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Exponential-Family Random Graph Models for Rank-Order Relational Data

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Biography

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Abstract

Rank-order relational data, in which each actor ranks the others according to some criterion, often arise from sociometric measurements of judgment or preference. We propose a general framework for representing such data, define a class of exponential-family models for rank-order relational structure, and derive sufficient statistics for interdependent ordinal judgments that do not require the assumption of comparability across raters. These statistics allow estimation of effects for a variety of plausible mechanisms governing rank structure, both in cross-sectional context and evolving over time. We apply this framework to model the evolution of liking judgments in an acquaintance process and to model recall of relative volume of interpersonal interaction among members of a technology education program.

1 Introduction

Rank-order sociometric data—in which each actor in a network ranks the others according to some criterion—have a long history in the social sciences. Among many other instances, Sampson (1968) famously asked each of 18 novitiates in a monastery to rank his three most liked novitiates among the other 17; Newcomb (1961) measured evolving rankings of each other by 17 men living in fraternity-style housing over the course of a semester; and, more recently, Wave I of National Longitudinal Study of Adolescent Health asked high school students to list, in order, up to 5 male and up to 5 female friends (Harris et al. 2003). While many network processes (e.g., diffusion, brokerage, exchange) are only sensibly posited for networks with categorical or ratio scale relationship states, many others—particularly those involving personal preferences (e.g., liking, advice-seeking)—are much more readily represented ordinally, and, indeed, may not even have interval, ratio, or categorical meaning across raters. This last can be true even when data is not collected in an explicitly ordinal fashion. For instance, Johnson, Boster, and Palinkas (2003) asked personnel in an isolated environment (the Amundsen–Scott South Pole Station) to rate their degree of interaction with each other on a 0–10 scale, with 0 indicating no interaction and 10 indicating a "great deal" of interaction. While it is reasonable to assume that such ratings are ordinally coherent within rater (e.g., if Bob rates Sally below Jill, then Bob regards himself as interacting more with Jill than with Sally), such ratings cannot be compared *across* raters: if Bob rates his interaction with Jill at 4, and Sally rates her interaction with Jill at 6, we have no basis for concluding that Sally's interaction with Jill is stronger than Bob's. Such interaction rating data is thus "local" to the rater, and must be analyzed in a manner that avoids cross-rater comparisons.

The most common approach taken to analyzing rank-valued network data in current practice is to dichotomize ranks into binary ties, defining a tie to be present if a given ego had ranked a given alter above a certain cut-off and absent otherwise. Many methods of dichotomizing have been proposed. For instance, cut-offs have been set at a particular rank (e.g., top 5) (Breiger, Boorman, and Arabie 1975; White, Boorman, and Breiger 1976; Arabie, Boorman, and Levitt 1978; Wasserman 1980; Pattison 1982; Harris et al. 2003, for example); at a particular quantile (e.g., top 50%) (Krackhardt and Handcock 2007); or have been found adaptively (Doreian, Kapuscinski, Krackhardt, and Szczypula 1996). Another common approach is to focus on rank correlations and on treating ranks on an additive scale (Newcomb 1956; Nakao and Romney 1993).

These approaches come with significant limitations. Dichotomizing ties requires a threshold point to be selected, inevitably discarding information and possibly introducing biases (Thomas and Blitzstein 2011), while methods like rank correlation are limited to simple comparisons and cannot, for example, be used to examine the strength of one social factor after controlling for the effects of another. *More importantly, these techniques implicitly assume that tie values can be equated across raters, an assumption that is often unjustified.* When the presence of an (i, j) tie has a different empirical meaning than the presence of a (k, j) tie, conventional network analytic techniques (e.g., centrality indices) may prove misleading.

Modeling frameworks explicitly designed for rank-order data would address these limitations, but to date work on model-based approaches to rank-order network data has been very limited. Gormley and Murphy (2008), for instance, use a generalization of the Plackett–Luce model (Plackett 1975) in a latent position framework to model what can be viewed as a bipartite rank-order network of affiliations from voters to candidates in Irish proportional representation through the single transferable vote elections. Null models for comparison of rank-order (or otherwise valued) data structures were developed by Hubert (1987), and model-based extensions of this approach for comparison of multiple structures have been introduced by Butts (2007b). (See also related work on null hypothesis testing in a network regression context, e.g., that of Krackhardt (1987) and Dekker, Krackhardt, and Snijders (2007).) This latter work is focused on modeling degrees of correspondence *between* relational structures, and does not attempt to model the internal properties of rank-valued networks themselves. This second problem is the focus of the present paper.

For modeling of internal network structure, exponential-family random graph (ERG) or p^* models (Holland and Leinhardt 1981; Wasserman and Pattison 1996; Robins, Pattison, and Wasserman 1999) are the currently favored approach. Models parameterized in this way have been applied to social network data in a variety of contexts, including dichotomized rank-order data (Krackhardt and Handcock 2007; Goodreau, Kitts, and Morris 2008b); used in this latter capacity, they inherit the difficulties with dichotomizing noted above. Robins et al. (1999) were the first to introduce a systematic treatment of ERGMs for categorically valued network data, along with procedures for approximate inference using pseudo-likelihood estimation. The model families they propose can be used when edge values are ordinal in an absolute sense, but assume (1) that values can be meaningfully equated across raters, and (2) that there is a well-defined zero-value indicating the absence of a tie, that is qualitatively different from other possible tie values (and, per (1), equivalent across raters). Building on this work, Krivitsky (2012) formulated a generalized framework for exponential-family models on networks whose ties have values (categorical or otherwise) and introduced Markov chain Monte Carlo (MCMC) methods for simulation and maximum-likelihood inference in this more general case. The Krivitsky formulation provides a basis for generalizing to models of the kind we consider here, but retains the assumption of “absolute” edge values that have a constant meaning across raters.

In this paper, we develop ERG models for “locally” ordinal relational data in which ratings cannot be directly compared across subjects; we focus on the foundational case of complete rankings, but introduce model terms that can be used with more general models (e.g., for partial orders). In Section 2, we discuss representation of ordinal relational data and introduce the probabilistic framework for exponential-family models for them, and in Section 3, we describe statistics that can be used to model common network properties within this framework. Two applications of this framework are demonstrated in Section 4, and some extensions of the framework are discussed in Section 5. Additional details involved in simulation and inference on these models are provided in Appendix A.

2 Exponential-Family Framework for Ordinal Relational Data

2.1 Actors, Rankings, and Comparisons

We begin this section by defining notation for representation of ordinal relational data and by establishing basic principles for using such data in a manner that respects its intrinsic measurement properties.

Consider a set of n actors, N, whom we index as $N = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \dots\}$. Each actor in N will be a rater in our network of interest; except as noted otherwise, the objects being rated by each rater are the other members of N (though we will consider other possibilities in Section 5.1). For various purposes, it will be helpful to have specific notation to refer to the set of possible p-tuples of distinct actors in N; we refer to this as the pth "distinct Cartesian power" of N. Recall that the pth (ordinary) Cartesian power of a set N is the set of all ordered p-tuples of individuals from N. For example, if $N = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$, then second Cartesian power of N is $N^2 = \{(\mathrm{A}, \mathrm{A}), (\mathrm{A}, \mathrm{B}), (\mathrm{A}, \mathrm{C}), (\mathrm{B}, \mathrm{A}), (\mathrm{B}, \mathrm{B}), (\mathrm{B}, \mathrm{C}), (\mathrm{C}, \mathrm{A}), (\mathrm{C}, \mathrm{B}), (\mathrm{C}, \mathrm{C})\}$. By analogy, we denote the pth distinct Cartesian power of N (the set containing all p-tuples of N whose elements do not repeat) by $N^{p\neq} \subset N^p$.[1] Following our example, then, the second *distinct* Cartesian power of N is given by $N^{2\neq} = \{\cancel{(\mathrm{A}, \mathrm{A})}, (\mathrm{A}, \mathrm{B}), (\mathrm{A}, \mathrm{C}), (\mathrm{B}, \mathrm{A}), \cancel{(\mathrm{B}, \mathrm{B})}, (\mathrm{B}, \mathrm{C}), (\mathrm{C}, \mathrm{A}), (\mathrm{C}, \mathrm{B}), \cancel{(\mathrm{C}, \mathrm{C})}\}$.

As discussed in the introduction, our data consist of observations in which each actor (*ego*) in the network $i \in N$ provides some ranking or ordering of the other actors (*alters*), and, possibly, of himself or herself. In other words, each actor i defines an ordinal relation $\overset{i}{\succ}$ over set N. This relation could represent "preferred to", "interacted more with than", "judged to be taller than", or any other judgment of interest. Importantly, we note that for two egos i and j, $\overset{i}{\succ}$ need not equal $\overset{j}{\succ}$, and the ratings of some alter k by i and j cannot be compared directly; we may, however, meaningfully ask whether, e.g. $(k \overset{i}{\succ} l) = (k \overset{j}{\succ} l)$—whether i and j's rankings of k and l are concordant—and our modeling framework is founded on exactly these distinctions.

A simple example of a ranking structure is provided in Figure 1, in which actor A (ego) ranks (alter) D above B and C and C above B. In our above notation, this corresponds to D $\overset{\mathrm{A}}{\succ}$ C, C $\overset{\mathrm{A}}{\succ}$ B, and D $\overset{\mathrm{A}}{\succ}$ B. The presence of corresponding structures associated with Egos B, C, and D (not illustrated, but shown in the rank matrix of Figure 1, right) results in a complete ordinal network.

[Figure 1 about here.]

In general, we make few assumptions regarding $\overset{i}{\succ}$. Assuming that ego may not include self in the ranking, we require that for all ego–alter–alter triples $(i, j, k) \in N^{3\neq}$, i reports either $j \overset{i}{\succ} k$ or $j \overset{i}{\not\succ} k$. With the additional assumption of transitivity ($j \overset{i}{\succ} k \wedge k \overset{i}{\succ} l \implies j \overset{i}{\succ} l$), the above formulation leads to a partial ordering of alters by each ego. If incomparability is also transitive ($j \overset{i}{\not\succ} k \wedge k \overset{i}{\not\succ} l \implies j \overset{i}{\not\succ} l$), a weak ordering results, which may be used to represent rank data in which "ties" are allowed. Finally, a further constraint on $\overset{i}{\succ}$ that $j \overset{i}{\not\succ} k \implies k \overset{i}{\succ} j$ results in a complete ordering, which may arise when an ego is forced to rank all of the alters with no equal ranks permitted. This is an important special case, and we consider it here in more detail.

We will, furthermore, focus on the case where the ego does not report a ranking for self and where the set of egos is the same as the set of alters (e.g., people ranking other people who rank them in turn, as opposed to e.g. consumers ranking brands or other objects), so relation $j \overset{i}{\succ} k$ is meaningful only for $(i, j, k) \in N^{3\neq}$. If an ego is permitted to rank itself amongst the others, it is also possible for j or k (but not both) to equal i: $(i, j, k) \in N \times N^{2\neq}$, and if we call the set of people in the people-ranking-objects scenario N_{e} and the set of objects N_{a}, then $(i, j, k) \in N_{\mathrm{e}} \times N_{\mathrm{a}}^{2\neq}$. This last case is discussed further in Section 5.1.

Rankings are not assumed to be comparable across egos: for each ego i, we may only say that one alter is "$\overset{i}{\succ}$" another (or that this does not hold). To concisely represent this fundamental operation in a specific ordinal network $\boldsymbol{y}$, we define an indicator

$$y_{i:\, j \succ k} \equiv \begin{cases} 1 & \text{if } j \overset{i}{\succ} k \text{ i.e., } i \text{ ranks } j \text{ above } k; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

This represents the basic distinction that can be unequivocally made from locally ordinal data. As we shall see in Section 3, being limited to such statements does not prevent us from specifying a very rich class of models.

2.2 Representations of Ordinal Networks

We make use of two numerical representations of the observed networks of rankings, which we illustrate on toy networks with $n = 4$: a network of complete orderings $\boldsymbol{y}$ in Figure 2a and a network of idiosyncratic ranking structures $\boldsymbol{y}^\star$ in Figure 3. Figure 2 also shows representations of two perturbations ((b) and (c)) of $\boldsymbol{y}$. We make use of them in subsequent sections on model terms and interpretations.

[Figure 2 about here.]

[Figure 3 about here.]

Firstly, a network of complete or a weak ordering $\boldsymbol{y}$ may be encoded simply as a row in an $n \times n$-matrix of ranks that we denote $\boldsymbol{y}_{\cdot,\cdot}$, with $y_{i,j}$ being the rank by ego i of alter j. Consider the ranking structure in network $\boldsymbol{y}$ in Figure 2a. As illustrated in Figure 1, the ranking reported by Ego A can be expressed by assigning the highest possible rank $n - 1 = 3$ to D, the highest-ranked alter, $n - 2 = 2$, to C, the alter with the next highest ranking, and 1 to B, the actor with the lowest ranking. This representation is slightly misleading, in that these ranks are not comparable across egos (rows). If the ordering is weak (i.e., there are ties), alters may share ranks, as with Ego A ranking Alters B and C equally in the idiosyncratic network $\boldsymbol{y}^\star$ in Figure 3. The diagonal of $\boldsymbol{y}_{\cdot,\cdot}$ is undefined, unless the egos also rank themselves in the data as Egos B and C do in Figure 3.

Secondly, we may represent the comparisons reported by an ego i, or implied by i's ranking, in a binary $n \times n$-matrix of pairwise comparisons, which we denote $\boldsymbol{y}_{i:\cdot \succ \cdot}$. Figure 1 shows how the reported complete ranking can be thus encoded: D is ranked over both B and C, therefore the corresponding elements in the matrix $\boldsymbol{y}_{\mathrm{A:D} \succ \mathrm{B}} = \boldsymbol{y}_{\mathrm{A:D} \succ \mathrm{C}} = 1$, while C is not ranked over D, so $\boldsymbol{y}_{\mathrm{A:D} \succ \mathrm{C}} = 0$. Since A does not rank self, the row and column corresponding to A are undefined, and since it is not meaningful to compare an alter to itself, so is the diagonal. (Notably, if A were allowed to rank self, A's row and column would be defined, and if A had then *chosen* to not rank self, they would be set to 0.) The collection of reported rankings by all the egos in N can then then be combined into a binary $n \times n \times n$-array.

As we have noted, because the framework itself requires relatively few assumptions regarding $\overset{i}{\succ}$, pairwise comparison matrices can encode a wider variety of ranking structures: for example, in $\boldsymbol{y}^{\star}$ in Figure 3, Ego C does not provide enough information to establish a weak order, while Ego D's reports violate transitivity of comparisons. This precludes their representation in the corresponding rows of $\boldsymbol{y}^{\star}_{\cdot,\cdot}$ but not their representation as $\boldsymbol{y}^{\star}_{\mathrm{C:}\cdot \succ \cdot}$ and $\boldsymbol{y}^{\star}_{\mathrm{D:}\cdot \succ \cdot}$

2.3 Model Formulation and Specification for ERGMs for Compete Rankings

Krivitsky (2012) suggests that a sample space of complete rankings of every actor in a network by every other actor can be represented by a directed network with no self-loops, whose set of observed relations $\mathbb{Y} = N^{2\neq}$ maps to dyad values $\mathbb{S} = \{1 \mathinner{..} n-1\}$, with the ranked nature of the data leading to a complex constraint: that an ego i *must* assign a unique rank to each possible alter. Formally,

$$\mathcal{Y} = \{\boldsymbol{y}' \in \mathbb{S}^{\mathbb{Y}} : \forall_{i \in N} \forall_{r \in \mathbb{S}} \exists!_{j \in N \setminus \{i\}} y'_{i,j} = r\}. \tag{2}$$

Again, this representation is slightly misleading in that elements of $\mathbb{S}$ have only ordinal and not interval or ratio meanings, and, as we noted above, are only ordered within the rankings of a given ego. That is, it makes sense to ask if for some $\boldsymbol{y} \in \mathcal{Y}$ and $(i, j, k) \in N^{3\neq}$, $y_{i,j} > y_{i,k}$—i.e., whether i ranks j over k—but not to evaluate the difference between ranks ($y_{i,j} - y_{i,k}$) or to compare ranks by different egos ($y_{j,i} > y_{k,i}$). It does, however, represent distinct complete rankings in a concise and convenient manner, so we make use of it, with the proviso that *the statistics evaluated on* $\boldsymbol{y} \in \mathcal{Y}$ *make use of no operation other than comparison within an ego's rankings* $y_{i:\, j \succ k}$.

Taking the set defined by (2) as our sample space, we can specify an exponential family for rank-order networks by defining a sufficient statistic $\boldsymbol{g}(\boldsymbol{y}; \boldsymbol{x})$, a function of a network $\boldsymbol{y} \in \mathcal{Y}$ that may also depend on exogenous covariates $\boldsymbol{x} \in \mathbb{X}$ (assumed fixed and known), for an exponential family and parametrized by $\boldsymbol{\theta} \in \mathbb{R}^p$. The probability associated with each network $\boldsymbol{y}$ in the sample space is then

$$\Pr_{\boldsymbol{\theta};\boldsymbol{g},\boldsymbol{x}}(\boldsymbol{Y} = \boldsymbol{y}) = \frac{\exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}; \boldsymbol{x})\}}{\kappa_{\boldsymbol{g},\boldsymbol{x}}(\boldsymbol{\theta})},\ \boldsymbol{y} \in \mathcal{Y} \tag{3}$$

with the normalizing factor

$$\kappa_{\boldsymbol{g},\boldsymbol{x}}(\boldsymbol{\theta}) = \sum_{\boldsymbol{y}' \in \mathcal{Y}} \exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}'; \boldsymbol{x})\}. \tag{4}$$

This is an exact parallel to the more familiar ERGMs for dichotomous data (e.g., Wasserman and Pattison 1996). For notational convenience, we will drop $\boldsymbol{x}$ from now on, unless the term in question uses it explicitly.

3 Terms and Parameters for Ordinal Relational Data

We now introduce and discuss a variety of sufficient statistics $\boldsymbol{g}(\cdot;\cdot)$ for the model of (3) ("model terms") that abide by the restrictions discussed in Section 2 while viably representing phenomena frequently observed in social networks. For each term, we discuss how it may be interpreted without assuming rankings to have more than ordinal meaning; numerical examples are provided in Appendix B.

3.1 Interpreting Model Terms Using "Promotion" Statistics

For binary ERGMs, Snijders et al. (2006), Hunter et al. (2008b), and others have used *change statistics* or *change scores*, the effect on the model of a toggling of a tie—an atomic change in the binary network structure—to aid in interpreting the model terms. For complete ordering data, an ego changing the ranking of one alter necessarily changes the ranking of at least one other, and the atomic change—one that affects the fewest alters—is to swap the rankings of two who are adjacently ranked. We thus use the effect of having ego i "promote" a promotable alter $j \in \{k \in N : k \neq i \wedge y_{i,k} < n-1\}$, swapping j's rank with that of the alter previously ranked immediately above j, as such a change. When it is clear from context (as below), we will use "j^+" to refer to the promoted-over alter. (See Butts (2007b) for a related use of pairwise permutations to assess model terms.)

Let $\boldsymbol{y}^{i:\, j \rightleftarrows j^+}$ represent the network $\boldsymbol{y}$ with i's ranking of j and j^+, the actor previously ranked by i immediately above j, swapped. We define a *promotion statistic* as

$$\boldsymbol{\Delta}^{\nearrow}_{i,j}\boldsymbol{g}(\boldsymbol{y}) \equiv \boldsymbol{g}(\boldsymbol{y}^{i:\, j\rightleftarrows j^{+}}) - \boldsymbol{g}(\boldsymbol{y}),$$

i.e., the change in $\boldsymbol{g}$ resulting from "promoting" j by one rank (and demoting the alter above him or her). An example of this is given Figure 2: starting with a network $\boldsymbol{y}$ (a), Ego A promotes Alter B (who had been ranked immediately below C in $\boldsymbol{y}$) over C to create $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$ (b). Its effect on $\boldsymbol{y}_{\cdot,\cdot}$ is to swap the rank values $y_{\mathrm{A,B}}$ and $y_{\mathrm{A,C}}$, and the effect on the pairwise comparison matrix $\boldsymbol{y}_{\mathrm{A:}\,\cdot\succ\cdot}$ is the smallest possible, subject to the complete ordering constraint: the indicators $y_{\mathrm{A:\,B}\succ\mathrm{C}}$ and $y_{\mathrm{A:\,C}\succ\mathrm{B}}$ are swapped. In contrast, if Ego A promotes Alter B over D, who had not been adjacently ranked to B in $\boldsymbol{y}$, producing $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{D}}$ (c), the effect on $\boldsymbol{y}_{\cdot,\cdot}$ is similar to before, but the effect on $\boldsymbol{y}_{\mathrm{A:}\,\cdot\succ\cdot}$ is profound.

Analogously to change scores, the promotion statistic emerges when considering the conditional probability of an ego i ranking an alter j over alter j^{+}, other rankings being the same. To see how, let $\boldsymbol{y}$ be a network that has i ranking $k \equiv j^{+}$ immediately over j, so that

$\boldsymbol{y}^{i:\,j\rightleftarrows k}$ is an otherwise identical network where i ranks j over k instead. Then,

$$\begin{aligned}
&\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:\,j\succ k} = 1 | \boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k} \vee \boldsymbol{Y} = \boldsymbol{y}) \\
&\qquad = \frac{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}\{Y_{i:\,j\succ k} = 1 \wedge (\boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k} \vee \boldsymbol{Y} = \boldsymbol{y})\}}{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k} \vee \boldsymbol{Y} = \boldsymbol{y})} \\
&\qquad = \frac{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k})}{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k}) + \Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y})} \\
&\qquad = \frac{\exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}^{i:\,j\rightleftarrows k})\}/\cancel{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}}{\exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}^{i:\,j\rightleftarrows k})\}/\cancel{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} + \exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y})\}/\cancel{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}} \\
&\qquad = \frac{\exp(\boldsymbol{\theta} \cdot \{\boldsymbol{g}(\boldsymbol{y}^{i:\,j\rightleftarrows k}) - \boldsymbol{g}(\boldsymbol{y})\})}{\exp\left[\boldsymbol{\theta} \cdot \{\boldsymbol{g}(\boldsymbol{y}^{i:\,j\rightleftarrows k}) - \boldsymbol{g}(\boldsymbol{y})\}\right] + 1} \\
&\qquad = \operatorname{logit}^{-1}\{\boldsymbol{\theta} \cdot \boldsymbol{\Delta}^{\nearrow}_{i,j}\boldsymbol{g}(\boldsymbol{y})\},
\end{aligned}$$

since, by construction, $k \equiv j^+$ in $\boldsymbol{y}$. Similarly, we may consider the conditional odds of i ranking j over $k \equiv j^+$ or the ratio between its probability and that of an otherwise identical network where k outranks j:

$$\frac{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:\,j\succ k} = 1 | \boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k} \vee \boldsymbol{Y} = \boldsymbol{y})}{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(y_{i:\,j\succ k} = 0 | \boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k} \vee \boldsymbol{Y} = \boldsymbol{y})} = \frac{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y}^{i:\,j\rightleftarrows k})}{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y} = \boldsymbol{y})} = \exp\{\boldsymbol{\theta} \cdot \boldsymbol{\Delta}^{\nearrow}_{i,j}\boldsymbol{g}(\boldsymbol{y})\}. \tag{5}$$

In this, the promotion statistic also reflects the conditional dependence structure of the model: if its form for a particular $\boldsymbol{g}(\cdot;\cdot)$ does not depend on a particular datum, using it in the model cannot induce conditional dependence on that datum. Thus, although we do not derive our model terms from a conditional dependence structure (cf. Robins et al. (1999)), we can use them to examine the conditional dependence structure of the model for each term we consider.

Note that promotion statistics are mainly useful for complete orderings: if the ordering is partial, it is possible for i to promote j without demoting j^+. (See Butts (2007a) for a parallel case involving models for one-to-one versus many-to-one assignments.)

3.2 Terms for Exogenous Covariates

We begin our quorum of substantively useful statistics by considering "exogenous" factors: those factors that would influence rankings by an ego i in a manner that is independent (in the probabilistic sense) of the rankings of all other egos $i' \in N \setminus \{i\}$. (Indeed, none of the promotion statistics in this section depend on any other rankings.) Substantively, these factors are exogenous to the ranking process in that they are not, at least on the time scale of the process, mutable, or in that they operate independently of an ego i being able to observe or infer rankings or other salient states or actions (that are endogenous to the model) of any other ego i'.

3.2.1 Attractiveness/Popularity Effects

For assessments of attractiveness, liking, and status, it is likely that egos' rankings will be influenced by some relatively stable (and exogenous) tendencies of particular alters to be rated more highly than others. For instance, assessments of physical attractiveness tend to be broadly consistent within a given cultural context, and such assessments correlate positively with physical attributes and performance characteristics (e.g., subtleties of dress and speech) that are usually difficult to alter over short time scales (Morse et al. 1974; Webster and Driskell 1983). Thus, we have the emic notion that some persons "are attractive," with the attribution regarded as a fixed trait of the person being assessed; while the reality is less trivial, stable factors governing attractiveness are sufficiently important that we may wish to capture them where possible. In other settings, institutionalized status characteristics (e.g., group membership, formal social roles) or the like may have similar effects (Berger, Cohen, and Zelditch 1972; Berger et al. 1977).

Regardless of source, we can treat these effects directly by positing some covariate vector $\boldsymbol{x} \in \mathbb{R}^n$, associated with a statistic of the general form

$$g_{\mathrm{A}}(\boldsymbol{y}; \boldsymbol{x}) = \sum_{(i,j,k) \in N^{3\neq}} y_{i:\, j \succ k}(x_j - x_k).$$

This statistic simply indexes the tendency for those with higher values on x to be ranked more highly than those with lower values. The promotion statistic associated with the above is

$$\Delta_{i,j}^{\nearrow} g_{\mathrm{A}}(\boldsymbol{y}; \boldsymbol{x}) = 2(x_j - x_{j^+}),$$

i.e., twice the difference between the attractiveness of j and the actor over whom j may be promoted. Therefore, the multiplicative effect of this term on the odds (5) of j being ranked over j^+, conditional on the other rankings, is $\exp(2\theta_{\mathrm{A}}(x_j - x_{j^+}))$.

This coefficient of 2 appears in many of the promotion statistics proposed, because every promotion of j over j^+ affects two pairwise comparisons: $y_{i:\, j \succ j^+}$ and $y_{i:\, j^+ \succ j}$. It is tempting to eliminate it by redefining these statistics, multiplying them by half to compensate for this "redundancy". However that would muddle interpretation of these statistics applied to partially or weakly ordered data, because for those, it is possible for a change to affect $y_{i:\, j \succ k}$ without affecting $y_{i:\, k \succ j}$ (by creating or breaking ties); and even within the completely ordered domain, promotion statistics of more complex terms like the local nonconformity introduced in Section 3.3.2 might not have this coefficient. Thus, in interpreting the parameter estimates of these models and the magnitudes of their effects, one must consider carefully the form of the corresponding promotion statistics.

As expressed, g_{A} treats $\boldsymbol{x}$ as at least an interval scale; in other cases, the subtraction operator would need to be replaced with a more appropriate function. While $\boldsymbol{x}$ may be an observed covariate, it is worth noting that this quantity is also a natural candidate for treatment via a random popularity effect (van Duijn, Snijders, and Zijlstra 2004).

3.2.2 Difference/Similarity Effects

Just as one may posit a differential tendency to "win" ranking contests overall, one may also posit that each actor i has exogenous characteristic $\boldsymbol{x}_i \in \mathbb{X}$ such that alters "close to" or "far from" ego will be more likely to be highly ranked than those with the reverse attributes. (Such assumptions are the basis of models such as e.g., spatial voting theory (Enelow and Hinich 1984).) This is a familiar application of homophily/heterophily to the rank order case, and the implementation is straightforward:

$$g_{\mathrm{H}}(\boldsymbol{y};\boldsymbol{x}) = \sum_{(i,j,k)\in N^{3\neq}} y_{i:\,j\succ k}\left[z(\boldsymbol{x}_i,\boldsymbol{x}_j) - z(\boldsymbol{x}_i,\boldsymbol{x}_k)\right], \tag{6}$$

where $z : \mathbb{X}^2 \to \mathbb{R}$ is any function that is monotone increasing in the difference between its arguments. Thus, where this statistic is enhanced we expect "far" actors to outrank "near" ones (from the point of actor i), with the reverse holding where this statistic is suppressed. The atomic effect of this term is simply

$$\Delta^{\nearrow}_{i,j} g_{\mathrm{H}}(\boldsymbol{y};\boldsymbol{x}) = 2\left[z(\boldsymbol{x}_i,\boldsymbol{x}_j) - z(\boldsymbol{x}_i,\boldsymbol{x}_{j^+})\right].$$

As with attractiveness, difference effects can be based either on observed covariates or on latent quantities.

3.2.3 Dyadic Covariates

We can extend the above logic to general dyadic covariates. For instance, we may consider a case in which a within-context ranking is made by actors having ongoing social relationships; we might expect, then, that actors engaged in positive long-term relationships would tend to give preference to their partners within the specific rating context. Statistics for this behavior can be produced like so:

$$g_{\text{Dyad}}(\boldsymbol{y};\boldsymbol{x}) = \sum_{(i,j,k)\in N^{3\neq}} y_{i:\, j\succ k} \left(x_{i,j} - x_{i,k}\right)$$

$$\Delta^{\nearrow}_{i,j} g_{\text{Dyad}}(\boldsymbol{y};\boldsymbol{x}) = 2\left(x_{i,j} - x_{i,j^{+}}\right).$$

Of course, the cases of attractiveness and difference described above are simply special cases of dyadic covariates, with particular structure imposed. (Notably, the matrix permutation family (ERGP) of Butts (2007b) has a somewhat similar structure.)

3.2.4 Comparison Covariates

Finally, in the framework of pairwise comparison, the most general exogenous covariate form assigns a weight to each pairwise comparison by each ego:

$$g_{\text{P}}(\boldsymbol{y};\boldsymbol{x}) = \sum_{(i,j,k)\in N^{3\neq}} y_{i:\, j\succ k} x_{i,j,k},$$

for some $\boldsymbol{x} \in \mathbb{R}^{N^{3\neq}}$—assigning to each distinct ordered triple (i, j, k) a covariate value $x_{i,j,k}$—resulting in

$$\Delta^{\nearrow}_{i,j} g_{\mathrm{P}}(\boldsymbol{y}; \boldsymbol{x}) = \left(x_{i,j,j^+} - x_{i,j^+,j}\right).$$

This statistic has all other exogenous statistics as special cases.

3.3 Terms for Endogenous Mechanisms

We now turn to factors that are endogenous in the sense that, unlike exogenous factors, their effect on the rankings by ego i does depend on rankings by other egos $i' \in N \setminus \{i\}$. Substantively, these are factors for phenomena that may plausibly arise in cases for which ego observes or is able to infer the rankings of others.

3.3.1 Global Conformity

In many settings where an ego is able to observe or infer the rankings of others, there is reason to presume that this will influence ego, so that he or she will tend to bring his or her own rankings into conformance with others'. This is certainly true in dominance or status rankings, where there is considerable evidence that individuals can and do infer status ordering from observation of third-party judgments (see, e.g., Anderson et al. 2006); this synchronization may even be explicit, as in certain types of gossip (wherein two or more parties "compare notes" on the relative status of their peers) (Dunbar 1997). The status of influence for relations such as relative liking is less clear, but still plausible: ego may take alter's evaluations of the relative merits of other alters into account in assessing his or her own preferences, just one can be influenced in one's judgment of the merits of food, art, or other experiential goods by the evaluations of others (Bordieu 1968). Finally, the mutual observability of rankings may produce in some settings a form of "conformity pressure" (Asch 1951, for example), such that those displaying deviant rankings anticipate (and are possibly exposed to) sanction. The importance of influence processes in such settings is well-documented.

To formalize influence in the ranking context, we must note that four elements are involved: ego's assessment of two alters (say, j and k), and the assessment of those same alters by a distinct third party (say, $l \notin \{i, j, k\}$). Denoting ego by i, we note that ego's assessment of j and k is in conformity with l's assessment of j and k when $y_{i:\, j \succ k} = y_{l:\, j \succ k}$ and $y_{i:\, k \succ j} = y_{l:\, k \succ j}$. A natural statistic to summarize the degree of ratings nonconformity, then, is

$$g_{\text{GNC}}(\boldsymbol{y}) = \sum_{(i,j,k,l) \in N^{4\neq}} y_{l:\, j \succ k} \left(1 - y_{i:\, j \succ k}\right). \tag{7}$$

The promotion statistic for nonconformity can be derived by observing that when i promotes j over j^+, the statistic is incremented by 2 every other ego l who has j^+ ranked over j and decremented by 2 for every l who has j ranked over j^+ (1 for i conforming/disconforming to l and 1 for l conforming/disconforming to i). Thus,

$$\Delta_{i,j}^{\nearrow} g_{\text{GNC}}(\boldsymbol{y}) = 2 \sum_{l \in N \setminus \{i, j, j^+\}} (y_{l:\, j^+ \succ j} - y_{l:\, j \succ j^+}), \tag{8}$$

a "vote" among the ls as to the relative ranking of j and j^+. Insofar as influence is active, g_{GNC} should be suppressed (and the associated parameter negative). In the typical case of total ordering within subjects, sufficiently strong suppression of g_{GNC} will force the population to converge to a universal consensus ranking; if the suppression is weaker, a looser but analogous set of states will be favored.

Unlike those of the terms for exogenous mechanisms, this promotion statistic is a summation. This is unsurprising, since it reflects the dependence among the egos' that it induces: where before promotion statistics did not depend on $\boldsymbol{y}$ at all, $\Delta^{\nearrow}_{i,j} g_{\text{GNC}}(\boldsymbol{y})$ depends specifically on how each of the other egos ranked the two specific alters j and j^+. Its form is also likely to affect its magnitude: where exogenous promotion statistics' magnitudes depend only on those of the covariates, the magnitude of $\Delta^{\nearrow}_{i,j} g_{\text{GNC}}(\boldsymbol{y})$ is likely to grow with network size, so care must be taken in interpreting how a particular value for the corresponding parameter estimate translates to the strength of the social force it models.

It should be noted that this form of influence and the random attractiveness effect, mentioned in Section 3.2.1, can both explain the same network feature: both heterogeneity in attractiveness and social influence induce an agreement in rankings, and the latter may be considered a marginal representation of the former, in a manner similar to that of a within-group correlation as a marginal reflection of a (conditional) random effects linear model.

3.3.2 Local Conformity

For global nonconformity g_{GNC}, the promotion statistic (8) implies that i weights agreement with every other l equally, regardless of how i had ranked l. In some cases, it may be plausible that the salience of l for i may depend upon i's ranking of him or her; for instance, i may be more likely to attend to (and to conform to) those whom he or she ranks highly than those whom he or she ranks lower. A possible formalization of this is the notion that i's ranking of j would be influenced by l only if i ranks l over j, so only actors ranked above j influence i's rankings involving j. As with global conformity, we define this statistic, the *local nonconformity*, negatively: counting the number of instances where an ego had ranked l over two alters j and k but then did not conform to l's ranking of j relative to k:

$$g_{\text{LNC}}(\boldsymbol{y}) = \sum_{(i,j,k,l)\in N^{4\neq}} y_{i:\,l\succ j} y_{i:\,l\succ k} y_{l:\,j\succ k}(1 - y_{i:\,j\succ k}). \tag{9}$$

The atomic effects for this statistic are somewhat complex:

$$\Delta^{\nearrow}_{i,j} g_{\text{LNC}}(\boldsymbol{y}) = \sum_{k\in N\setminus\{i,j,j^+\}} (y_{i:\,k\succ j^+} y_{k:\,j^+\succ j} - y_{i:\,k\succ j^+} y_{k:\,j\succ j^+} \tag{10a}$$

$$+ y_{k:\,i\succ j^+} y_{k:\,j^+\succ j} - y_{k:\,i\succ j} y_{k:\,j\succ j^+} \tag{10b}$$

$$+ y_{j:\,k\succ j^+} y_{i:\,j^+\succ k} - y_{j^+:\,k\succ j} y_{i:\,j\succ k}). \tag{10c}$$

They have, however, a meaningful interpretation. The two terms (10a) represent the effect of i bringing his or her ordering of j and j^+ into conformance (or disconformance) with those of some actor k whom i had ranked higher than at least one of them. The pair (10b) represents the situation where some actor k ranks i over j and/or j^+, so i promoting j over j^+ either creates or eliminates disconformance on the part of k. The pair (10c) represent the notion that nonconformity is also created if actors i and j disagree on the ordering of j^+ and some actor k, so i promoting j over j^+ creates disconformance by making j's ordering of j^+ salient to i. (Ego i can resolve this tension either by changing the ranking of j^+ and k to conform with j (affecting (10a)) or by demoting j to make his or her ranking less salient.)

This promotion statistic is also unusual among the others in that it does not have the coefficient of 2 first discussed in Section 3.2.1. In this case, it serves to underscore that the indicators $y_{i:\,j\succ k}$ and $y_{i:\,k\succ j}$ being coupled due to the complete ordering constraint does not imply that the act of ranking j over k has equivalent significance in the social process as the act of ranking k over j.

3.3.3 Deference Aversion

Influence, as defined above, deals with the mutual adjustment among raters regarding their relative assessments of third parties. When ego is a party to the rating in question, the situation becomes more complex. By assumption, ego does not explicitly self-rate; thus, ego cannot adjust towards alter's impression of him or her. In many settings, however, another mechanism may be active that will make alter's ranking of ego salient for ego's ranking of alter. In particular, consider the case in which higher rankings are associated with positive evaluation, such that being ranked below others is aversive. Moreover, let us assume that ego infers his or her own status via an *implicit transitivity mechanism*, such that if alter l ranks j above ego (i) and ego ranks l above j, then ego is for social purposes ranking himself or herself below l. Under such circumstances, *deference aversion* may lead ego to resist ranking l above j.

To capture this notion with a statistic, we propose the following:

$$g_{\mathrm{D}}(\boldsymbol{y}) = \sum_{(i,j,l)\in N^{3\neq}} y_{l:\, j \succ i} y_{i:\, l \succ j}. \tag{11}$$

We expect this statistic to be suppressed where deference aversion is present. The promotion statistic is incremented if j^+ had ranked i over j, since it creates a deference of j^+ to i via j; or if j had ranked j^+ over i since it creates a deference of i to j via k; and it is decremented if j^+ had ranked j over i, as it would eliminate the deference of i to j^+ via j;

or if j had ranked i over j^+, as it would eliminate the deference of j to i via j^+. Thus,

$$\Delta^{\nearrow}_{i,j} g_{\mathrm{D}}(\boldsymbol{y}) = 2(y_{j^+ : i \succ j} + y_{j : j^+ \succ i} - 1).$$

It is interesting to note that the principal effect of suppressing this statistic is actually to bring ego's rankings in line with those of alter, somewhat akin to the reciprocity or mutuality in binary relations. Specifically, if there are r persons ranked by alter as being above ego, then ego will also tend to rank those same r persons as being above alter. Where a total order is present, ego and alter will thus tend to give each other the same rank (and, indeed, to agree on those persons having higher ranks). Of course, applying this logic to all pairs suggests pressure towards equality, which is impossible to achieve in the total order case (but not necessarily for others). Even in the case of total orders, however, considerable variation in g_{D} is possible, with lower values indicating rating structures in which agreement between raters on high-ranked alters is maximized.

3.4 Consistency Across Settings

When ranking the same alters among multiple settings—across time or across rubrics—there is reason to expect that ego will tend to exhibit consistency in alter ratings. Across time, this is an exogenous effect, because earlier rankings cannot be influenced by later rankings. Across rubrics, it may be endogenous. Here, we assume two rating structures, $\boldsymbol{y}$ and $\boldsymbol{y}'$, on vertex sets N and N', such that some set $N_s = N \cap N'$ of actors are involved in both networks. For convenience in notation, we take the labeling of the members of N_s to be the same in both N and N'. Given this, our statistic measuring inconsistency is simply

$$g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}') = \sum_{(i,j,k)\in N_s^{3\neq}} \Big[y_{i:\,j\succ k}(1-y'_{i:\,j\succ k}) + \big(1-y_{i:\,j\succ k}\big)\, y'_{i:\,j\succ k} \Big], \tag{12}$$

with promotion statistic being simply

$$\Delta^{\nearrow}_{i,j} g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}') = 2(y'_{i:\,j^{+}\succ j} - y'_{i:\,j\succ j^{+}}).$$

As g_{I} measures the discordant pairs of rankings in $\boldsymbol{y}$ versus $\boldsymbol{y}'$, suppressing it implies higher levels of cross-context consistency.

The statistic (12) treats all disagreements between $\boldsymbol{y}$ and $\boldsymbol{y}'$ as equivalent. It may be the case, however, that only some disagreements are of interest, or disagreements themselves need to be modeled. This can be facilitated by a more general form of g_{I}. Given weights $\boldsymbol{x} \in \mathbb{R}^{N^{3\neq}}$, (symmetric for complete orderings, such that $x_{i,j,k} \equiv x_{i,k,j}$), let the *weighted inconsistency* of $\boldsymbol{y}$ versus $\boldsymbol{y}'$ be defined as

$$g_{\mathrm{I}}(\boldsymbol{y};\boldsymbol{y}',\boldsymbol{x}) = \sum_{(i,j,k)\in N_s^{3\neq}} \Big[y_{i:\,j\succ k}(1-y'_{i:\,j\succ k}) + \big(1-y_{i:\,j\succ k}\big)\, y'_{i:\,j\succ k} \Big]\, x_{i,j,k}. \tag{13}$$

Note that $\boldsymbol{x}$ can, itself, be parametrized to model factors affecting disagreement between two rankings, making it a potentially interesting basis for hierarchical modeling.

These statistics are utilized extensively in the example of Section 4.2, where they are employed to examine the accuracy of informants' self-reported interaction frequencies.

4 Examples

4.1 Dynamics of the Acquaintance Process

From 1953 to 1956, a research group led by Theodore Newcomb (1961) conducted an experimental study of acquaintance and friendship formation. In each of the two study years, 17 men attending the University of Michigan—all transfer students with no prior acquaintance among them—were recruited to live in off-campus fraternity-style housing. Demographic, attitudinal, and sociometric information was collected about the subjects. In particular, in the second year, at each of 15 weekly time points (with week 9 being missing), each participant was asked to rate all other participants on "favorableness of feeling", with ratings forced to be distinct and converted to ranks. (1961:32–34) These data represent an example of longitudinal data of complete ranks, and, particularly the data from the second year of the study, have been used to study the formation of interpersonal relationships by Newcomb (1956), Breiger et al. (1975), White et al. (1976), Arabie et al. (1978), Wasserman (1980), Pattison (1982), Nakao and Romney (1993), Doreian et al. (1996), Krackhardt and Handcock (2007), and many others.

We use ERGMs for rank-order data to study this network, examining the social forces relevant to its structure and its evolution over time. We take two distinct approaches: cross-sectional, where each time point's network structure is modeled on its own and dynamic, where each time point but the first is effectively modeled as a change from the previous time point's rankings.

4.1.1 Cross-sectional Analysis

Demographic data, including age, religion, and political views of the subjects were gathered. Also, within the house, the subjects were assigned to rooms, spread over two floors of the house—three one-occupant rooms, four two-occupant, and two three-occupant rooms (Newcomb 1961:67–68). Furthermore, while some were assigned to rooms at random, others were assigned with an aim to maximize (for some rooms) and minimize (for other rooms) the roommates' compatibility as understood by the researchers (1961:216–220). If available, all of these factors could be used as predictors in our modeling framework via terms introduced in Section 3.2. Sadly, however, none of these elements of the Newcomb data survive, leaving us to focus on endogenous effects (although the "birds of a feather or friend of a friend" (Goodreau et al. 2008b) caveat applies). For each of the 15 networks, we model deference aversion (via $g_{\text{D}}(\boldsymbol{Y}^t)$ (11)) and global (via $g_{\text{GNC}}(\boldsymbol{Y}^t)$ (7)) and local (via $g_{\text{LNC}}(\boldsymbol{Y}^t)$ (9)) conformity. Per Section 3.3.1, the suppressing of the global nonconformity statistic produces the same effect as latent actor attractiveness, so it may, in this case, be viewed as modeling latent heterogeneity in popularity.

[Figure 4 about here.]

[Table 1 about here.]

We report the maximum likelihood estimates for each of the terms over time in Table 1 and plot them in Figure 4. Deference aversion is significant (the coefficient on the deference statistic is negative) throughout the evolution of rankings, starting with the first point of observation. This is consistent with the finding of Newcomb (1956), Doreian et al. (1996), and others that "friendships are reciprocated immediately". Our analysis, however, suggests deepening deference aversion over time, not reaching its ultimate magnitude until week 4 or 7. (Informally, the estimated Kendall's rank correlation between the parameter estimate and week number is $\hat{\tau} = -0.41$, significant with P-value $= 0.036$.) One explanation for this difference is that prior analyses, including that of Doreian et al. (1996), dichotomized the dyads in the network. Our approach uses the entire ranking and may thus be more precise.

The local nonconformity term is also significant for all but the first observation point, although its effects seem to emerge more gradually than those of deference aversion (Kendall's $\hat{\tau} = -0.64$, P-value < 0.001), leveling off around Week 7 or perhaps even later. This does agree with Doreian et al. (1996), although others have suggested earlier and later times when the network stabilized.

In the presence of the local nonconformity term, the global nonconformity term is not significant: there does not appear to be a significant overall consensus in ratings, although Newcomb (1956) reports that three specific subjects were generally disliked by everyone, including each other, so perhaps failing to detect this is a result of lack of power and presence of the local nonconformity term. Notably, the estimated correlation between the local and global nonconformity parameters (within each week's fit) is strongly negative and consistent (-0.87–-0.77). This suggests that these terms explain similar behavior, though, significance of one and not the other indicates that conformity is primarily to those more liked.

4.1.2 Dynamic Analysis

We now turn to modeling the evolution of the rankings over time. For our dynamic analysis, we use a simple Markov formulation similar to those of Krackhardt and Handcock (2007) and Hanneke, Fu, and Xing (2010) for the evolution of rankings over time:

$$\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}^t = \boldsymbol{y}^t | \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}) = \frac{\exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}^t; \boldsymbol{y}^{t-1})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}, \boldsymbol{y}^{t-1})},$$

having the normalizing constant

$$\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}, \boldsymbol{y}^{t-1}) = \sum_{\boldsymbol{y}' \in \mathcal{Y}} \exp\{\boldsymbol{\theta} \cdot \boldsymbol{g}(\boldsymbol{y}'; \boldsymbol{y}^{t-1})\}.$$

For each of the 14 *transitions* between successive networks, in addition to the three terms used in the cross-sectional analysis, we model inconsistency over time via $g_{\mathrm{I}}(\boldsymbol{Y}^t; \boldsymbol{Y}^{t-1})$ (12). This term effectively absorbs the structure in the network at t that is due to inertia and to social forces operating in the time periods prior to $t-1$, and thus the other terms model the social forces affecting only the *changes* in the rankings over time.

Because we seek to examine the strengths of the factors over time, we use time-varying parameters (although Krivitsky and Handcock (2014) show that the approach of Hunter and Handcock (2006) can be applied to series of networks or transitions as well). Week 9 rankings were not reported. Because of this, for Week 10, we fit the parameters for transition from Week 8.

[Figure 5 about here.]

The maximum likelihood estimates for each transition are reported in Table 2 and visualized in Figure 5. The estimates for the transition from Week 8 to Week 10 do not appear to be qualitatively different from those for nearby transitions. In particular, inconsistency does not appear to be higher over this particular two-week period.

[Table 2 about here.]

The clear downward trend ($\hat{\tau} = -0.54$, P-value $= 0.007$) in inconsistency over successive weeks suggests that the rankings are initially in flux as the acquaintance process takes place, solidifying over time. As before, global nonconformity is not a significant factor. The parameter estimates for the other two factors still appear to be, on the whole, significant, but are uniformly smaller in magnitude compared to those of the corresponding weeks in the cross-sectional analysis and are less precisely estimated (as represented by uniformly greater standard errors). This is because rather than embodying the structure of the whole network, they embody only the structure of *changes* in the network over the week, and information to infer their strength is drawn only from those changes. This means that some "instant" social effects such as friendship reciprocation have been absorbed into the Week 0 observation, which is not modeled in the dynamic analysis.

In contrast to the cross-sectional analysis, neither deference nor local nonconformity appear to have a significant monotone trend over time (both correlations have P-value ≥ 0.5). That is, while they are time-varying when viewed cross-sectionally, the effects of these social forces over and above inertia are fairly consistent over time, at least after the initial time point. This suggests that this modeling approach may be successfully isolating social forces as they affect actors' behavior over time from the effects of preexisting configurations.

4.2 Informant Accuracy

In the late 1970s, Bernard et al. (1984) conducted a series of studies to assess the accuracy of retrospective sociometric surveys of several types. In each study, respondents in a social network—deaf teletype users; amateur radio operators; office workers at a firm; students in a fraternity; and faculty, graduate students, and staff in an academic program—had their social interactions observed or recorded and were asked, in retrospect, to indicate others in their network with whom they interacted, allowing recalled and observed network structures to be compared. In the latter study—conducted in a graduate program in technology education at West Virginia University—the 34 subjects had the frequency of their interactions recorded by a team of observers over the course of a week, and then each subject was asked to provide a complete ranking of the other subjects on "most to least communication that week" (Bernard and Killworth 1977). This produces a complete ranking, suitable for analysis using our methods.

4.2.1 Modeling inconsistency

In this application, we use models with sufficient statistics of the form (12) and (13) to assess factors that appear to affect accuracy of rankings. Let $\boldsymbol{y} \in \mathcal{Y}$ be the reported rankings; and let $\boldsymbol{y}^{\text{ct.}} \in \mathbb{N}_0^{\mathbb{Y}}$ be a weighted symmetric graph of the observed frequencies of interaction, so that $y_{i,j}^{\text{ct.}}$ is the number of times i and j were observed interacting. Like $\boldsymbol{y}^{t-1}$ in the previous example, it is exogenous in our framework.

For notational convenience, we reexpress (3) and (13) as

$$\Pr_{\boldsymbol{\theta};\boldsymbol{g},\boldsymbol{x}}(\boldsymbol{Y}=\boldsymbol{y}|\boldsymbol{Y}^{\text{ct.}}=\boldsymbol{y}^{\text{ct.}}) = \frac{\exp\left[\sum_{(i,j,k)\in N^{3\neq}}\{y_{i:\,j\succ k}(1-y^{\text{ct.}}_{i:\,j\succ k})+(1-y_{i:\,j\succ k})y^{\text{ct.}}_{i:\,j\succ k}\}w_{i,j,k}(\boldsymbol{\theta};\boldsymbol{x})\right]}{\kappa_{\boldsymbol{g},\boldsymbol{x}}(\boldsymbol{\theta})}, \quad \text{(14a)}$$

where

$$w_{i,j,k}(\boldsymbol{\theta};\boldsymbol{x}) = \boldsymbol{\theta}\cdot\boldsymbol{x}_{i,j,k}, \quad \text{(14b)}$$

for covariate $n \times n \times n \times p$-array $\boldsymbol{x} \in \mathbb{R}^{N^{3\neq} \times \{1\,..\,p\}}$, so that $\boldsymbol{x}_{i,j,k}$ is the p-vector of covariates associated with i's comparison of j and k, and, as before, $\boldsymbol{x}_{i,j,k} \equiv \boldsymbol{x}_{i,k,j}$. This allows us to model inconsistency in a form semblant of logistic regression. Unlike logistic regression for accuracy of pairwise comparisons, this model takes into account the dependence between the comparisons that is induced by the structure of the sample space. (I.e., that $y_{i:\, j \succ k} \wedge y_{i:\, k \succ l} \implies y_{i:\, j \succ l}$.)

Because there are "ties" among the observed interaction frequencies (i.e., where $y^{\text{ct.}}_{i,j} = y^{\text{ct.}}_{i,k}$, so i interacted with j and k equally often), while the reported rankings are forced to be complete (no ties were allowed), there is no configuration of rankings $\boldsymbol{y} \in \mathcal{Y}$, such that statistic (12) is 0—that the reported rankings are completely consistent with those observed. This reveals an interesting property of the proposed class of models: because the comparisons that are tied in the observed frequencies simply add a constant to their sufficient statistic, their effect on the likelihood is canceled by the normalizing constant. That is, the model and the estimation are only affected by those inconsistent comparisons that had the possibility of being consistent under the model in the first place.

For convenience, further let $y^{\text{obs.}}_{i,j}$ be the *observed* rank of j among those with whom i had interacted, with 33 being code for the highest frequency, 1 being code for the lowest frequency, and ranks for tied alters (i.e., $y^{\text{ct.}}_{i,j} = y^{\text{ct.}}_{i,k}$, for $k \neq j$) being computed by averaging the ranks that these alters share.

4.2.2 Effect of Frequency of Interaction

The first question we address is whether the magnitude of the difference in the frequency of interaction affects the accuracy. That is, if i's frequency of interaction with j differs from i's frequency of interactions with k by more than i's frequency of interaction with j differs from i's frequency of interaction with l, is i more likely to rank j and k accurately than j and l?

To answer this, we begin by fitting a simple model with two covariates: $x_{i,j,k,1} = 1$ in the form (14), equivalent to plain inconsistency (12); and $x_{i,j,k,2} = \left|y^{\text{ct.}}_{i,j} - y^{\text{ct.}}_{i,k}\right|$, the absolute difference between the interaction frequency of i with j and k. We report the results in Table 3. Greater difference in interaction frequency of two alters does appear to lead to greater accuracy (i.e., lower inaccuracy) in reporting their relative ranks.

[Table 3 about here.]

We also fit a similar model where we replace frequency difference with frequency rank difference: $x_{i,j,k,2} = \left|y^{\text{obs.}}_{i,j} - y^{\text{obs.}}_{i,k}\right|$. We find that the effect of rank difference is, as expected, negative, but not statistically significantly so. This lack of significance may be counterintuitive, but it is, in fact, a consequence of the constraints imposed by the sample space. Intuitively, given that i ranks j and k adjacently, an inaccurate reporting of the pairwise comparison of these alters (e.g., $y^{\text{ct.}}_{i:\, j \succ k}$ but $y_{i:\, k \succ j}$) does not entail misreporting any other pairwise comparisons, including those involving j or k; but if j and k are some $d > 1$ ranks apart in $\boldsymbol{y}^{\text{ct.}}$—they have $d - 1$ other alters between them—then inaccurately reporting the pairwise comparison of j and k entails inaccurately reporting the pairwise comparisons $y^{\text{ct.}}_{i:\, j \succ l}$ and/or $y^{\text{ct.}}_{i:\, k \succ l}$ for every l ranked between j and k. It can be shown easily that any configuration $\boldsymbol{y}$ in which $y_{i:\, k \succ j} \neq y^{\text{ct.}}_{i:\, j \succ k}$ must also misreport at least $d - 1$ such comparisons. Thus, even a model with no rank difference effect and only a baseline inconsistency effect would already heavily penalize inaccurate reporting of comparisons between distantly-ranked alters.

4.2.3 Effect of Salience

The second question that we address is whether the accuracy of reported ranking is affected by the positions of those being ranked. Can an ego i better discern the ranking of those with whom he or she interacts the most? Is he or she more accurate at the extremes?

To answer this, we fit a model for inconsistency that is a quadratic polynomial in rank values. More concretely, in the form (14), $w_{i,j,k}(\boldsymbol{\theta}; \boldsymbol{x})$ has the covariate vector

$$\boldsymbol{x}_{i,j,k} = \left[1,\ y_{i,j}^{\text{obs.}} + y_{i,k}^{\text{obs.}},\ (y_{i,j}^{\text{obs.}})^2 + (y_{i,k}^{\text{obs.}})^2,\ y_{i,j}^{\text{obs.}} y_{i,k}^{\text{obs.}}\right], \tag{15}$$

inducing a model in which the inconsistency of reported with observed is modeled as a second-degree polynomial function of the observed ranks of the alters being compared; and this function is symmetric for these two alters. The statistics in (15) represent baseline inconsistency, linear effect of ranks of the alters being compared, their quadratic effect, and their interaction effect, respectively.

In fitting this model, we found that it suffers from collinearity, which impedes inference, so to improve its numeric conditioning, we fit an equivalent model, using rescaled and centered quantiles, evaluating $y_{i,j}^{\text{q.}} \equiv (y_{i,j}^{\text{obs.}} - 17)/32$ and substituting $\boldsymbol{y}^{\text{q.}}$ for $\boldsymbol{y}^{\text{obs.}}$ in (15).

We report results from this fit in Table 4. All four covariates appear to be highly significant. Somewhat surprisingly, the higher-ranked alters appear to have slightly higher inconsistency between observed and reported. However, the negative coefficient on the quadratic term suggests that the middle ranks are reported with the least accuracy of all. We show the predicted inconsistency weight $w_{i,j,k}(\hat{\boldsymbol{\theta}}; \boldsymbol{x})$ as a function of their observed ranks in Figure 6. From this, it appears that, indeed, for alters whose observed rankings are close together, the accuracy is lowest if the alters are ranked in the middle. (For alters whose observed rankings are far apart, the accuracy is higher.)

[Table 4 about here.]

[Figure 6 about here.]

5 Discussion

As befits the long history of ordinal data analysis in the social sciences, many extensions and applications of the present framework are possible. Here, we briefly discuss two such directions: the extension of the current framework to "bipartite" rank data; and the use of our framework in settings that imply particular kinds of ordinal constraints (e.g., partial orders, semi-orders, incompletely observed total orders, etc.).

5.1 Extension to "Bipartite" Rank Data

Although our focus here has been on the classic sociometric setting in which a group of individuals are asked to rank each other with respect to some dimension (e.g., liking), our modeling framework can easily accommodate other cases as well. One such setting is the case of "bipartite" rank data, in which members of a given group are asked to rank a set of objects not including the group members themselves. Examples of such data include preference rankings of political candidates, organizations, or policy positions; ordinal judgments regarding physical objects or perceptual stimuli; liking or other rankings of non-group members, etc. Although such data is widely analyzed throughout the social sciences using traditional techniques, the contribution of our approach is the ability to model *interdependence among raters* in a natural way. For instance, the global conformity statistic of

Section 3.3.1 can be used to capture a general tendency of group members to converge on a common rating of a set of objects; likewise, the consistency effects of Section 3.4 have the same meaning in the "bipartite" setting as in a standard sociometric setting. The exogenous effects of Section 3.2.1 can be used to capture differences in the net attractiveness of objects to members of the rating group, and dyadic covariates (Section 3.2.3) can be used to measure effects related to the tendency of particular raters or subgroups of raters to give higher/lower ratings to particular objects. On the other hand, statistics that depend upon the ratings *by* the object (e.g., local conformity) are clearly meaningless in a bipartite context, and should not be employed.

The bipartite case suggests certain statistics that may prove especially useful for assessing influence in structured groups. For instance, consider the case in which our set of raters, N_{e}, interact via a known, fixed social network with adjacency matrix $\boldsymbol{x}$. Let N_{a} be the set of objects to be ranked. We may then capture the tendency for those adjacent in $\boldsymbol{x}$ to rate objects in N_{a} (dis)similarly via the *dyadic nonconformity statistic*,

$$g_{\mathrm{DNC}}(\boldsymbol{y}) = \sum_{(i,l)\in N_{\mathrm{e}}^{2\neq}} x_{i,l} \sum_{(j,k)\in N_{\mathrm{a}}^{2\neq}} y_{l:\,j\succ k}\left(1 - y_{i:\,j\succ k}\right).$$

To the extent that adjacent actors influence each other to form similar views of the object set, g_{DNC} will be suppressed; the associated parameter is hence analogous (up to a sign change) to the autocorrelation parameter in a standard linear network autocorrelation model (LNAM) (Cliff and Ord 1973; Doreian 1990). As with the LNAM, the adjacency structure (analogous to the weight matrix) need not be dichotomous, and can contain continuous measures of exposure, proximity, similarity, group co-membership, or the like. Also like the LNAM, the adjacency/weight matrix is taken to be exogenous and fixed; models in which object ratings and interpersonal relationships co-evolve would be a promising direction for future research in this area.

Simulation and estimation for bipartite rank data requires modifying the support of the associated model to include only the observable ranks, and eliminating impermissible rating triads from the proposals in Algorithm 1 in Appendix A.2. These are straightforward changes to the base implementation, and are not discussed in detail here.

5.2 Considerations Relating to Types of Orderings

It should be noted that, in our development, we focus on the case of orderings that are defined from the psychological process in question, rather than data that are ordinal simply due to limitations in measurement (e.g., count or continuous data observed only as ranks). Although the present framework may in some cases be useful for such data (e.g., to avoid having to model the count distribution), the assumptions involved, e.g., in the choice of sufficient statistics (and their interpretation), may be quite different.

While we have focused on complete orderings, the above distinction gains further importance when considering partial orderings: the case of partial orderings being a property of underlying psychological phenomena is substantively different from the case of incomplete orderings arising from measurement itself. A well-known example of the latter is a frequently used sociometric survey design that asks each ego to rank their top k alters with respect to some criterion (liking, interaction frequency, etc.). Cases of unobserved ranking are better handled by means of a latent variable framework in which a probability distribution is placed on the complete data, and in which likelihood is assessed via the marginalization of the complete data conditional on that observed. (Plackett–Luce models for top-k ranking data (Plackett 1975) do this implicitly.) Such a strategy has been used in the traditional ERGM framework by Handcock and Gile (2010), and can be employed here as well.

In contrast, orderings in which the alters may be *substantively* tied or incomparable with each other pose different challenges. A close examination of Sampson's data reveals that some of the novitiates were, in fact, recorded assigning equal ranks to some of those they nominated; and these would be substantively tied in an underlying partial ordering. Tied ratings in data like those of Johnson et al. (2003) could be interpreted either way: either that the scale of 0–10 was not sufficiently granular to encode small differences in degrees of interaction, so which of the tied alters was actually higher is unobserved; or that an ego's choice to rate two alters equally means that their degrees of interaction were substantively the same.

In the complete ordering case, there exists a distribution of rank-orderings that is unambiguously uniform and can thus serve as the baseline distribution (*reference measure*) for the exponential family (see, e.g. Barndorff-Nielsen 1978:115–116). In the partially ordered case, there is no natural baseline that is unambiguously uniform. For example, an ego may rank all $n - 1$ alters equally (with only one such "ranking" possible) or he or she may assign a distinct rank to each (with $(n - 1)!$ possible rankings), or anything in between. Modeling of partial orderings, therefore, requires modeling not only the actors' ranking propensities but also their choices whether to rank or not to rank, and, in particular, the baseline distribution of these choices, when $\boldsymbol{\theta} = \mathbf{0}$. How to capture such behavior in a cognitively realistic way is not currently known.

Despite our focus on complete orderings, our techniques can (with appropriate choice of reference measure) be generalized to the weaker case: all of the model statistics that we present in Section 3 can be applied to partially ordered network data without modification, because they only make references to the network of interest through the indicator $y_{\cdot:\cdot\succ\cdot}$ (1).

6 Conclusion

Rank-order data are a cornerstone of sociometric measurement, but principled treatment of such data in an interpersonal context poses significant statistical challenges. Here, we have shown how statistical exponential families may be used to generalize the now well-known ERGM framework to the rank-order case. We have also introduced a corresponding set of sufficient statistics that are appropriate for use when only within-ego ordinal judgments are psychologically meaningful, a restriction that is important when modeling such data. As with conventional ERGMs, a wide range of statistics may be posited to capture alternative psychological and social mechanisms; the ability to evaluate and compare competing models based on such distinct alternatives is one of the strengths of the statistical approach.

In our assumption that the network must be modeled through $y_{\cdot:\cdot\succ\cdot}$, we have discarded any information associated with the rank as such. Where measurements are *truly* ordinal, this is the appropriate choice; nevertheless, it has been noted e.g. by Levine (1993) that putatively ordinal data can in practice contain more information than a strict ordinal interpretation would allow. As we show in our example in Section 4.2, rank values and rank differences can be incorporated into the models as well, though that example only uses them exogenously. Our framework does not, fundamentally, preclude incorporating such effects, and, in fact, it permits separating and testing their significance over and above the purely comparison-based effects. This, also, is a subject for ongoing work.

Finally, we note that development of ERGMs for rank-order data opens the way to a rich family of novel statistical models for phenomena such as interdependent choice behavior in group context, social influence on preferences, etc. Particularly because of their suitability for data collected in observational settings, rank-order ERGMs provide a useful tool for considering both new and classic problems in social psychology and the study of decision making.

Endnotes

[1]In formal notation, this can be written as $N^{p\neq} \equiv \{\boldsymbol{v} \in N^p : \forall_{i \in \{1\,..\,p\}} \forall_{j \in \{1\,..\,p\}} i \neq j \implies v_i \neq v_j\}$.

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Appendices

A Inference and Implementation

The proposed class of models is a finite exponential family over a finite sample space. Its natural parameter space $\Theta_{\text{N}} = \{\boldsymbol{\theta}' : \kappa_{\boldsymbol{g}}(\boldsymbol{\theta}') < \infty\} = \mathbb{R}^p$, an open set, so the models from this family are regular and possess all of the inferential properties of such families (Brown 1986:1–2). Here, we briefly discuss practical issues associated with fitting the models of this class.

A.1 Implementation

The sample space of a complete ranking ERGM is finite but large ($|\mathcal{Y}| = \{(n-1)!\}^n$), so even for networks of modest size, evaluating the normalizing constant (4) is not computationally feasible. However, provided a method for sampling from this ERGM distribution $\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\cdot)$ exists, the Monte Carlo MLE method can be used to fit the model. (Geyer and Thompson 1992; Hunter and Handcock 2006; Krivitsky 2012) We review this technique and describe a simple algorithm for sampling from aa complete ordering ERGM of Section 2 in A.2. Bayesian inference using exchange algorithms is also possible, if more computationally expensive. (Caimo and Friel 2011)

We base our implementation on the valued ERGM extensions of Krivitsky (2012) to the R (R Core Team 2014) package `ergm` (Hunter et al. 2008b; Handcock et al. 2014), and we have publicly released our extensions in a new package, `ergm.rank`.

Lacking a general procedure such as the maximum pseudo-likelihood estimation (MPLE) (Strauss and Ikeda 1990) to use as a starting point for optimization, we initiated the MCMC

MLE optimization at $\boldsymbol{\theta} = \mathbf{0}$ in each of our applications. This parameter configuration corresponds to a uniform distribution on the space of possible rankings, and it is therefore the safest configuration to start with. Although a poor starting value can cause MCMC MLE to fail, techniques such as the Stepping Algorithm of (Hummel, Hunter, and Handcock 2012) can be used to ameliorate its effects, and a form of it is used here as well. To confirm the effectiveness of these techniques even with a poor starting value, we performed a small parameter recovery simulation study, which we report in the Supplement, along with MCMC diagnostics for all of the fits we report below.

Given the current wave of interest in "large" networks, computationally scalable estimation has been an increasing focus of research in the social network community. Relative to population-scale networks, the networks considered here have been fairly small (i.e., 10s of vertices), and our focus has been on developing methods that work well on this latter scale. This focus is motivated by the fact that sociometric data of the form treated here is typically of interest only in group or organizational settings in which all members of the network are salient and well-known to each other; since such networks are by nature fairly small (e.g., rarely larger than 20–50 persons), highly scalable techniques are less critical for rank-order models than for binary networks. That said, computationally scalable estimation (particularly for the above-described top-k design) is an interesting challenge for future research in this area.

A.2 Overview of Monte Carlo MLE and a simple sampling algorithm for a complete ranking ERGM

In this appendix, we review the basic algorithm of Geyer and Thompson (1992), its application to ERGMs for dichotomous data by Hunter and Handcock (2006), and its extension to valued ERGMs by Krivitsky (2012), and outline the Markov chain Monte Carlo sampling algorithm that it requires for rank data.

The challenge in fitting the model specified by (3) is that the likelihood contains an intractable normalizing constant (4), which is a summation over a large, finite sample space of cardinality $\{(n-1)!\}^n$. This summation can be approximated using importance sampling Monte Carlo integration. In practice, this is highly inefficient. For the purposes of likelihood maximization, however, it suffices to approximate the ratio of the normalizing constants, i.e., $\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}')/\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})$ for some two parameter configurations $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}$: maximizing

$$\frac{\Pr_{\boldsymbol{\theta}';\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})}{\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})} = \frac{\exp\{\boldsymbol{\theta}'\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}')}\Bigg/\frac{\exp\{\boldsymbol{\theta}\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} = \frac{\exp\{\boldsymbol{\theta}'\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\exp\{\boldsymbol{\theta}\cdot\boldsymbol{g}(\boldsymbol{y})\}}\Bigg/\frac{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}')}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} \tag{A1}$$

with respect to $\boldsymbol{\theta}'$ will produce the MLE as well as maximizing $\Pr_{\boldsymbol{\theta}';\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})$ itself would.

In estimating this ratio, in particular, for a given ERGM configuration $\boldsymbol{\theta}'$, it makes sense to use a proposal distribution from the same family, with a similar $\boldsymbol{\theta}$, which, for an exponential family in particular, simplifies as follows:

$$\begin{aligned}\frac{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}')}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} &= \frac{\sum_{\boldsymbol{y}\in\mathcal{Y}}\exp\{\boldsymbol{\theta}'\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} = \frac{\sum_{\boldsymbol{y}\in\mathcal{Y}}\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot\boldsymbol{g}(\boldsymbol{y})\}\exp\{\boldsymbol{\theta}\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}\\ &= \sum_{\boldsymbol{y}\in\mathcal{Y}}\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot\boldsymbol{g}(\boldsymbol{y})\}\frac{\exp\{\boldsymbol{\theta}\cdot\boldsymbol{g}(\boldsymbol{y})\}}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})}.\end{aligned} \tag{A2}$$

Now, $\exp\{\boldsymbol{\theta}\cdot\boldsymbol{g}(\boldsymbol{y})\}/\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})$ is just $\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\boldsymbol{Y}=\boldsymbol{y})$, making (A2) an expectation of $\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot\boldsymbol{g}(\boldsymbol{Y})\}$ under $\boldsymbol{Y}\sim\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\cdot)$, so the ratio can be approximated simply with

$$\frac{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta}')}{\kappa_{\boldsymbol{g}}(\boldsymbol{\theta})} = \mathrm{E}_{\boldsymbol{\theta};\boldsymbol{g}}\left[\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot\boldsymbol{g}(\boldsymbol{Y})\}\right] \approx \frac{1}{S}\sum_{s=1}^{S}\exp\{(\boldsymbol{\theta}'-\boldsymbol{\theta})\cdot\boldsymbol{g}(\boldsymbol{Y}^{(s)})\}, \tag{A3}$$

given a sample $\boldsymbol{Y}^{(1)},\ldots,\boldsymbol{Y}^{(S)}$ drawn from $\Pr_{\boldsymbol{\theta};\boldsymbol{g}}(\cdot)$ for a given guess $\boldsymbol{\theta}$. The accuracy of approximation (A3) decreases as $\boldsymbol{\theta}'$ draws father from $\boldsymbol{\theta}$, because the greater the difference between them, the greater the variance of the quantity in the exponent of (A3), so, in practice, a starting guess $\boldsymbol{\theta}^{(0)}$ is selected, sampled from, used to maximize (A1) to produce a new guess $\boldsymbol{\theta}^{(1)}$, which is in turn used to draw a new sample, and so on until convergence.

Notice that (A3) only depends on $\boldsymbol{Y}^{(s)}$ through its vector of sufficient statistics $\boldsymbol{g}(\boldsymbol{Y}^{(s)})$. This means that once the sample is drawn, the maximization is agnostic to the nature of the data and to the structure of the sample space $\mathcal{Y}$, depending only on the sufficient statistics. In practice, this reduces its storage requirements and greatly simplifies convergence and other diagnostics, and it means that developments such as those of Hummel et al. (2012) to improve stability and accuracy of the estimation can be applied directly. We use them here as well.

To obtain such a sample, we use a Metropolis sampling algorithm (Algorithm 1). At every iteration, an ego is chosen at random and a symmetric proposal to swap the rankings of two of its alters (also chosen at random) is made. Any permutation of $n-1$ alters for each ego can be reached from any other permutation in at most $n-2$ swaps, and the sample space is finite, guaranteeing ergodicity.

Algorithm 1 Sampling from a complete rank ERGM

Let:

$\mathrm{RandomChoose}(A)$ return a random element of a set A
$\mathrm{Uniform}(a, b)$ return a random draw from the $\mathrm{Uniform}(a, b)$ distribution

Require: $\boldsymbol{y}^{(0)} \in \mathcal{Y}$, S sufficiently large, N, $\boldsymbol{g}(\cdot)$, $\boldsymbol{\theta}$
Ensure: a draw from a complete ranking ERGM $\mathrm{Pr}_{\boldsymbol{\theta};\boldsymbol{g}}(\cdot)$

for $s \leftarrow \{1 \mathinner{..} S\}$ **do**
  $i \leftarrow \mathrm{RandomChoose}(N)$ {Select an ego at random.}
  $j \leftarrow \mathrm{RandomChoose}(N \backslash \{i\})$ {Propose one alter.}
  $j' \leftarrow \mathrm{RandomChoose}(N \backslash \{i, j\})$ {Propose another alter.}
  $\boldsymbol{y}^* \leftarrow \left(\boldsymbol{y}^{(s-1)}\right)^{i:\, j \rightleftarrows j'}$ {Propose a swap.}
  $r \leftarrow \exp\left[\boldsymbol{\theta} \cdot \{\boldsymbol{g}(\boldsymbol{y}^*) - \boldsymbol{g}(\boldsymbol{y}^{(s-1)})\}\right]$
  $u \leftarrow \mathrm{Uniform}(0, 1)$
  if $u < r$ **then**
    $\boldsymbol{y}^{(s)} \leftarrow \boldsymbol{y}^*$ {Accept the proposal.}
  else
    $\boldsymbol{y}^{(s)} \leftarrow \boldsymbol{y}^{(s-1)}$ {Reject the proposal.}
return $\boldsymbol{y}^{(S)}$

A.3 Degeneracy

Degeneracy, in its many meanings, is often a concern in applying ERGMs. Despite recent progress (Rinaldo, Fienberg, and Zhou 2009; Butts 2011; Schweinberger 2011, for example), few general and unambiguous diagnostics for degeneracy exist beyond simulated goodness-of-fit measures (Hunter, Goodreau, and Handcock 2008a) and diagnostics of symptoms such as poor convergence of the MLE-finding procedure and, in particular, poor mixing of the MCMC sampler (Goodreau, Handcock, Hunter, Butts, and Morris 2008a), often due to multimodality of the distribution of networks under the model (Snijders et al. 2006; Rinaldo et al. 2009; Krivitsky 2012).

In our case, there is some cause for concern due to the high order of some of the sufficient statistics we propose. For example, g_{GNC} is a summation of indicators over a set of cardinality $\left|N^{4\neq}\right| = O(n^4)$. Its promotion statistic $\Delta_{i,j}^{\nearrow} g_{\text{GNC}}$ is a sum over a set of $O(n)$. Thus, there may exist configurations for which a small change in the ranking leads to a massive change in the value of this statistic and thus in the likelihood. Such statistics can induce excessively strong dependence and lead to asymptotic degeneracy (Butts 2011; Schweinberger 2011).

At the same time, a number of factors are likely to ameliorate such problems for the specific case of rank data. A network of ranks simply contains more information than a network of binary outcomes of the same size. More concretely, a uniform distribution over binary directed networks of size n with no self-loops has entropy of $n(n-1)$ bits, while a uniform distribution over rank networks has entropy of $\log_2\{(n-1)!\}^n = n\log_2\{(n-1)!\}$ bits, which grows much faster in n. This can reduce the risk of pathological model behavior due to poor identification in the small-n case. Also, while there exists the potential for problematic statistics and configurations, the constrained nature of the sample space makes degeneracy more difficult to achieve than is the case for typical graphs or digraphs: an ego cannot promote one alter without demoting another, and while some statistics, like g_{GNC}, have an extreme case of all actors agreeing, others, like g_{D}, cannot be reduced to 0.

In all of the following examples, we found no symptoms of degeneracy: for every fit reported, the estimation algorithm had converged without issues to an estimate for which the expectation of the sufficient statistic from the penultimate MCMC MLE step was close to its observed value, MCMC diagnostics (found in the Supplement) showed no sign of poor mixing or multimodality, and MCMC sample sizes used were sufficient to render the error due to the use of MCMC to approximate the likelihood (Hunter and Handcock 2006: eq. 3.11) negligible compared to the standard error. That these MLEs were found and converged to despite the $\mathbf{0}$ starting point, again, suggests that models using the terms introduced here are quite stable. In practice, we recommend simulation from fitted models (as in the case of binary ERGMs) as a useful tool for verifying non-degeneracy.

B Change Statistic Examples

Section 3 introduces a variety of terms for ordinal relational data, all of which are defined in terms of the comparison relation of Section 2.1. To further fix ideas (and to provide a resource for those seeking to implement these terms), we here provide numerical examples of change statistic calculations for several of these terms. All examples are based on the illustrative ranking structure of Section 3.

Attractiveness/Popularity Effects: As an illustrative example of attractiveness, we consider these statistics in the cases of the sample complete-ordering networks $\boldsymbol{y}$ and $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$ from Figure 2. For purposes of illustration, we also assume that each sample network is associated with a covariate vector $\boldsymbol{x} = [1, 2, 3, 4]$. Note that the attractiveness statistic g_{A} is equal to the sum of the signed differences in the elements of $\boldsymbol{x}$ for all those ordered pairs having non-zero entries in $y_{\cdot:\cdot\succ\cdot}$. These entries are depicted in Figure 2; summing the associated differences gives us $g_{\mathrm{A}}(\boldsymbol{y}; \boldsymbol{x}) = 6$ and $g_{\mathrm{A}}(\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}; \boldsymbol{x}) = 4$. Intuitively, the ratings of $\boldsymbol{y}$ are more strongly aligned with the attribute values in $\boldsymbol{x}$ than those of $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$. Note that the difference between $g_{\mathrm{A}}(\boldsymbol{y}; \boldsymbol{x})$ and $g_{\mathrm{A}}(\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}; \boldsymbol{x})$ is necessarily equal to the promotion statistic for a swap of B and C in A's rating structure, i.e. $2(x_{\mathrm{B}} - x_{\mathrm{C}}) = 2(2 - 3) = -2$. In the case of attractiveness, we thus have the intuitive notion that promoting an actor with a lower attribute value over one whose attribute value is higher results in a decline in the associated statistic.

Difference/Similarity Effects: Returning to our two sample cases, let us consider how the distance/similarity statistic g_{H} varies (again taking $\boldsymbol{x} = [1, 2, 3, 4]$, and for simplicity taking z equal to the absolute difference). In the case of g_{A}, our statistic reduced to the sum of attribute value differences for ordered pairs having a judgment in $y_{\cdot:\,\cdot\succ\cdot}$; the present case can be recognized as essentially similar, with the only distinction being that the relevant attribute for $y_{i:\,j\succ k}$ is now the proximity to the rater (i) vis-a-vis $\boldsymbol{x}$. Summing over the ordinal judgments in each case gives us $g_{\mathrm{H}}(\boldsymbol{y};\boldsymbol{x}) = 6$, $g_{\mathrm{H}}(\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}};\boldsymbol{x}) = 4$, and $\Delta^{\nearrow}_{\mathrm{A,B}} g_{\mathrm{H}}(\boldsymbol{y}) = 2(|x_{\mathrm{A}} - x_{\mathrm{B}}| - |x_{\mathrm{A}} - x_{\mathrm{C}}|) = 2(|1-2| - |1-3|) = -2$.

Global Conformity: For our sample networks, $\boldsymbol{y}$ and $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$, summing over all tetrads gives us respectively $g_{\mathrm{A}}(\boldsymbol{y}) = 8$ and $g_{\mathrm{A}}(\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}) = 6$. As this implies, $\Delta^{\nearrow}_{\mathrm{A,B}} g_{\mathrm{GNC}}(\boldsymbol{y}) = 2\sum_{l\in N\setminus\{\mathrm{A,B,C}\}}(y_{l:\,\mathrm{C}\succ\mathrm{B}} - y_{l:\,\mathrm{B}\succ\mathrm{C}}) = 2(y_{\mathrm{D:\,C}\succ\mathrm{B}} - y_{\mathrm{D:\,B}\succ\mathrm{C}}) = 2(0-1) = -2$. Intuitively, this implies that the ratings of $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$ are in greater consistency with one another than those of $\boldsymbol{y}$; as the promotion statistic indicates, this is because A's switching from C $\overset{\mathrm{A}}{\succ}$ B to B $\overset{\mathrm{A}}{\succ}$ C brings A view of B and C into conformity with the view of D. To the extent that greater conformity pressure is present (i.e., a negative coefficient associated with g_{A}, such changes are (*ceteris paribus*) favored.

Local Conformity: For our sample networks $\boldsymbol{y}$ and $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$, calculation of the local conformity statistic shows that $\boldsymbol{y}$ exhibits a slightly higher level of non-conformity than $\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}$: $g_{\mathrm{LNC}}(\boldsymbol{y}) = 2$, versus $g_{\mathrm{LNC}}(\boldsymbol{y}^{\mathrm{A:\,B}\rightleftarrows\mathrm{C}}) = 1$. To appreciate the origin of this difference we may examine the promotion statistic:

$$\begin{aligned}\Delta^{\nearrow}_{\mathrm{A,B}} g_{\mathrm{LNC}}(\boldsymbol{y}) &= \big(y_{\mathrm{A:\,D\succ C}} y_{\mathrm{D:\,C\succ B}} - y_{\mathrm{A:\,D\succ C}} y_{\mathrm{D:\,B\succ C}} \\ &\quad + y_{\mathrm{D:\,A\succ C}} y_{\mathrm{D:\,C\succ B}} - y_{\mathrm{D:\,A\succ B}} y_{\mathrm{D:\,B\succ C}} \\ &\quad + y_{\mathrm{A:\,C\succ D}} y_{\mathrm{B:\,D\succ C}} - y_{\mathrm{A:\,B\succ D}} y_{\mathrm{C:\,B\succ D}}\big) \\ &= (1 \times 0 - 1 \times 1 \\ &\quad + 0 \times 0 - 0 \times 1 \\ &\quad + 0 \times 1 - 0 \times 1) = -1.\end{aligned}$$

In promoting B over C, A conformed with D—the only actor A ranked over both B and C—so local nonconformity declined (per Equation (10a)). D did not rank A highly enough to experience conformity pressure in ranking B and C, and the only actor who ranked A highly is C, but since actors are not permitted to rank themselves, C has nothing to say about C's own ranking relative to B (see Equation (10b)). Promoting C over B made B salient in comparisons about C, but not about D, so no disconformance was created or eliminated in this respect (see Equation (10c)). As this case illustrates, local nonconformity involves both consistency of ratings and status: to the extent that pressure for local conformity is present, actors will seek to adapt their rankings to match those of individuals they rate highly, with less regard to mismatches with those to whom they assign a low rank.

Deference Aversion: While we saw that $\boldsymbol{y}^{\mathrm{A:\,B\rightleftarrows C}}$ showed lower local nonconformity than $\boldsymbol{y}$, we see the opposite for deference aversion: applying the above definition to the two networks gives us $g_{\mathrm{D}}(\boldsymbol{y}) = 6$ and $g_{\mathrm{D}}(\boldsymbol{y}^{\mathrm{A:\,B\rightleftarrows C}}) = 8$, implying that $\boldsymbol{y}$ has fewer instances in which an individual tacitly places him or herself below others in the network. Examining the promotion statistic, we see that

$$\begin{aligned}\Delta^{\nearrow}_{\mathrm{A,B}} g_{\mathrm{D}}(\boldsymbol{y}) &= 2(y_{\mathrm{C:\,A \succ B}} + y_{\mathrm{B:\,C \succ A}} - 1) \\ &= 2 \times (1 + 1 - 1) = 2.\end{aligned}$$

Intuitively, C had ranked A over B, so in ranking C over B, A was not deferring to either party. By contrast, B had ranked C over A, so in promoting B over C, A implicitly deferred to B. In settings for which deference aversion is operative, such changes are *ceteris paribus* unfavorable.

List of Figures

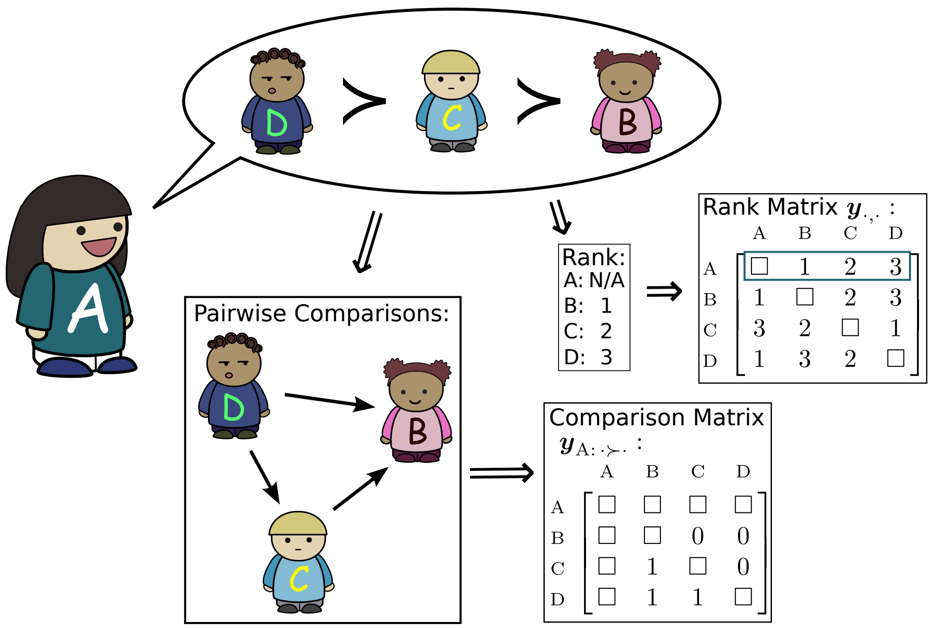

Figure 1: Ego A's report regarding her ranking of B, C, and D can be encoded as a rank ordering or as pairwise comparisons. Here, Ego A's response ranks D highest, then C, then B. We may encode this report by assigning a rank of 1 to B, 2 to C, and 3 to D resulting in a row of rank matrix $\boldsymbol{y}_{\cdot,\cdot}$; or we may consider all pairwise comparisons implied by A: that D is ranked over C, D is ranked over B, and C is ranked over B, and the opposite does not hold, resulting in a binary matrix of comparison indicators $\boldsymbol{y}_{\mathrm{A}:\cdot\succ\cdot}$. Here, boxes (□) denote matrix entries that are unobservable and/or meaningless: in this case, these include A's row and column entries in the comparison matrix and the diagonal of the rank matrix, because A is not permitted to rank herself among the others; and the diagonal of the comparison matrix, because it is meaningless to compare an alter with himself or herself.

$\boldsymbol{y}$ Ranking:

$$\begin{array}{ll} \mathrm{A}: & \mathrm{D} \succ \mathrm{C} \succ \mathrm{B} \\ \mathrm{B}: & \mathrm{D} \succ \mathrm{C} \succ \mathrm{A} \\ \mathrm{C}: & \mathrm{A} \succ \mathrm{B} \succ \mathrm{D} \\ \mathrm{D}: & \mathrm{B} \succ \mathrm{C} \succ \mathrm{A} \end{array}$$

$$\boldsymbol{y}_{\cdot,\cdot} = \begin{bmatrix} \square & 1 & 2 & 3 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix}$$

$$\boldsymbol{y}_{\mathrm{A}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 0 & 0 \\ \square & 1 & \square & 0 \\ \square & 1 & 1 & \square \end{bmatrix} \quad \boldsymbol{y}_{\mathrm{B}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & \square & 0 & 0 \\ \square & \square & \square & \square \\ 1 & \square & \square & 0 \\ 1 & \square & 1 & \square \end{bmatrix} \quad \boldsymbol{y}_{\mathrm{C}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & 1 & \square & 1 \\ 0 & \square & \square & 1 \\ \square & \square & \square & \square \\ 0 & 0 & \square & \square \end{bmatrix} \quad \boldsymbol{y}_{\mathrm{D}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & 0 & 0 & \square \\ 1 & \square & 1 & \square \\ 1 & 0 & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$$

(a) $\boldsymbol{y}$: original network

$\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{C}}$ Ranking:

$$\begin{array}{ll} \mathrm{A}: & \mathrm{D} \succ \mathrm{B} \succ \mathrm{C} \\ \mathrm{B}: & \mathrm{D} \succ \mathrm{C} \succ \mathrm{A} \\ \mathrm{C}: & \mathrm{A} \succ \mathrm{B} \succ \mathrm{D} \\ \mathrm{D}: & \mathrm{B} \succ \mathrm{C} \succ \mathrm{A} \end{array}$$

$$\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{C}}_{\cdot,\cdot} = \begin{bmatrix} \square & 2 & 1 & 3 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix} \quad \boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{C}}_{\mathrm{A}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 1 & 0 \\ \square & 0 & \square & 0 \\ \square & 1 & 1 & \square \end{bmatrix}$$

(b) $\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{C}}$: Ego A swapped B and C

$\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{D}}$ Ranking:

$$\begin{array}{ll} \mathrm{A}: & \mathrm{B} \succ \mathrm{C} \succ \mathrm{D} \\ \mathrm{B}: & \mathrm{D} \succ \mathrm{C} \succ \mathrm{A} \\ \mathrm{C}: & \mathrm{A} \succ \mathrm{B} \succ \mathrm{D} \\ \mathrm{D}: & \mathrm{B} \succ \mathrm{C} \succ \mathrm{A} \end{array}$$

$$\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{D}}_{\cdot,\cdot} = \begin{bmatrix} \square & 3 & 2 & 1 \\ 1 & \square & 2 & 3 \\ 3 & 2 & \square & 1 \\ 1 & 3 & 2 & \square \end{bmatrix} \quad \boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{D}}_{\mathrm{A}:\,\cdot\succ\cdot} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 1 & 1 \\ \square & 0 & \square & 1 \\ \square & 0 & 0 & \square \end{bmatrix}$$

(c) $\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{D}}$: Ego A swapped B and D

Figure 2: Representations of complete rankings $\boldsymbol{y}$ and illustration of $\boldsymbol{y}^{i:\,j\rightleftarrows k}$ (defined in Section 3.1) for $n = 4$. Here, boxes ($\square$) denote matrix entries that are unobservable and/or meaningless. In $\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{C}}$ (b) and $\boldsymbol{y}^{\mathrm{A}:\,\mathrm{B}\rightleftarrows\mathrm{D}}$ (c), pairwise comparison matrices $\boldsymbol{y}_{i:\,\cdot\succ\cdot}$ for i other than A are unchanged from those of $\boldsymbol{y}$ (a).

$\boldsymbol{y}^\star$ Ranking:

$$
\begin{array}{ll}
\text{A}: & \text{D} \cong \text{C} \succ \text{B} \\
\text{B}: & \text{A} \succ \text{B} \succ \text{C} \succ \text{D} \\
\text{C}: & \text{B} \succ \text{C},\ \text{A} \succ \text{D} \\
\text{D}: & \text{A} \succ \text{B},\ \text{B} \succ \text{C},\ \text{C} \succ \text{A}
\end{array}
\qquad
\boldsymbol{y}^\star_{\cdot,\cdot} = \begin{bmatrix} \square & 1 & 2 & 2 \\ 4 & 3 & 2 & 1 \\ & \text{N/A} & & \\ & \text{N/A} & & \end{bmatrix}
$$

$$
\boldsymbol{y}^\star_{\text{A}:\cdot\succ\cdot} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 0 & 0 \\ \square & 1 & \square & 0 \\ \square & 1 & 0 & \square \end{bmatrix}
\quad
\boldsymbol{y}^\star_{\text{B}:\cdot\succ\cdot} = \begin{bmatrix} \square & 1 & 1 & 1 \\ 0 & \square & 1 & 1 \\ 0 & 0 & \square & 1 \\ 0 & 0 & 0 & \square \end{bmatrix}
\quad
\boldsymbol{y}^\star_{\text{C}:\cdot\succ\cdot} = \begin{bmatrix} \square & 0 & 0 & 1 \\ 0 & \square & 1 & 0 \\ 0 & 0 & \square & 0 \\ 0 & 0 & 0 & \square \end{bmatrix}
\quad
\boldsymbol{y}^\star_{\text{D}:\cdot\succ\cdot} = \begin{bmatrix} \square & 1 & 0 & \square \\ 0 & \square & 1 & \square \\ 1 & 0 & \square & \square \\ \square & \square & \square & \square \end{bmatrix}
$$

Figure 3: Representations of idiosyncratic ordering structures for $n = 4$. Here, boxes ($\square$) denote matrix entries that are unobservable and/or meaningless. Ego A does not rank C or D over one another but ranks both above B, for a partial ordering; Ego B is permitted to rank self along with the others; Ego C is permitted to rank self, but does not establish a weak order, since sets $\{\text{A}, \text{D}\}$ and $\{\text{B}, \text{C}\}$ are incomparable; and Ego D reports comparisons that violate transitivity. Notice that not all these can be represented using a rank matrix $\boldsymbol{y}^\star_{\cdot,\cdot}$, but all can be represented using the comparison operation $y_{\cdot:\cdot\succ\cdot}$ and sample space constraints.

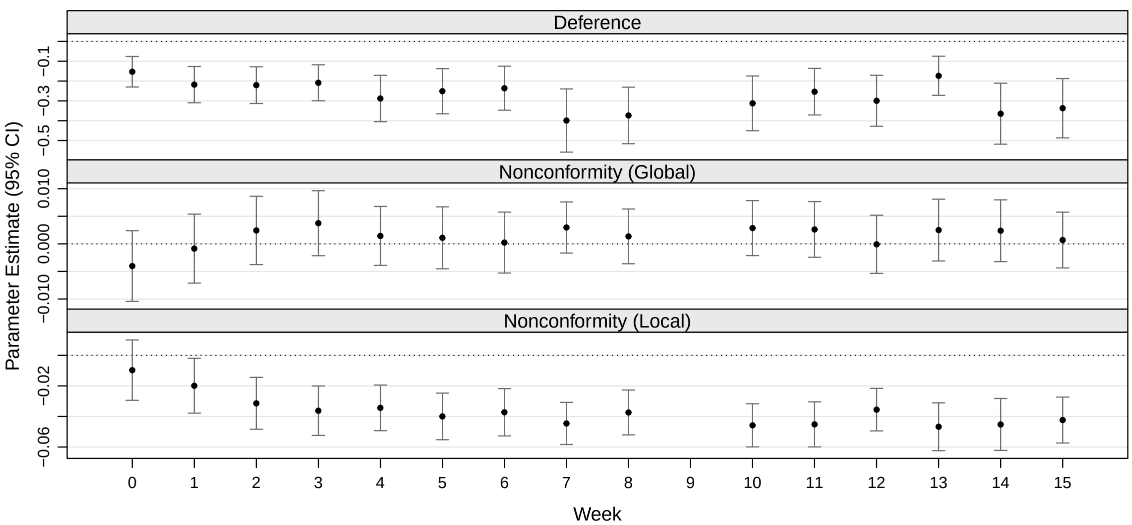

Figure 4: Estimated coefficients for the cross-sectional model fit to each week's rankings in the Newcomb's Fraternity. Error bars are at 95% confidence.

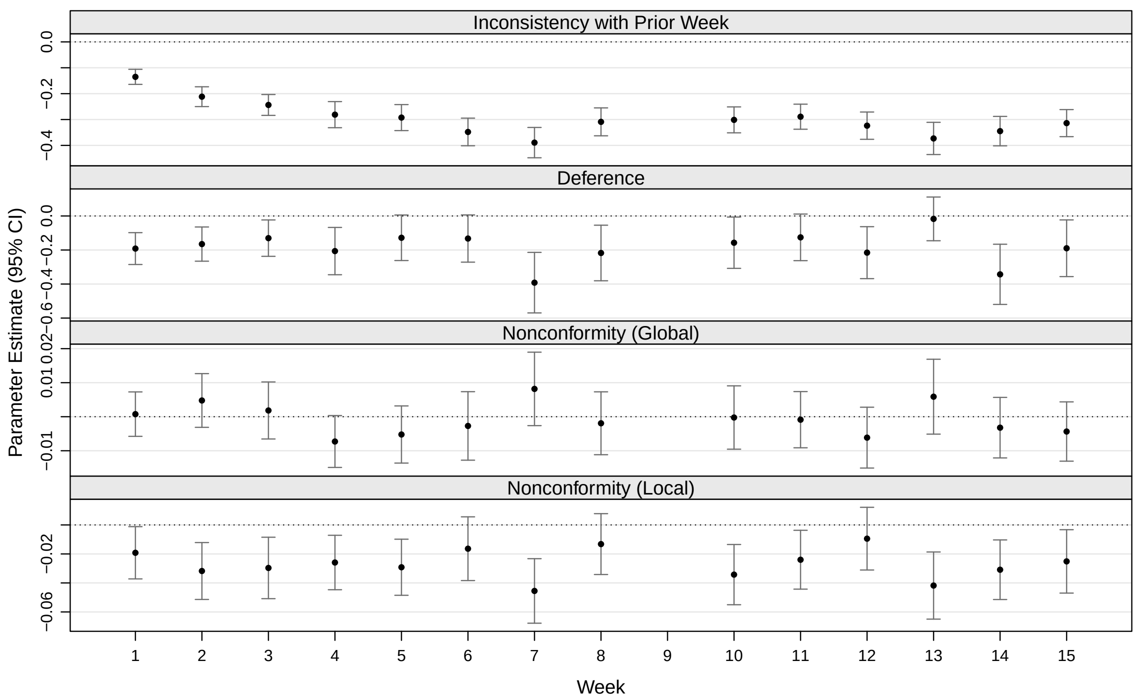

Figure 5: Estimated coefficients for the longitudinal model fit to each week's rankings in the Newcomb's Fraternity. Error bars are at 95% confidence.

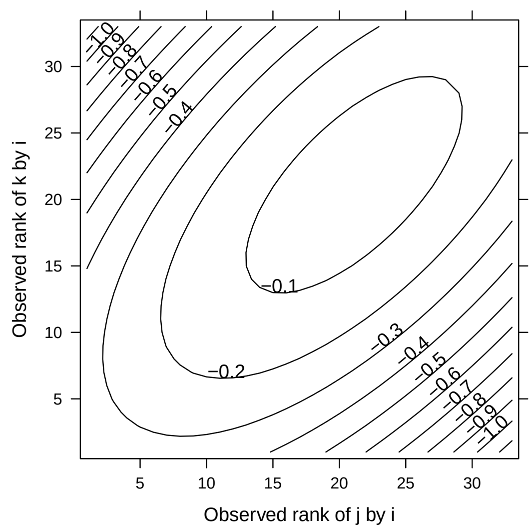

Figure 6: Predicted effect of alter positions on reporting inaccuracy

List of Tables

Table 1: Results for cross-sectional analysis of the Newcomb's data

	Estimates (Std. Errors)		
		Nonconformity	
Week	Deference	Global	Local
0	-0.153 (0.039)***	-0.004 (0.003)	-0.010 (0.010)
1	-0.218 (0.047)***	-0.001 (0.003)	-0.020 (0.009)*
2	-0.221 (0.047)***	0.002 (0.003)	-0.031 (0.009)***
3	-0.209 (0.046)***	0.004 (0.003)	-0.036 (0.008)***
4	-0.288 (0.060)***	0.001 (0.003)	-0.034 (0.008)***
5	-0.251 (0.058)***	0.001 (0.003)	-0.040 (0.008)***
6	-0.236 (0.057)***	0.000 (0.003)	-0.037 (0.008)***
7	-0.399 (0.081)***	0.003 (0.002)	-0.045 (0.007)***
8	-0.373 (0.073)***	0.001 (0.003)	-0.037 (0.007)***
10	-0.312 (0.070)***	0.003 (0.003)	-0.046 (0.007)***
11	-0.254 (0.060)***	0.003 (0.003)	-0.045 (0.008)***
12	-0.299 (0.066)***	-0.000 (0.003)	-0.036 (0.007)***
13	-0.174 (0.050)***	0.002 (0.003)	-0.047 (0.008)***
14	-0.365 (0.078)***	0.002 (0.003)	-0.045 (0.009)***
15	-0.337 (0.076)***	0.001 (0.003)	-0.042 (0.008)***

Significance levels: $0.05 \geq^{\star} > 0.01 \geq^{\star\star} > 0.001 \geq^{\star\star\star}$

Table 2: Results for dynamic analysis of the Newcomb's data

Week Transition	Estimates (Std. Errors)			
	Inconsistency with Prior Week	Deference	Nonconformity	
			Global	Local
$0 \to 1$	-0.135 (0.015)***	-0.192 (0.048)***	0.001 (0.003)	-0.019 (0.009)*
$1 \to 2$	-0.212 (0.020)***	-0.165 (0.051)**	0.005 (0.004)	-0.032 (0.010)**
$2 \to 3$	-0.244 (0.021)***	-0.130 (0.055)*	0.002 (0.004)	-0.030 (0.011)**
$3 \to 4$	-0.281 (0.026)***	-0.206 (0.071)**	-0.007 (0.004)	-0.026 (0.010)**
$4 \to 5$	-0.292 (0.026)***	-0.128 (0.068)	-0.005 (0.004)	-0.029 (0.010)**
$5 \to 6$	-0.348 (0.027)***	-0.133 (0.071)	-0.003 (0.005)	-0.016 (0.011)
$6 \to 7$	-0.389 (0.030)***	-0.392 (0.091)***	0.008 (0.006)	-0.046 (0.011)***
$7 \to 8$	-0.309 (0.028)***	-0.218 (0.083)**	-0.002 (0.005)	-0.013 (0.011)
$8 \to 10$	-0.301 (0.026)***	-0.157 (0.077)*	-0.000 (0.005)	-0.034 (0.011)**
$10 \to 11$	-0.289 (0.025)***	-0.126 (0.070)	-0.001 (0.004)	-0.024 (0.010)*
$11 \to 12$	-0.324 (0.027)***	-0.216 (0.078)**	-0.006 (0.005)	-0.009 (0.011)
$12 \to 13$	-0.373 (0.032)***	-0.017 (0.066)	0.006 (0.006)	-0.042 (0.012)***
$13 \to 14$	-0.345 (0.029)***	-0.343 (0.090)***	-0.003 (0.005)	-0.031 (0.010)**
$14 \to 15$	-0.314 (0.027)***	-0.190 (0.085)*	-0.004 (0.004)	-0.025 (0.011)*

Significance levels: $0.05 \geq^{\star} > 0.01 \geq^{\star\star} > 0.001 \geq^{\star\star\star}$

Table 3: Effect of frequency of interaction on reporting inaccuracy

Term	Estimates (Std. Errors)	
	Frequency	Rank
Intercept	-0.066 $(0.020)^{\star\star}$	-0.159 $(0.023)^{\star\star\star}$
Frequency difference	-0.018 $(0.003)^{\star\star\star}$	
Frequency rank difference		-0.003 (0.003)

Significance levels: $0.05 \geq^{\star} > 0.01 \geq^{\star\star} > 0.001 \geq^{\star\star\star}$

Table 4: Effects of alter positions on reporting inaccuracy

Term	Estimate (Std. Err.)
Intercept	-0.068 $(0.024)^{\star\star}$
Linear	0.107 $(0.018)^{\star\star\star}$
Quadratic	-1.300 $(0.296)^{\star\star\star}$
Interaction	1.765 $(0.512)^{\star\star\star}$

Sig.: $0.05 \geq^{\star} > 0.01 \geq^{\star\star} > 0.001 \geq^{\star\star\star}$