{\sigma}. \) Then, we apply it to the \(I_{2}\) and obtain

\[ I_{2} = \int_{0}^{\tau - \tau_{M}} \frac{Ke^{-r(\tau - \tau_{M} - \xi)}}{\sqrt{2\pi(\tau - \tau_{M} - \xi)\sigma^{2}}} \cdot e^{-\frac{[y - P(\xi)]^{2}}{2(\tau - \tau_{M} - \xi)}} \cdot \left\{ \frac{1}{2}\sigma^{2}A(\tau_{M} + \xi) \right\} n + \frac{1}{2}[r - D_{0} - \frac{1}{2}\sigma^{2} + \frac{\ln(K_{n}) - \ln(S)}{\tau - \tau_{M} - \xi}] \, d\xi \]

\[ = \int_{0}^{\tau - \tau_{M}} \frac{Ke^{-r(\tau - \tau_{M} - \xi)}}{\sqrt{2\pi(\tau - \tau_{M} - \xi)\sigma^{2}}} \cdot e^{-\frac{[y - P(\xi)]^{2}}{2(\tau - \tau_{M} - \xi)}} \cdot \left\{ \frac{1}{2}\sigma^{2}A(\tau_{M} + \xi) \right\} n + \frac{1}{2}[r - D_{0} - \frac{1}{2}\sigma^{2} + \frac{\ln(K_{n}) - \ln(S)}{\tau - \tau_{M} - \xi}] \, d\xi \]

\[ = \int_{0}^{\tau - \tau_{M}} \frac{Ke^{-r(\tau - \tau_{M} - \xi)}}{\sqrt{2\pi(\tau - \tau_{M} - \xi)\sigma^{2}}} \cdot e^{-\frac{[y - P(\xi)]^{2}}{2(\tau - \tau_{M} - \xi)}} \]
Then is the I₃

\[ I₃ = \int_{\tau - \tau_M}^{\tau} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)^3}} \cdot e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi)}{2\sigma(\tau - \tau_M - \xi)} \cdot \left\{ \frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \right\} d\xi \]
As the same method as before, we define

\[
    h_3(S, \xi) \triangleq \frac{[r-D_0 - \frac{1}{2}\sigma^2](\tau - \tau_M - \xi) + \ln(S) - \ln(S_0(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)}
\]

\[
= \frac{1}{2(\tau - \tau_M - \xi)} \left[ \frac{r-D_0 - \frac{1}{2}\sigma^2}{\sigma}(\tau - \tau_M - \xi) + \ln(S) - \ln(S_0(\tau_M + \xi)) \right]^2
\]

\[
= \frac{1}{2(\tau - \tau_M - \xi)} \left[ \frac{r-D_0 - \frac{1}{2}\sigma^2}{\sigma}(\tau - \tau_M) + \ln(S) - \ln(S_0(\tau_M + \xi)) \right]^2
\]

\[
= \frac{1}{2(\tau - \tau_M - \xi)}[x - J(\xi)]^2,
\]  

where \( x = \frac{(r-D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma} \) and \( J(\xi) = \frac{(r-D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_0(\tau_M + \xi))}{\sigma} \).

It is interesting to notice that \( J'(\xi) = \frac{1}{\sigma} \left[ \frac{S_0'(\tau_M + \xi)}{S_0(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2 \right] \). Then, we apply it to the \( I_3 \) and obtain

\[
I_3 = \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \left\{ J'(\xi) - \frac{1}{2\sigma} (r-D_0 - \frac{1}{2}\sigma^2) \right\} d\xi
\]

\[
= \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \left\{ \frac{x - J(\xi)}{2(\tau - \tau_M - \xi)} \right\} d\xi
\]

\[
= \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \left\{ \frac{2J'(\xi)(\tau - \tau_M - \xi) - x + J(\xi)}{2(\tau - \tau_M - \xi)\sqrt{\tau - \tau_M - \xi}} \right\} d\xi
\]

\[
= \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \left\{ \frac{2J'(\xi)\sqrt{\tau - \tau_M - \xi} - (x - J(\xi))\frac{1}{\sqrt{\tau - \tau_M - \xi}}}{2(\tau - \tau_M - \xi)} \right\} d\xi
\]

\[
= - \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \frac{\partial}{\partial \xi} \left\{ \frac{x - J(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right\} d\xi
\]

\[
= - \int_0^{\tau - \tau_M} \frac{Me^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{(x - J(\xi))^2}{2(\tau - \tau_M - \xi)^2}} \cdot \frac{x - J(\xi)}{\sqrt{\tau - \tau_M - \xi}} d\xi.
\]  

(B.53)

Then, we substitute the expression of \( I_2 \) and \( I_3 \) into Equation (B.47), and obtain

\[
V_1(S, \tau) = \int_{\ln(S_0(\tau_M))}^{\ln(S_M)} V_1(e^u, \tau_M) \cdot e^{-r(\tau - \tau_M)} \cdot \frac{e^{-\frac{(r-D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_0(\tau_M + \xi))}{2\sigma^2(\tau - \tau_M - \xi)}}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} \cdot du
\]

\[
+ \int_0^{\tau - \tau_M} Ke^{-r(\tau - \tau_M - \xi)} \cdot \frac{e^{-\frac{(r-D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_0(\tau_M + \xi))}{2\sigma^2(\tau - \tau_M - \xi)}}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} \cdot \frac{\sigma A(\tau_M + \xi)}{2\sqrt{\tau - \tau_M - \xi}} d\xi
\]
\[
- \int_{0}^{\tau_{M}} Ke^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{y - P(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi \\
+ \int_{0}^{\tau_{M}} Me^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{x - J(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi
\]

\[
= \int_{\ln(S_P(t_{M}))} \frac{V_1(e^u, \tau_{M})}{\sqrt{2\pi \sigma^2(\tau_{M} - \tau)}} \frac{e^{-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau_{M} - \xi) + \ln(S) - \ln(S_P(t_{M} + \xi))}{2\sigma^2(\tau_{M} - \tau)}}}{\sqrt{\tau_{M} - \tau}} du
\]

\[
+ \int_{0}^{\tau_{M}} Ke^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{y - P(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi \\
+ \int_{0}^{\tau_{M}} Me^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{x - J(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi
\]

\[
= \int_{\ln(S_P(t_{M}))} \frac{V_1(e^u, \tau_{M})}{\sqrt{2\pi \sigma^2(\tau_{M} - \tau)}} \frac{e^{-\frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau_{M} - \xi) + \ln(S) - \ln(S_P(t_{M} + \xi))}{2\sigma^2(\tau_{M} - \tau)}}}{\sqrt{\tau_{M} - \tau}} du
\]

\[
+ \int_{0}^{\tau_{M}} Ke^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{y - P(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi \\
+ \int_{0}^{\tau_{M}} Me^{-r(\tau_{M} - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{M}\left\{ \frac{x - J(\xi)}{\sqrt{\tau_{M} - \xi}} \right\} d\xi
\]
\[-K_{1_{S=n}}^1 + M_{1_{S=S_p(\tau)}}\], \hspace{1cm} (B.54)

where

\[1_{S=n}^1 = \begin{cases} 
\frac{1}{2} & S = \frac{K}{n}, \\
0 & S < \frac{K}{n}, 
\end{cases} \hspace{1cm} (B.55)\]

\[1_{S=S_p(\tau)}^1 = \begin{cases} 
\frac{1}{2} & S = S_p(\tau), \\
1 & S > S_p(\tau). 
\end{cases} \hspace{1cm} (B.56)\]
Appendix C

Appendix for Chapter 5

C.1 Appendix C.1

Recall the PDE system (5.11)

\[
\begin{aligned}
- \frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} + (r - D_0 - \frac{1}{2} \sigma^2) \frac{\partial V}{\partial x} - rV &= 0, \\
V(x,0) &= \max\{n_1 e^x, Z\}, \\
V(\ln(S_r), \tau) &= F(\tau), \\
\frac{\partial V}{\partial x}(\ln(S_r), \tau) &= S_r \cdot G(\tau), \\
V(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau), \\
\frac{\partial V}{\partial x}(\ln(S_c(\tau)), \tau) &= n_1 \cdot S_c(\tau),
\end{aligned}
\] (C.1)

and the definition of incomplete Fourier transform, Equation (5.10),

\[
\mathcal{F}\{V(x, \tau)\} = \int_{\ln(S_r)}^{\ln(S_c(\tau))} V(x, \tau) \cdot e^{i\omega x} dx \triangleq \hat{V}(\omega, \tau).
\] (C.2)

Applying the incomplete Fourier transform on the PDE leads to

\[
- \mathcal{F}\left\{ \frac{\partial V}{\partial \tau} \right\} + \frac{1}{2} \sigma^2 \mathcal{F}\left\{ \frac{\partial^2 V}{\partial x^2} \right\} + (r - D_0 - \frac{1}{2} \sigma^2) \mathcal{F}\left\{ \frac{\partial V}{\partial x} \right\} - r \mathcal{F}\{V\} = 0,
\] (C.3)

and clearly we need to calculate every term involved to the above equation. In particular, we have

\[
\begin{aligned}
\mathcal{F}\left\{ \frac{\partial V}{\partial \tau}(x, \tau) \right\} &= \int_{\ln(S_r)}^{\ln(S_c(\tau))} \frac{\partial V}{\partial \tau}(x, \tau) \cdot e^{i\omega x} dx \\
&= \frac{\partial}{\partial \tau} \int_{\ln(S_r)}^{\ln(S_c(\tau))} V(x, \tau) \cdot e^{i\omega x} dx - \frac{S'_c(\tau)}{S_c(\tau)} V(x, \tau) \cdot e^{i\omega x}\bigg|_{\ln(S_r(\tau))} \\
&= \frac{\partial}{\partial \tau}(\omega, \tau) - \frac{S'_c(\tau)}{S_c(\tau)} F(\tau) \cdot e^{i\omega \ln(S_c(\tau))}.
\end{aligned}
\] (C.4)
\[ \mathcal{F}\left\{ \frac{\partial V}{\partial x}(x, \tau) \right\} = \int_{\ln(S_0)}^{\ln(S_1)} \frac{\partial V}{\partial x}(x, \tau) \cdot e^{i\omega x} dx \]

\[ = V(x, \tau) \cdot e^{i\omega \ln(S_0)} - V(x, \tau) \cdot e^{i\omega \ln(S_1)} - i\omega \int_{\ln(S_0)}^{\ln(S_1)} V(x, \tau) \cdot e^{i\omega x} dx \]

\[ \mathcal{F}\left\{ \frac{\partial^2 V}{\partial x^2} \right\} = \int_{\ln(S_0)}^{\ln(S_1)} \frac{\partial^2 V}{\partial x^2}(x, \tau) \cdot e^{i\omega x} dx \]

\[ = \frac{\partial V}{\partial x}(x, \tau) \cdot e^{i\omega \ln(S_0)} - \frac{\partial V}{\partial x}(x, \tau) \cdot e^{i\omega \ln(S_1)} - i\omega \int_{\ln(S_0)}^{\ln(S_1)} \frac{\partial V}{\partial x}(x, \tau) \cdot e^{i\omega x} dx \]

\[ = n_1 S_c(\tau) \cdot e^{i\omega \ln(S_0)} - S_r G(\tau) \cdot e^{i\omega \ln(S_1)} - F(\tau) \cdot e^{i\omega \ln(S_1)} - i\omega \hat{V}(\omega, \tau) \]

\[ = (1 - i\omega)n_1 S_c(\tau) \cdot e^{i\omega \ln(S_0)} - [S_r G(\tau) - i\omega F(\tau)] \cdot e^{i\omega \ln(S_1)} - \omega^2 \hat{V}(\omega, \tau). \]  

(C.5)

Combining all the equations above yields the ODE system (5.12).

C.2 Appendix C.2

Equation (5.16) can be rewritten as

\[ \hat{V}(\omega, \tau) = \hat{V}(\omega, 0) \cdot e^{-B(\omega)\tau} + \int_0^\tau [F_1(\omega, \xi) - F_2(\omega, \xi)] \cdot e^{-B(\omega)(\tau - \xi)} d\xi, \]  

(C.7)

with

\[ B(\omega) = \frac{1}{2} \sigma^2 \omega^2 + (r - D_0) - \frac{1}{2} \sigma^2 i\omega + r, \]

(C.8)

\[ F_1(\omega, \tau) = \left[ \frac{S_c^\prime(\tau)}{S_c(\tau)} - \frac{1}{2} \sigma^2 i\omega + (r - D_0) \right] \cdot n_1 S_c(\tau) e^{i\omega \ln(S_0)}, \]

(C.9)

\[ F_2(\omega, \tau) = \left[ \frac{1}{2} \sigma^2 G(\tau) S_r - \frac{1}{2} \sigma^2 i\omega F(\tau) + (r - D_0) - \frac{1}{2} \sigma^2 \right] F(\tau) \cdot e^{i\omega \ln(S_1)}. \]  

(C.10)

In addition, the Fourier Inversion transform can be specified as

\[ V(x, \tau) = \mathcal{F}^{-1}\{\hat{V}(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, \tau) \cdot e^{-i\omega x} d\omega. \]  

(C.11)

Thus, if the Fourier inversion transform (C.11) is applied on Equation (C.7), we can obtain

\[ V(x, \tau) = \mathcal{F}^{-1}\{\hat{V}(\omega, \tau)\} \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega \]
being calculated as

\[ I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot e^{-[1/2\sigma^2\omega^2 + (r-D_0 - 1/2\sigma^2)i\omega + r]\tau} \cdot e^{-i\omega x} d\omega \]

\[ \triangleq I_1 + I_2 - I_3, \tag{C.12} \]

implying that we need to compute these three integrals one by one. First, \(I_1\) can be rearranged as

\[ I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot e^{-B(\omega)\tau} \cdot e^{-i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot e^{-\left[1/2\sigma^2\omega^2 + (r-D_0 - 1/2\sigma^2)i\omega + r\right]\tau} \cdot e^{-i\omega x} d\omega \]

\[ \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot G(\omega, \tau) \cdot e^{-i\omega x} d\omega, \tag{C.13} \]

where \(G(\omega, \tau) = e^{-\left[1/2\sigma^2\omega^2 + (r-D_0 - 1/2\sigma^2)i\omega + r\right]\tau}\), with the Fourier inversion of \(G(\omega, \tau)\) being calculated as

\[ g(x, \tau) = \mathcal{F}^{-1}\{G(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega, \tau) \cdot e^{-i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[1/2\sigma^2\omega^2 + (r-D_0 - 1/2\sigma^2)i\omega + r\right]\tau} \cdot e^{-i\omega x} d\omega \]

\[ = e^{-\tau r} \left[ \int_{-\infty}^{\infty} e^{-\left[1/2\sigma^2\omega^2 - \left[\left(r-D_0 - 1/2\sigma^2\right)i\omega \right] + x\right]\tau} \cdot e^{-i\omega x} d\omega \right] \]

\[ = e^{-\tau r} \cdot \frac{\left[(r-D_0 - 1/2\sigma^2)i\omega + x\right]^{-1}}{\sqrt{2\pi\sigma^2\tau}} \cdot \sqrt{\frac{2\pi}{\sigma^2\tau}} \]

\[ = \frac{e^{-\tau r}}{\sqrt{2\pi\sigma^2\tau}} \cdot \frac{\sqrt{2\pi}}{(2\delta^2)} \cdot \max\{n_1 e^u, Z\}\frac{du}{2\sigma^2\tau}, \tag{C.14} \]

we can get the expression of \(I_1\) below

\[ I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}(\omega, 0) \cdot G(\omega, \tau) \cdot e^{-i\omega x} d\omega \]

\[ = V(x, \tau) \ast g(x, \tau) \]

\[ = \frac{e^{-\tau r}}{\sqrt{2\pi\sigma^2\tau}} \int_{\text{in}(S_r)} e^{-\left[(r-D_0 - 1/2\sigma^2)i\omega + x\right]\tau} \cdot \max\{n_1 e^u, Z\}\frac{du}{2\sigma^2\tau}, \tag{C.15} \]

according to the convolution theorem.

On the other hands, \(I_2\) and \(I_3\) can be respectively computed as

\[ I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\tau} F_1(\omega, \xi) \cdot e^{-B(\omega)(\tau - \xi)} d\xi \cdot e^{-i\omega x} d\omega \]
\[ I_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_2(\omega, \xi) \cdot e^{-B(\omega)(\tau-\xi)} d\xi \cdot e^{-i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma^2 G(\xi) S_\tau - \frac{1}{2} \sigma^2 i\omega F(\xi) + (r-D_0 - \frac{1}{2} \sigma^2) F(\xi) \]

\[ \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-r(\tau-\xi)} d\xi \cdot e^{-i\omega x} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma^2 G(\xi) S_\tau \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-r(\tau-\xi)} d\xi \cdot e^{-i\omega x} d\omega \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (r-D_0 - \frac{1}{2} \sigma^2) - \frac{1}{2} \sigma^2 i\omega F(\xi) \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-r(\tau-\xi)} d\xi \cdot e^{-i\omega x} d\omega d\xi \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma^2 G(\xi) S_\tau \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-i\omega x} d\omega d\xi \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (r-D_0 - \frac{1}{2} \sigma^2) - \frac{1}{2} \sigma^2 i\omega F(\xi) \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-i\omega x} d\omega d\xi \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma^2 G(\xi) S_\tau \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-i\omega x} d\omega d\xi \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (r-D_0 - \frac{1}{2} \sigma^2) - \frac{1}{2} \sigma^2 i\omega F(\xi) \cdot e^{i\omega \ln(S_\tau)} \cdot e^{-i\omega x} d\omega d\xi \]
\[
\frac{1}{2} \sigma^2 G(\xi) S_r \cdot e^{-r(\tau - \xi)} \cdot \frac{e^{-r(\tau - \xi)} - [r - D_0 - \frac{1}{2} \sigma^2(\tau - \xi) + x - \ln(S_r)]i\omega d\omega d\xi}{2\pi \sigma^2(\tau - \xi)} \cdot \sqrt{\frac{2\pi}{\sigma^2(\tau - \xi)}} d\xi \\
+ \int_0^\tau F(\xi) \cdot e^{-r(\tau - \xi)} \cdot \frac{e^{-r(\tau - \xi)} - [r - D_0 - \frac{1}{2} \sigma^2(\tau - \xi) + x - \ln(S_r)]^2}{2\sigma^2(\tau - \xi)} \cdot \sqrt{\frac{2\pi}{\sigma^2(\tau - \xi)}} d\xi \\
\cdot \{(r - D_0 - \frac{1}{2} \sigma^2 + (r - D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + x - \ln(S_r))d\xi
\]

where a simple identity is employed in the derivation process

\[
\int_{-\infty}^{\infty} e^{-\rho \omega^2 - q\omega} \cdot \omega^n d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{\rho^2}{4q}}.
\]

Hence, we can finally arrive at the integral equation representations, (5.17), if we use the original variable $S$ to replace $x$.

C.3 Appendix C.3

Firstly, we rewrite Equation (5.17) as

\[
V(S, \tau) = \frac{e^{-r\tau}}{2\pi \sigma^2 \tau} \int_{\ln(S_r)}^{\ln(S_c(\xi))} e^{-r(\tau - \xi)} \cdot \frac{e^{-r(\tau - \xi) - \frac{1}{2} \sigma^2(\tau - \xi) + x - \ln(S_r)]^2}{2\sigma^2(\tau - \xi)}} \cdot \max\{n_1 e^{\omega}, Z\} d\omega \\
+ \int_0^\tau n_1 S_c(\xi) \cdot \frac{e^{-r(\tau - \xi)} - [r - D_0 - \frac{1}{2} \sigma^2(\tau - \xi) + x - \ln(S_r)]^2}{2\sigma^2(\tau - \xi)} \cdot \sqrt{\frac{2\pi}{\sigma^2(\tau - \xi)}} d\xi \\
\cdot \{(r - D_0 + \frac{1}{2} \sigma^2(\tau - \xi) - \ln(S) + \ln(S_c(\xi))\}
\]
\begin{equation}
\int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_\nu \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2(\tau - \xi)}} \cdot e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}
- \int_0^\tau F(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2(\tau - \xi)}} \cdot e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}
\cdot \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) - \ln(S) + \ln(S_\nu)}{2(\tau - \xi)}
\end{equation}

\begin{equation}
\Delta \frac{e^{-r\tau}}{\sqrt{2\pi \sigma^2} \cdot \int_{\ln(S_\nu)}^{\ln(S(0))} e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}} \cdot \max\{n_1 e^u, Z\} du

- \int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_\nu \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2(\tau - \xi)}} \cdot e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}
\cdot \frac{(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) - \ln(S) + \ln(S_\nu)}{2(\tau - \xi)}
\end{equation}

\begin{equation}
+ I, \quad (C.19)
\end{equation}

where

\begin{equation}
I = \int_0^\tau n_1 S_\nu(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2(\tau - \xi)}} \cdot e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}
\cdot \frac{S_\nu'(\xi)}{S_\nu(\xi)} + \frac{\{r-D_0 + \frac{1}{2} \sigma^2\}(\tau - \xi) - \ln(S) + \ln(S_\nu)}{2(\tau - \xi)} d\xi. \quad (C.20)
\end{equation}

This demonstrates that the remaining task is to work out $I$. If we define

\begin{equation}
h(\xi) \triangleq \frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}
= \frac{1}{2(\tau - \xi)} \frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{\sigma}
= \frac{1}{2(\tau - \xi)} \frac{[(r-D_0 + \frac{1}{2} \sigma^2)\tau + \ln(S) - (r-D_0 - \frac{1}{2} \sigma^2)\xi + \ln(S_\nu)]^2}{\sigma}
\triangleq \frac{1}{2(\tau - \xi)} [y - P(\xi)]^2, \quad (C.21)
\end{equation}

where $y = \frac{(r-D_0 - \frac{1}{2} \sigma^2)\tau + \ln(S)}{\sigma}$ and $P(\xi) = \frac{(r-D_0 + \frac{1}{2} \sigma^2)\xi + \ln(S_\nu)}{\sigma}$, and noticing the fact that $P'(\xi) = \frac{1}{\sigma} [r-D_0 - \frac{1}{2} \sigma^2 + \frac{S_\nu'(\xi)}{S_\nu(\xi)}]$, we can obtain

\begin{equation}
I = \int_0^\tau n_1 S_\nu(\xi) \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi \sigma^2(\tau - \xi)}} \cdot e^{\frac{[(r-D_0 - \frac{1}{2} \sigma^2)(\tau - \xi) + \ln(S) - \ln(S_\nu)]^2}{2\sigma^2(\tau - \xi)}}
\end{equation}
\[ \begin{align*}
\int_0^\tau & n_1 S_c(\xi) \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} \cdot e^{\frac{(y-P(\xi))^2}{2(\tau-\xi)}} \cdot \{P'(\xi) - \frac{r-D_0 - \frac{1}{2} \sigma^2}{\sigma}(\tau-\xi) + \ln(S) - \ln(S_c(\xi)) \}d\xi \\
& = \int_0^\tau n_1 S_c(\xi) \cdot \frac{e^{\frac{(y-P(\xi))^2}{2(\tau-\xi)}}}{\sqrt{2\pi(\tau-\xi)}} \cdot \{P'(\xi) - \frac{r-D_0 - \frac{1}{2} \sigma^2}{\sigma}(\tau-\xi) + \ln(S) - \ln(S_c(\xi)) \}d\xi \\
& = \int_0^\tau n_1 S_c(\xi) \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} \cdot e^{\frac{(y-P(\xi))^2}{2(\tau-\xi)}} \cdot \{P'(\xi) - \frac{y-P(\xi) + \sigma(\tau-\xi) - \frac{\sigma}{2}}{\frac{1}{2\sqrt{\tau-\xi}}} \}d\xi \\
& = \int_0^\tau -n_1 S_c(\xi) \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi}} \cdot e^{\frac{(y-P(\xi))^2}{2(\tau-\xi)}} \cdot \frac{\partial}{\partial \xi} \left( \frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}} \right)d\xi \\
& = \int_0^\tau -n_1 S_c(\xi) \cdot e^{-r(\tau-\xi)} \cdot e^{(r-D_0)(\tau-\xi) + \ln(S) - \ln(S_c(\xi))} \\
& \quad \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2} \sigma^2}{\sigma^2(\tau-\xi)} \right) d\xi
\end{align*} \]

Using the integration by parts yields

\[ \begin{align*}
I &= \int_0^\tau D_0 n_1 S_c(\xi) \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2} \sigma^2}{\sigma^2(\tau-\xi)} \right) d\xi \\
& \quad -n_1 S_c \cdot D_0(\tau-\xi) \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2} \sigma^2}{\sigma^2(\tau-\xi)} \right) |_{0}^{\tau} \\
& = \int_0^\tau D_0 n_1 S_c(\xi) \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2} \sigma^2}{\sigma^2(\tau-\xi)} \right) d\xi \\
& \quad -n_1 S_c \cdot \mathcal{L}_{S=S_c(\tau)} + n_1 S_c \cdot D_0(\tau-\xi) \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2} \sigma^2}{\sigma^2(\tau-\xi)} \right), \quad \text{(C.23)}
\end{align*} \]
where
\[ L_{S=S_c(\tau)} = \begin{cases} 
0, & S < S_c(\tau), \\
\frac{1}{2}, & S = S_c(\tau).
\end{cases} \] (C.24)

Here, \( \mathcal{N}(\cdot) \) is the cumulative distribution function of the standard normal distribution. This has completed the proof.

### C.4 Appendix C.4

To prove Proposition 5.3.1, we start with the PDE system (5.7) and make the transform \( \tau = T - t \), which yields
\[
-\frac{\partial V}{\partial \tau} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0,
\] (C.25)

with the initial condition as \( V(S, 0) = \max \{ n_1 S, Z \} \). Considering the continuity of \( V(S, \tau) \) and \( V(S, 0) = \max \{ n_1 S, Z \} \), for any \( S_1 < Z/n_1 \), there always exists a small number \( \delta > 0 \), such that
\[
V(S_1, \tau) > n_1 S_1, \quad \forall \tau < \delta,
\] (C.26)

which further leads to
\[
S_c(0) = \lim_{\tau \to 0^+} S_c(\tau) > S_1.
\] (C.27)

As a result, the arbitrariness of \( S_1 \) gives
\[
S_c(0) \geq Z/n_1.
\] (C.28)

On the other hand, if we assume \( S_c(0) > Z/n_1 \), there exists \( S \) such that \( S_c(0) > S > Z/n_1 \). This implies \( V(S, 0) = n_1 S \), as a result of \( n_1 S > Z \), and thus
\[
\frac{\partial V}{\partial \tau}(S, 0) = -D_0 n_1 S < 0,
\] (C.29)

after the substitution of \( V(S, 0) \) into (C.25). In this case, there exists \( \tau > 0 \) being small enough such that
\[
V(S, \tau) < n_1 S,
\] (C.30)

which contradicts to \( V \geq n_1 S \). Therefore,
\[
S_c(0) \leq Z/n_1.
\] (C.31)

Combining Equation (C.28) and (C.31) yields the desired result.
C.5 Appendix C.5

To prove Proposition 5.3.2, we only need to show that there exists \( \tau_\ast \), s.t. \( G(\tau) < 0 \) for any \( 0 < \tau < \tau_\ast \), as \( G(\tau) \) is the first-order derivative of the bond price, \( \frac{\partial V}{\partial S}(S, \tau) \), at \( S = S_r \).

Recall Equation (5.21)

\[
\frac{1}{2} F(\tau) = \frac{e^{-\frac{\tau}{\sigma}}}{\sqrt{2\pi} \sigma^2} \int_{\ln(S_r)}^{\ln(S_c(0))} \frac{e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - u^2}{2\sigma^2}}}{\sqrt{2\pi} \sigma^2} \cdot \max\{n_1 e^{u}, Z\} du \\
+ n_1 S_r \cdot e^{-D_0 \tau} \cdot \mathcal{N}\left( \frac{r - D_0 + \frac{\sigma^2}{2} \ln(S_r) - \ln(S_c(0))}{\sqrt{\sigma^2 \tau}} \right) \\
\quad + \int_{0}^{\tau} D_0 n_1 S_r \cdot e^{-D_0 (\tau - \xi)} \cdot \mathcal{N}\left( \frac{r - D_0 + \frac{\sigma^2}{2} \ln(S_r) - \ln(S_c(\xi))}{\sqrt{\sigma^2 (\tau - \xi)}} \right) d\xi \\
\quad - \int_{0}^{\tau} \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-\frac{(\tau - \xi)}{\sigma^2} \ln(S_r) - \ln(S_c(\xi))}}{\sqrt{2\pi} \sigma^2 (\tau - \xi)} \cdot e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - \ln(S_c(\xi))}{2\sigma^2 (\tau - \xi)}} d\xi \\
\quad - \int_{0}^{\tau} F(\xi) \cdot \frac{e^{-\frac{(\tau - \xi)}{\sigma^2} \ln(S_r) - \ln(S_c(\xi))}}{\sqrt{2\pi} \sigma^2 (\tau - \xi)} \cdot \frac{e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - \ln(S_c(\xi))}{2\sigma^2 (\tau - \xi)}}}{\sigma} d\xi. \tag{C.32}
\]

With \( S_c(\tau)|_{\tau=0} = S_r \), the first term involved in the above equation is eliminated. If we rearrange the equation by moving the integral involving \( G(\tau) \) to the left hand side while moving all the other terms to the right hand side, we can obtain

\[
\int_{0}^{\tau} \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-\frac{(\tau - \xi)}{\sigma^2} \ln(S_r) - \ln(S_c(\xi))}}{\sqrt{2\pi} \sigma^2 (\tau - \xi)} \cdot e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - \ln(S_c(\xi))}{2\sigma^2 (\tau - \xi)}} d\xi \\
= n_1 S_r \cdot e^{-D_0 \tau} \cdot \mathcal{N}\left( \frac{r - D_0 + \frac{\sigma^2}{2} \ln(S_r)}{\sigma} \right) \\
+ \int_{0}^{\tau} D_0 n_1 S_r \cdot e^{-D_0 (\tau - \xi)} \cdot \mathcal{N}\left( \frac{r - D_0 + \frac{\sigma^2}{2} \ln(S_r) - \ln(S_c(\xi))}{\sqrt{\sigma^2 (\tau - \xi)}} \right) d\xi \\
\quad - \frac{1}{2} F(\tau) - \int_{0}^{\tau} F(\xi) \cdot \frac{e^{-\frac{(\tau - \xi)}{\sigma^2} \ln(S_r) - \ln(S_c(\xi))}}{\sqrt{2\pi} \sigma^2 (\tau - \xi)} \cdot \frac{e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - \ln(S_c(\xi))}{2\sigma^2 (\tau - \xi)}}}{\sigma} \cdot \frac{r - D_0 + \frac{\sigma^2}{2} \ln(S_r) - \ln(S_c(\xi))}{\sigma} d\xi. \tag{C.33}
\]

It should be noted that the pricing domain is the range of the underlying asset of the resettable bond before conversion or reset takes place; once either of these actions has taken place, the bond value is known already. Thus, we have \( S_r \leq S \leq S_c(\tau) \), or simply \( S_c(\tau) \geq S_r \), from which it is not difficult to obtain

\[
\int_{0}^{\tau} \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-\frac{(\tau - \xi)}{\sigma^2} \ln(S_r) - \ln(S_c(\xi))}}{\sqrt{2\pi} \sigma^2 (\tau - \xi)} \cdot e^{-\frac{(\tau - D_0 + \frac{\sigma^2}{2}) \ln(S_r) - \ln(S_c(\xi))}{2\sigma^2 (\tau - \xi)}} d\xi.
\]
\[ \leq I - \frac{1}{2} F(\tau) - \int_0^\tau F(\xi) : \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi \sigma^2(\tau-\xi)}} \cdot e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)\xi}{2\sigma^2}} \cdot \frac{r-D_0 - \frac{1}{2}\sigma^2}{2} d\xi, \] (C.34)

where

\[ I = n_1 S_r \cdot e^{-D_0 \tau} \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma} \sqrt{\tau} \right) + \int_0^\tau D_0 n_1 S_r \cdot e^{-D_0(\tau-\xi)} \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma} \sqrt{\tau-\xi} \right) d\xi. \]

In fact, by making use of the integration by parts, \( I \) can be computed through

\[ I = n_1 S_r \cdot e^{-D_0 \tau} \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma} \sqrt{\tau} \right) + n_1 S_r \cdot e^{-D_0(\tau-\xi)} \cdot \mathcal{N} \left( \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma} \sqrt{\tau-\xi} \right) \int_0^\tau d\xi 

+ \int_0^\tau n_1 S_r \cdot e^{-D_0(\tau-\xi)} \cdot \frac{(r-D_0+\frac{1}{2}\sigma^2)^2(\tau-\xi)}{2\sigma^2} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma^2} d\xi 

= \frac{1}{2} n_1 S_r + \int_0^\tau n_1 S_r \cdot e^{-D_0(\tau-\xi)} \cdot \frac{(r-D_0+\frac{1}{2}\sigma^2)^2(\tau-\xi)}{2\sigma^2} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma^2} d\xi. \] (C.35)

Thus, Equation (C.34) can further lead to

\[ \int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi \sigma^2(\tau-\xi)}} \cdot e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)\xi}{2\sigma^2}} d\xi \]

\[ \leq \frac{1}{2} n_1 S_r + \int_0^\tau n_1 S_r \cdot e^{-D_0(\tau-\xi)} \cdot \frac{(r-D_0+\frac{1}{2}\sigma^2)^2(\tau-\xi)}{2\sigma^2} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma^2} d\xi 

- \frac{1}{2} F(\tau) - \int_0^\tau F(\xi) \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi \sigma^2(\tau-\xi)}} \cdot e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)\xi}{2\sigma^2}} \cdot \frac{r-D_0 - \frac{1}{2}\sigma^2}{2} d\xi. \] (C.36)

Now, if we make the transformation of \( x = \sqrt{\tau-\xi} \) for the right hand side, we obtain

\[ \int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi \sigma^2(\tau-\xi)}} \cdot e^{-\frac{(r-D_0-\frac{1}{2}\sigma^2)\xi}{2\sigma^2}} d\xi \]

\[ \leq \frac{1}{2} n_1 S_r + \int_0^\sqrt{\tau} n_1 S_r \cdot e^{-D_0 x^2} \cdot \frac{(r-D_0+\frac{1}{2}\sigma^2)^2 x^2}{2\sigma^2} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma^2} dx 

- \frac{1}{2} F(\tau) - \int_0^\sqrt{\tau} F(\tau-x^2) \cdot e^{-r x^2} \cdot \frac{(r-D_0-\frac{1}{2}\sigma^2)^2 x^2}{2\sigma^2} \cdot \frac{r-D_0 - \frac{1}{2}\sigma^2}{\sigma^2} dx \]

\[ \leq \frac{1}{2} n_1 S_r + \int_0^\sqrt{\tau} n_1 S_r \cdot e^{-D_0 x^2} \cdot \frac{(r-D_0+\frac{1}{2}\sigma^2)^2 x^2}{2\sigma^2} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma^2} |dx| 

- \frac{1}{2} F(\tau) + \int_0^\sqrt{\tau} F(\tau-x^2) \cdot e^{-r x^2} \cdot \frac{(r-D_0-\frac{1}{2}\sigma^2)^2 x^2}{2\sigma^2} \cdot \frac{r-D_0 - \frac{1}{2}\sigma^2}{\sigma^2} |dx| \]

\[ \leq \frac{1}{2} n_1 S_r + n_1 S_r \cdot \frac{\sqrt{\tau}}{\sqrt{\sigma}} \cdot \frac{r-D_0 + \frac{1}{2}\sigma^2}{\sigma}. \]
\[-\frac{1}{2} F(\tau) + \max\{F(\xi) | \xi \in [0, \tau]\} \cdot \frac{\sqrt{\tau}}{\sqrt{2\pi}} \cdot |\frac{r - D_0 - \frac{1}{2}\sigma^2}{\sigma}|, \quad (C.37)\]

From the definition of the function $F(\tau)$, we know that $F(\tau) \geq n_2 S_r > n_1 S_r$ holds for any $\tau$, and thus we have

\[
\int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi\sigma^2(\tau - \xi)}} \cdot e^{-\frac{(r - D_0 - \frac{1}{2}\sigma^2)^2(\tau - \xi)}{2\sigma^2}} d\xi < \frac{1}{2} (n_1 - n_2) S_r + \max\{F(\xi) | \xi \in [0, \tau]\} \cdot \frac{2\sqrt{\tau}}{\sqrt{2\pi}} \cdot \max\{|\frac{r - D_0 - \frac{1}{2}\sigma^2}{\sigma}|, |\frac{r - D_0 + \frac{1}{2}\sigma^2}{\sigma}|\}.
\]

Considering the fact that there exists $\tau_0$, s.t.

\[
\sqrt{\tau} \cdot \max\{F(\xi) | \xi \in [0, \tau]\} < \frac{\sqrt{\pi}(n_2 - n_1) S_r}{2 \sqrt{2} \max\{|\frac{r - D_0 - \frac{1}{2}\sigma^2}{\sigma}|, |\frac{r - D_0 + \frac{1}{2}\sigma^2}{\sigma}|\}} \quad (C.38)
\]

for any $0 < \tau < \tau_0$, it is straightforward that

\[
\int_0^\tau \frac{1}{2} \sigma^2 G(\xi) S_r \cdot \frac{e^{-r(\tau - \xi)}}{\sqrt{2\pi\sigma^2(\tau - \xi)}} \cdot e^{-\frac{(r - D_0 - \frac{1}{2}\sigma^2)^2(\tau - \xi)}{2\sigma^2}} d\xi < 0 \quad (C.39)
\]

for any $0 < \tau < \tau_0$. Therefore, one can certainly reach the conclusion that there exists $0 < \tau_* \leq \tau_0$, s.t. $G(\tau) < 0$ for any $0 < \tau < \tau_*$. This has completed the proof.
Appendix D

Appendix for Chapter 6

D.1 Appendix D.1

Recall the FDE in our system

\[
\frac{\partial V_{i,j}^{(n)}}{\partial \tau} = a_j \delta_{xx} V_{i,j}^{(n)} + b_j \delta_{nn} V_{i,j}^{(n)} + c_j \delta_{nn} V_{i,j}^{(n)} + [d_j + \lambda_j] \delta_i V_{i,j}^{(n)} + e_j \delta_{i} V_{i,j}^{(n)} - r V_{i,j}^{(n)}, \quad (D.1)
\]

with the implicit Euler scheme applied to the time derivative function, we can obtain

\[
\frac{V_{i,j}^{(n+1)} - V_{i,j}^{(n)}}{\Delta \tau} = a_j \delta_{xx} V_{i,j}^{(n+1)} + b_j \delta_{nn} V_{i,j}^{(n+1)} + c_j \delta_{nn} V_{i,j}^{(n+1)} + [d_j + \lambda_j] \delta_i V_{i,j}^{(n+1)} + e_j \delta_{i} V_{i,j}^{(n+1)} - r V_{i,j}^{(n+1)}. \quad (D.2)
\]

If we define

\[
A_0 = \Delta \tau \cdot c_j \delta_{xy}, \quad (D.3)
\]

\[
A_1 = \Delta \tau [a_j \delta_{xx} + (d_j + \lambda_j) \delta_i - \frac{r}{2} I], \quad (D.4)
\]

\[
A_2 = \Delta \tau [b_j \delta_{xy} + e_j \delta_{i} - \frac{r}{2} I], \quad (D.5)
\]

then the PDE can be derived

\[
[\mathbb{I} - (A_0 + A_1 + A_2)] V_{i,j}^{(n+1)} = V_{i,j}^{(n)} + O((\Delta \tau)^2). \quad (D.6)
\]

Similarly, if the explicit Euler scheme used instead of the implicit one, we obtain

\[
[\mathbb{I} + (A_0 + A_1 + A_2)] V_{i,j}^{(n)} = V_{i,j}^{(n+1)} + O((\Delta \tau)^2). \quad (D.7)
\]

Therefore, the weighted average of these two scheme can be displayed

\[
[\mathbb{I} - \phi (A_0 + A_1 + A_2)] V_{i,j}^{(n+1)} = [\mathbb{I} + (1 - \phi) (A_0 + A_1 + A_2)] V_{i,j}^{(n)} + O((\Delta \tau)^2). \quad (D.8)
\]
It should be noted that when $\phi$ equals to zero and one, the above equation is as same as the explicit Euler scheme and the implicit Euler scheme, respectively. In addition, the Crank-Nicolson scheme is derived when $\phi$ equals to $1/2$.

Now, adding $\phi^2 A_1 A_2 V_{i,j}^{(n+1)}$ to both sides of the above equation

\[
[\mathbb{I} - \phi A_0 - \phi A_1 - \phi A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n+1)} = \mathbb{I} [\mathbb{I} + (1 - \phi) A_0 + (1 - \phi) A_1 + (1 - \phi) A_2] V_{i,j}^{(n)} + \phi^2 A_1 A_2 V_{i,j}^{(n+1)} + O((\Delta \tau)^2)
\]

\[
\Rightarrow [\mathbb{I} - \phi A_0 - \phi A_1 - \phi A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n+1)} = \mathbb{I} [\mathbb{I} + (1 - \phi) A_0 + (1 - \phi) A_1 + (1 - \phi) A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n)} + \phi^2 A_1 A_2 (V_{i,j}^{(n+1)} - V_{i,j}^{(n)}) + O((\Delta \tau)^2)
\]

\[
\Rightarrow [\mathbb{I} - \phi A_0 - \phi A_1 - \phi A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n+1)} = \mathbb{I} [\mathbb{I} + A_0 + (1 - \phi) A_1 + (1 - \phi) A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n)} + \phi A_0 (V_{i,j}^{(n+1)} - V_{i,j}^{(n)}) + O((\Delta \tau)^2)
\]

\[
\Rightarrow [\mathbb{I} - \phi A_1 - \phi A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n+1)} = \mathbb{I} + A_0 + (1 - \phi) A_1 + (1 - \phi) A_2 + \phi^2 A_1 A_2] V_{i,j}^{(n)} + \phi A_0 (V_{i,j}^{(n+1)} - V_{i,j}^{(n)}) + O((\Delta \tau)^2)
\]

\[
\Rightarrow (\mathbb{I} - \phi A_1)(\mathbb{I} - \phi A_2) V_{i,j}^{(n+1)} = \mathbb{I} + A_0 + (1 - \phi) A_1 + A_2] V_{i,j}^{(n)} - (\mathbb{I} - \phi A_1) \phi A_2 V_{i,j}^{(n)},
\]

where two mergers appear since $\phi^2 A_1 A_2 (V_{i,j}^{(n+1)} - V_{i,j}^{(n)}) \sim O((\Delta \tau)^3)$ and $\phi A_0 (V_{i,j}^{(n+1)} - V_{i,j}^{(n)}) \sim O((\Delta \tau)^2)$.

In summary, the linear operators $A_0, A_1$ and $A_2$ at the $(n+1)$th time step are

\[
A_0 = \Delta \tau \cdot c_j \delta_{xv}
\]

\[
= \Delta \tau \cdot [\rho \sigma v_j - \sigma^2 v_j \xi_j] \delta_{xv},
\]

\[
A_1 = \Delta \tau [\sigma c_{\chi} + (d_j + \lambda_j) \delta_h - \frac{r}{2} \mathbb{I}]
\]

\[
= \Delta \tau [(\frac{1}{2} v_j + \frac{1}{2} \sigma^2 v_j \xi_j - \rho \sigma v_j \xi_j) \delta_{xx} + (\frac{1}{2} v_j + \frac{1}{2} \sigma^2 v_j \xi_j - \frac{1}{2} \sigma^2 v_j \beta_j + r - D_0 - \kappa (\eta - v_j) \xi_j + \lambda_j) \delta_h - \frac{r}{2} \mathbb{I}],
\]

\[
A_2 = \Delta \tau [\beta_j \delta_{hv} + c_j \delta_h - \frac{r}{2} \mathbb{I}]
\]

\[
= \Delta \tau [\frac{1}{2} \sigma^2 v_j \delta_{hv} + \kappa (\eta - v_j) \delta_h - \frac{r}{2} \mathbb{I}],
\]

where

\[
\xi_j = \frac{\delta_t (\phi S_{f}^{(n+1)}(j) + (1 - \phi) S_{f}^{(n)}(j))}{\phi S_{f}^{(n+1)}(j) + (1 - \phi) S_{f}^{(n)}(j)},
\]
\[ \beta_j = \frac{\delta v_i (\phi S^{(n+1)}_j) + (1 - \phi) S^{(n)}_j(j)}{\phi S^{(n+1)}_j(j) + (1 - \phi) S^{(n)}_j(j)}, \]
\[ \lambda_j = \frac{S^{(n+1)}_j(j) - S^{(n)}_j(j)}{(\phi S^{(n+1)}_j(j) + (1 - \phi) S^{(n)}_j(j)) \Delta \tau}. \]

D.2 Appendix D.2

\[ \Delta Y_j = P_j + \mathbb{B} x_j, \]

with

\[ \mathbb{A} = \begin{pmatrix}
1 + \phi \left( \frac{2 a \Delta \tau}{\Delta x^2} \right) & -\phi \left( \frac{a \Delta \tau}{\Delta x^2} + \frac{d_j \Delta \tau}{2 \Delta x} \right) \\
-\phi \left( \frac{a \Delta \tau}{\Delta x^2} - \frac{d_j \Delta \tau}{2 \Delta x} \right) & 1 + \phi \left( \frac{2 a \Delta \tau}{\Delta x^2} + \frac{d_j \Delta \tau}{2 \Delta x} \right) \\
\ddots & \ddots \\
-\phi \left( \frac{a \Delta \tau}{\Delta x^2} - \frac{d_j \Delta \tau}{2 \Delta x} \right) & 1 + \phi \left( \frac{2 a \Delta \tau}{\Delta x^2} + \frac{d_j \Delta \tau}{2 \Delta x} \right)
\end{pmatrix}_{N_x \times N_x}, \]

\[ Y_j = (Y_{1,j}, Y_{2,j}, \ldots, Y_{N_x - 1,j})^T, \]

\[ P_j = (P_{1,j}, P_{2,j}, \ldots, P_{N_x - 1,j})^T, \]

\[ \mathbb{B} x_j = \begin{pmatrix}
\phi \left( \frac{a \Delta \tau}{\Delta x^2} - \frac{d_j \Delta \tau}{2 \Delta x} \right) Y_{0,j} \\
\vdots \\
\phi \left( \frac{a \Delta \tau}{\Delta x^2} + \frac{d_j \Delta \tau}{2 \Delta x} \right) Y_{N_x - 1,j}
\end{pmatrix}_{N_x \times 1}, \]

\[ P_{i,j} = [\mathbb{I} + A_0 + (1 - \phi) A_1 + A_2] V^{(n)}_{i,j}, \]

\[ = V^{(n)}_{i,j} + \frac{c_j \Delta \tau}{4 \Delta x \Delta v} (V^{(n)}_{i+1,j+1} - V^{(n)}_{i+1,j-1} - V^{(n)}_{i-1,j+1} + V^{(n)}_{i-1,j-1}) + (1 + \phi) \left( \frac{a \Delta \tau}{(\Delta v)^2} - \frac{d_j \Delta \tau}{2 \Delta x} \right) V^{(n)}_{i,j} + (1 - \phi) \left( - \frac{a \Delta \tau}{(\Delta x)^2} - \frac{r \Delta \tau}{2} \right) V^{(n)}_{i,j} + (1 - \phi) \left( \frac{a \Delta \tau}{(\Delta v)^2} + \frac{d_j \Delta \tau}{2 \Delta x} \right) V^{(n)}_{i,j+1} + (1 + \phi) \left( \frac{a \Delta \tau}{(\Delta v)^2} + \frac{r \Delta \tau}{2} \right) V^{(n)}_{i,j+1} + \left( \frac{b \Delta \tau}{(\Delta v)^2} - \frac{r \Delta \tau}{2} \right) V^{(n)}_{i,j+1} + \left( \frac{b \Delta \tau}{(\Delta v)^2} + \frac{r \Delta \tau}{2} \right) V^{(n)}_{i,j+1}, \]

where \( d_j' = d_j + \lambda_j \).

D.3 Appendix D.3

\[ \mathbb{C} V^{(n+1)}_{i} = Q_i + \mathbb{B} V_i, \]
with

$$C = \begin{pmatrix}
1 + \phi \left( \frac{2b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) & -\phi \left( \frac{b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) \\
-\phi \left( \frac{b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) & 1 + \phi \left( \frac{2b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) \\
\vdots & \\
-\phi \left( \frac{b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) & -\phi \left( \frac{b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right) & 1 + \phi \left( \frac{2b_j \Delta \tau}{(\Delta \nu)^2} + \frac{\Delta \tau}{2} \right)
\end{pmatrix}_{N_r \times N_r}$$

\[D.24\]

$$V_i^{(n+1)} = (V_i^{(n+1)}, V_i^{(n+1)}, \ldots, V_i^{(n+1)})^T, \quad \text{and} \quad \text{D.23}\]

\[Q_i^{(n+1)} = (q_i, q_i, \ldots, q_i, \ldots)^T, \quad \text{D.25}\]

\[\mathbb{B} v = \left( \begin{array}{c}
\phi \frac{b_j \Delta \tau - e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n+1)} \\
\vdots \\
\phi \frac{b_j \Delta \tau + e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n+1)}
\end{array} \right)_{N_r \times 1}, \quad \text{D.26}\]

\[q_{i,j} = y_{i,j} - \phi \left( \frac{b_j \Delta \tau - e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n)} \right) - \phi \left( \frac{b_j \Delta \tau + e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n)} \right) - \phi \left( \frac{b_j \Delta \tau - e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n)} \right) - \phi \left( \frac{b_j \Delta \tau + e_j \Delta \tau}{2(\Delta \nu)} V_i^{(n)} \right) V_i^{(n)} + r_{i,j}. \quad \text{D.27}\]

### D.4 Appendix D.4

\[A_0 = \Delta \tau \cdot c_j \delta_{\epsilon r} \]

\[= \Delta \tau \cdot [\sigma \rho \xi \sqrt{\tau_j} - \xi^2 r_j \xi_j] \delta_{\epsilon r}, \quad \text{D.28}\]

\[A_1 = \Delta \tau [a_j \delta_{r x} + (d_j + \lambda_j) \delta_r - \frac{r_j}{2}] \]

\[= \Delta \tau \left( \frac{1}{2} \sigma^2 + \frac{1}{2} \xi^2 r_j \xi_j^2 - \sigma \rho \xi \sqrt{\tau_j} \xi_j \right) \delta_{r x} + \left( - \frac{1}{2} \sigma^2 + \frac{1}{2} \xi^2 r_j \xi_j^2 - \frac{1}{2} \xi^2 r_j \beta + r_j - D_0 - \kappa (\eta - r_j) \xi_j + \lambda_j \right) \delta_r - \frac{r_j}{2} \right], \quad \text{D.29}\]

\[A_2 = \Delta \tau [b_j \delta_{rr} + e_j \delta_r - \frac{r_j}{2}] \]

\[= \Delta \tau \left( \frac{1}{2} \xi^2 r_j \delta_{rr} + \kappa (\eta - r_j) \delta_r - \frac{r_j}{2} \right), \quad \text{D.30}\]

where

\[\xi_j = \frac{\delta_{r} (\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j))}{\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j)} \]

\[\beta_j = \frac{\delta_{r} (\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j))}{\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j)} \]

\[\lambda_j = \frac{\delta_{r} (\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j))}{\phi S^{(n+1)}_c(j) + (1 - \phi) S^{(n)}_c(j)} \Delta \tau. \quad \text{D.31}\]
D.5 Appendix D.5

\[ AY_j = P_j + \mathbb{B}x_j, \]  
(D.34)

with

\[
A = \begin{pmatrix}
1 + \phi(\frac{2a_i\Delta r}{h^2} + \frac{\Delta r}{r}) & -\phi(\frac{2a_i\Delta r}{h^2} + \frac{\Delta r}{r}) & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) \\
-\phi(\frac{a_i\Delta t}{h^2} - \frac{d_j\Delta t}{h^2}) & 1 + \phi(\frac{2a_i\Delta r}{h^2} + \frac{\Delta r}{r}) & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\phi(\frac{a_i\Delta t}{h^2} - \frac{d_j\Delta t}{h^2}) & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) & 1 + \phi(\frac{2a_i\Delta r}{h^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) \\
-\phi(\frac{a_i\Delta t}{h^2} - \frac{d_j\Delta t}{h^2}) & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) & -\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r}) & 1 + \phi(\frac{2a_i\Delta r}{h^2} + \frac{\Delta r}{r}) & \cdots
\end{pmatrix}_{N_t \times N_t}
\]

\[ Y_j = (Y_{1,j}, Y_{2,j}, \cdots, Y_{N_t,j})^T, \]  
(D.35)

\[ P_j = (P_{1,j}, P_{1,j}, \cdots, P_{N_t,j})^T, \]  
(D.36)

\[
\mathbb{B}x_j = \begin{pmatrix}
\phi(\frac{a_i\Delta t}{h^2} - \frac{d_j\Delta t}{h^2})Y_{0,j} \\
\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r})Y_{0,j} \\
\phi(\frac{a_i\Delta t}{h^2} - \frac{d_j\Delta t}{h^2})Y_{0,j} \\
\phi(\frac{a_i\Delta t}{h^2} + \frac{\Delta r}{r})Y_{0,j}
\end{pmatrix}_{N_t-1 \times 1},
\]

(D.37)

\[ P_{t,j} = \llbracket \mathbb{I} + A_0 + (1 - \phi)A_1 + A_2 \rrbracket |V_{i,j}^{(n)}| 
= V_{i,j}^{(n)} + \frac{c_j\Delta \tau}{4\Delta \tau h} (V_{i,j+1}^{(n)} - V_{i,j-1}^{(n)}) - V_{i-1,j+1}^{(n)} + V_{i-1,j-1}^{(n)} + V_{i,j+1}^{(n)} + V_{i,j-1}^{(n)} 
+ (1 + \phi)\left(\frac{a_i\Delta t}{h^2} \Delta r + \frac{d_j\Delta t}{h^2}\right) V_{i-1,j}^{(n)} + (1 - \phi)\left(-\frac{a_i\Delta t}{h^2} \Delta r + \frac{\Delta r}{h^2}\right) V_{i,j}^{(n)} 
+ (1 - \phi)\left(\frac{b_j\Delta t}{(\Delta r)^2} \Delta r + \frac{d_j\Delta t}{h^2}\right) V_{i+1,j}^{(n)} + (\frac{b_j\Delta t}{(\Delta r)^2} \Delta r - \frac{c_j\Delta t}{(\Delta r)^2} \Delta r) V_{i,j+1}^{(n)} + \Delta r \Delta \tau \frac{2}{r^2} V_{i,j}^{(n)} + \Delta r \Delta \tau \frac{2}{r^2} V_{i,j+1}^{(n)},
\]

(D.38)

where \( d_j^2 = d_j + \lambda_j \).

D.6 Appendix D.6

\[ \mathbb{C}V_{i}^{(n+1)} = Q_i + \mathbb{B}r_i, \]  
(D.40)

with

\[
C = \begin{pmatrix}
1 + \phi(\frac{2b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) \\
-\phi(\frac{b_i\Delta t}{(\Delta r)^2} - \frac{c_i\Delta t}{(\Delta r)^2}) & 1 + \phi(\frac{2b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\phi(\frac{b_i\Delta t}{(\Delta r)^2} - \frac{c_i\Delta t}{(\Delta r)^2}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & 1 + \phi(\frac{2b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & \cdots & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) \\
-\phi(\frac{b_i\Delta t}{(\Delta r)^2} - \frac{c_i\Delta t}{(\Delta r)^2}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & -\phi(\frac{b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & 1 + \phi(\frac{2b_i\Delta t}{(\Delta r)^2} + \frac{\Delta r}{r}) & \cdots
\end{pmatrix}_{N_t \times N_t}
\]
\[
V_{i}^{(n+1)} = (V_{i,1}^{(n+1)}, V_{i,2}^{(n+1)}, \ldots, V_{i,N_r-1}^{(n+1)})^T,
\]
(D.41)

\[
Q_{i}^{(n+1)} = (q_{i,1}, q_{i,2}, \ldots, q_{i,N_r-1})^T,
\]
(D.42)

\[
E v_i = \begin{pmatrix}
\phi \left( \frac{b_i \Delta \tau}{\Delta r} - \frac{e_j \Delta \tau}{2\Delta r} \right) V_{i,0}^{(n+1)} \\
\vdots \\
\phi \left( \frac{b_i \Delta \tau}{\Delta r} + \frac{e_j \Delta \tau}{2\Delta r} \right) V_{i,N_r}^{(n+1)}
\end{pmatrix},
\]
(D.43)

\[
q_{i,j} = Y_{i,j} - \phi \left( \frac{b_j \Delta \tau}{(\Delta r)^2} - \frac{e_j \Delta \tau}{2\Delta r} \right) V_{i,j-1}^{(n)} - \phi \left( -2 \frac{b_j \Delta \tau}{(\Delta r)^2} - \frac{r \Delta \tau}{2} \right) V_{i,j}^{(n)} - \phi \left( \frac{b_j \Delta \tau}{(\Delta r)^2} + \frac{e_j \Delta \tau}{2\Delta r} \right) V_{i,j+1}^{(n)}.
\]
(D.44)