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A study of weighted Bergman projections

Trong Thuc Phung
University of Wollongong

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WEIGHTED BERGMAN PROJECTIONS

PHUNG TRONG THUC

Supervisors

Dr. Jiakun Liu and Dr. Tran Vu Khanh

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Declaration

I, *Phung Trong Thuc*, declare that this thesis submitted in fulfilment for the requirements of the degree Doctor of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Phung Trong Thuc

August 2018

To my son, Phung Ha Nguyen

and my wife, Ha Nguyen Thuy Linh.

Abstract

The aim of this thesis is to study certain weighted Bergman projections associated to a bounded pseudoconvex domain in $\mathbb{C}^n$. We obtain three new results as follows:

The first result concerns estimates of weighted Bergman kernels and the holomorphic Hardy norm of the Bergman kernel. The main tools are L^2-estimates of the $\overline{\partial}$-equation and the relation between the Bergman kernel and the pluricomplex Green function. In particular, these improve the result of Chen and Fu [CF11] on the comparison of the Szegö and Bergman kernels for the class of pseudoconvex domains admitting a plurisubharmonic defining function, such as convex domains, strongly pseudoconvex domains and Kohn special domains.

The second result extends the work of Čučković and McNeal [CM06]; and of Abate, Raissy and Saracco [ARS12], on the gain L^p regularity of the Bergman-Toeplitz operators

$$f \longrightarrow T_{K^{-\alpha}}(f)(z) := \int\limits_{\Omega} K(z,w)\, K^{-\alpha}(w,w)\, f(w)\, dV(w)\,, \tag{0.0.1}$$

for a class of weakly pseudoconvex smooth domains in $\mathbb{C}^n$, which we call sharp $\mathcal{B}$-type domains, that contains such as strongly pseudoconvex domains in $\mathbb{C}^n$, convex domains of finite type in $\mathbb{C}^n$ and finite-type domains in $\mathbb{C}^2$.

For the third result, we obtain the L^p mapping properties of the Bergman-Toeplitz operators (0.0.1) for a model of non-smooth domains, namely, the fat Hartogs triangles

$$\Omega_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1 \right\}, \quad k \in \mathbb{Z}^+.$$

As a result, our work generalises the recent work by Edholm and McNeal [EM16].

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I am grateful to my family, especially my parents and my wife, who always encourage and believe in my ability. This thesis is also devoted to my son, who really enjoys doing mathematics.

Contents

Chapter 1

Introduction

1.1 Motivation

The study of the Bergman kernel and the Bergman projection has a long history since it was first introduced by Stefan Bergman in the 1920s. The definition of the Bergman kernel is quite simple and natural. It is the reproducing kernel for the space of square-integrable holomorphic functions. However, in general, it is not an easy task to obtain an explicit formula for the Bergman kernel, even with simple domains such as ellipsoids. Nevertheless, progress on the study of the $\overline{\partial}$-Neumann problem has advanced the understanding of the Bergman kernel and brought new insights to the topic. Particularly exciting is the close connection between the Bergman kernel and many important problems in complex analysis, such as the extension of biholomorphic mappings, the study of invariant metrics, and the theory of holomorphic peak functions. We refer readers to the book of Krantz [Kra13, Kra06] for further details and discussion.

Motivated by the work of Diederich and Ohsawa [DO95] in the study of lower bounds for the Bergman distance, Błocki [Bło05] introduced a method in which the relation between the pluricomplex Green function and the Bergman distance is derived directly under the L^2 theory of the $\overline{\partial}$-problem. The remarkable fact is that the pluricomplex Green function, which comes from the pluripotential theory, can be applied in a very flexible way to obtain various estimates on the Bergman kernel. The argument basically relies on two key facts about the pluricomplex Green function: the log-singularity property and the maximality. On the other hand, the problem of characterising the Bergman completeness motivates the study of quantitative estimates on sub-level sets of the pluricomplex Green function, which had also been studied earlier by several authors [DH00, Her99, BP98, Che99].

In the spirit of weighted estimates for the $\overline{\partial}$ operator, Chen and Fu [CF11] obtained estimates on the ratio between the Szegö kernel and the Bergman kernel.

The method is a clever use of tools from the $\overline{\partial}$-Neumann problem and properties of the pluricomplex Green function. The novelty is that the method works under very mild assumptions on the geometry of domains. For example, they showed that for any C^2-bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and any $a \in (0,1)$,

$$\frac{S(z,z)}{K(z,z)} \lesssim \delta(z) \left|\log \delta(z)\right|^{n/a}. \tag{1.1.1}$$

Here S and K denote the Szegö kernel and the Bergman kernel of Ω respectively, and $\delta : \Omega \to \mathbb{R}^+$ is the distance-to-boundary function. The lower bound is also provided as

$$\frac{S(z,z)}{K(z,z)} \gtrsim \delta(z) \left|\log \delta(z)\right|^{-1/\eta}, \tag{1.1.2}$$

given that the domain admits a defining function ρ such that $i(1/\rho)\,\partial\overline{\partial}\rho \leqslant i\partial\overline{\partial}\varphi$, for some plurisubharmonic function φ on Ω. Here η is a Diederich-Fornæss exponent of Ω. The latter condition is satisfied for the class of pseudoconvex domains of D'Angelo finite type and pseudoconvex domains having a plurisubharmonic defining function on the boundary.

The study of lower estimates for the ratio between the Szegö and Bergman kernels is closely related to a conjecture made by Ohsawa on the lower bound of the Szegö kernel– that is, $S \gtrsim \delta^{-1}$ for any bounded pseudoconvex domain. At this point, one may ask whether (1.1.2) can be improved further by the L^2-boundary integral estimates of $K(\cdot, z)$, namely

$$\left\|K(\cdot, z)\right\|_{L^2(\partial\Omega)} \leqslant A(z), \forall z \in \Omega, \tag{1.1.3}$$

for some function $A : \Omega \to \mathbb{R}^+$. Recall that

$$S(z,z) = \sup\left\{\left|f(z)\right|^2 : f \text{ is holomorphic on } \Omega, \left\|f\right\|_{L^2(\partial\Omega)} \leqslant 1\right\}.$$

Therefore (1.1.3) would imply in particular the estimate

$$\frac{S(z,z)}{K(z,z)} \geqslant \frac{1}{A(z)}.$$

A part of Chapter 3 is devoted to studying this question. We would like to remark that weighted L^2 methods, which have been developed through the study of the $\overline{\partial}$-equation, recently turn out to be essential to answer some open problems in one and several complex variables, such as the openness conjecture [Ber15, GZ15] and the Suita conjecture [Bło13, BL16]. The latter is related to the study of optimal constants in the Ohsawa-Takegoshi extension theorem.

One of fundamental problems in several complex analysis is the regularity of the Bergman projection. A basic question is under what conditions on a domain

Ω, the Bergman projection P associated to Ω is bounded from $L^p(\Omega)$ to $A^p(\Omega)$, for a given number $p \neq 2$. Here $A^p(\Omega)$ is the closed subspace of holomorphic functions in $L^p(\Omega)$. It seems that there is no complete answer to this question so far. An observation is that for domains with less regularity, like the Hartogs triangle domains $\Omega_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$ with $\gamma > 0$, $\gamma \notin \mathbb{Q}$, the Bergman projection might be bounded only if $p = 2$, see [EM17]. Even in the case of smoothly bounded pseudoconvex domains, the range of L^p-boundedness of the Bergman projection is still not well-understood. For a class of smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^n$, P is bounded for any $1 < p < \infty$; see for instance [MS94, PS77, McN89, CD06, KLT18a]. However, in [BS12], a smoothly bounded pseudoconvex domain with only limited range of L^p boundedness has been shown. On the other hand, regularity properties of the Bergman projection in Sobolev spaces have also been studied by many authors, see e.g. [BS99, Boa87, BS91, Bar92, BC00, BS12, KP08, KPS16]. Various models of domains have been used in the hope of having a better understanding of the regularity behaviour of the Bergman projection. However, the whole picture is still not completely understood.

It is reasonable to expect that the Bergman projection cannot gain the L^p regularity. Moreover, it is of particular interest to produce a holomorphic function with better regularity from an input function in the L^p space. To do this, one may study certain weighted Bergman projections. In [CM06], Čučković and McNeal obtained explicitly the L^p-L^q mapping properties of a special Bergman-Toeplitz operator on strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. Let ψ be a Lebesgue measurable function on Ω, the Bergman-Toeplitz operator with symbol (weight) ψ, denoted by T_ψ, is defined as

$$f \longrightarrow T_\psi(f)(z) := \int_\Omega K(z,w)\,\psi(w)\,f(w)\,dV(w). \tag{1.1.4}$$

The authors considered the case $\psi = \delta^\eta$, for $\eta > 0$. Recall that δ is the distance function to the boundary. Since the singularity of the Bergman kernel occurs only on the boundary, the operators T_{δ^η} should gain their regularity properties, for any $\eta > 0$. The main result in [CM06] is that if $1 < p \leqslant q < \infty$ and $\eta \geqslant (n+1)((1/p) - (1/q))$ then T_{δ^η} is bounded from $L^p(\Omega)$ to $L^q(\Omega)$. The paper also ends with the question of whether this result is sharp. The answer is affirmative, as was shown by Abate, Raissy and Saracco [ARS12], see also Proposition 3.3.5. Combining these facts together, we know exactly how much degeneracy one needs to impose on the weight $\psi = \delta^\eta$ in order to establish the L^p-L^q boundedness of T_ψ for any strongly pseudoconvex domain in $\mathbb{C}^n$.

In the setting of strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$, many nice results related to the Bergman kernel stem from its uniform behaviour near the boundary. Hörmander [Hör65] proved that

$$K(z,z)\,\delta^{n+1}(z) \to D(z_0)\,(n!/4\pi^n)\,, \text{ as } z \to z_0 \in \partial\Omega. \tag{1.1.5}$$

Here $D(z_0)$ is the determinant of the Levi form at z_0. The proof used the L^2-technique of the $\overline{\partial}$ operator. Later, more delicate analysis on the Bergman kernel of strongly pseudoconvex domains was obtained by Fefferman [Fef74] and also Boutet de Monvel and Sjöstrand [BdMS76]. However, beyond the case of strongly pseudoconvex domains, the situation is more complicated. An asymptotic estimate of the Bergman kernel, like in the case of strongly pseudoconvex domains, has not been known in general. Nevertheless, the problem has been studied for some special cases of smoothly bounded pseudoconvex domains of finite type, such as convex domains of finite type [McN94b], finite-type pseudoconvex domains in $\mathbb{C}^2$ [McN89, NRSW89a], decoupled domains of finite type [McN91], or tube domains of finite type in $\mathbb{C}^2$ [Kam98]. The boundary limit of the Bergman kernel for h-extendible domains, which includes a large class of finite-type domains, was also obtained in [BSY95, DH97].

It can be seen that the boundary behaviour of the Bergman kernel on domains of finite type depends heavily on *the type* of each boundary point. Therefore, in order to obtain sharp estimates for the Bergman-Toeplitz operators T_ψ, the choice of the weight $\psi = \delta^\eta$ seems not to be suitable. For the case of strongly pseudoconvex domains, (1.1.5) implies that K behaves like δ^{-n-1} near a boundary point, with the uniform exponent $-(n+1)$. It also suggests that in the finite-type case, a reasonable choice of ψ, which reflects sharply geometric information, should be $\psi(z) = K^\eta(z,z)$, as one can observe from the fact that for each $z_0 \in \partial\Omega$, $K(z,z) \approx \delta^{-\tau(z_0)}(z)$ as $z \approx z_0$, where $\tau(z_0)$ is a quantity depending on the type of z_0. Indeed, the number $\tau(z_0)$, which is called the growth exponent of the Bergman kernel ([DH93]), has been studied for some classes of finite-type pseudoconvex domains. For example, for domains of finite type in $\mathbb{C}^2$, Catlin [Cat89] showed that $\tau(z_0) = 2 + 2/m$, where m is the type of z_0. It is known that $\tau(z_0)$ is not necessarily rational for a point of finite type z_0, see [Her83]. Taking into account this observation, in Chapter 4 we will focus on the sharp L^p-mapping properties of the Bergman-Toeplitz operators T_{K^η} for a class of smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^n$.

It will be important to study mapping properties of the Bergman-Toeplitz operators for non-smooth pseudoconvex domains as well. Indeed, singularities of a domain can certainly distort the regularity of the operators. For example, it is known that for any bounded pseudoconvex domain Ω with Lipschitz boundary, it is always possible to choose a number q bigger than 2 and a number p large enough (depending on q) such that the Bergman projection associated to Ω is bounded from $L^p(\Omega)$ to $L^q(\Omega)$. However, in general, the same situation turns out not to be true for non-Lipschitz pseudoconvex domains, see Corollary 3.2.7.

A very old and important example of non-smooth pseudoconvex domains is the Hartogs triangle

$$\mathbb{H} := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \right\},$$

discovered by Fritz Hartogs in his 1905 thesis. The boundary of $\mathbb{H}$ is very irregular at the origin, as it is even not continuous at $(0, 0)$ as a graph. The Hartogs triangle is a bounded pseudoconvex domain in which one cannot approximate it from outside by pseudoconvex domains. Recall that, in contrast, any pseudoconvex domain can always be approximated from inside by pseudoconvex domains. This fact leads to the study of pseudoconvex domains having a Stein neighbourhood basis, see [DF77a, DF77b, BF78]. A typical example in this regard is the smooth worm domain, constructed by Diederich and Fornæss [DF77a], whose closure does not have a Stein neighbourhood basis.

In [CZ16, CS13], the regularity of the Bergman projection and of the $\bar{\partial}$-equation on the Hartogs triangle have been obtained. Interestingly, the Bergman projection is bounded in $L^p(\mathbb{H})$ if and only if $p \in \left(\frac{4}{3}, 4\right)$. Recently, the generalised Hartogs triangles

$$\mathbb{H}_\gamma := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1 \right\}, \gamma \in \mathbb{R}^+,$$

have been studied by Edholm and McNeal [EM16, EM17]. In particular, the Bergman projection is bounded in $L^p(\mathbb{H}_k)$, for $k \in \mathbb{Z}^+$ if and only if $p \in \left(\frac{2k+2}{k+2}, \frac{2k+2}{k}\right)$. These domains are called the fat Hartogs triangles. Using this model as an example for non-smooth pseudoconvex domains, we study in Chapter 5 the L^p-mapping behaviour of the Bergman-Toeplitz operators T_{K^η} on the fat Hartogs triangles. We remark that the non-symmetric setting of the Bergman-Toeplitz operators, due to the presence of weights, makes many situations harder to work with; for example, the duality argument cannot be applied. The case when γ is rational is basically similar to handle as in the integer case. It would be interesting to study the gain L^p regularity of the Bergman-Toeplitz operators T_{K^η} on $\mathbb{H}_\gamma$, with $\gamma \notin \mathbb{Q}$.

By combining the results of Chapter 4 (for a class of smooth domains) and Chapter 5 (for a model of non-smooth domains), if the Bergman-Toeplitz operator $T_{K^{-\alpha}}$ is bounded from L^p to L^q then $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$, for any pseudoconvex domain in these classes. It is therefore reasonable to expect that this is true for any bounded pseudoconvex domain. I hope to address this matter in my future work.

1.2 Publications

This thesis contains the material from the following papers:

1. Phung Trong Thuc. A note on L^2-boundary integrals of the Bergman kernel, arXiv: 1803.09393 (to appear, International Journal of Mathematics, 2019).

2. Tran Vu Khanh, Jiakun Liu and Phung Trong Thuc, Bergman-Toeplitz operators on fat Hartogs triangles, *Proc. Amer. Math. Soc.* (2018), https://doi.org/10.1090/proc/14218.

3. Tran Vu Khanh, Jiakun Liu and Phung Trong Thuc, Bergman-Toeplitz operators on weakly pseudoconvex domains, *Math. Z.* (2018), DOI 10.1007/s00209-018-2096-z.

1.3 Notation

Throughout this thesis, Ω will be a bounded pseudoconvex domain in $\mathbb{C}^n$. We use $|\Omega|$ to denote the Lebesgue measure of Ω. Unless otherwise stated, we will always consider Ω as a general pseudoconvex domain without any assumption on the regularity of the boundary. The function $\delta : \Omega \to \mathbb{R}^+$ is the distance function to the boundary

$$\delta(z) := \inf\{|z-w| : w \in \partial\Omega\},$$

where $\partial\Omega$ is the boundary of Ω. The set of all holomorphic functions on Ω is denoted by $\mathcal{O}(\Omega)$. We denote by $\mathcal{O}(\Omega, \Omega')$ the set of all holomorphic mappings from Ω into Ω'.

By $\mathbb{D}$ we denote the unit disk in $\mathbb{C}$, and we use $\mathbb{H}$ for the Hartogs triangle

$$\mathbb{H} := \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\right\}.$$

For $z \in \mathbb{C}^n$ and $r > 0$, $B(z, r) := \{w \in \mathbb{C}^n : |w - z| < r\}$ denotes the open ball of radius r centred at z in $\mathbb{C}^n$.

We shall use S and K to denote the Szegö kernel and the Bergman kernel of Ω respectively. These kernels are related to the Bergman space $A^2(\Omega)$ and the *holomorphic* Hardy space $H^2(\Omega)$, defined by

$$A^2(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\},$$

$$H^2(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \int_{\partial\Omega} |f(z)|^2 \, d\sigma(z) < \infty \right\}.$$

Here dV and $d\sigma$ denote Lebesgue measure and the standard surface measure respectively. Also, for $p \in [1, \infty]$,

$$A^p(\Omega) := \{f \in \mathcal{O}(\Omega) : f \in L^p(\Omega)\}.$$

We equip $A^p(\Omega)$ with the norm induced from $L^p(\Omega)$. We denote by $(A^p(\Omega))^\star$ the dual space of $A^p(\Omega)$. Each element of $(A^p(\Omega))^\star$ is a continuous linear functional on $A^p(\Omega)$. When there is no confusion, we may also write $\int_\Omega |f|^p$ instead of $\int_\Omega |f(z)|^p\, dV(z)$. The *harmonic* Hardy space $h^2(\Omega)$ is defined by

$$h^2(\Omega) := \left\{ f \text{ is harmonic on } \Omega : \int_{\partial\Omega} |f(z)|^2\, d\sigma(z) < \infty \right\}.$$

The Hilbert spaces $L^2(\Omega)$ and $L^2(\partial\Omega)$ carry the standard inner products

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega f\overline{g}\, dV;\ \forall f, g \in L^2(\Omega),$$

$$\langle f, g\rangle_{L^2(\partial\Omega)} := \int_{\partial\Omega} f\overline{g}\, d\sigma;\ \forall f, g \in L^2(\partial\Omega).$$

Recall that (see also Chapter 2) the mapping

$$L^2(\partial\Omega) \ni f \longmapsto \int_{\partial\Omega} S(z, w) f(w)\, d\sigma(w)$$

is the Szegö projection, and the mapping

$$L^2(\Omega) \ni f \longmapsto \int_\Omega K(z, w) f(w)\, dV(w)$$

is the Bergman projection. We denote by $P(f)$ the Bergman projection of f.

Let ψ be a Lebesgue measurable function on Ω. By $L^2\left(\Omega, e^\psi\right)$ we denote the Hilbert space of measurable functions associated with the norm

$$\|f\|_{L^2(\Omega, e^\psi)} := \sqrt{\int_\Omega |f|^2\, e^\psi dV}.$$

The weighted Bergman projection P_ψ is the orthogonal projection of $L^2\left(\Omega, e^\psi\right)$ onto $A^2\left(\Omega, e^\psi\right)$, the subspace of holomorphic functions in $L^2\left(\Omega, e^\psi\right)$. We then denote by K_{e^ψ} the corresponding weighted Bergman kernel. Note that we may abuse notation and also write K_U, for a domain U in $\mathbb{C}^n$, to indicate the ordinary Bergman kernel of U.

We will use the notation $X \lesssim Y$ (resp. $X \gtrsim Y$) to denote the estimate $|X| \leqslant CY$ (resp. $X \geqslant C|Y|$), for some positive constant C. We use $X \approx Y$ for the fact $X \lesssim Y \lesssim X$.

Chapter 2

Background

In this chapter we provide some standard facts concerning the Bergman kernel, the pluricomplex Green function and the $\overline{\partial}$-equation. Since the theory is by now a vast subject, we only discuss here a background that is relevant for our purposes. This chapter serves as an introduction to several notations and results that are used later in this thesis. We refer readers to the book of Krantz [Kra13] for further discussions on the Bergman theory; the book of Hörmander [Hör90] for tools related to the $\overline{\partial}$-equation and the article by Błocki [Bło14a] on the pluripotential aspect.

2.1 The Bergman kernel and Bergman projection

The Bergman projection P is the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$. It can be represented as an integral operator via the Bergman kernel K,

$$L^2(\Omega) \ni f \longmapsto P(f)(z) := \int_{\Omega} K(z,w) f(w)\, dV(w)\,. \tag{2.1.1}$$

The Bergman kernel $K : \Omega \times \Omega \to \mathbb{C}$ is holomorphic in the first variable (when the second variable is fixed), moreover

$$K(z,w) = \overline{K(w,z)}; \quad \forall z, w \in \Omega.$$

Since Ω is bounded, $K(z,z) > 0$ for any $z \in \Omega$. On the other hand,

$$K(z,z) = \sup\left\{ |f(z)|^2 : f \in A^2(\Omega), \|f\|_{L^2(\Omega)} \leqslant 1 \right\}. \tag{2.1.2}$$

We usually use the equation (2.1.2) to obtain a lower bound for the Bergman kernel on the diagonal. For example, take $f \equiv |\Omega|^{-1/2}$ then we have

$$K(z,z) \geqslant \frac{1}{|\Omega|}, \quad \forall z \in \Omega. \tag{2.1.3}$$

When $\Omega = B(0,1)$ is the unit ball in $\mathbb{C}^n$, the Bergman kernel of Ω is

$$K(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - z\overline{w})^{n+1}}.$$

Thus (2.1.3) becomes an equality at $z = 0$. However, as seen from the case of the unit ball, $K(z,z)$ blows up as z approaches the boundary. Therefore, non-trivial estimates should be seen as $z \to \partial\Omega$. It is proved by S. Fu [Fu94] that for any bounded pseudoconvex domain Ω with C^2-boundary in $\mathbb{C}^n$,

$$K(z,z) \gtrsim \frac{1}{\delta^2(z)}, \forall z \in \Omega. \tag{2.1.4}$$

Here δ is the distance function to the boundary. It is known that the exponent 2 in the inequality (2.1.4) is sharp for any C^2 bounded pseudoconvex domain in $\mathbb{C}^n$. The following lemma is an alternative characterisation for the Bergman kernel on the diagonal.

Lemma 2.1.1. *For any bounded pseudoconvex domain Ω in $\mathbb{C}^n$ and any $z \in \Omega$,*

$$K(z,z) = \frac{1}{\inf\left\{\int_\Omega |f|^2 : f \in A^2(\Omega), f(z) = 1\right\}}.$$

Proof. For any function $f \in A^2(\Omega)$ with $f(z) = 1$, consider the function

$$g(\cdot) := \frac{f(\cdot)}{\sqrt{\int_\Omega |f|^2}}.$$

Then we have $g \in A^2(\Omega)$ and $\|g\|_{L^2(\Omega)} = 1$. Thus

$$K(z,z) \geqslant |g(z)|^2 = \frac{1}{\int_\Omega |f|^2}.$$

On the other hand, if we choose

$$f(\cdot) := \frac{K(\cdot,z)}{K(z,z)}$$

then f is holomorphic on Ω, $f(z) = 1$ and

$$K(z,z) = \frac{1}{\int_\Omega |f|^2}.$$

$\square$

A simple and effective way to calculate the Bergman projection $P(f)$ of a given function $f \in L^2(\Omega)$ is to use an orthonormal basis of $A^2(\Omega)$. Let $\{\varphi_j\}_{j=0}^{\infty}$ be an orthonormal basis of $A^2(\Omega)$. Since $P(f) \in A^2(\Omega)$,

$$\begin{aligned} P(f) &= \sum_{j=0}^{\infty} \langle P(f), \varphi_j \rangle \varphi_j \\ &= \sum_{j=0}^{\infty} \left(\int_{\Omega} \left(\int_{\Omega} K(\xi, w) f(w)\, dV(w) \right) \overline{\varphi_j(\xi)} dV(\xi) \right) \varphi_j \\ \text{(by Fubini's theorem)} &= \sum_{j=0}^{\infty} \left(\int_{\Omega} \overline{\left(\int_{\Omega} K(w, \xi) \varphi_j(\xi)\, dV(\xi) \right)} f(w)\, dV(w) \right) \varphi_j \\ &= \sum_{j=0}^{\infty} \langle f, \varphi_j \rangle \varphi_j . \end{aligned}$$

Example 2.1.2. Let $\Omega = \mathbb{D}$ be the unit disk in $\mathbb{C}$. We consider the function $f : \mathbb{D} \to \mathbb{C}$ defined by $f(0) = 1$ and $f(z) = \overline{z}(-\log|z|)^{\frac{2}{3}}$ for $z \neq 0$. Then

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 dV(z) &= \int_{\mathbb{D}} |z|^2 (-\log|z|)^{\frac{4}{3}} dV(z) \\ &= \int_0^{2\pi} d\theta \int_0^1 r^3 (-\log r)^{\frac{4}{3}} dr \\ &= \frac{\pi \Gamma\left(\frac{1}{3}\right)}{18\sqrt[3]{4}} . \end{aligned}$$

Thus $f \in L^2(\mathbb{D})$. To compute $P(f)$, recall that $\left\{ \varphi_j(z) := \frac{\sqrt{j+1} z^j}{\sqrt{\pi}} \right\}_{j=0}^{\infty}$ is an orthonormal basis of $A^2(\mathbb{D})$. We have

$$\begin{aligned} \langle f, \varphi_j \rangle &= \int_{\mathbb{D}} \overline{z} (-\log|z|)^{\frac{2}{3}} \frac{\sqrt{j+1}}{\sqrt{\pi}} \overline{z^j} dV(z) \\ &= \frac{\sqrt{j+1}}{\sqrt{\pi}} \int_0^{2\pi} \int_0^1 r^{j+2} e^{-i\theta(j+1)} (-\log r)^{\frac{2}{3}} dr d\theta \\ &= \frac{\sqrt{j+1}}{\sqrt{\pi}} \left(\int_0^{2\pi} e^{-i\theta(j+1)} d\theta \right) \left(\int_0^1 r^{j+2} (-\log r)^{\frac{2}{3}} dr \right) \end{aligned}$$

$$= 0, \forall j = 0, 1, \cdots .$$

Therefore $P(f) = \sum_{j=0}^{\infty} \langle f, \varphi_j \rangle \varphi_j = 0$.

The representation (2.1.1) allows us to extend the domain of definition of the Bergman projection. For any measurable function f, we say that $P(f)$ is well-defined if for almost every $z \in \Omega$, the function $w \mapsto K(z,w) f(w)$ is integrable.

Example 2.1.3. Consider the function $g_0 : z \mapsto 1/z$, then $g_0 \in L^p(\mathbb{D})$ for any $1 \leqslant p < 2$, and $g_0 \notin L^2(\mathbb{D})$. Since $w \mapsto K_{\mathbb{D}}(z,w)$ is a bounded function on $\mathbb{D}$, for any fixed $z \in \mathbb{D}$, the Bergman projection $P(g_0)$ is well-defined, even $g_0 \notin L^2(\mathbb{D})$.

Let us compute $P(g_0)$. Note that $g_0 \notin L^2(\mathbb{D})$, so it is not quite legitimate to proceed our computation by using an orthonormal basis as in Example 2.1.2. Nevertheless, for a fixed $z \in \mathbb{D}$ and $\varepsilon > 0$ small, the series

$$\sum_{j=0}^{\infty} (j+1) \frac{(z\overline{w})^j}{w}$$

is uniformly convergent in w on $\mathbb{D} \backslash \{|\cdot| < \varepsilon\}$. On the other hand, for any $j \geqslant 0$,

$$\int_{\mathbb{D}\backslash\{|\cdot|<\varepsilon\}} \frac{(z\overline{w})^j}{w} dV(w) = \int_{\varepsilon}^{1} \int_{0}^{2\pi} z^j r^j e^{-i(j+1)\theta} d\theta dr = 0.$$

It follows that

$$\begin{aligned}\int_{\mathbb{D}\backslash\{|\cdot|<\varepsilon\}} K_{\mathbb{D}}(z,w) \frac{1}{w} dV(w) &= \int_{\mathbb{D}\backslash\{|\cdot|<\varepsilon\}} \sum_{j=0}^{\infty} \frac{(j+1)}{\pi} \frac{(z\overline{w})^j}{w} dV(w) \\ &= \sum_{j=0}^{\infty} \int_{\mathbb{D}\backslash\{|\cdot|<\varepsilon\}} \frac{(j+1)}{\pi} \frac{(z\overline{w})^j}{w} dV(w) \\ &= 0.\end{aligned}$$

Letting $\varepsilon \to 0$, we have $P(g_0) = 0$.

Example 2.1.4. Let $\mathbb{H}$ be the Hartogs triangle

$$\mathbb{H} := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \right\},$$

and consider the function $f_0(z) := 1/z_2^2$. Then we have

$$\|f_0\|_{L^p(\mathbb{H})}^p = \int_{\mathbb{H}} \frac{1}{|z_2|^{2p}} dV(z) = \int_{\mathbb{D}\backslash\{0\}} \frac{1}{|z_2|^{2p}} \left(\int_{B(0,|z_2|)} dV(z_1) \right) dV(z_2)$$

$$= \pi \int_{\mathbb{D}\setminus\{0\}} \frac{1}{|z_2|^{2p-2}} dV(z_2)$$
$$= \pi \int_0^{2\pi} d\theta \int_0^1 r^{3-2p} dr.$$

Thus $f_0 \in L^p(\mathbb{H})$ for any $1 \leqslant p < 2$. By Proposition 2.1.7, $K_{\mathbb{H}}(z, \cdot) \in L^q(\mathbb{H})$ for any $1 \leqslant q < 4$. Using Hölder's inequality, the function $w \mapsto K(z,w) f_0(w)$ is integrable, therefore $P(f_0)$ is well-defined, but $f_0 \notin L^2(\mathbb{H})$.

By definition, we always have $P(f) = f$, for any $f \in A^p(\mathbb{H})$, with $p \geqslant 2$. One may ask whether this is true for $f \in A^p(\mathbb{H})$ with $p < 2$, given that $P(f)$ is well-defined. Since $f_0 \in A^p(\mathbb{H})$ for any $1 \leqslant p < 2$, let us compute $P(f_0)$. Note that $f_0 \notin L^2(\mathbb{H})$, however we can compute $P(f_0)$ directly by definition. Recall that (see e.g. [EM16]) the Bergman kernel of $\mathbb{H}$ is

$$K_{\mathbb{H}}(z,w) = \frac{z_2 \overline{w_2}}{\pi^2 (1 - z_2 \overline{w_2})^2 (z_2 \overline{w_2} - z_1 \overline{w_1})^2}. \tag{2.1.5}$$

Therefore, by the change of variable $v := w_1 / w_2$, we have

$$\begin{aligned}
P(f_0)(z) &= \int_{\mathbb{H}} \frac{z_2 \overline{w_2}}{\pi^2 (1 - z_2 \overline{w_2})^2 (z_2 \overline{w_2} - z_1 \overline{w_1})^2} \times \frac{1}{w_2^2} dV(w) \\
&= \int_{\mathbb{D}\setminus\{0\}} \frac{\overline{w_2}}{\pi^2 z_2 (1 - z_2 \overline{w_2})^2 |w_2|^4} \left(\int_{\{|\cdot|<|w_2|\}} \left(1 - \frac{z_1}{z_2} \frac{\overline{w_1}}{\overline{w_2}}\right)^{-2} dw_1 \right) dw_2 \\
&= \int_{\mathbb{D}\setminus\{0\}} \frac{\overline{w_2}}{\pi^2 z_2 (1 - z_2 \overline{w_2})^2 |w_2|^4} \left(|w_2|^2 \int_{\{|\cdot|<|1|\}} \left(1 - \frac{z_1}{z_2} \overline{v}\right)^{-2} dv \right) dw_2 \\
&= \frac{1}{\pi^2 z_2} \int_{\mathbb{D}\setminus\{0\}} \frac{1}{(1 - z_2 \overline{w_2})^2 w_2} \left(\int_{\{|\cdot|<|1|\}} \left(1 - \frac{z_1}{z_2} \overline{v}\right)^{-2} dv \right) dw_2 \\
&= 0.
\end{aligned}$$

Here, the last equality follows by Example 2.1.3. We conclude that $P(f_0) \neq f_0$.

It is of interest to determine under what conditions one has $P(f) = f$ for $f \in A^p(\Omega)$. It turns out that the question is related to the *integrability index* of the Bergman kernel.

Definition 2.1.5. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. The *integrability index* of the Bergman kernel of Ω is defined as

$$\beta(\Omega) := \sup\left\{\beta \geqslant 2 : K(\cdot, w) \in L^\beta(\Omega) \text{ for any } w \in \Omega\right\}.$$

Proposition 2.1.6. *Assume that $A^2(\Omega)$ is dense in $A^p(\Omega)$ for some $p > \beta/(\beta-1)$, where β is the integrability index of Ω. Then $P(f) = f$ for any $f \in A^p(\Omega)$.*

Remark. If $\beta = \infty$ then the condition on p in Proposition 2.1.6 is understood as $p > 1$.

Proof. Let $f \in A^p(\Omega)$ and $z \in \Omega$, choose a sequence $\{f_j\}$ in $A^2(\Omega)$ such that $f_j(z) \to f(z)$ and $\|f_j - f\|_{L^p(\Omega)} \to 0$. By Hölder's inequality

$$\begin{aligned}
|P(f_j)(z) - P(f)(z)| &\leqslant \left|\int_\Omega K(z,w)(f_j(w) - f(w))\, dV(w)\right| \\
&\leqslant \|K(\cdot, z)\|_{L^q(\Omega)} \|f_j - f\|_{L^p(\Omega)},
\end{aligned}$$

here q is the dual exponent of p. The last term goes to zero since $\|K(\cdot, z)\|_{L^q(\Omega)}$ is finite by $p > \beta/(\beta-1)$. The conclusion follows by the fact $P(f_j)(z) = f_j(z)$ and $f_j(z) \to f(z)$. $\square$

For the unit ball $B(0,1)$, the integrability index is infinity since $K(\cdot, w) \in C^\infty\left(\overline{B(0,1)}\right)$. This is also true for any smoothly bounded pseudoconvex domain of finite type, see [Boa87, Bel86]. Let us compute the integrability index of the Hartogs triangle.

Proposition 2.1.7. $\beta(\mathbb{H}) = 4$.

Proof. For any $p \geqslant 2$,

$$\begin{aligned}
\int_{\mathbb{H}} |K_{\mathbb{H}}(z,w)|^p\, dV(z) &\approx \int_{\mathbb{H}} \frac{|z_2\overline{w_2}|^p}{|1 - z_2\overline{w_2}|^{2p}\, |z_2\overline{w_2} - z_1\overline{w_1}|^{2p}} dV(z) \\
&\approx \int_{\mathbb{D}\setminus\{0\}} \frac{|z_2\overline{w_2}|^p}{|1 - z_2\overline{w_2}|^{2p}} \left(\int_{\{|\cdot|<|z_2|\}} \frac{1}{|z_2\overline{w_2} - z_1\overline{w_1}|^{2p}} dz_1 \right) dz_2 \\
&\approx \int_{\mathbb{D}\setminus\{0\}} \frac{|z_2\overline{w_2}|^{-p}}{|1 - z_2\overline{w_2}|^{2p}} \left(\int_{\{|\cdot|<|z_2|\}} \left|1 - \frac{z_1}{z_2}\frac{\overline{w_1}}{\overline{w_2}}\right|^{-2p} dz_1 \right) dz_2.
\end{aligned}$$

Let E be the expression in $(\cdots)$, then we have

$$\begin{aligned} E &= \int_{\mathbb{D}} \left|1 - v\frac{\overline{w_1}}{\overline{w_2}}\right|^{-2p} |z_2|^2 \, dv \quad (v := z_1/z_2) \\ &= |z_2|^2 \int_{\mathbb{D}} |1 - va|^{-2p} \, dv \quad (a := \overline{w_1}/\overline{w_2}) \\ &\approx |z_2|^2 , \end{aligned}$$

where the last line follows by the convergence of $\int |1 - va|^{-2p} \, dv$. It continues as

$$\begin{aligned} \int_{\mathbb{H}} |K_{\mathbb{H}}(z,w)|^p \, dV(z) &\approx \int_{\mathbb{D}\setminus\{0\}} \frac{|z_2\overline{w_2}|^{-p}}{|1 - z_2\overline{w_2}|^{2p}} |z_2|^2 \, dz_2 \\ &\approx \int_{\mathbb{D}\setminus\{0\}} |z_2|^{-p+2} \, dz_2 \\ &\approx \int_0^1 r^{-p+3} dr. \end{aligned}$$

Thus $K_{\mathbb{H}}(\cdot, w) \in L^p(\mathbb{H})$ if and only if $p < 4$. □

By Example 2.1.4, Proposition 2.1.6 and Proposition 2.1.7, we conclude that $A^2(\mathbb{H})$ is not dense in $A^p(\mathbb{H})$, for any $4/3 < p < 2$. In general, it might be difficult to determine whether $A^q(\Omega)$ is dense in $A^p(\Omega)$, for $p < q$. A related question is what geometric information of a domain implies that $(A^p(\Omega))^\star \equiv A^{p'}(\Omega)$, where $1/p + 1/p' = 1$. These in turn are closely connected with the L^p regularity of the Bergman projection. We also remark that the density problem in Bergman spaces and estimates for the integrability index have been obtained recently in [Che17a].

Let $\psi : \Omega \to \mathbb{R}$ be a measurable function. The weight function ψ is called *admissible* if $A^2(\Omega, e^\psi)$ is a closed subspace of $L^2(\Omega, e^\psi)$ and for any $z \in \Omega$, $f \mapsto f(z)$ is a continuous linear functional on $A^2(\Omega, e^\psi)$. The Riesz representation theorem guarantees that there is a reproducing kernel $K_{e^\psi}(z,w)$ for $A^2(\Omega, e^\psi)$; that is

$$f(z) = \int_\Omega K_{e^\psi}(z,w) f(w) e^{\psi(w)} dV(w), \quad \forall f \in A^2(\Omega, e^\psi).$$

When $\psi \equiv 0$, K_{e^ψ} is the usual Bergman kernel. Sufficient conditions for the existence of admissible weights were studied in [PW90]. In particular, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ then ψ is admissible. It may happen that ψ is admissible and $A^2(\Omega, e^\psi)$

is trivial, i.e. $A^2\left(\Omega, e^{\psi}\right) = \{0\}$. For example, take $\psi(z) = -\log(\delta(z))$ then ψ is admissible. On the other hand, it is known that $A^2\left(\Omega, \delta^{-1}\right) \equiv \{0\}$, see [Che14].

B-Y. Chen has recently found a connection between the weighted Bergman kernels $K_{\delta^{-s}}$ and the Szegö kernel. More specifically, in [Che14], he proved that

$$\frac{K_{\delta^{-s}}(z,w)}{1-s} \xrightarrow{s\to 1^-} S(z,w)\,,\ \text{locally uniform in } z,w \in \Omega,$$

for any bounded domain Ω with C^2-smooth boundary in $\mathbb{C}^n$. Recall that the Szegö kernel S is the reproducing kernel for $H^2(\Omega)$; that is

$$f(z) = \int_{\partial\Omega} S(z,w)\, f(w)\, d\sigma(w)\,, \forall f \in H^2(\Omega)\,.$$

The relation between the weighted Bergman kernels $K_{\delta^{-s}}$ and the ordinary Bergman kernel K, and the comparison between the ordinary Bergman kernel and the Szegö kernel have also been studied by the same author in [Che06, CF11]. For example, the results imply in particular that for any strongly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$ with C^2-smooth boundary,

$$\frac{K_{\delta^{-s}}(z,z)}{K(z,z)} \approx \delta^s(z)\,; \forall s < 1,$$

and

$$\frac{S(z,z)}{K(z,z)} \approx \delta^{-1}(z)\,.$$

We will discuss further these relations in Chapter 3. In particular, we will also show that for any C^2 strongly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$,

$$\frac{\|K(\cdot,z)\|^2_{L^2(\partial\Omega)}}{K(z,z)} \approx \delta^{-1}(z)\,. \tag{2.1.6}$$

Let us examine (2.1.6) in the case of the unit disk $\mathbb{D}$.

$$\begin{aligned}
\|K(\cdot,z)\|^2_{L^2(\partial\mathbb{D})} &= \frac{1}{\pi^2}\int_{\partial\mathbb{D}} \frac{1}{|1-w\bar{z}|^4}\, d\sigma(w) \\
&= \frac{1}{\pi^2}\int_0^{2\pi} \frac{1}{\left|1-|z|\, e^{i\theta}\right|^4}\, d\theta \\
&= \frac{1}{\pi^2}\int_0^{2\pi} \frac{1}{\left(1+|z|^2 - 2|z|\cos\theta\right)^2}\, d\theta
\end{aligned}$$

$$= \frac{2\left(1+|z|^2\right)}{\pi\left(1-|z|^2\right)^3}.$$

Therefore

$$\frac{\|K(\cdot,z)\|^2_{L^2(\partial\mathbb{D})}}{K(z,z)} = \frac{2\left(1+|z|^2\right)}{1-|z|} = \frac{2\left(1+|z|^2\right)}{\delta(z)}.$$

It follows

$$\frac{2}{\delta(z)} \leqslant \frac{\|K(\cdot,z)\|^2_{L^2(\partial\mathbb{D})}}{K(z,z)} \leqslant \frac{4}{\delta(z)}, \forall z \in \mathbb{D}.$$

2.2 The pluricomplex Green function

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. The pluricomplex Green function with a pole $w \in \Omega$ is defined by

$$G(\cdot,w) := \sup\left\{u(\cdot) : u \in PSH^-(\Omega), \limsup_{z\to w}\left(u(z) - \log|z-w|\right) < \infty\right\}.$$

Here $PSH^-(\Omega)$ denotes the set of all negative plurisubharmonic functions on Ω. The pluricomplex Green function was first introduced by Klimek [Kli85] and has been studied by many authors. Demailly [Dem87] showed that if Ω is hyperconvex then $G(\cdot,w)$ is the unique solution of the complex Monge-Ampère equation

$$\begin{cases} (dd^c u)^n = \delta_w \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, \\ \limsup_{z\to w}\left(u(z) - \log|z-w|\right) < \infty, \\ u \in PSH^-(\Omega) \cap C\left(\overline{\Omega}\backslash\{w\}\right). \end{cases} \tag{2.2.1}$$

Here δ_w is the Dirac measure at w. Recall that a domain is called *hyperconvex* if it admits a negative plurisubharmonic exhaustion function. The pluricomplex Green function with several poles was introduced by Lelong [Lel89] and has also attracted further study recently, see e.g. [Com00, Bło01, Wik03, TT03]. The following properties of the pluricomplex Green function are used later in this thesis. We refer the reader to the book of Klimek [Kli91] and also [Kli95] for details.

Proposition 2.2.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $w \in \Omega$. Then the following properties hold:*

1. $G(\cdot,w) \in PSH^-(\Omega)$ *and* $\limsup_{z\to w}\left(G(z,w) - \log|z-w|\right) < \infty$.

2. *If* $\overline{B(w,r)} \subset \Omega \subset B(w,R)$ *then*

$$\log\left(\frac{|z-w|}{R}\right) \leqslant G(z,w) \leqslant \log\left(\frac{|z-w|}{r}\right), \forall z \in \Omega.$$

3. $(dd^c G(\cdot, w))^n \equiv 0$ *on* $\Omega \backslash \{w\}$.
4. *If* Ω *is hyperconvex then* $G(\cdot, w)$ *is continuous in* Ω *and* $\lim_{z \to z_0} G(z, w) = 0$, *for any* $z_0 \in \partial\Omega$.
5. *If* $F \in \mathcal{O}(\Omega, \Omega')$ *then* $G_{\Omega'}(F(z), F(w)) \leqslant G_\Omega(z, w)$, $\forall z \in \Omega$.
6. *If* $\Omega' \backslash \Omega$ *is pluripolar then* $G_\Omega(z, w) = G_{\Omega'}(z, w)$, $\forall z \in \Omega$.
7. *If* $\Omega \subset \mathbb{C}^n$ *and* $\Omega' \subset \mathbb{C}^m$ *are pseudoconvex domains then*

$$G_{\Omega \times \Omega'}((z_1, z_1'), (z_2, z_2')) = \max\{G_\Omega(z_1, z_2), G_{\Omega'}(z_1', z_2')\},$$

for any $(z_1, z_1'), (z_2, z_2') \in \Omega \times \Omega'$.

Example 2.2.2. For the unit disk in $\mathbb{C}$,

$$G_{\mathbb{D}}(z, w) = \log\left|\frac{z - w}{1 - z\overline{w}}\right|; \forall z, w \in \mathbb{D}.$$

This can be seen as the unique solution of the complex Monge-Ampère equation (2.2.1). The pluricomplex Green function of the bidisc $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ can be obtained by using Proposition 2.2.1(7.),

$$G_{\mathbb{D}\times\mathbb{D}}((z_1, z_2), (w_1, w_2)) = \max\left\{\log\left|\frac{z_1 - w_1}{1 - z_1\overline{w_1}}\right|, \log\left|\frac{z_2 - w_2}{1 - z_2\overline{w_2}}\right|\right\}, \tag{2.2.2}$$

for any (z_1, z_2), (w_1, w_2) in $\mathbb{D} \times \mathbb{D}$.

The pluricomplex Green function of the Hartogs triangle $\mathbb{H}$ is

$$G_{\mathbb{H}}(z, w) = \log\max\left\{\left|\frac{\frac{z_1}{z_2} - \frac{w_1}{w_2}}{1 - \frac{z_1}{z_2}\frac{\overline{w_1}}{\overline{w_2}}}\right|, \left|\frac{z_2 - w_2}{1 - z_2\overline{w_2}}\right|\right\}, \tag{2.2.3}$$

for any $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{H}$. To see this, we follow the argument in [CCW99]. Let $E := \{(z_1, 0) \in \mathbb{C}^2 : |z_1| < 1\}$, then

$$\begin{aligned} F : (\mathbb{D} \times \mathbb{D}) \backslash E &\longmapsto \mathbb{H} \\ (z_1, z_2) &\longmapsto (z_1 z_2, z_2) \end{aligned}$$

is biholomorphic. By Proposition 2.2.1(5.), $G_{\mathbb{H}}(F(z), F(w)) = G_{(\mathbb{D}\times\mathbb{D})\backslash E}(z, w)$. On the other hand, since E is pluripolar ($E = \{u = -\infty\}$, with $u(z) := \log|z_2|$), it follows that $G_{(\mathbb{D}\times\mathbb{D})\backslash E}(z, w) = G_{\mathbb{D}\times\mathbb{D}}(z, w)$ by Proposition 2.2.1(6.). The conclusion can be seen from (2.2.2).

Note that for $k \geqslant 2$, $\mathbb{H}_k := \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1\right\}$ is not biholomorphic to $(\mathbb{D} \times \mathbb{D}) \setminus E$. Therefore the above argument cannot be applied directly. However, one might still be able to estimate the pluricomplex Green function by using a holomorphic mapping, see Chapter 5.

By (2.2.3) and Proposition 2.2.1(4.), we can see that $\mathbb{H}$ is not hyperconvex by showing that $G_{\mathbb{H}}(\varepsilon z, w) \nrightarrow 0$ as $\varepsilon \to 0$. Nevertheless, by (2.1.5), $\mathbb{H}$ is Bergman exhaustive, that is, $K_{\mathbb{H}}(z, z) \to \infty$ as $z \to \partial\mathbb{H}$. That is to say, the Bergman exhaustiveness does not imply the hyperconvexity. However, Ohsawa [Ohs93] proved that any bounded hyperconvex domain in $\mathbb{C}^n$ is Bergman exhaustive. It is known that bounded pseudoconvex domains with Lipschitz boundary are hyperconvex [Dem87, Har08]. Recently, in [AHP15], it has been shown that if $\partial\Omega \in C^\alpha$, for $0 < \alpha < 1$ then the pseudoconvex domain Ω is hyperconvex.

Quantitative estimates for sub-level sets of the pluricomplex Green function are important for many applications. Let us first consider the simplest case of the unit disk.

Lemma 2.2.3. *For any $t > 0$ and $w \in \mathbb{D}$,*

$$\{G_{\mathbb{D}}(\cdot, w) < -t\} \subset \left\{\frac{e^t - 1}{e^t + 1}(1 - |w|) \leqslant 1 - |\cdot| \leqslant \frac{e^t + 1}{e^t - 1}(1 - |w|)\right\}. \tag{2.2.4}$$

It is clear that (2.2.4) is a direct consequence of the following claim.

Claim. If $a, b \in \mathbb{D}$, $t > 0$ and $|a - b| / \left|1 - a\bar{b}\right| < e^{-t}$ then

$$\frac{e^t - 1}{e^t + 1}(1 - |b|) \leqslant 1 - |a| \leqslant \frac{e^t + 1}{e^t - 1}(1 - |b|). \tag{2.2.5}$$

Proof of the claim. By the elementary inequality (by expansion, it is equivalent to $(1 - |a|)(1 - |b|)\left(2|ab| - a\bar{b} - b\bar{a}\right) \geqslant 0$)

$$\frac{|a - b|}{\left|1 - a\bar{b}\right|} \geqslant \frac{|a| - |b|}{1 - |ab|},$$

we obtain that

$$|a| \leqslant \frac{e^{-t} + |b|}{1 + e^{-t}|b|}.$$

It continues as

$$\begin{aligned} 1 - |a| &\geqslant 1 - \frac{e^{-t} + |b|}{1 + e^{-t}|b|} = \frac{1 - e^{-t}}{1 + e^{-t}|b|}(1 - |b|) \\ &\geqslant \frac{e^t - 1}{e^t + 1}(1 - |b|). \end{aligned}$$

We have proved the LHS of (2.2.5). The RHS then follows by symmetry. □

For strongly pseudoconvex domains, a similar result is stated as follows.

Proposition 2.2.4. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2-boundary. Then there are positive constants c_1, c_2 and c_3 such that*

$$\{G(\cdot, w) < -1\} \subset \{c_1 \delta(w) \leqslant \delta(\cdot) \leqslant c_2 \delta(w)\},$$

for any $w \in \Omega$ with $\delta(w) < c_3$. Here δ denotes the distance function to the boundary.

By the Lempert theorem, the pluricomplex Green function of any bounded convex domain Ω is symmetric, i.e. $G(z, w) = G(w, z)$, $\forall z, w \in \Omega$. On the other hand, Bedford and Demailly [BD88] showed a strongly pseudoconvex domain with C^∞-boundary in which the pluricomplex Green function is not symmetric.

The proof of Proposition 2.2.4 was given in [DH00]. In fact, the authors only proved the part $\delta(\cdot) \lesssim \delta(w)$. However, the other part can be shown using the same argument. Let me present the proof here. The idea is to map the domain into the unit disk under a "good" holomorphic peak function, which in turn is known by the following result due to Graham [Gra75].

Lemma 2.2.5. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2-smooth boundary. Then there exist positive constants ε_1, ε_2, ε_3 and a function $F : \overline{\Omega} \times \partial\Omega \to \mathbb{D}$ such that the following properties are true:*

1. *For any $z \in \partial\Omega$, $F(\cdot, z)$ is a holomorphic peak function at z.*
2. *If $w \in \Omega$ and $\delta(w) < \varepsilon_1$ then $|1 - F(w, \pi(w))| < \varepsilon_2 \delta(w)$. Here π is the orthogonal projection to the boundary.*
3. *For any $z \in \partial\Omega$, $1 - |F(\cdot, z)| > \varepsilon_3 \delta(\cdot)$ on Ω.*

Recall that a function f is called a holomorphic peak function at $z_0 \in \partial\Omega$ if $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$, $f(z_0) = 1$ and $|f(z)| < 1$ for any $z \in \overline{\Omega} \setminus \{z_0\}$.

Proof of Proposition 2.2.4. We first assume that $\delta(z) < \varepsilon_1$, where ε_1 is as in Lemma 2.2.5. By Proposition 2.2.1(5.),

$$G(z, w) \geqslant G_{\mathbb{D}}(F(z, \pi(z)), F(w, \pi(z))), \forall w \in \Omega. \tag{2.2.6}$$

Now, combining this with Lemma 2.2.3, we have

$$1 - |F(z, \pi(z))| \approx 1 - |F(w, \pi(z))|,$$

for $z \in \{G(\cdot, w) < -1\}$. On the other hand, we also have

$$1 - |F(z, \pi(z))| \leqslant |1 - F(z, \pi(z))| < \varepsilon_2 \delta(z)$$

and

$$\varepsilon_3\delta(w) \leqslant 1 - |F(w, \pi(z))|.$$

We conclude that $\delta(w) \lesssim \delta(z)$, for $z \in \{G(\cdot, w) < -1\}$ and $\delta(z) < \varepsilon_1$. In the case $\delta(z) \geqslant \varepsilon_1$, it is obvious that $\delta(w) \lesssim \delta(z)$, for w close enough to the boundary.

Similarly, in order to prove $\delta(\cdot) \lesssim \delta(w)$ on $\{G(\cdot, w) < -1\}$, we simply replace (2.2.6) by

$$G(z, w) \geqslant G_{\mathbb{D}}(F(z, \pi(w)), F(w, \pi(w))).$$

□

The following result of Błocki [Bło05] (see also [CF11]) provides estimates on the sub-level set $\{G(\cdot, w) < -1\}$ for a more general class of pseudoconvex domains.

Theorem 2.2.6. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that there exist a plurisubharmonic function φ on Ω and positive constants a, b, c_1, c_2 such that*

$$c_1\delta^a(z) \leqslant -\varphi(z) \leqslant c_2\delta^b(z), \forall z \in \Omega. \tag{2.2.7}$$

Then there are positive constants ε_0 and C such that

$$\{G(\cdot, w) < -1\} \subset \left\{\frac{1}{C}\delta^{\frac{a}{b}}(w)\, |\log\delta(w)|^{-\frac{1}{b}} \leqslant \delta(\cdot) \leqslant C\delta^{\frac{b}{a}}(w)\, |\log\delta(w)|^{\frac{n}{a}}\right\}, \tag{2.2.8}$$

for any $w \in \Omega$ with $\delta(w) < \varepsilon_0$.

It is known that for any bounded pseudoconvex domain with Lipschitz boundary, the Diederich-Fornæss index is positive ([Har08]). Therefore, for this class of domains, there always exists a plurisubharmonic function φ that satisfies (2.2.7). In the case of convex domains, the appearance of the logarithmic terms as in (2.2.8) can be removed and even a more precise estimate can be obtained (compare with Lemma 2.2.3).

Theorem 2.2.7 (Błocki [Bło05])**.** *Let Ω be a bounded convex domain in $\mathbb{C}^n$. Then*

$$G(z, w) \geqslant \log\left|\frac{\delta(z) - \delta(w)}{\delta(z) + \delta(w)}\right|; \quad \forall z, w \in \Omega.$$

In particular, for any $t > 0$ and $w \in \Omega$,

$$\{G(\cdot, w) < -t\} \subset \left\{\frac{e^t - 1}{e^t + 1}\delta(w) \leqslant \delta(\cdot) \leqslant \frac{e^t + 1}{e^t - 1}\delta(w)\right\}.$$

There is an interesting relation between the Bergman kernel of the sub-level set $\{G(\cdot, w) < -t\}$ and the ordinary Bergman kernel.

Proposition 2.2.8. *Let Ω be a bounded pseudoconvex domain and let t be any positive number. Then for any $w \in \Omega$,*

$$K_{\{G(\cdot,w)<-t\}}(w,w) \leqslant e^{2nt} K(w,w). \tag{2.2.9}$$

The proof of Proposition 2.2.8 relies on a rather delicate argument, which in fact leads to a stronger statement that the function $t \mapsto \log K_{\{G(\cdot,w)<-t\}}(w,w)$ is convex on $[0,\infty)$, see [Bło14a]. Combining Theorem 2.2.7 and Proposition 2.2.8, we obtain the following inequality.

Proposition 2.2.9. *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and let $s > 0$. Then*

$$\frac{K_{\delta^s}(w,w)}{K(w,w)} \leqslant \frac{c(n,s)}{\delta^s(w)}, \quad \forall w \in \Omega. \tag{2.2.10}$$

Here

$$c(n,s) := \inf\left\{ e^{2nt}\left(\frac{e^t+1}{e^t-1}\right)^s : t > 0\right\}.$$

Proof. Note that $\dim\left(A^2(\Omega,\delta^s)\right) = \infty$ for any $s > 0$ since the space of complex polynomials is contained in $A^2(\Omega,\delta^s)$. By Lemma 2.1.1,

$$\int\limits_{\{G(\cdot,w)<-t\}} \left|\frac{K_{\delta^s}(z,w)}{K_{\delta^s}(w,w)}\right|^2 dV(z) \geqslant \frac{1}{K_{\{G(\cdot,w)<-t\}}(w,w)},$$

for any $t > 0$ and $w \in \Omega$. Thus

$$\begin{aligned} \int\limits_{\{G(\cdot,w)<-t\}} |K_{\delta^s}(z,w)|^2 dV(z) &\geqslant \frac{K^2_{\delta^s}(w,w)}{K_{\{G(\cdot,w)<-t\}}(w,w)} \\ &\geqslant \frac{K^2_{\delta^s}(w,w)}{e^{2nt}K(w,w)}, \end{aligned}$$

where the last inequality follows from Proposition 2.2.8. On the other hand, by Theorem 2.2.7,

$$\begin{aligned} \int\limits_{\{G(\cdot,w)<-t\}} |K_{\delta^s}(z,w)|^2 dV(z) &\leqslant \left(\left(\frac{e^t+1}{e^t-1}\right)^s \delta^{-s}(w)\right) \int\limits_{\{G(\cdot,w)<-t\}} |K_{\delta^s}(z,w)|^2 \delta^s(z)\, dV(z) \\ &\leqslant \left(\left(\frac{e^t+1}{e^t-1}\right)^s \delta^{-s}(w)\right) \int\limits_{\Omega} |K_{\delta^s}(z,w)|^2 \delta^s(z)\, dV(z) \\ &= \left(\frac{e^t+1}{e^t-1}\right)^s \delta^{-s}(w)\, K_{\delta^s}(w,w). \end{aligned}$$

We therefore obtain the desired inequality (2.2.10). ∎

The following inequality is also a nice consequence of Proposition 2.2.8.

Proposition 2.2.10. *Let $\Omega \Subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞-boundary. Then*

$$\int_{\partial\Omega} \frac{|K(z,w)|^2}{|\nabla G(z,w)|} d\sigma(z) \leqslant 2nK(w,w), \forall w \in \Omega. \tag{2.2.11}$$

Remark 2.2.11. In our context, it is known that $K(\cdot,w)$ and $|\nabla G(\cdot,w)| \in C(\partial\Omega)$, see e.g. [Ker72, Bło00]. Moreover, by Hopf's lemma, $|\nabla G(z,w)| \neq 0, \forall z \in \partial\Omega$. Therefore the left-hand side of (2.2.11) is well-defined.

Proof. By Lemma 2.1.1 and Proposition 2.2.8,

$$\begin{aligned}\int_{\{G(\cdot,w)<-t\}} |K(z,w)|^2 dV(z) &\geqslant \frac{K^2(w,w)}{K_{\{G(\cdot,w)<-t\}}(w,w)} \\ &\geqslant e^{-2nt}K(w,w).\end{aligned}$$

Using the co-area formula (see e.g. [EG15]), it follows

$$\begin{aligned}\int_{\partial\Omega} \frac{|K(z,w)|^2}{|\nabla G(z,w)|} d\sigma(z) &= \lim_{t\to 0+} \frac{\int_{\{G(\cdot,w)>-t\}} |K(z,w)|^2 dV(z)}{t} \\ &= \lim_{t\to 0+} \frac{K(w,w) - \int_{\{G(\cdot,w)\leqslant -t\}} |K(z,w)|^2 dV(z)}{t} \\ &\leqslant \lim_{t\to 0+} \frac{K(w,w) - e^{-2nt}K(w,w)}{t} \\ &= 2nK(w,w).\end{aligned}$$

This completes the proof. ∎

By (2.1.3),

$$|\Omega| \geqslant \frac{1}{K(w,w)}; \forall w \in \Omega,$$

and from (2.2.9) we also have

$$e^{2nt} |\{G(\cdot,w) < -t\}| \geqslant \frac{1}{K(w,w)}; \forall t > 0, \forall w \in \Omega. \tag{2.2.12}$$

It is an open question ([BZ15]) whether the right-hand side of the following is also true:

$$\frac{1}{K(w,w)} \leqslant e^{2nt} \left|\{G(\cdot, w) < -t\}\right| \leqslant |\Omega| ; \forall t > 0, \forall w \in \Omega.$$

The inequality (2.2.12) has been generalized to the following by Guan [Gua17]

$$e^{c(\varphi,w)t} \left|\{\varphi < -t\}\right| \geqslant \frac{1}{K(w,w)},$$

for any $t > 0$, $w \in \Omega$ and $\varphi \in PSH^{-}(\Omega)$. Here $c(\varphi, w)$ is the log canonical threshold of φ at w, defined as

$$c(\varphi, w) := \sup\left\{c > 0 : e^{-c\varphi} \text{ is locally integrable on a neighbourhood of } w\right\}.$$

2.3 L^2-estimates for the $\bar{\partial}$-equation

In this section, we recall some useful L^2-estimates for the $\bar{\partial}$-equation. These estimates can be obtained from classical Hörmander's estimate by *shifting* a solution under various weighted L^2-spaces. The upshot of this technique is that strong enough estimates for our purposes can be derived by a simple (and clever!) choice of weight functions. Roughly speaking, the more intricate part that involves techniques from functional analysis and geometric information of the domain is implicit in Hörmander's estimate.

Theorem 2.3.1 (Hörmander's estimate)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi \in C^2(\Omega)$ be a plurisubharmonic function. Then for any closed $(0,1)$-form $\alpha = \sum \alpha_j d\bar{z}_j \in L^2_{loc,(0,1)}(\Omega)$, there exists $u \in L^2_{loc}(\Omega)$ such that $\bar{\partial} u = \alpha$ and*

$$\int_\Omega |u|^2 e^{-\varphi} dV \leqslant \int_\Omega |\alpha|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} dV. \tag{2.3.1}$$

Here $|\alpha|^2_{i\partial\bar{\partial}\varphi} = \sum_{j,k} \varphi^{j\bar{k}} \bar{\alpha}_j \alpha_k$, and $\left(\varphi^{j\bar{k}}\right)$ is the inverse matrix of $\left(\varphi_{j\bar{k}}\right) = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}\right)$.

Remark 2.3.2. As noted by Błocki [Bło05], Hörmander's estimate can be applied for any plurisubharmonic function φ, not necessarily be C^2. In this regard, $|\alpha|^2_{i\partial\bar{\partial}\varphi}$ is the least function H satisfying $i\bar{\alpha} \wedge \alpha \leqslant H i\partial\bar{\partial}\varphi$ as currents.

If $u \in \left(\ker\left(\bar{\partial}\right)\right)^{\perp}$ in $L^2(\Omega, e^{-\varphi})$ solves the equation $\bar{\partial} u = \alpha$, then we call it the $L^2(\Omega, e^{-\varphi})$-minimal solution of $\bar{\partial} u = \alpha$. The estimate (2.3.1) also implies that for any $u \in \left(\ker\left(\bar{\partial}\right)\right)^{\perp}$ in $L^2(\Omega, e^{-\varphi})$,

$$\int_\Omega |u|^2 e^{-\varphi} dV \leqslant \int_\Omega \left|\bar{\partial} u\right|^2_{i\partial\bar{\partial}\varphi} e^{-\varphi} dV.$$

It is beneficial to extend Hörmander's estimate to more general weight functions, for example $e^{\psi-\varphi}$ instead of $e^{-\varphi}$. To do this, one might need to impose some condition on ψ. Using an argument due to Berndtsson, Błocki [Bło04] obtained the following estimate.

Theorem 2.3.3. *Let Ω be a bounded pseudoconvex domain and let $\varphi \in PSH(\Omega)$. Let α be a $\overline{\partial}$-closed $(0,1)$-form and let $\psi \in PSH(\Omega)$ such that $\left|\overline{\partial}\psi\right|^2_{i\partial\overline{\partial}\psi} \leqslant r$, for some $0 < r < 1$. Then there is a solution v of $\overline{\partial}v = \alpha$ satisfying*

$$\int_\Omega |v|^2 e^{\psi-\varphi} dV \leqslant \frac{4r}{(1-r)^2} \int_\Omega |\alpha|^2_{i\partial\overline{\partial}\psi} e^{\psi-\varphi} dV, \tag{2.3.2}$$

provided that the right-hand side of (2.3.2) *is finite.*

Remark 2.3.4. Błocki [Bło15] proved that the constant $\frac{4r}{(1-r)^2}$ in (2.3.2) is sharp if $\varphi \equiv 0$.

As a consequence of Theorem 2.3.3, the following estimate, due to Donnelly and Fefferman [DF83], is very useful.

Theorem 2.3.5. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let φ and ψ be plurisubharmonic functions such that $i\partial\psi \wedge \overline{\partial}\psi \leqslant i\partial\overline{\partial}\psi$ as currents. Then for any $\overline{\partial}$-closed form $\alpha \in L^2_{loc,(0,1)}(\Omega)$, there exists $u \in L^2_{loc}(\Omega)$ satisfying $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u|^2 e^{-\varphi} dV \leqslant 4 \int_\Omega |\alpha|^2_{i\partial\overline{\partial}\psi} e^{-\varphi} dV.$$

Chapter 3

Weighted Bergman projections and applications [1]

3.1 Introduction

The aim of this chapter is to study weighted Bergman projections by using tools from the $\bar{\partial}$-equation. We obtain estimates for the L^2-boundary norm of the Bergman kernel from a weighted estimate of the Bergman projection. The relation between the Bergman kernel and the pluricomplex Green function is the key feature here, which provides various applications and will be discussed in this chapter.

Throughout this chapter, Ω will be a bounded pseudoconvex domain in $\mathbb{C}^n$, and $\delta : \Omega \to \mathbb{R}^+$ is the distance function to the boundary, $\delta(z) := \inf\{\|z - w\| : w \in \partial\Omega\}$. The Bergman kernel $K(z, w)$ of Ω is the reproducing kernel for the space of square-integrable holomorphic functions. That is, $K : \Omega \times \Omega \to \mathbb{C}$ is the function such that

$$f(z) = \int_{\Omega} K(z, w) f(w)\, dV(w), \forall f \in A^2(\Omega).$$

Here $A^2(\Omega)$ denotes the space of square-integrable holomorphic functions on Ω, and dV is Lebesgue measure.

It was suggested by Catlin ([Cat]) for studying the boundary behavior of the Bergman kernel, by starting with simple domains such as ellipsoids. In this chapter, we are also interested in obtaining estimates on the L^2-boundary norm of the Bergman kernel $\|K(\cdot, w)\|_{L^2(\partial\Omega)}$, as w varies in Ω.

It is known that for a class of pseudoconvex domains of finite type with C^∞-boundary (such as: convex domains of finite type, finite-type domains in $\mathbb{C}^2$, de-

[1] This chapter contains the material from the paper: P.T. Thuc, A note on L^2-boundary integrals of the Bergman kernel, International Journal of Mathematics, to appear.

coupled domains of finite type or finite-type domains whose Levi form has only one degenerate eigenvalue or comparable eigenvalues), pointwise estimates near the boundary of the Bergman kernel have been obtained by several authors, see e.g. [McN89, McN94b, McN91, Cat89, Cho96, Cho02]. These results are useful for many important problems such as the regularity of the Bergman projection ([MS94, PS77]), the regularity of Bergman-Toeplitz operators ([CM06, KLT18a]) and estimates of invariant metrics ([Cat89, McN92]).

It is possible to bound the L^2-boundary norm of the Bergman kernel using these good estimates. However, it seems that such sharp estimates are still not available in the context of a general pseudoconvex domain, or at least without the assumption of finite type. One aim of this chapter is to establish estimates on the L^2-boundary norm of the Bergman kernel for any pseudoconvex domain admitting a plurisubharmonic defining function. Due to the work of Michel-Shaw [MS01] (see also [BS91]), for any C^2-smooth domain Ω in this class, $K(\cdot, w)$ belongs to the holomorphic Hardy space $H^2(\Omega)$. Therefore it is important to study the behavior of $\|K(\cdot, w)\|_{L^2(\partial\Omega)}$, which is the Hardy norm of $K(\cdot, w)$, especially as w approaches to the boundary.

Since a general convex domain is the simplest example, to illustrate the method, let us focus on this case of domains. The more general statement is stated in Theorem 3.4.4.

The main result is as follows.

Theorem 3.1.1. *Let Ω be a bounded convex domain with C^2 boundary in $\mathbb{C}^n$. Then*

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)}} \leqslant \|K(\cdot, w)\|_{L^2(\partial\Omega)} \leqslant C_2\sqrt{\frac{K(w,w)}{\delta(w)}}, \tag{3.1.1}$$

for any $w \in \Omega$. Here C_1 is a positive constant depending on Ω, and $C_2 = \sqrt{4en+1}$.

Our method is indeed elementary, basically being based on a weighted estimate of the Bergman projection by Berndtsson and Charpentier ([BC00]), and relations between the pluricomplex Green function and the Bergman kernel. Our approach is motivated by the work of Chen and Fu ([CF11]) on the comparison of the Bergman and Szegö kernels. In fact, by the definition of the Szegö kernel

$$S(z,z) = \sup\left\{|f(z)|^2 : f \text{ is holomorphic on } \Omega, \|f\|_{L^2(\partial\Omega)} \leqslant 1\right\},$$

from Theorem 3.1.1 we obtain that for any C^2-bounded convex domain Ω in $\mathbb{C}^n$,

$$S(z,z)\, K(w,w) \geqslant \frac{|K(z,w)|^2\, \delta(w)}{4en+1}; \quad \forall z, w \in \Omega.$$

In particular, we also have

$$\frac{S(z,z)}{K(z,z)} \geqslant \frac{\delta(z)}{4en+1}; \quad \forall z \in \Omega.$$

We note that in [CF11], however, the authors obtained lower estimates on S/K for a much larger class of domains called δ-regular, which includes pseudoconvex domains of finite type and pseudoconvex domains having plurisubharmonic defining functions at boundary points.

In Section 3.2, we provide a weighted estimate of the Bergman projection. We illustrate some applications of the pluricomplex Green function in Section 3.3. The proof of Theorem 3.1.1 is given in Section 3.4.

3.2 A weighted estimate of the Bergman projection

Let ψ be a Lebesgue measurable function on Ω. By $L^2(\Omega, e^{\psi})$ we denote the Hilbert space of measurable functions associated with the norm

$$\|f\|_{L^2(\Omega,e^{\psi})} := \sqrt{\int_{\Omega} |f|^2 e^{\psi} dV}.$$

Let P be the ordinary Bergman projection associated to Ω. It can be represented as

$$P(f)(z) := \int_{\Omega} K(z,w) f(w) dV(w), \forall f \in L^2(\Omega). \tag{3.2.1}$$

The formula (3.2.1) allows us to extend the domain of definition of P. Let f be a Lebesgue measurable function on Ω, we say that $P(f)$ is well-defined on Ω if for almost every $z \in \Omega$, we have $K(z,\cdot) f(\cdot) \in L^1(\Omega)$. For example, when Ω is a smoothly bounded pseudoconvex domain of finite type, $P(f)$ is well-defined for any $f \in L^p(\Omega)$, with $p \geqslant 1$ (see, e.g. [Boa87, Bel86]).

The main purpose of this section is to establish the following result.

Proposition 3.2.1. *Let Ω be a bounded pseudoconvex domain. Let $0 < r < 1$ and let ψ be a locally bounded, plurisubharmonic function on Ω such that $ri\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi$ as currents. Then for any Lebesgue measurable function f so that $P(f)$ is well-defined, the following inequality holds*

$$\int_{\Omega} |P(f)(z)|^2 e^{\psi(z)} dV(z) \leqslant \frac{1}{1-r} \int_{\Omega} |f(z)|^2 e^{\psi(z)} dV(z). \tag{3.2.2}$$

Remark 3.2.2. The inequality (3.2.2) was already stated in [BC00] with an implicit constant in the right-hand side. It turns out that the constant $1/(1-r)$ here allows us to establish the estimates in Theorem 3.1.1.

We also recall that the condition $r\imath\partial\overline{\partial}\psi \geqslant \mathrm{i}\partial\psi \wedge \overline{\partial}\psi$ is equivalent to the statement that $-e^{-\psi/r}$ is plurisubharmonic on Ω.

Proof. It suffices to consider the case that the right hand side of (3.2.2) is finite. We employ an approach similar to that in [BC00]. The idea is to shift the standard L^2-space to weighted ones by Kohn's formula and by being more careful with the use of weighted Bergman projections.

We first assume that $\psi \in C^2\left(\overline{\Omega}\right)$ and $f \in L^2(\Omega)$. By Kohn's formula, e.g. [Koh63, CS01] and also the formula (2), page 29 in [BoSt90],

$$P(f) = e^{-\psi}P_{-\psi}\left(e^{\psi}f\right) - \overline{\partial}^{\star}N\left(\overline{\partial}\left(e^{-\psi}P_{-\psi}\left(e^{\psi}f\right)\right)\right), \tag{3.2.3}$$

where P_{ψ} denotes the Bergman projection in $L^2\left(\Omega, e^{\psi}\right)$. Note that since $\psi \in C^2\left(\overline{\Omega}\right)$, $L^2\left(\Omega, e^{-\psi}\right)$ and $L^2(\Omega)$ are identical as sets; though, they have different inner products.

Set $g := e^{-\psi}P_{-\psi}\left(e^{\psi}f\right)$ and $u := \overline{\partial}^{\star}N\left(\overline{\partial}\left(e^{-\psi}P_{-\psi}\left(e^{\psi}f\right)\right)\right)$. Since $P_{-\psi}\left(e^{\psi}f\right) \in A^2(\Omega)$, we have $\int_{\Omega} u\overline{g}e^{\psi} = 0$. Thus

$$\int_{\Omega} |P(f)|^2 e^{\psi} = \int_{\Omega} |g-u|^2 e^{\psi} = \int_{\Omega} |g|^2 e^{\psi} + \int_{\Omega} |u|^2 e^{\psi}. \tag{3.2.4}$$

For the first term of (3.2.4), we get

$$\int_{\Omega} |g|^2 e^{\psi} = \int_{\Omega} \left|P_{-\psi}\left(e^{\psi}f\right)\right|^2 e^{-\psi} \leqslant \int_{\Omega} \left|e^{\psi}f\right|^2 e^{-\psi} = \int_{\Omega} |f|^2 e^{\psi}. \tag{3.2.5}$$

Note that $u = \overline{\partial}^{\star}N\left(-g \wedge \overline{\partial}\psi\right)$, and ue^{ψ} belongs to the orthogonal complement of $\ker\overline{\partial}$ in $L^2\left(\Omega, e^{-\psi}\right)$. Thus we obtain (see Section 2.3)

$$\int_{\Omega} \left|ue^{\psi}\right|^2 e^{-\psi} \leqslant \int_{\Omega} \left|\overline{\partial}\left(ue^{\psi}\right)\right|^2_{\imath\partial\overline{\partial}\psi} e^{-\psi}. \tag{3.2.6}$$

It continues

$$\begin{aligned}\int_{\Omega} |u|^2 e^{\psi} &\leqslant \int_{\Omega} \left|\overline{\partial}u + \overline{\partial}\psi \wedge u\right|^2_{\imath\partial\overline{\partial}\psi} e^{\psi} \\ &= \int_{\Omega} \left|-g \wedge \overline{\partial}\psi + \overline{\partial}\psi \wedge u\right|^2_{\imath\partial\overline{\partial}\psi} e^{\psi} \end{aligned} \tag{3.2.7}$$

$$\leqslant r \int_{\Omega} \left(|u|^2 + |g|^2\right) e^{\psi},$$

here the last inequality follows by $\int_\Omega u\overline{g}e^{\psi} = 0$ and $\left|\overline{\partial}\psi\right|^2_{i\partial\overline{\partial}\psi} \leqslant r$. Therefore

$$\int_{\Omega} |u|^2 e^{\psi} \leqslant \frac{r}{1-r} \int_{\Omega} |g|^2 e^{\psi}. \tag{3.2.8}$$

From (3.2.4), (3.2.5) and (3.2.8), we get the estimate (3.2.2).

When $f \in L^2(\Omega)$, but ψ is just a locally bounded, plurisubharmonic on Ω. Consider a sequence of pseudoconvex domains $\{\Omega_j\}$ such that $\overline{\Omega}_j \Subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. For $\varepsilon > 0$, denote $\psi_\varepsilon := -r \log\left(e^{-\psi/r} \star \eta_\varepsilon\right)$. Here $\star$ denotes the standard convolution. For each j, we can choose ε_j such that $0 < \varepsilon_j < \operatorname{dist}(\Omega_j, \partial\Omega)$ and

$$\int_{\Omega_j} |f|^2 \left(e^{\psi_{\varepsilon_j}} - e^{\psi}\right) \leqslant \frac{1}{j}. \tag{3.2.9}$$

This is due to the monotone convergence theorem and the fact $f \in L^2(\Omega) \cap L^2\left(\Omega, e^{\psi}\right)$. Since $ri\partial\overline{\partial}\psi_{\varepsilon_j} \geqslant i\partial\psi_{\varepsilon_j} \wedge \overline{\partial}\psi_{\varepsilon_j}$, by applying the previous argument, we get that

$$\int_{\Omega_j} \left|P_{\Omega_j}(f)\right|^2 e^{\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r} \int_{\Omega_j} |f|^2 e^{\psi_{\varepsilon_j}}.$$

Since $\psi_{\varepsilon_j} \geqslant \psi$ on Ω_j and (3.2.9), it continues

$$\int_{\Omega_j} \left|P_{\Omega_j}(f)\right|^2 e^{\psi} \leqslant \frac{1}{1-r} \int_{\Omega_j} |f|^2 e^{\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r} \left(\frac{1}{j} + \int_{\Omega} |f|^2 e^{\psi}\right). \tag{3.2.10}$$

We conclude that for each fixed $k \in \mathbb{Z}^+$, the sequence $\left\{P_{\Omega_j}(f)\right\}_{j=k}^{\infty}$ is uniformly bounded in $L^2(\Omega_k)$. By Cantor's diagonal argument, we can assume, by passing to a subsequence, that $P_{\Omega_j}(f)$ converges weakly to a function v in $L^2(\Omega, \text{loc})$. It is clear that $P_{\Omega_j}(f)\, e^{\psi/2}$ also converges weakly to $ve^{\psi/2}$ in $L^2(\Omega, \text{loc})$. Thus for each $K \Subset \Omega$, by (3.2.10)

$$\int_{K} |v|^2 e^{\psi} \leqslant \liminf_{j\to\infty} \int_{K} \left|P_{\Omega_j}(f)\right|^2 e^{\psi} \leqslant \frac{1}{1-r} \int_{\Omega} |f|^2 e^{\psi}.$$

It follows that

$$\int_{\Omega} |v|^2 e^{\psi} \leqslant \frac{1}{1-r} \int_{\Omega} |f|^2 e^{\psi}.$$

We now prove that $v = P_\Omega(f)$. First, since $\overline{\partial}\left(P_{\Omega_j}(f)\right) = 0$ and $P_{\Omega_j}(f) \to v$ weakly, v is holomorphic in Ω. It remains to show that

$$\int_\Omega |f-v|^2 \leqslant \int_\Omega |f-h|^2, \forall h \in A^2(\Omega). \tag{3.2.11}$$

To see this, fix any $K \Subset \Omega$, we have

$$\begin{aligned}\int_K |f-v|^2 &\leqslant \liminf_{j\to\infty} \int_K \left|f - P_{\Omega_j}(f)\right|^2 \\ &\leqslant \liminf_{j\to\infty} \int_{\Omega_j} \left|f - P_{\Omega_j}(f)\right|^2 \\ &\leqslant \liminf_{j\to\infty} \int_{\Omega_j} |f-h|^2 \\ &\leqslant \int_\Omega |f-h|^2.\end{aligned}$$

So (3.2.11) follows.

Finally, we consider the case when we only require that $\int_\Omega |f|^2 e^\psi$ is finite and $P(f)$ is well-defined. Set $f_k := \chi_{\Omega_k} f$, where the sequence $\{\Omega_k\}$ is the same as above, and χ_{Ω_k} is the indicator function of Ω_k. We have $f_k \in L^2(\Omega) \cap L^2\left(\Omega, e^\psi\right)$ and

$$\int_\Omega |f_k - f|^2 e^\psi = \int_{\Omega\backslash\Omega_k} |f|^2 e^\psi \to 0 \text{ as } k \to \infty.$$

By the previous estimate,

$$\int_\Omega |P(f_k)|^2 e^\psi \leqslant \frac{1}{1-r}\int_\Omega |f_k|^2 e^\psi.$$

It follows that $\{P(f_k)\}$ is a Cauchy sequence in $L^2\left(\Omega, e^\psi\right)$ and so converges to a function v in $L^2\left(\Omega, e^\psi\right)$. Thus

$$\int_\Omega |v|^2 e^\psi \leqslant \frac{1}{1-r}\int_\Omega |f|^2 e^\psi.$$

Now, we only need to prove that $v = P(f)$. Since we can choose a subsequence of the $\{P(f_k)\}$ that converges pointwise to v, it suffices to show that $P(f_k)$ converges pointwise to $P(f)$. For each $z \in \Omega$,

$$\begin{aligned} |P(f_k)(z) - P(f)(z)| &= \left| \int_\Omega K(z,w) f_k(w)\, dV(w) - \int_\Omega K(z,w) f(w)\, dV(w) \right| \\ &= \left| \int_{\Omega \setminus \Omega_k} K(z,w) f(w)\, dV(w) \right|. \end{aligned}$$

The last integral goes to zero as $k \to \infty$ since $K(z,\cdot) f(\cdot) \in L^1(\Omega)$.

□

Remark 3.2.3. The constant $1/(1-r)$ is not sharp for any $r \in (0,1)$. More specifically, it is not sharp for $r \approx 0$. To see this, due to Theorem 2.3.3, for any function $v \perp \ker \overline{\partial}$ in $L^2(\Omega, e^{-\psi})$, we have

$$\int_\Omega |v|^2 e^{-\psi} \leqslant \frac{4r}{(1-r)^2} \int_\Omega \left|\overline{\partial} v\right|^2_{i\partial\overline{\partial}\psi} e^{-\psi}. \tag{3.2.12}$$

Now, apply (3.2.12) with $v := \overline{\partial}^\star N\left(-g \wedge \overline{\partial}\psi\right) e^{\psi}$, then we can replace the inequality (3.2.6) by

$$\int_\Omega \left|u e^{\psi}\right|^2 e^{-\psi} \leqslant \frac{4r}{(1-r)^2} \int_\Omega \left|\overline{\partial}\left(u e^{\psi}\right)\right|^2_{i\partial\overline{\partial}\psi} e^{-\psi}.$$

Continue the argument there, we get that

$$\int_\Omega |P(f)|^2 e^{\psi} \leqslant \frac{1}{1 - \frac{4r^2}{(1-r)^2}} \int_\Omega |f|^2 e^{\psi},$$

provided $0 < r < \frac{1}{3}$. Note that the constant $1/\left(1 - 4r^2/(1-r)^2\right)$ is sharper than $1/(1-r)$ for $0 < r < 3 - 2\sqrt{2} \approx 0.17$.

Nevertheless, if we call $C(r)$ the sharp constant for the estimate (3.2.2), that is, given $0 < r < 1$, $C(r)$ is the least constant such that for any pseudoconvex domain Ω and $ri\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi$, we have

$$\int_\Omega |P(f)|^2 e^{\psi} \leqslant C(r) \int_\Omega |f|^2 e^{\psi},$$

then we can show that

$$\lim_{r \to 1} \frac{C(r)}{\frac{1}{1-r}} = 1. \tag{3.2.13}$$

Therefore, the constant $1/(1-r)$ is sharp in the use of $r \to 1$. To see (3.2.13), choose $\Omega = \mathbb{D}$ the unit disc in $\mathbb{C}$, $f = (-\log|z|)^r$ and $\psi = -r\log(-\log|z|)$. We can easily check that

$$\frac{\int_\Omega |P(f)|^2 e^{\psi}}{\int_\Omega |f|^2 e^{\psi}} = \frac{\pi r}{\sin(\pi r)}.$$

Thus

$$\frac{\pi r}{\sin(\pi r)} \leqslant C(r) \leqslant \frac{1}{1-r},$$

and (3.2.13) follows.

Remark 3.2.4. By Proposition 3.2.1 and a duality argument, under the same hypothesis as in Proposition 3.2.1, the reader can verify that the following inequality is also true

$$\int_\Omega |P(f)(z)|^2 e^{-\psi(z)} dV(z) \leqslant \frac{1}{1-r} \int_\Omega |f(z)|^2 e^{-\psi(z)} dV(z). \tag{3.2.14}$$

The inequality (3.2.14) can also be obtained directly by using the same trick as in the proof of Proposition 3.2.1. To see this, we need the following result.

Proposition 3.2.5. *[Che16, Corollary 2.3] Let Ω be a bounded pseudoconvex domain and let $\varphi \in PSH(\Omega)$. Let α be a $\bar\partial$-closed $(0,1)$-form and let $\psi \in PSH(\Omega)$ such that $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \leqslant r$, for some $0 < r < 1$. Then the $L^2(\Omega, e^{-\varphi})$-minimal solution of $\bar\partial v = \alpha$ satisfies*

$$\int_\Omega |v|^2 e^{-\psi-\varphi} dV \leqslant \frac{1}{1-r} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{-\psi-\varphi} dV. \tag{3.2.15}$$

To prove (3.2.14), first consider the case that $f \in L^2(\Omega)$ and $\psi \in C^2(\overline{\Omega})$. Set

$$g := e^{\psi} P_\psi\left(e^{-\psi} f\right) \text{ and } u := \bar\partial^\star N\left(\bar\partial\left(e^{\psi} P_\psi\left(e^{-\psi} f\right)\right)\right).$$

The estimate for $\int_\Omega |g|^2 e^{-\psi}$ can be done similarly as (3.2.5). The estimate for $\int_\Omega |u|^2 e^{-\psi}$ is obtained by

$$\begin{aligned} \int_\Omega |u|^2 e^{-\psi} &\leqslant \int_\Omega \left|\bar\partial^\star N\left(g \wedge \bar\partial\psi\right)\right|^2 e^{-\psi} \\ &\leqslant \frac{1}{1-r} \int_\Omega |g \wedge \bar\partial\psi|^2_{i\partial\bar\partial\psi} e^{-\psi}. \end{aligned}$$

The latter inequality follows from Proposition 3.2.5 (by applying $\varphi \equiv 0$ there). We therefore obtain (3.2.14).

In the case that $f \in L^2(\Omega)$ and ψ is a locally bounded, plurisubharmonic function on Ω. Consider the approximants $\psi_{\varepsilon_j} := -r\log\left(e^{-\psi/r} \star \eta_{\varepsilon_j}\right)$, with $0 < \varepsilon_j < \operatorname{dist}(\Omega_j, \partial\Omega)$. For each $k \in \mathbb{Z}^+$, we have

$$\int_{\Omega_k} \left|P_{\Omega_j}(f)\right|^2 e^{-\psi_{\varepsilon_k}} \leqslant \int_{\Omega_j} \left|P_{\Omega_j}(f)\right|^2 e^{-\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r}\int_{\Omega_j} |f|^2 e^{-\psi_{\varepsilon_j}} \leqslant \frac{1}{1-r}\int_{\Omega} |f|^2 e^{-\psi},$$

for any $j \geqslant k$. Thus, there is a subsequence, say still $P_{\Omega_j}(f)$, converges weakly to $v \in L^2(\Omega, \mathrm{loc})$. For each $K \Subset \Omega$ and $k \in \mathbb{Z}^+$,

$$\int_K |v|^2 e^{-\psi_{\varepsilon_k}} \leqslant \liminf_{j\to\infty} \int_K \left|P_{\Omega_j}(f)\right|^2 e^{-\psi_{\varepsilon_k}} \leqslant \frac{1}{1-r}\int_\Omega |f|^2 e^{-\psi}.$$

Letting $k \to \infty$, by the monotone convergence theorem, we get that

$$\int_K |v|^2 e^{-\psi} \leqslant \frac{1}{1-r}\int_\Omega |f|^2 e^{-\psi},$$

and so we have

$$\int_\Omega |v|^2 e^{-\psi} \leqslant \frac{1}{1-r}\int_\Omega |f|^2 e^{-\psi}.$$

The rest of our argument goes through in the same way as in the proof of Proposition 3.2.1.

By choosing $f = (-\log|z|)^{-r}$, $\psi = -r\log(-\log|z|)$ and $\Omega = \mathbb{D}$, the sharp constant $C(r)$ for the estimate (3.2.14) also satisfies

$$\lim_{r\to 1} \frac{C(r)}{\frac{1}{1-r}} = 1.$$

Proposition 3.2.1, together with an idea of Chen [Che17a], gives the following result.

Corollary 3.2.6. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with a positive Diederich-Fornæss index η. Then for any $0 < t < \eta$ and $1 \leqslant q < 4n/(2n-t)$, the Bergman projection P associated to Ω is bounded from $L^2(\Omega, \delta^{-t})$ to $L^q(\Omega)$.*

Proof. Since $4n/(2n-t) > 2$, it suffices to assume that $q > 2$. Recall that the Diederich-Fornæss index of Ω is defined by

$$\eta(\Omega) := \sup\left\{\alpha \in [0,1] : \exists h \in PSH(\Omega) \text{ and } C > 0 \text{ such that } \frac{1}{C}\delta^\alpha < -h < C\delta^\alpha\right\}.$$

Thus, we can choose $t' \in (t, \eta)$ and $h \in PSH(\Omega)$ such that $\frac{1}{C}\delta^{t'} < -h < C\delta^{t'}$, for some positive constant C. Set $\psi := -(t/t')\log(-h)$, then $\psi \in L^{\infty}_{\text{loc}}(\Omega)$ and

$$\frac{t}{t'} i\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi.$$

For any measurable function f such that $\int_{\Omega} |f|^2 \delta^{-t}$ is finite (which also implies $f \in L^2(\Omega)$), by applying Proposition 3.2.1, we get that

$$\int\limits_{\Omega} |P(f)|^2 \delta^{-t} \lesssim \int\limits_{\Omega} |P(f)|^2 e^{\psi} \lesssim \int\limits_{\Omega} |f|^2 e^{\psi} \lesssim \int\limits_{\Omega} |f|^2 \delta^{-t}. \tag{3.2.16}$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned} \int\limits_{\delta \leqslant \varepsilon} |P(f)|^2 &\leqslant \varepsilon^t \int\limits_{\delta \leqslant \varepsilon} |P(f)|^2 \delta^{-t} \leqslant \varepsilon^t \int\limits_{\Omega} |P(f)|^2 \delta^{-t} \\ &\lesssim \varepsilon^t \int\limits_{\Omega} |f|^2 \delta^{-t}. \end{aligned}$$

Moreover, by the mean value inequality,

$$\begin{aligned} |P(f)(z)|^2 &\lesssim \delta^{-2n}(z) \int\limits_{B(z,\delta(z))} |P(f)|^2 \\ &\lesssim \delta^{-2n}(z) \int\limits_{\delta \leqslant 2\delta(z)} |P(f)|^2 \\ &\lesssim \delta^{-2n+t}(z) \int\limits_{\Omega} |f|^2 \delta^{-t}. \end{aligned}$$

It follows that for each $k \in \mathbb{Z}^+$,

$$\begin{aligned} \int\limits_{2^{-k-1} < \delta \leqslant 2^{-k}} |P(f)|^q &\lesssim 2^{-(k+1)(q-2)\left(-n+\frac{t}{2}\right)} \left(\int\limits_{\Omega} |f|^2 \delta^{-t} \right)^{\frac{q}{2}-1} \int\limits_{\delta \leqslant 2^{-k}} |P(f)|^2 \\ &\lesssim 2^{-k\left(t-(q-2)\left(n-\frac{t}{2}\right)\right)} \left(\int\limits_{\Omega} |f|^2 \delta^{-t} \right)^{\frac{q}{2}}. \end{aligned}$$

Similarly,

$$\int\limits_{\delta > 1/2} |P(f)|^q \lesssim \left(\frac{1}{2}\right)^{(q-2)\left(-n+\frac{t}{2}\right)} \left(\int\limits_{\Omega} |f|^2 \delta^{-t} \right)^{\frac{q}{2}-1} \int\limits_{\Omega} |P(f)|^2$$

$$\lesssim \left(\int_\Omega |f|^2 \delta^{-t}\right)^{\frac{q}{2}-1} \int_\Omega |f|^2$$
$$\lesssim \left(\int_\Omega |f|^2 \delta^{-t}\right)^{\frac{q}{2}}.$$

Therefore

$$\begin{aligned}\int_\Omega |P(f)|^q &\leqslant \int_{\delta>1/2} |P(f)|^q + \sum_{k=1}^{\infty} \int_{2^{-k-1}<\delta\leqslant 2^{-k}} |P(f)|^q \\ &\lesssim \left(1+\sum_{k=1}^{\infty} 2^{-k\left(t-(q-2)\left(n-\frac{t}{2}\right)\right)}\right)\left(\int_\Omega |f|^2 \delta^{-t}\right)^{\frac{q}{2}} \\ &\lesssim \left(\int_\Omega |f|^2 \delta^{-t}\right)^{\frac{q}{2}},\end{aligned}$$

where the last inequality follows by the hypothesis $q < 4n/(2n-t)$. □

Note that we do not impose any regularity assumption on the boundary of Ω in Corollary 3.2.6. In the case when $\partial\Omega$ is Lipschitz, it is known that $\eta(\Omega)$ is positive, see [Har08]. Moreover, using the Lipschitz property, we can conclude that $\delta^{-\alpha} \in L^1(\Omega)$ for any $\alpha < 1$ (see e.g. [Gri11]). Thus, using Hölder's inequality, we get that if $p \in (2,\infty)$ and $tp/(p-2) < 1$ then

$$\int_\Omega |f|^2 \delta^{-t} \leqslant \left(\int_\Omega |f|^p\right)^{\frac{2}{p}} \left(\int_\Omega \delta^{\frac{-tp}{p-2}}\right)^{\frac{p-2}{p}} \lesssim \left(\int_\Omega |f|^p\right)^{\frac{2}{p}}.$$

Combining this with Corollary 3.2.6, for a given $q \in [2, 4n/(2n-\eta))$, if there exists t such that

$$\eta > t > 2n - \frac{4n}{q} \text{ and } 1 - \frac{2}{p} > t,$$

then the Bergman projection P maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously. This requirement on t is equivalent to $p > 2q/(q+2n(2-q))$.

We therefore arrive at the following result (see also [HZ17, Section 4]):

Corollary 3.2.7. *Let Ω be a bounded pseudoconvex domain with Lipschitz boundary. Let η be the Diederich-Fornæss index of Ω. Then for any $q \in [2, 4n/(2n-\eta))$, the Bergman projection associated to Ω is bounded from $L^p(\Omega)$ to $L^q(\Omega)$, provided that $p > 2q/(q+2n(2-q))$.*

Remark 3.2.8. Corollary 3.2.7 says that for domains with Lipschitz boundary, one can always gain the regularity of the output space to an exponent bigger than 2 (i.e. $L^q, q > 2$), given that the regularity of the input is high enough. This is not the case for non-Lipschitz domains. For instance, consider the Hartogs triangle domain $\Omega_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$, with $\gamma > 0$ and $\gamma \notin \mathbb{Q}$. This is a non-Lipschitz pseudoconvex domain (see e.g. [Zwo99]). It is known that for any $p \in (1, \infty)$ and $q > 2$, the Bergman projection associated to Ω_γ cannot be bounded from $L^p(\Omega_\gamma)$ to $L^q(\Omega_\gamma)$, see [EM17, p. 2681].

3.3 Some applications of the pluricomplex Green function

In this section, we give some applications of the pluricomplex Green function in the study of weighted Bergman kernels and weighted Bergman projections.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Recall that the pluricomplex Green function with a pole $w \in \Omega$ is defined by

$$G(\cdot, w) := \sup\left\{u(\cdot) : u \in PSH^-(\Omega), \limsup_{z \to w}(u(z) - \log|z - w|) < \infty\right\}.$$

Here $PSH^-(\Omega)$ denotes the set of all negative plurisubharmonic functions on Ω.

By Proposition 2.2.8,

$$K_{\{G(\cdot,w)<-t\}}(w, w) \leqslant e^{2nt} K(w, w), \tag{3.3.1}$$

for any $w \in \Omega$ and $t > 0$. Given $f \in A^2(\Omega)$, using Lemma 2.1.1 we have

$$\frac{1}{K_{\{G(\cdot,w)<-t\}}(w, w)} \leqslant \int_{\{G(\cdot,w)<-t\}} \left|\frac{f(z)}{f(w)}\right|^2 dV(z), \tag{3.3.2}$$

provided that $f(w) \neq 0$. From (3.3.1) and (3.3.2), we obtain that for any bounded pseudoconvex domain Ω in $\mathbb{C}^n$,

$$\int_{\{G(\cdot,w)<-t\}} |f(z)|^2 dV(z) \geqslant e^{-2nt} \frac{|f(w)|^2}{K(w, w)}, \tag{3.3.3}$$

for any $f \in A^2(\Omega)$, $w \in \Omega$ and $t > 0$.

The inequality (3.3.3) suggests a generalisation to weighted Bergman kernels.

Proposition 3.3.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let ψ be a locally bounded, plurisubharmonic function on Ω. Then there is a positive constant C_n depending only on n such that*

$$\int_{\{G(\cdot,w)<-1\}} |f(z)|^2 e^{-\psi(z)} dV(z) \geqslant C_n \frac{|f(w)|^2}{K_{-\psi}(w,w)}, \tag{3.3.4}$$

for any holomorphic function f on Ω and any $w \in \Omega$. Here $K_{-\psi}$ is the weighted Bergman kernel of $L^2(\Omega, e^{-\psi})$.

Remark 3.3.2. The hypothesis in Proposition 3.3.1 guarantees that ψ is admissible (see Section 2.1) and $K_{-\psi}(w,w) > 0$, see [Che06, Theorem 3.2].

Proof. Assume that the left-hand side of (3.3.4) is finite. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $\chi(t) := 0$ when $t \leqslant 0$ and $\chi(t) := \int_0^t e^{-ne^x} dx$ when $t > 0$. By the estimate of Donnelly-Fefferman (Theorem 2.3.5), we can find a function $u \in L^2_{loc}(\Omega)$ such that

$$\bar\partial u = -f \bar\partial (\chi(\log(-G(\cdot,w))))$$

and

$$\int_\Omega |u|^2 e^{-(2nG(\cdot,w)+\psi)} \leqslant 4 \int_\Omega \left| f\bar\partial(\chi(\log(-G(\cdot,w))))\right|^2_{i\partial\bar\partial(-\log(-G(\cdot,w)))} e^{-(2nG(\cdot,w)+\psi)}. \tag{3.3.5}$$

Set $v := u + f\chi(\log(-G(\cdot,w)))$, then v is holomorphic on Ω. Since

$$\left|\bar\partial(\chi(\log(-G(\cdot,w))))\right|^2_{i\partial\bar\partial(-\log(-G(\cdot,w)))} \leqslant |\dot\chi(\log(-G(\cdot,w)))|^2,$$

we obtain that

$$\begin{aligned}
\int_\Omega \frac{|v|^2}{2} e^{-\psi} &\leqslant \int_\Omega |f\chi(\log(-G(\cdot,w)))|^2 e^{-\psi} + \int_\Omega |u|^2 e^{-\psi} \\
&\leqslant \int_{\{G(\cdot,w)<-1\}} c_1 |f|^2 e^{-\psi} + \int_\Omega |u|^2 e^{-(2nG(\cdot,w)+\psi)} \\
&\leqslant \int_{\{G(\cdot,w)<-1\}} c_1 |f|^2 e^{-\psi} + 4 \int_{\{G(\cdot,w)<-1\}} |f|^2 e^{-\psi} \\
&= (4+c_1) \int_{\{G(\cdot,w)<-1\}} |f|^2 e^{-\psi}.
\end{aligned}$$

Here $c_1 := \left(\int_0^\infty e^{-ne^x} dx\right)^2 \in \mathbb{R}^+$. Since $\psi \in L^\infty_{loc}(\Omega)$ and

$$\int_\Omega |u|^2 e^{-(2nG(\cdot,w)+\psi)} \leqslant 4 \int_{\{G(\cdot,w)<-1\}} |f|^2 e^{-\psi} < \infty,$$

it follows that $u(w) = 0$. Thus $v(w) = \sqrt{c_1} f(w)$. Therefore

$$K_{-\psi}(w,w) \geqslant \frac{|v(w)|^2}{\int_\Omega |v|^2 e^{-\psi}} \geqslant \left(\frac{c_1}{8+2c_1}\right) \frac{|f(w)|^2}{\int_{\{G(\cdot,w)<-1\}} |f|^2 e^{-\psi}}.$$

We get the desired claim. □

An application of Proposition 3.3.1 is the following estimates for bounded homogeneous domains. Recall that a domain Ω is called homogeneous if $\mathrm{Aut}(\Omega)$ acts transitively on Ω. That is, for any $z, w \in \Omega$, there is $\Phi \in \mathrm{Aut}(\Omega)$ such that $\Phi(z) = w$. Here $\mathrm{Aut}(\Omega)$ is the group of all biholomorphic mappings of Ω. It is known that any bounded homogeneous domain is hyperconvex, see [KO07].

Corollary 3.3.3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded homogeneous domain. Then there is a positive constant C_n depending only on n such that*

$$\int_{\{G(\cdot,w)<-1\}} |f(z)|^2 K^{-\alpha}(z,z)\, dV(z) \geqslant \left(C_n \frac{K^{\alpha+1}(w_0,w_0)}{K_{K^{-\alpha}}(w_0,w_0)}\right) \frac{|f(w)|^2}{K^{\alpha+1}(w,w)}, \quad (3.3.6)$$

for any $f \in \mathcal{O}(\Omega)$, $\alpha > 0$ and $w_0, w \in \Omega$. Here $K_{K^{-\alpha}}$ denotes the weighted Bergman kernel of $L^2(\Omega, K^{-\alpha})$. In particular, we have

$$\int_{\{G(\cdot,w)<-1\}} |f(z)|^2 K^{-\alpha}(z,z)\, dV(z) \gtrsim \frac{|f(w)|^2}{K^{\alpha+1}(w,w)},$$

where $\gtrsim$ is up to a constant depending on Ω, α and n.

Proof. The inequality (3.3.6) can be obtained from Proposition 3.3.1 by applying $\psi(z) := \alpha \log(K(z,z))$ and using the fact that

$$K_{-\psi}(w,w) = \frac{K_{-\psi}(w_0,w_0)}{K^{1+\alpha}(w_0,w_0)} K^{1+\alpha}(w,w)\,; \forall w, w_0 \in \Omega. \quad (3.3.7)$$

To see (3.3.7), choose $\Phi \in \mathrm{Aut}(\Omega)$ such that $\Phi(w) = w_0$. By [HW15, Lemma 2.1],

$$K_{-\psi}(w,w) = K_{-\psi}(w_0,w_0)\, |\det J\Phi(w)|^{2(1+\alpha)},$$

where $J\Phi(w)$ is the Jacobian of Φ at w. On the other hand, by the transformation formula

$$|\det J\Phi(w)|^2 = \frac{K(w,w)}{K(w_0,w_0)}.$$

Thus (3.3.7) follows. □

Proposition 3.3.1 and Proposition 3.2.1 together give the following estimates for the comparison of weighted Bergman kernels and the ordinary Bergman kernel. This is an improvement of [Che06, Theorem 1.4]. Even though the underlying method also involves the use of the pluricomplex Green function and L^2 estimates of the $\overline{\partial}$-equation, the proof here is different from that in the paper of Chen.

Proposition 3.3.4. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain with a positive Diederich-Fornæss exponent η. Then there are positive constants C_1, C_2 and ε_0 depending on Ω and η such that the following statements are true.*

(a) For any $w \in \Omega$ with $\delta(w) < \varepsilon_0$ and any $s \in \mathbb{R}$,

$$\frac{K_s(w,w)}{K(w,w)} \leqslant C_1 \begin{cases} \delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}}, & s \geqslant 0, \\ \delta^{-s}(w)\,|\log\delta(w)|^{-\frac{ns}{\eta}}, & s < 0. \end{cases} \tag{3.3.8}$$

(b) For any $w \in \Omega$ with $\delta(w) < \varepsilon_0$ and any $s \in (-\eta, \infty)$,

$$\frac{K_s(w,w)}{K(w,w)} \geqslant C_2 \begin{cases} \delta^{-s}(w)\,|\log\delta(w)|^{-\frac{ns}{\eta}}, & s \geqslant 0, \\ \left(1+\frac{s}{\eta}\right)\delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}}, & -\eta < s < 0. \end{cases} \tag{3.3.9}$$

Here K_s denotes the weighted Bergman kernel of $L^2(\Omega, \delta^s)$.

Recall that $0 < \eta \leqslant 1$ is a Diederich-Fornæss exponent of Ω if there exists a negative plurisubharmonic function φ on Ω such that

$$c_1\delta^\eta(z) \leqslant -\varphi(z) \leqslant c_2\delta^\eta(z), \forall z \in \Omega, \tag{3.3.10}$$

for some positive constants c_1 and c_2.

Proof. By Theorem 2.2.6,

$$\{G(\cdot,w) < -1\} \subset \left\{\frac{1}{C}\delta(w)\,|\log\delta(w)|^{-\frac{1}{\eta}} \leqslant \delta(\cdot) \leqslant C\delta(w)\,|\log\delta(w)|^{\frac{n}{\eta}}\right\}, \tag{3.3.11}$$

for any $w \in \Omega$ with $\delta(w) < \varepsilon_0$. Here C and ε_0 are positive constants depending on Ω and η.

For $s \geqslant 0$, using Lemma 2.1.1, we obtain that

$$\int_{\{G(\cdot,w)<-1\}} \left|\frac{K_s(z,w)}{K_s(w,w)}\right|^2 dV(z) \geqslant \frac{1}{K_{\{G(\cdot,w)<-1\}}(w,w)}.$$

Thus

$$\begin{aligned}\int_{\{G(\cdot,w)<-1\}} |K_s(z,w)|^2 dV(z) &\geqslant \frac{K_s^2(w,w)}{K_{\{G(\cdot,w)<-1\}}(w,w)} \\ &\gtrsim \frac{K_s^2(w,w)}{K(w,w)}, \qquad (3.3.12)\end{aligned}$$

where the second inequality follows by Proposition 2.2.8. On the other hand, by (3.3.11)

$$\begin{aligned}\int_{\{G(\cdot,w)<-1\}} |K_s(z,w)|^2 dV(z) &\lesssim \delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}} \int_{\{G(\cdot,w)<-1\}} |K_s(z,w)|^2 \delta^s(z)\, dV(z) \\ &\lesssim \delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}} \int_{\Omega} |K_s(z,w)|^2 \delta^s(z)\, dV(z) \\ &\lesssim \delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}} K_s(w,w). \qquad (3.3.13)\end{aligned}$$

From (3.3.12) and (3.3.13), we conclude that

$$\frac{K_s(w,w)}{K(w,w)} \lesssim \delta^{-s}(w)\,|\log\delta(w)|^{\frac{s}{\eta}}.$$

For $s < 0$, the estimate can be obtained similarly. We only need to replace the inequality (3.3.13) to

$$\int_{\{G(\cdot,w)<-1\}} |K_s(z,w)|^2 dV(z) \lesssim \delta^{-s}(w)\,|\log\delta(w)|^{-\frac{ns}{\eta}} K_s(w,w).$$

Therefore the claim of (3.3.8) follows. Now, we turn to the inequalities (3.3.9).

First, we consider the case $s \geqslant 0$. Applying Proposition 3.3.1 to $\psi = -s\log\delta$, we have

$$\int_{\{G(\cdot,w)<-1\}} |f(z)|^2 \delta^s(z)\, dV(z) \gtrsim \frac{|f(w)|^2}{K_s(w,w)},$$

for any holomorphic function f on Ω and any $w \in \Omega$. Thus

$$\frac{K^2(w,w)}{K_s(w,w)} \lesssim \int_{\{G(\cdot,w)<-1\}} |K(z,w)|^2 \delta^s(z)\, dV(z)$$

$$
\begin{aligned}
&\lesssim \delta^s(w)\left|\log\delta(w)\right|^{\frac{ns}{\eta}} \int\limits_{\{G(\cdot,w)<-1\}} |K(z,w)|^2\, dV(z)\\
&\lesssim \delta^s(w)\left|\log\delta(w)\right|^{\frac{ns}{\eta}} \int\limits_{\Omega} |K(z,w)|^2\, dV(z)\\
&\lesssim \delta^s(w)\left|\log\delta(w)\right|^{\frac{ns}{\eta}} K(w,w).
\end{aligned}
$$

It follows that

$$
\frac{K_s(w,w)}{K(w,w)} \gtrsim \delta^{-s}(w)\left|\log\delta(w)\right|^{-\frac{ns}{\eta}}.
$$

Finally, we need to consider the case $0 > s > -\eta$. Using Proposition 3.2.1 with $\psi(z) = (s/\eta)\log(-\varphi(z))$, where the function φ comes from (3.3.10). It is clear that $(-s/\eta)\, i\partial\overline{\partial}\psi \geqslant i\partial\psi \wedge \overline{\partial}\psi$. Therefore

$$
\int\limits_{\Omega} |P(f)|^2 e^{\psi} \leqslant \frac{1}{1+\frac{s}{\eta}} \int\limits_{\Omega} |f|^2 e^{\psi}. \tag{3.3.14}
$$

Inserting $f(z) = \chi_{\{G(\cdot,w)<-1\}}(z)\, K_{\{G(\cdot,w)<-1\}}(z,w)$ into (3.3.14), and by noting that $P(f)(z) = K(z,w)$, we have

$$
\int\limits_{\Omega} |K(z,w)|^2 (-\varphi(z))^{\frac{s}{\eta}}\, dV(z) \leqslant \frac{1}{1+\frac{s}{\eta}} \int\limits_{\{G(\cdot,w)<-1\}} \left|K_{\{G(\cdot,w)<-1\}}(z,w)\right|^2 (-\varphi(z))^{\frac{s}{\eta}}\, dV(z).
$$

Thus it continues as

$$
\begin{aligned}
\int\limits_{\Omega} |K(z,w)|^2 \delta^s(z)\, dV(z) &\lesssim \int\limits_{\Omega} |K(z,w)|^2 (-\varphi(z))^{\frac{s}{\eta}}\, dV(z)\\
&\lesssim \frac{1}{1+\frac{s}{\eta}} \int\limits_{\{G(\cdot,w)<-1\}} \left|K_{\{G(\cdot,w)<-1\}}(z,w)\right|^2 (-\varphi(z))^{\frac{s}{\eta}}\, dV(z)\\
&\lesssim \frac{1}{1+\frac{s}{\eta}} \int\limits_{\{G(\cdot,w)<-1\}} \left|K_{\{G(\cdot,w)<-1\}}(z,w)\right|^2 \delta^s(z)\, dV(z)\\
&\lesssim \frac{1}{1+\frac{s}{\eta}} \delta^s(w)\left|\log\delta(w)\right|^{-\frac{s}{\eta}} K_{\{G(\cdot,w)<-1\}}(w,w)\\
&\lesssim \frac{1}{1+\frac{s}{\eta}} \delta^s(w)\left|\log\delta(w)\right|^{-\frac{s}{\eta}} K(w,w).
\end{aligned}
$$

Thus, it follows from the last inequality and the basic property

$$
K_s(w,w) = \sup\left\{ |f(w)|^2 : f \in \mathcal{O}(\Omega),\ \int\limits_{\Omega} |f|^2 \delta^s \leqslant 1 \right\}
$$

that
$$\frac{K_s(w,w)}{K(w,w)} \gtrsim \left(1+\frac{s}{\eta}\right)\delta^{-s}(w)\left|\log\delta(w)\right|^{\frac{s}{\eta}}.$$
This completes the proof of Proposition 3.3.4.

□

We close this section with an application to Bergman-Toeplitz operators.

Proposition 3.3.5. *Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary and let $\alpha \in \mathbb{R}$. If the Bergman-Toeplitz operator T_α, defined by*

$$f \to T_\alpha(f)(z) := \int_\Omega K(z,w) f(w)\delta^\alpha(w)\,dV(w),$$

is bounded from $L^p(\Omega)$ to $L^q(\Omega)$, with $1 < p \leqslant q < \infty$ then

$$\alpha \geqslant (n+1)\left(\frac{1}{p}-\frac{1}{q}\right).$$

Remark 3.3.6. This result has been obtained in [ARS12] by using several estimates of Kobayashi balls and θ-Carleson measures in Bergman spaces. The proof given below is a direct consequence of Theorem 2.2.6 and Proposition 2.2.8. Note that the converse statement is also true, i.e. if $\alpha \geqslant (n+1)((1/p)-(1/q))$ then T_α maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously, see [CM06].

Proof. Without loss of generality we may assume that $\alpha \geqslant 0$. First, it is well-known that for any strongly pseudoconvex domain Ω with smooth boundary, there is a positive constant $C(\Omega)$ such that

$$K(z,z) \geqslant C\delta^{-n-1}(z), \tag{3.3.15}$$

for any $z \in \Omega$. Moreover, for any $p > 1$, there is a constant $C(p,\Omega)$ such that

$$\|K(z,\cdot)\|_{L^p(\Omega)} \leqslant C\delta^{-(n+1)\left(1-\frac{1}{p}\right)}(z), \tag{3.3.16}$$

for any $z \in \Omega$, see e.g. [ARS12, Theorem 2.7], also [Li92, CM06].

Now, using properties of the Bergman projection, we have

$$\begin{aligned}\int_\Omega |K(z,w)|^2\delta^\alpha(w)\,dV(w) &= \int_\Omega K(w,z)\delta^\alpha(w)K(z,w)\,dV(w)\\ &= \int_\Omega K(w,z)\delta^\alpha(w)\left(\int_\Omega K(z,\xi)K(\xi,w)\,d\xi\right)dV(w)\end{aligned}$$

$$= \int_{\Omega} \left(\int_{\Omega} K(\xi, w)\, \delta^{\alpha}(w)\, K(w, z)\, dw \right) K(z, \xi)\, dV(\xi).$$

By Hölder's inequality, it follows

$$\begin{aligned} \int_{\Omega} |K(z,w)|^2 \delta^{\alpha}(w)\, dV(w) &\leqslant \left\| \int_{\Omega} K(\cdot, w)\, \delta^{\alpha}(w)\, K(w,z)\, dV(w) \right\|_{L^q(\Omega)} \|K(z,\cdot)\|_{L^{q'}(\Omega)} \\ &\lesssim \|K(z,\cdot)\|_{L^p(\Omega)} \|K(z,\cdot)\|_{L^{q'}(\Omega)} \\ &\lesssim \delta^{-(n+1)\left(1-\frac{1}{p}+\frac{1}{q}\right)}(z)\,. \end{aligned} \tag{3.3.17}$$

Here, the second inequality comes from the boundedness of T_α, the third follows from (3.3.16), and q' is the dual exponent of q, i.e. $1/q + 1/q' = 1$. On the other hand, by using Theorem 2.2.6, Proposition 2.2.8 and (3.3.15), we obtain

$$\begin{aligned} \int_{\Omega} |K(z,w)|^2 \delta^{\alpha}(w)\, dV(w) &\geqslant \int_{\{G(\cdot,z)<-1\}} |K(z,w)|^2 \delta^{\alpha}(w)\, dV(w) \\ &\gtrsim \delta^{\alpha}(z)\, |\log \delta(z)|^{-\alpha} \int_{\{G(\cdot,z)<-1\}} |K(z,w)|^2\, dV(w) \\ &\gtrsim \delta^{\alpha}(z)\, |\log \delta(z)|^{-\alpha} K(z,z) \\ &\gtrsim \delta^{\alpha-n-1}(z)\, |\log \delta(z)|^{-\alpha}\,. \end{aligned}$$

From this and (3.3.17), the conclusion follows by letting $z \to \partial\Omega$. □

Remark 3.3.7. A similar approach has been used in the proof of Theorem 5.3.3 for Hartogs triangle domains.

3.4 Proof of Theorem 3.1.1

Let us first recall some facts from the theory of Hardy spaces. We refer readers to the books by Stein [Ste72] and Krantz [Kra01] for details.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. The Hardy space $h^2(\Omega)$ is defined by

$$h^2(\Omega) := \left\{ f \text{ harmonic on } \Omega : \int_{\partial\Omega} |f(z)|^2\, d\sigma(z) := \limsup_{\varepsilon \to 0^+} \int_{\delta=\varepsilon} |f(z)|^2\, d\sigma(z) < \infty \right\}.$$

There always exists a positive constant ε_0 depending on Ω such that the following norms are equivalent

$$\left(\int_{\partial\Omega} |f(z)|^2 \, d\sigma(z)\right)^{\frac{1}{2}} \quad \text{and} \quad \left(\sup_{0<\varepsilon<\varepsilon_0} \int_{\delta=\varepsilon} |f(z)|^2 \, d\sigma(z)\right)^{\frac{1}{2}},$$

for $f \in h^2(\Omega)$. Moreover, there is a constant C depending on Ω such that

$$\int_{\Omega} |f(z)|^2 \, dV(z) \leqslant C \int_{\partial\Omega} |f(z)|^2 \, d\sigma(z), \tag{3.4.1}$$

for any $f \in h^2(\Omega)$. That is, the L^2-norm is dominated by the Hardy space norm for functions in $h^2(\Omega)$. Finally, we will need the following result, see [CF11, Lemma 2.2].

Lemma 3.4.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. For any harmonic function u on Ω,*

$$\limsup_{\varepsilon\to 0^+} \int_{\delta=\varepsilon} |u(z)|^2 \, d\sigma(z) = \limsup_{r\to 1^-} (1-r) \int_{\Omega} |u(z)|^2 \, \delta^{-r}(z) \, dV(z). \tag{3.4.2}$$

We are ready to proceed Theorem 3.1.1.

Proof of Theorem 3.1.1. Let ε_0 and c_1 be positive constants such that

$$\sup_{0<\varepsilon<\varepsilon_0} \int_{\delta=\varepsilon} |f(z)|^2 \, d\sigma(z) \leqslant c_1 \int_{\partial\Omega} |f(z)|^2 \, d\sigma(z), \forall f \in h^2(\Omega).$$

We first assume that $c_0\delta(w) < \varepsilon_0$, with $c_0 := (e+1)/(e-1)$. By Theorem 2.2.6,

$$\begin{aligned} \int_{\{G(\cdot,w)<-1\}} |K(z,w)|^2 \, dV(z) &\leqslant \int_{\{\delta(\cdot)\leqslant c_0\delta(w)\}} |K(z,w)|^2 \, dV(z) \\ &\leqslant \int_0^{c_0\delta(w)} \left(\int_{\delta=\varepsilon} |K(z,w)|^2 \, d\sigma(z)\right) d\varepsilon \qquad (3.4.3) \\ &\leqslant c_0 c_1 \delta(w) \int_{\partial\Omega} |K(z,w)|^2 \, d\sigma(z). \end{aligned}$$

By using the inequality (3.3.3),

$$\int_{\{G(\cdot,w)<-1\}} |K(z,w)|^2 \, dV(z) \geqslant e^{-2n} K(w,w). \tag{3.4.4}$$

Combining (3.4.3) with (3.4.4), we conclude that

$$\|K(\cdot,w)\|_{L^2(\partial\Omega)} \geqslant C_1\sqrt{\frac{K(w,w)}{\delta(w)}},$$

for a positive constant C_1 depending on Ω.

For the case $c_0\delta(w) \geqslant \varepsilon_0$, using (3.4.1) we have

$$\begin{aligned}\int_{\partial\Omega} |K(z,w)|^2\, d\sigma(z) &\geqslant C\int_{\Omega} |K(z,w)|^2\, dV(z)\\ &= CK(w,w)\\ &\geqslant \frac{C\varepsilon_0}{c_0}\frac{K(w,w)}{\delta(w)}.\end{aligned}$$

Therefore we have proved the left-hand side of (3.1.1).

We now turn to the proof of the right-hand side. Since Ω is convex, the function $-\delta$ is convex on Ω, and is thus also plurisubharmonic on Ω, see e.g. [AK85]. By applying Proposition 3.2.1 with $\psi(z) := -r\log(\delta(z))$, we have

$$(1-r)\int_{\Omega} |P(f)(z)|^2\,\delta^{-r}(z)\, dV(z) \leqslant \int_{\Omega} |f(z)|^2\,\delta^{-r}(z)\, dV(z)\,, \tag{3.4.5}$$

for any $0 < r < 1$ and any measurable function f. Inserting

$$f(z) := \chi_{\{G(\cdot,w)<-t\}}(z)\, K_{\{G(\cdot,w)<-t\}}(z,w)$$

into (3.4.5), we obtain that

$$(1-r)\int_{\Omega} |K(z,w)|^2\,\delta^{-r}(z)\, dV(z) \leqslant \int_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 \delta^{-r}(z)\, dV(z)\,, \tag{3.4.6}$$

for any $t > 0$. Using Lemma 3.4.1, it continues as

$$\begin{aligned}\int_{\partial\Omega} |K(z,w)|^2\, d\sigma(z) &= \limsup_{r\to1^-}(1-r)\int_{\Omega} |K(z,w)|^2\,\delta^{-r}(z)\, dV(z)\\ &\leqslant \int_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 \delta^{-1}(z)\, dV(z)\\ &\leqslant \frac{e^t+1}{e^t-1}\delta^{-1}(w)\int_{\{G(\cdot,w)<-t\}} \left|K_{\{G(\cdot,w)<-t\}}(z,w)\right|^2 dV(z)\end{aligned}$$

$$= \frac{e^t+1}{e^t-1}\delta^{-1}(w)\, K_{\{G(\cdot,w)<-t\}}(w,w)$$
$$\leqslant \frac{e^t+1}{e^t-1}e^{2nt}\frac{K(w,w)}{\delta(w)}.$$

The desired inequality then follows by noting that

$$\inf\left\{\left(e^t+1\right)e^{2nt}/\left(e^t-1\right) : t>0\right\} < 4en+1.$$

□

Remark 3.4.2. Since the constant $C_2 = \sqrt{4en+1}$ depends only on the dimension n, it suggests for example a study of the sharp estimates in Theorem 3.1.1.

Remark 3.4.3. The idea of plugging the function f into the Bergman projection follows from a remark in [Che17a]. It is clear that the method used in the proof of Theorem 3.1.1 can be extended to domains having a plurisubharmonic defining function, such as strongly pseudoconvex domains and Kohn special domains defined by

$$\Omega_F := \left\{z\in\mathbb{C}^n : |f_1(z)|^2+\ldots+|f_m(z)|^2<1\right\},$$

where $F=(f_1,\ldots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ is a holomorphic map, see [CF12]. It is known that for strongly pseudoconvex domains and Kohn special domains, $\delta(z)\approx\delta(w)$ for $z\in\{G(\cdot,w)<-1\}$, see [DH00, CF12, CF11], also Proposition 2.2.4. As a result, the estimate $\|K(\cdot,w)\|_{L^2(\partial\Omega)}\approx(K(w,w)/\delta(w))^{1/2}$ holds true for these domains. For a general domain admitting a plurisubharmonic defining function, we may use the estimates (2.2.8) in Theorem 2.2.6, which involve the logarithmic terms. To be precise, let us state these as the following theorem.

Theorem 3.4.4. *Let Ω be a bounded domain with C^∞-boundary.*

1. *If Ω is either a strongly pseudoconvex domain or a Kohn special domain then there exist positive constants C_1 and C_2 such that for any $w\in\Omega$,*

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)}}\leqslant\|K(\cdot,w)\|_{L^2(\partial\Omega)}\leqslant C_2\sqrt{\frac{K(w,w)}{\delta(w)}}.$$

2. *If Ω is a pseudoconvex domain having a plurisubharmonic defining function then there exist positive constants C_1 and C_2 such that for any $w\in\Omega$,*

$$C_1\sqrt{\frac{K(w,w)}{\delta(w)\,|\log\delta(w)|^n}}\leqslant\|K(\cdot,w)\|_{L^2(\partial\Omega)}\leqslant C_2\sqrt{\frac{K(w,w)\,|\log\delta(w)|}{\delta(w)}}.$$

Chapter 4

Bergman-Toeplitz operators on weakly pseudoconvex domains [1]

4.1 Introduction

In this chapter we study L^p regularity of Bergman-Toeplitz operators. We prove that for certain classes of pseudoconvex domains of finite type, the Bergman-Toeplitz operator T_ψ with symbol $\psi = K^{-\alpha}$ maps from L^p to L^q continuously with $1 < p \leqslant q < \infty$ if and only if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$, where K is the Bergman kernel on diagonal. This work generalises the results on strongly pseudoconvex domains by Čučković and McNeal, and Abate, Raissy and Saracco.

Let Ω be a bounded domain in $\mathbb{C}^n$ with the boundary $b\Omega$. A fundamental object associated to Ω is the Bergman projection P, that is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace of square-integrable holomorphic functions on Ω. The Bergman projection can be expressed via the integral representation

$$Pu(z) = \int_\Omega K(z,w)u(w)dV(w)\,,$$

where dw is the Lebesgue measure on Ω and the integral kernel K is called the Bergman kernel. It is well known that the Bergman projection is bounded from $L^p(\Omega)$ to $L^p(\Omega)$ with $1 < p < \infty$ on some classes of pseudoconvex domains of finite type such as strongly pseudoconvex domains [PS77], convex domains of finite type [MS94], pseudoconvex domains of finite type in $\mathbb{C}^2$ [NRSW89b] (see also [McN94a, KR]).

[1] This chapter contains the material from the paper: T.V. Khanh, J. Liu and P.T. Thuc, Bergman-Toeplitz operators on weakly pseudoconvex domains, *Math. Z.*, 2018: DOI 10.1007/s00209-018-2096-z.

Let $\psi \in L^\infty(\Omega)$, the Bergman-Toeplitz operator with symbol ψ is defined by

$$T_\psi(f)(z) := P(\psi f)(z) = \int_\Omega K(z,w)\psi(w) f(w)\, dV(w).$$

The study of the Bergman-Toeplitz operators has become a central topic since it is at the interface of many important fields in algebra and analysis, e.g., C^*-algebra, operator theory, harmonic analysis, pseudodifferential operators, and several complex variables (see [BS06, SSU89, Upm96] and references therein). In this work, we focus on the "gain" L^p-estimate property of these operators by the effect of their symbols. It is clear that $T_\psi : L^p(\Omega) \to L^p(\Omega)$ continuously if $P : L^p(\Omega) \to L^p(\Omega)$ continuously, as $\psi \in L^\infty(\Omega)$. In order to improve the regularity of T_ψ in L^p spaces, one should compensate by choosing ψ such as $\psi(z) \to 0$ as $z \to b\Omega$. Working on strongly pseudoconvex domains, Čučković and McNeal [CM06] study this gain property by choosing $\psi = \delta^\eta$ with $\eta > 0$, where $\delta(\cdot) = d(\cdot, \partial\Omega)$ is the Euclidean distance from the boundary. Precisely, they prove the following result:

Theorem 4.1.1 (Čučković and McNeal [CM06])**.** *Let Ω be a smooth, bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ and let $\eta \geqslant 0$.*

1. *If $0 \leqslant \eta < n+1$ and $1 < p < \infty$, then*
   - *(i) if $\frac{n+1}{n+1-\eta} < \frac{p}{p-1}$, then $T_{\delta^\eta} : L^p(\Omega) \to L^q(\Omega)$ continuously, where $\frac{1}{q} = \frac{1}{p} - \frac{\eta}{n+1}$;*
   - *(ii) if $\frac{n+1}{n+1-\eta} \geqslant \frac{p}{p-1}$, then $T_{\delta^\eta} : L^p(\Omega) \to L^q(\Omega)$ continuously, for all $p \leqslant q < \infty$.*
2. *If $\eta \geqslant n+1$, then $T_{\delta^\eta} : L^1(\Omega) \to L^\infty(\Omega)$ continuously.*

Later on, Abate, Raissy and Saracco [ARS12] show that the gain exponents in Part (1.i) of Theorem 4.1.1 are also optimal by using geometric characterisation of Carleson measures in term of the intrinsic Kobayashi geometry. In fact, they prove:

Theorem 4.1.2 (Abate, Raissy and Saracco [ARS12])**.** *Let Ω be a smooth, bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ and let $1 < p < q < \infty$ and $\eta \geqslant 0$. Then $T_{\delta^\eta} : L^p(\Omega) \to L^q(\Omega)$ continuously if and only if $\frac{\eta}{n+1} \geqslant \frac{1}{p} - \frac{1}{q}$.*

A crucial ingredient in the proof of Theorem 4.1.1 is the precise information on the Bergman kernel established by Fefferman [Fef74] on strongly pseudoconvex domains. Although, the authors in [CM06] commented *"our proof of Theorem 4.1.1 goes through, with minimal changes, on other classes of domains where good estimates on the Bergman kernel are known, e.g., finite type domains in $\mathbb{C}^2$, convex* *domains of finite type in $\mathbb{C}^n$*", we observe that for weakly pseudoconvex domains, the "good estimates" on Bergman kernels (if established) depend on the multi-type of boundary points, then the symbol should also depend on this multi-type (the multi-type is uniform on strongly pseudocovex domains). Thus the symbol δ^η with a fixed η is not a suitable candidate for the study of gain L^p estimates of Toeplitz operators on weakly pseudoconvex domains. For this reason, we shall work on the symbol $K^{-\alpha}(z,z)$ for some constant $\alpha > 0$ instead of the symbol $\delta^\eta(z)$, and then generalise Čučković and McNeal's result in [CM06] and Abate, Raissy and Saracco's result in [ARS12] for a large class of pseudocovex domains of finite type (see Theorem 4.1.3). In particular, Bergman kernels in this class have good estimates, called "sharp $\mathcal{B}$-type". Additionally, we also provide an upper-bound for the norm $\|T_\psi\|_{L^p(\Omega)\to L^q(\Omega)}$. The upper-bound for the norm of the Bergman projection has been given by Zhu [Zhu06] on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, that is

$$\|P\|_{L^p(\mathbb{B})\to L^p(\mathbb{B})} \leqslant C\frac{p^2}{p-1}, \quad \text{for } 1 < p < \infty.$$

Recently, Čučković [Cuc17] obtains this upper-bound for strongly pseudoconvex domains for a different constant C. Although we will not study it in this thesis, we would like to remark that the control of L^p norms for the Bergman projection can help us to obtain further endpoint regularity results.

Our main result of this chapter is stated as follows:

Theorem 4.1.3. *Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies one of the following conditions:*

(a) Ω is a strongly pseudoconvex domain;

(b) Ω is a pseudoconvex domain of finite type and $n = 2$;

(c) Ω is a convex domain of finite type;

(d) Ω is a decoupled domain of finite type;

(e) Ω is a pseudoconvex domain of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues.

Then we have the following conclusions:

1. *The Bergman-Toeplitz operator T_ψ with symbol $\psi(z) = K^{-\alpha}(z,z)$ maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously with $1 < p \leqslant q < \infty$ if and only if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$. Additional, if $T_\psi : L^p(\Omega) \to L^q(\Omega)$ continuously, then*

$$\|T_\psi\|_{L^p(\Omega)\to L^q(\Omega)} \leqslant C_\Omega \left(\frac{p}{p-1} + q\right)^{1-\frac{1}{p}+\frac{1}{q}},$$

where the constant C_Ω depends only on the domain Ω.

2. *The Bergman-Toeplitz operator T_ψ with symbol $\psi(z) = K^{-1}(z,z)$ maps from $L^1(\Omega)$ to $L^\infty(\Omega)$ continuously.*

When Ω is a strongly pseudoconvex domain with smooth boundary, from the asymptotic estimates of the Bergman kernel by Fefferman [Fef74], one has

$$C_1|r(z)|^{-(n+1)} \leqslant K(z,z) \leqslant C_2|r(z)|^{-(n+1)} \quad \text{for all } z \in \Omega, \tag{4.1.1}$$

where r is a defining function of Ω satisfying $c_1\delta(z) \leqslant |r(z)| \leqslant c_2\delta(z)$, $\forall z \in \Omega$, for two constants c_1, c_2. Using (4.1.1), one can see that for strongly pseudoconvex domains, the conclusions for case (a) in Theorem 4.1.3 are equivalent to those in Theorems 4.1.1 and 4.1.2.

The proof of Theorem 4.1.3 is a consequence of the results of Theorems 4.2.7, 4.3.1 and 4.4.1 below. In Theorems 4.2.7 and 4.3.1, we give the statement in an abstract setting of domains. To be more specific, if the Bergman kernel satisfies the good estimate (named *sharp $\mathcal{B}$-type*), we prove in Theorems 4.2.7 that the Bergman-Toeplitz operator T_ψ with the symbol $\psi(z) \leqslant K^{\frac{1}{q}-\frac{1}{p}}(z,z)$ (almost everywhere) maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously with $1 < p \leqslant q < \infty$. By an additional geometric hypothesis, in Theorem 4.3.1 we prove conversely that if $T_\psi : L^p(\Omega) \to L^q(\Omega)$ continuously with $\psi(z) = K^{-\alpha}(z,z)$, then $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$. Finally in Theorem 4.4.1, we verify that all domains listed in Theorem 4.1.3 satisfy the hypotheses of both Theorems 4.2.7 and 4.3.1 by using the work of Fefferman [Fef74], Catlin [Cat89], McNeal [McN94b, McN91], McNeal and Stein [MS94], and Cho [Cho96, Cho02]. We remark that our work may extend to other classes of pseudoconvex domains, e.g., the h-extendible domains [Yu94]. However, to the best of our knowledge, all the domains whose Bergman kernel is of the desired "good estimates" have been listed in Theorem 4.1.3.

Notations: Throughout this chapter, we use letter c and C to denote universal positive constants that only depend on the domain Ω (e.g. n and the type of Ω) and $\|\psi\|_{L^\infty(\Omega)}$, but may change from place to place. We also denote by $r_\Omega(z)$ and K_Ω the negative distance function and the Bergman kernel associated to Ω, respectively.

4.2 L^p-L^q boundedness of Bergman-Toeplitz operators

In this section, we introduce the notion of "sharp $\mathcal{B}$-type". Heuristically, if a Bergman kernel is of this type, then it has good estimates that ensure the Bergman projection is self-bounded in L^p. This sharp $\mathcal{B}$-type condition unifies all good estimates established by many authors on strongly pseudoconvex domains, pseudoconvex domains of finite type in $\mathbb{C}^2$, convex domains of finite type, and etc.

Let Ω' be a bounded domain in $\mathbb{C}^n$. For $z' \in \overline{\Omega'}$ near the boundary $b\Omega'$, a family of functions $\mathcal{B} = \{b_j(z', \cdot)\}_{j=1}^n$ is called a *$\mathcal{B}$-system* at z', if there exist a neighbourhood U of z', a positive integer $m \geqslant 2$ such that $\forall\ w' \in U$,

$$b_1(z', w') := \frac{1}{\delta(z', w')} \quad \text{and} \quad b_j(z', w') := \sum_{k=2}^{m} \left(\frac{A_{jk}\left(z'\right)}{\delta(z', w')} \right)^{\frac{1}{k}}, \quad \text{for } j = 2, ..., n,$$

where $\{A_{jk} : U \to \mathbb{R}^{\geqslant 0}\}$ are bounded functions, and $\delta(z', w')$ is the pseudo-distance between z' and w', given by

$$\delta(z', w') = |r_{\Omega'}(z')| + |r_{\Omega'}(w')| + |z'_1 - w'_1| + \sum_{l=2}^{n} \sum_{s=2}^{m} A_{ls}(z') \left|z'_l - w'_l\right|^s, \quad (4.2.1)$$

under a proper system of coordinates, see [MS94].

Let us start with the definition of sharp $\mathcal{B}$-type at a point near the boundary.

Definition 4.2.1. The Bergman kernel $K_{\Omega'}$ is said to be of *$\mathcal{B}$-type at $z' \in \overline{\Omega'}$ near the boundary* $b\Omega'$ if there exist positive constants c and C such that for any $w' \in \Omega' \cap \mathbb{B}(z', c)$,

$$|K_{\Omega'}(z', w')| \leqslant C \prod_{j=1}^{n} b_j^2(z', w').$$

We also say that $K_{\Omega'}$ is of *sharp $\mathcal{B}$-type at $z' \in \overline{\Omega'}$ near the boundary* $b\Omega'$ if $K_{\Omega'}$ is of $\mathcal{B}$-type and has the sharp lower-bound on diagonal, i.e.,

$$C^{-1} \prod_{j=1}^{n} b_j^2(z', z') \leqslant K_{\Omega'}(z', z') \leqslant C \prod_{j=1}^{n} b_j^2(z', z').$$

Next we give the definition of (global) sharp $\mathcal{B}$-type on a domain Ω.

Definition 4.2.2. We say that a kernel K_Ω associated to a domain Ω is of sharp $\mathcal{B}$-type if

(i) $K_\Omega \in C(\overline{\Omega} \times \overline{\Omega} \backslash \Delta_b)$, where $\Delta_b := \{(z,z) : z \in b\Omega\}$; and

(ii) For any $z \in \overline{\Omega}$ near the boundary $b\Omega$, there exists a biholomorphism Φ_z whose holomorphic Jacobian is uniformly nonsingular in the sense that

$$C^{-1} \leqslant |\det J_{\mathbb{C}}\Phi_z(w)| \leqslant C,$$

for all w in a neighbourhood of z, so that the Bergman kernel $K_{\Omega'}$ associated to the domain $\Omega' := \Phi_z(\Omega)$ is of sharp $\mathcal{B}$-type at $z' := \Phi_z(z)$.

Remark 4.2.3. The $\mathcal{B}$-type kernel condition is originally introduced by McNeal and Stein [MS94]. In [MS94], they prove that if the Bergman kernel associated to a convex domain of finite type is of $\mathcal{B}$-type, then the Bergman projection is bounded in L^p for $1 < p < \infty$.

The sharp $\mathcal{B}$-type implies the following integral estimate, which will be used subsequently.

Proposition 4.2.4. *Let Ω be a domain in $\mathbb{C}^n$. Assume that the Bergman kernel K_Ω is of sharp $\mathcal{B}$-type. Then, for each $z_0 \in b\Omega$, there is a neighbourhood U of z_0 such that for any $a \geqslant 1$ and $-1 < b < 2a-2$,*

$$\begin{aligned} I_{a,b}(z) &:= \int\limits_{\Omega \cap U} |K_\Omega(z,w)|^a |r_\Omega(w)|^b \, dV(w) \\ &\leqslant C \frac{2a-1}{(2a-2-b)(b+1)} K_\Omega(z,z)^{a-1} |r_\Omega(z)|^b \end{aligned} \tag{4.2.2}$$

for every $z \in \Omega \cap U$.

Remark 4.2.5. To obtain (4.2.2) for $a = 1$, it suffices to assume that K_Ω is of $\mathcal{B}$-type, instead of sharp $\mathcal{B}$-type.

Proof. We choose U a small neighbourhood of z_0 such that $\Phi_z(U) \subset \mathbb{B}(z',c)$ for any $z \in U$, where the ball $\mathbb{B}(z',c)$ and the biholomorphism Φ_z are given in Definition 4.2.1 and 4.2.2, respectively. By the invariant formula of the Bergman kernel

$$K_\Omega(z,w) = \det J_{\mathbb{C}}\Phi_z(z) K_{\Phi_z(\Omega)}(\Phi_z(z), \Phi_z(w)) \overline{\det J_{\mathbb{C}}\Phi_z(w)}, \tag{4.2.3}$$

where $C^{-1} \leqslant |\det J_{\mathbb{C}}\Phi_z(w)| \leqslant C$ for all $w \in U$, we have

$$I_{a,b}(z) \leqslant C \int\limits_{\Omega' \cap \mathbb{B}(z',c)} |K_{\Omega'}(z',w')|^a |r_{\Omega'}(w')|^b dV(w'),$$

where $\Omega' = \Phi_z(\Omega)$, $z' = \Phi_z(z)$, $w' = \Phi_z(w)$ and $r_{\Omega'}(w') = r_\Omega(\Phi_z^{-1}(w'))$. Thus, in order to show (4.2.2), it suffices to show that

$$\begin{aligned} I'_{a,b}(z') &:= \int\limits_{\Omega' \cap \mathbb{B}(z',c)} |K_{\Omega'}(z',w')|^a \, |r_{\Omega'}(w')|^b \, dV(w') \\ &\leqslant C\frac{2a-1}{(2a-2-b)(b+1)} K_{\Omega'}(z',z')^{a-1} \, |r_{\Omega'}(z')|^b, \end{aligned}$$

in which $K_{\Omega'}$ is a kernel of sharp $\mathcal{B}$-type at z'. Here and in what follows in this proof, we omit the subscript Ω' and the superscript *prime* for convenience.

It is clear that $b_j(z,w) \leqslant b_j(z,z)$ and $b_1(z,w) \leqslant \dfrac{1}{|r(z)| + |r(w)|}$ and hence

$$|K(z,w)| \leqslant CK(z,z)\left(\frac{|r(z)|}{|r(z)| + |r(w)|}\right)^2, \quad \text{for all } w \in \Omega \cap \mathbb{B}(z,c). \tag{4.2.4}$$

Since $a \geqslant 1$, from (4.2.4) one has

$$I_{a,b}(z) \leqslant C|r(z)|^{2a-2} K^{a-1}(z,z) J_{a,b}(z), \tag{4.2.5}$$

where

$$J_{a,b}(z) = \int\limits_{\Omega} \frac{|r(w)|^b}{(|r(z)| + |r(w)|)^{2a-2}} |K(z,w)| dV(w).$$

We remark that in the special case when $a = 1$, the inequality (4.2.5) is trivial. The sharp $\mathcal{B}$-type hypothesis is merely needed in obtaining (4.2.4). Hereafter, we only require the $\mathcal{B}$-type hypothesis.

We shall use the polar coordinates in $z_k - w_k$ with $\rho_k := |z_k - w_k|$ for $k = 2, \ldots, n$ and the change of variables $\rho_1 := -r(w)$, $\xi := |z_1 - w_1|$, to estimate the integral $J_{a,b}$. First, we define the increasing sequence $\{M_j(z,\rho)\}_{j=1}^n$ by induction: $M_1(z,\rho) = |r(z)| + \rho_1$ and

$$M_j(z,\rho) = M_{j-1}(z,\rho) + \sum_{k=2}^m A_{jk}(z)\, \rho_j^k, \quad \text{for} \quad j = 2, \ldots, n.$$

We remark that M_j is independent of ρ_l for $l = j+1, j+2, \ldots, n$. Then, observe that $b_j(z,w) \leqslant m\rho_j^{-1}$ and also

$$b_j(z,w) = \sum_{k=2}^m \left(\frac{A_{jk}(z)}{\delta(z,w)}\right)^{\frac{1}{k}} \leqslant \sum_{k=2}^m \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)}\right)^{\frac{1}{k}},$$

since $\delta(z,w) = \xi + M_n(z,\rho)$ is greater than both $A_{jk}\rho_j^k$ and $M_{j-1}(z,\rho)$. Therefore, the integral $J_{a,b}$, after changing coordinates, can be estimated as

$$\begin{aligned} J_{a,b}(z) &\leqslant C \int_0^\infty \cdots \int_0^\infty \frac{\rho_1^b}{M_1(z,\rho)^{2a-2}} \prod_{j=2}^n \left[\sum_{k=2}^m \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)} \right)^{\frac{1}{k}} \right] \\ &\quad \times \frac{1}{(\xi + M_n(z,\rho))^2} d\rho_1 \cdots d\rho_n d\xi \\ &\leqslant C \int_0^\infty \cdots \int_0^\infty \frac{\rho_1^b}{M_1(z,\rho)^{2a-2}} \prod_{j=2}^n \left[\sum_{k=2}^m \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)} \right)^{\frac{1}{k}} \right] \frac{1}{M_n(z,\rho)} d\rho_1 \cdots d\rho_n, \end{aligned}$$

where the second inequality follows by integrating with respect to ξ. We compute this integral by the following claim.

Claim: *For $j = 2, \ldots, n$, we have*

$$\left[\sum_{k=2}^m \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)} \right)^{\frac{1}{k}} \right] \int_0^\infty \frac{d\rho_j}{M_j(z,\rho)} \leqslant \frac{C}{M_{j-1}(z,\rho)}. \tag{4.2.6}$$

Proof of the claim. Since $M_j(z,\rho) \geqslant M_{j-1}(z,\rho) + A_{jk}(z)\rho_j^k$ for all $k = 2, \ldots, m$, the LHS of (4.2.6) is bounded by

$$\sum_{k=2}^m \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)} \right)^{\frac{1}{k}} \int_0^\infty \frac{d\rho_j}{M_{j-1}(z,\rho) + A_{jk}(z)\rho_j^k}.$$

For each $k = 2, \ldots, m$, if $A_{jk}(z) = 0$ then there is nothing to do; otherwise, we use the change of the coordinate $x := A_{jk}^{\frac{1}{k}}(z)\rho_j$ to get

$$\begin{aligned} \left(\frac{A_{jk}(z)}{M_{j-1}(z,\rho)} \right)^{\frac{1}{k}} \int_0^\infty \frac{d\rho_j}{M_{j-1}(z,\rho) + A_{jk}(z)\rho_j^k} &= \frac{1}{(M_{j-1}(z,\rho))^{\frac{1}{k}}} \int_0^\infty \frac{dx}{M_{j-1}(z,\rho) + x^k} \\ &\leqslant \frac{C}{M_{j-1}(z,\rho)}. \end{aligned}$$

This proves the claim. □

Coming back to the computation of $J_{a,b}(z)$, by the claim (4.2.6) and an induction argument from $j = n$ to $j = 2$, we have

$$J_{a,b}(z) \leqslant C \int_0^\infty \frac{\rho_1^b d\rho_1}{M_1(z,\rho)^{2a-1}} = C \int_0^\infty \frac{\rho_1^b d\rho_1}{(|r(z)| + \rho_1)^{2a-1}} = C|r(z)|^{b-2a+2}\kappa_{a,b}, \tag{4.2.7}$$

where

$$\kappa_{a,b} = \int_0^\infty \frac{t^b dt}{(1+t)^{2a-1}} \leqslant \int_0^1 t^b dt + \int_1^\infty t^{b-2a+1} dt = \frac{2a-1}{(2a-2-b)(b+1)}.$$

By (4.2.5) and (4.2.7), the conclusion of Proposition 4.2.4 follows. □

The following corollary is a combination of a generalised Schur's test (Theorem 4.5.1 below) and Proposition 4.2.4.

Corollary 4.2.6. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $1 < p \leqslant q < \infty$. Assume that the Bergman kernel K_Ω is of sharp $\mathcal{B}$-type and $\psi : \Omega \to \mathbb{C}$ satisfies*

$$|\psi(z)| \leqslant K_\Omega^{\frac{1}{q}-\frac{1}{p}}(z,z)$$

almost everywhere. Then, for each $z_0 \in b\Omega$ there exists a neighbourhood U of z_0 such that the Bergman-Toeplitz operator $T_{\psi,U}$ defined by

$$(T_{\psi,U}u)(z) := \int\limits_{\Omega\cap U} K_\Omega(z,w)\psi(w)u(w)dV(w), \quad \text{for } z \in \Omega \cap U,$$

maps from $L^p(\Omega \cap U)$ to $L^q(\Omega \cap U)$ continuously and

$$\|T_{\psi,U}u\|_{L^q(\Omega\cap U)} \leqslant C\left(\frac{p}{p-1} + q\right)^{1-\frac{1}{p}+\frac{1}{q}} \|u\|_{L^p(\Omega\cap U)}$$

for all $u \in L^p(\Omega \cap U)$, where C is independent of p, q and u.

Proof. Let $\alpha = \frac{1}{p'} = 1 - \frac{1}{p}$ and $0 < \beta < \min\{\frac{1}{p'}, \frac{1}{q}\}$. Apply Proposition 4.2.4 twice for the pair (a,b) replaced by $(1, -\beta p') = (\alpha p', -\beta p')$ and by $(\frac{q}{p}, -\beta q) = ((1-\alpha)q, -\beta q)$, we have

$$\begin{aligned} I_{(1,-\beta p')}(z) &= \int\limits_{\Omega\cap U} |K_\Omega(z,w)|^{\alpha p'} \left(|r_\Omega(w)|^{-\beta}\right)^{p'} dV(w) \leqslant C_1 \left(|r_\Omega(z)|^{-\beta}\right)^{p'}, \\ I_{(\frac{q}{p},-\beta q)}(w) &= \int\limits_{\Omega\cap U} |K_\Omega(z,w)|^{(1-\alpha)q} \left(|r_\Omega(z)|^{-\beta}\right)^{q} dV(z) \\ &\leqslant C_2 \left(K_\Omega(w,w)^{\frac{(1-\alpha)q-1}{q}} |r_\Omega(w)|^{-\beta}\right)^q, \end{aligned}$$

where

$$C_1 = \frac{C}{\beta p'(1-\beta p')} \quad \text{and} \quad C_2 = C\frac{\frac{2q}{p} - 1}{(\frac{2q}{p} - 2 + \beta q)(1-\beta q)}.$$

Notice that, by the hypothesis of ψ,

$$(|r_\Omega(w)|^{-\beta})^{-1}K_\Omega(w,w)^{\frac{(1-\alpha)q-1}{q}}|r_\Omega(w)|^{-\beta}\psi(w) = \psi(w)K_\Omega(w,w)^{\frac{1}{p}-\frac{1}{q}} \leqslant 1$$

for all $w \in \Omega$. Thus, the hypothesis of Theorem 4.5.1 holds for

$$X = Y = \Omega \cap U, \qquad h_1(w) = |r_\Omega(w)|^{-\beta},$$

$$h_2(w) = K_\Omega(w,w)^{\frac{(1-\alpha)q-1}{q}}|r_\Omega(w)|^{-\beta}, \quad \text{and} \quad g(z) = |r_\Omega(z)|^{-\beta}.$$

Therefore $T_{\psi,U} : L^p(\Omega \cap U) \to L^q(\Omega \cap U)$ continuously and

$$\|T_{\psi,U}u\|_{L^q(\Omega\cap U)} \leqslant C \inf_{0<\beta<\min\{\frac{1}{p'},\frac{1}{q}\}}\{\tau(\beta)\}\|u\|_{L^p(\Omega\cap U)},$$

where

$$\tau(\beta) = \left(\frac{1}{\beta p'(1-\beta p')}\right)^{\frac{1}{p'}}\left(\frac{\left(\frac{2q}{p}-2\right)+1}{\left(\left(\frac{2q}{p}-2\right)+\beta q\right)(1-\beta q)}\right)^{\frac{1}{q}}.$$

We finish the proof of this corollary by showing that

$$\inf_{0<\beta<\min\{\frac{1}{p'},\frac{1}{q}\}}\tau(\beta) \leqslant 4\left(\frac{p}{p-1}+q\right)^{1-\frac{1}{p}+\frac{1}{q}}. \tag{4.2.8}$$

We first get rid of term $\left(\frac{2q}{p}-2\right)$ in $(\cdots)^{\frac{1}{q}}$ by the inequality $\frac{x+a}{x+b} \leqslant \frac{a}{b}$ for $a \geqslant b > 0$ and $x \geqslant 0$ to obtain

$$\tau(\beta) \leqslant \left(\frac{1}{\beta p'(1-\beta p')}\right)^{\frac{1}{p'}}\left(\frac{1}{\beta q(1-\beta q)}\right)^{\frac{1}{q}}.$$

Then we choose $\beta = \frac{1}{p'+q} \leqslant \min\{\frac{1}{p'},\frac{1}{q}\}$. It follows that

$$\frac{1}{\beta p'(1-\beta p')} = \frac{1}{\beta q(1-\beta q)} = \frac{(p'+q)^2}{p'q} = \left(\frac{1}{p'}+\frac{1}{q}\right)(p'+q).$$

Therefore,

$$\tau\left(\frac{1}{p'+q}\right) \leqslant \left[\left(\frac{1}{p'}+\frac{1}{q}\right)(p'+q)\right]^{\frac{1}{p'}+\frac{1}{q}} \leqslant 4\,(p'+q)^{\frac{1}{p'}+\frac{1}{q}} = 4\left(\frac{p}{p-1}+q\right)^{1-\frac{1}{p}+\frac{1}{q}},$$

where the second inequality follows by $x^x \leqslant 4$ for $x = \frac{1}{p'}+\frac{1}{q} \in [0,2]$. This proves (4.2.8). ∎

The main result of this section is the following theorem, in which we prove the gain L^p-L^q estimate of Bergman-Toeplitz operators T_ψ.

Theorem 4.2.7. *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that the Bergman kernel K_Ω is of sharp $\mathcal{B}$-type.*

1. *If $|\psi(z)| \leqslant \left(K_\Omega(z,z)\right)^{\frac{1}{q}-\frac{1}{p}}$ almost everywhere with $1 < p \leqslant q < \infty$, then $T_\psi : L^p(\Omega) \to L^q(\Omega)$ continuously. Furthermore,*
$$\|T_\psi\|_{L^p(\Omega)\to L^q(\Omega)} \leqslant C_{\Omega,\psi}\left(\frac{p}{p-1}+q\right)^{1-\frac{1}{p}+\frac{1}{q}},$$
where $C_{\Omega,\psi}$ is independent of p and q.
2. *If $|\psi(z)| \leqslant \left(K_\Omega(z,z)\right)^{-1}$ almost everywhere, then $T_\psi : L^1(\Omega) \to L^\infty(\Omega)$ continuously.*

Proof. We choose a partition of unity $\{\chi_j\}_{j=0}^N$ and a covering $\{U_j\}_{j=0}^N$ to $\overline{\Omega}$ so that $\operatorname{supp}(\chi_j) \Subset U_j$, $U_0 \Subset \Omega$, $b\Omega \subset \bigcup_{j=1}^N U_j$, and the integral estimates in Proposition 4.2.4 hold on U_j for all $j = 1, \dots, N$. Denote by $\mathbf{1}_A$ the characteristic function of $A \subset \Omega$. So we can decompose $T_\psi u$ as

$$T_\psi u = \sum_{j=0}^N \chi_j T_\psi u = \chi_0 T_\psi u + \sum_{j=1}^N \chi_j T_\psi(u\mathbf{1}_{\Omega\cap U_j}) + \sum_{j=1}^N \chi_j T_\psi(u\mathbf{1}_{\Omega\setminus U_j}).$$

It follows

$$\|T_\psi u\|_{L^q(\Omega)} \leqslant \|\chi_0 T_\psi u\|_{L^q(\Omega)} + \sum_{j=1}^N \|\chi_j T_\psi(u\mathbf{1}_{\Omega\cap U_j})\|_{L^q(\Omega)} + \sum_{j=1}^N \|\chi_j T_\psi(u\mathbf{1}_{\Omega\setminus U_j})\|_{L^q(\Omega)}. \tag{4.2.9}$$

In order to estimate $\|\chi_0 T_\psi u\|_{L^q(\Omega)}$ and $\|\chi_j T_\psi(u\mathbf{1}_{\Omega\setminus U_j})\|_{L^q(\Omega)}$ with $j \geqslant 1$, we use the continuity up to the off-diagonal boundary of the Bergman kernel. Indeed,

$$K_\Omega \in C\left(\left(\overline{\Omega}\times\overline{\Omega}\right)\setminus\{(z,z) : z \in b\Omega\}\right)$$

implies that there exists a positive constant C such that

$$|K_\Omega(z,w)| \leqslant C \text{ for all } (z,w) \in \left(\bigcup_{j=1}^N \left((\operatorname{supp}(\chi_j)) \times (\Omega\setminus\bar{U}_j)\right) \bigcup (\operatorname{supp}(\chi_0)) \times \Omega)\right).$$

Thus, for $j = 1, \dots, N$, and $z \in \Omega$, we have

$$\begin{aligned}\left|\left(\chi_j T_\psi(u\mathbf{1}_{\Omega\setminus U_j})\right)(z)\right| &= \left|\int_\Omega \chi_j(z)K(z,w)\mathbf{1}_{\Omega\setminus U_j}u(w)\psi(w)dV(w)\right| \\ &\leqslant C\int_\Omega |u(w)|dV(w),\end{aligned}$$

and hence

$$\|\chi_j T_\psi(u\mathbf{1}_{\Omega\setminus U_j})\|_{L^q(\Omega)} \leqslant C\|u\|_{L^p(\Omega)}, \tag{4.2.10}$$

for all $1 \leqslant p \leqslant q \leqslant \infty$, where C is independent of p and q. Analogously,

$$\|\chi_0 T_\psi(u)\|_{L^q(\Omega)} \leqslant C\|u\|_{L^p(\Omega)}. \tag{4.2.11}$$

To estimate the norm $\|\chi_j T_\psi(u\mathbf{1}_{\Omega\cap U_j})\|_{L^q(\Omega)}$ for $j = 1, \dots, N$, we combine the fact that

$$\begin{aligned}\|\chi_j T_\psi(u\mathbf{1}_{U_j})\|_{L^q(\Omega)} &\leqslant \left(\int_{\Omega\cap U_j}\left(\int_{\Omega\cap U_j}|K_\Omega(z,w)u(w)\psi(w)|dV(w)\right)^q dV(z)\right)^{\frac{1}{q}} \\ &= \|T_{\psi,U_j}u\|_{L^q(\Omega\cap U_j)}\end{aligned}$$

and Corollary 4.2.6 to yield

$$\|\chi_j T_\psi(u\mathbf{1}_{U_j})\|_{L^q(\Omega)} \leqslant C\left(\frac{p}{p-1}+q\right)^{1-\frac{1}{p}+\frac{1}{q}}\|u\|_{L^p(\Omega)}, \tag{4.2.12}$$

for given $1 < p \leqslant q < \infty$. From (4.2.9), (4.2.10), (4.2.11) and (4.2.12), we have the desired inequality

$$\|T_\psi u\|_{L^q(\Omega)} \leqslant C\left(\frac{p}{p-1}+q\right)^{1-\frac{1}{p}+\frac{1}{q}}\|u\|_{L^p(\Omega)},$$

for the given $1 < p \leqslant q < \infty$, provided that $u \in L^p(\Omega)$. This proves Part (1) in Theorem 4.2.7.

Similarly, the proof of Part (2) follows by (4.2.9), (4.2.10), (4.2.11) (for the choice $p = 1, q = \infty$) and

$$\|\chi_j T_\psi(u\mathbf{1}_{U_j})\|_{L^\infty(\Omega)} \leqslant \|u\|_{L^1(\Omega)}$$

since

$$|\psi(w)K_\Omega(z,w)| \leqslant |K_\Omega(z,w)|\, K_\Omega(z,z)^{-1} \leqslant C, \quad \text{(by (4.2.4))}$$

for any $z, w \in \Omega \cap U_j$ and $j = 1, \ldots, N$. This completes the proof of Theorem 4.2.7. □

The following corollary gives the L^p boundedness of the Bergman projection that follows immediately from Theorem 4.2.7 and using Remark 4.2.5 to avoid the sharp condition of the Bergman kernel.

Corollary 4.2.8. *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that the Bergman kernel K_Ω is of $\mathcal{B}$-type, that means: (i) $K_\Omega \in C(\overline{\Omega} \times \overline{\Omega} \backslash \Delta_b)$, and (ii) for any $z \in \overline{\Omega}$ near the boundary $b\Omega$ there exists a biholomorphism Φ_z depending on z but with holomorphic Jacobian uniformly nonsingular so that the Bergman kernel $K_{\Omega'}$ associated to the domain $\Omega' := \Phi_z(\Omega)$ is of $\mathcal{B}$-type at $z' := \Phi_z(z)$.*

Then the Bergman projection P is bounded in L^p, for $1 < p < \infty$, with the norm estimate

$$\|P\|_{L^p(\Omega)\to L^p(\Omega)} \leqslant C_\Omega \frac{p^2}{p-1},$$

where C_Ω is independent of p.

4.3 Sharp estimates of Bergman-Toeplitz operators

As we have already showed in §4.2 that $T_{K^{-\alpha}}$ is bounded from $L^p(\Omega)$ to $L^q(\Omega)$ if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$. In this section we show that this gain boundedness is sharp with an additional hypothesis to the $\mathcal{B}$-system.

Denote by

$$P_\lambda(z') = \{w' \in \mathbb{C}^n : |w'_j - z'_j| b_j(z', z') \leqslant \lambda, \text{for all } j = 1, 2 \ldots n\}$$

a $\mathcal{B}$-polydisc with the centre z' associated to the $\mathcal{B}$-system defined in §4.2.

Theorem 4.3.1. *Let Ω be a bounded smooth pseudoconvex domain in $\mathbb{C}^n$ such that the Bergman kernel is of sharp $\mathcal{B}$-type. Assume further that there are universal constants λ and C such that for any $z \in \Omega$ near the boundary $b\Omega$, after mapping by the biholomorphism Φ_z (in Definition 4.2.2), the $\mathcal{B}$-system associated to $z' := \Phi_z(z)$ has the property:*

$$P_\lambda(z') \subset \Omega' := \Phi_z(\Omega) \text{ and } K_{\Omega'}(w', w') \leqslant C K_{\Omega'}(z', z'),\ \forall w' \in P_\lambda(z').$$

Then, if $T_{K_\Omega^{-\alpha}} : L^p(\Omega) \to L^q(\Omega)$ continuously with $1 < p \leqslant q < \infty$ then $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$.

Proof. We may assume $\alpha > 0$. The proof is based on upper and lower estimates of $\int_\Omega |K_\Omega(w,z)|^2 K_\Omega(w,w)^{-\alpha} dV(w)$ for z approaching to the boundary.

We first give the upper bound of $\int_\Omega |K_\Omega(w,z)|^2 K_\Omega(w,w)^{-\alpha} dV(w)$ by using the sharp $\mathcal{B}$-type condition and the assumption $T_{K_\Omega^{-\alpha}} : L^p(\Omega) \to L^q(\Omega)$ continuously. Since $K_\Omega(w,z)$ is holomorphic in $w \in \Omega$,

$$K_\Omega(w,z) = P(K_\Omega(\cdot,z))(w) = \int\limits_\Omega K_\Omega(w,\xi) K_\Omega(\xi,z) dV(\xi)$$

and hence

$$\overline{K_\Omega(w,z)} = \overline{\int\limits_\Omega K_\Omega(w,\xi) K_\Omega(\xi,z) dV(\xi)} = \int\limits_\Omega K_\Omega(\xi,w) K_\Omega(z,\xi) dV(\xi).$$

Thus, we have

$$\begin{aligned}
\int\limits_\Omega \left|K_\Omega\left(w,z\right)\right|^2 K_\Omega^{-\alpha}(w,w) dw &= \int\limits_\Omega K_\Omega(w,z) K_\Omega^{-\alpha}(w,w) \left(\int\limits_\Omega K_\Omega(\xi,w) K_\Omega(z,\xi) d\xi \right) dw \\
&= \int\limits_\Omega \left(\int\limits_\Omega K_\Omega(\xi,w) K_\Omega(w,z) K_\Omega^{-\alpha}(w,w) dw \right) K_\Omega(z,\xi) d\xi \\
&= \int\limits_\Omega \Big(T_{K_\Omega^{-\alpha}}(K_\Omega(\cdot,z))(\xi) \Big) K_\Omega(z,\xi) d\xi \\
&\leqslant \|T_{K_\Omega^{-\alpha}}(K_\Omega(\cdot,z))\|_{L^q(\Omega)} \|K_\Omega(z,\cdot)\|_{L^{q'}(\Omega)},
\end{aligned} \tag{4.3.1}$$

where the last inequality follows from Hölder's inequality and $\frac{1}{q'} + \frac{1}{q} = 1$. Since $T_{K_\Omega^{-\alpha}}$ maps from $L^p(\Omega)$ to $L^q(\Omega)$ continuously then (4.3.1) continues as

$$\begin{aligned}
\int\limits_\Omega \left|K_\Omega\left(w,z\right)\right|^2 K_\Omega^{-\alpha}(w,w) dV(w) &\leqslant C \|K_\Omega(\cdot,z)\|_{L^p(\Omega)} \|K_\Omega(\cdot,z)\|_{L^{q'}(\Omega)} \\
&\leqslant C \left(K_\Omega(z,z)^{p-1}\right)^{\frac{1}{p}} \left(K_\Omega(z,z)^{q'-1}\right)^{\frac{1}{q'}} \qquad (4.3.2) \\
&= C K_\Omega(z,z)^{1-\frac{1}{p}+\frac{1}{q}}.
\end{aligned}$$

Here, the second inequality follows by using Proposition 4.2.4 twice for

$$(a,b) = (p,0) \text{ and } (a,b) = (q',0).$$

In order to get the lower bound of $\int_\Omega |K_\Omega(w,z)|^2 K_\Omega(w,w)^{-\alpha} dw$, we first use the invariant formula (4.2.3) and $|\det J_{\mathbb{C}}\Phi_z(w)| \geqslant C$ uniformly to get

$$\int_\Omega |K_\Omega(w,z)|^2 K_\Omega^{-\alpha}(w,w) dV(w) \geqslant C \int_{\Omega'} |K_{\Omega'}(w',z')|^2 K_{\Omega'}^{-\alpha}(w',w') dV(w').$$

By the hypothesis: there exists $\lambda > 0$ such that, if $w' \in P_\lambda(z')$ then $w' \in \Omega'$ and $K_{\Omega'}(w',w') \lesssim K_{\Omega'}(z',z')$, it follows

$$\begin{aligned}\int_{\Omega'} |K_{\Omega'}(w',z')|^2 K_{\Omega'}^{-\alpha}(w',w') dV(w') &\geqslant \int_{P_\lambda(z')} |K_{\Omega'}(w',z')|^2 K_{\Omega'}^{-\alpha}(w',w') dV(w') \\ &\gtrsim K_{\Omega'}^{-\alpha}(z',z') \int_{P_\lambda(z')} |K_{\Omega'}(w',z')|^2 dV(w') \\ &\gtrsim K_{\Omega'}^{-\alpha}(z',z') K_{\Omega'}^2(z',z') \mathrm{Vol}(P_\lambda(z')) \\ &\gtrsim K_{\Omega'}^{1-\alpha}(z',z').\end{aligned}$$

Here the third inequality follows by the sub-mean property and the last follows by

$$\begin{aligned}\mathrm{Vol}(P_\lambda(z')) &= \pi^n \lambda^{2n} \left(\prod_{j=1}^n b_j(z',z') \right)^{-2} \\ &\approx (K_{\Omega'}(z',z'))^{-1}.\end{aligned}$$

Since $K_\Omega(z,z) = |\det J_{\mathbb{C}}\Phi_z(z)|^2 K_{\Omega'}(z',z')$ (by the invariant formula) and

$$|\det J_{\mathbb{C}}\Phi_z(z)| \approx 1,$$

we get the lower bound

$$\int_\Omega |K_\Omega(w,z)|^2 K_\Omega^{-\alpha}(w,w) dV(w) \geqslant C K_\Omega^{1-\alpha}(z,z). \tag{4.3.3}$$

Therefore, by (4.3.3) and (4.3.2), we obtain that

$$K_\Omega^{1-\frac{1}{p}+\frac{1}{q}}(z,z) \geqslant C K_\Omega^{1-\alpha}(z,z),$$

and hence

$$K_\Omega^{\alpha-\frac{1}{p}+\frac{1}{q}}(z,z) \geqslant C.$$

Letting $z \to b\Omega$, we conclude that

$$\alpha \geqslant \frac{1}{p} - \frac{1}{q},$$

since $K_\Omega(z,z) \to \infty$ as $z \to b\Omega$. This proves Theorem 4.3.1. □

4.4 Proof of Theorem 4.1.3

As mentioned in §4.1, the proof of Theorem 4.1.3 is complete if we can verify all domains listed in Theorem 4.1.3 satisfy the hypothesis of Theorem 4.2.7 and 4.3.1. The main goal of this section is the following theorem.

Theorem 4.4.1. *Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies one of the following settings:*

(a) Ω is a strongly pseudoconvex domain;

(b) Ω is a pseudoconvex domain of finite type and $n = 2$;

(c) Ω is a convex domain of finite type;

(d) Ω is a decoupled domain of finite type;

(e) Ω is a pseudoconvex domain of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues.

Then the Bergman kernel $K_\Omega \in C^\infty(\overline{\Omega} \times \overline{\Omega} \backslash \Delta_b)$ and for each point z near the boundary, there is a biholomorphism Φ_z depending on z but with holomorphic Jacobian uniformly nonsingular and a $\mathcal{B}$-system associated to $z' := \Phi_z(z)$ such that the Bergman kernel $K_{\Omega'}$ associated to $\Omega' := \Phi_z(\Omega)$ has the following properties:

1. *$|K_{\Omega'}(z', w')| \lesssim \prod_{j=1}^n b_j^2(z', w')$, for $w' \in \Omega \cap \mathbb{B}(z', c)$. Here c is a universal constant.*

2. *$K_{\Omega'}(z', z') \approx \prod_{j=1}^n b_j^2(z', z')$.*

3. *There exist universal constants λ and C such that the $\mathcal{B}$-polysdisc $P_\lambda(z') \Subset \Omega'$ and $K_{\Omega'}(w', w') \leqslant C K_{\Omega'}(z', z')$, for any $w' \in P_\lambda(z')$.*

Proof. Inspired by the work of Kerzman [Ker72], Boas [Boa87] proves that the Bergman kernel associated to smooth, bounded, pseudoconvex domains of finite type is smooth up to off-diagonal boundary, i.e, $K_\Omega \in C^\infty(\bar{\Omega} \times \bar{\Omega} \backslash \Delta_b)$. Thus, we only need to verify the local properties after the bihomomorphic mapping Φ_z.

The details of the proof of conclusion (1)-(2) can be found in the papers by McNeal [McN94b, McN94a], Cho [Cho96, Cho02], Catlin [Cat89]. For example, the proof of the upper-bound estimate of the Bergman kernel, i.e. (1), has been established on strongly pseudoconvex domains [Fef74], pseudoconvex domains of finite type in $\mathbb{C}^2$ [NRSW89b, Theorem 3.1], convex domains of finite type [McN94b, Theorem 5.2], decoupled domains [McN91, Theorem 2], and pseudoconvex domains of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues [Cho96, Theorem 1] and [Cho02, Theorem 1.1]. Moreover, the sharp estimate of the Bergman kernel on the diagonal has been shown on strongly pseudoconvex domains [Fef74], pseudoconvex domains of finite type in $\mathbb{C}^2$ [Cat89, Theorem 2], convex domains of finite type [McN94b, Theorem 3.4 and Theorem 5.2], decoupled domains [McN91, Theorem 2], and pseudoconvex domains of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues [Cho94, Theorem 1]. The proof of (3) can be given as follows.

By the characteristic of domains listed in this theorem, one can construct a biholomorphism Φ_z associated to a given point z near the boundary of Ω such that the conclusion (1) and (2) hold for the $\mathcal{B}$-system $\{b_j(z',\cdot)\}_{j=1}^n$ associated to $z' = \Phi_z(z)$ defined as in §4.2 by

$$A_{jk}(z') = \sum_{k_1+k_2=k, k_1,k_2>0} \left| \frac{\partial^k r_{\Omega'}(z')}{\partial z'^{k_1}_j \partial \bar{z}'^{k_2}_j} \right|.$$

Here $r_{\Omega'} = r_\Omega \circ \Phi_z^{-1}$. The construction of Φ hinging on the nice geometric properties on these domains also gives us

$$\sum_{i,j=2}^n \sum_{k=2}^m \sum_{k_1+k_2=k, k_1,k_2>0} \left| \frac{\partial^k r_{\Omega'}(z')}{\partial z'^{k_i}_i \partial \bar{z}'^{k_j}_j} \right| |w'_i - z'_i|^{k_i} |w'_j - z'_j|^{k_j} \leqslant C \sum_{j=2}^n \sum_{k=2}^m A_{jk}(z') |w'_j - z'_j|^k, \tag{4.4.1}$$

for $w' \in \mathbb{B}(z', c)$. For the details of the construction of Φ_z, we refer to [McN03, Section 3] for strongly pseudoconvex domains, [Cat89, Section 1] for pseudoconvex domains of finite type in $\mathbb{C}^2$, [McN94b, Section 2] for convex domains of finite type, [McN91] for decoupled domains of finite type, [Cho96, Section 2] for pseudoconvex domains of finite type whose Levi-forms has only one degenerate eigenvalue, and [Cho02, Section 2] for pseudoconvex domains with comparable Levi-form.

Combining the Taylor expansion and (4.4.1), we can obtain

$$|r_{\Omega'}(w') - r_{\Omega'}(z')| \leqslant C \left(|w'_1 - z'_1| + \sum_{j=2}^n \sum_{k=2}^m A_{jk}(z') |w'_j - z'_j|^k \right), \tag{4.4.2}$$

for $w' \in \mathbb{B}(z', c)$, for a sufficiently small $c > 0$. Notice that by the definition (4.2.1), when $z' = w'$ on diagonal, we have $\delta(z', z') = 2|r_{\Omega'}(z')|$. Thus, if we restrict $w' \in P_\lambda(z')$, the definitions of b_j's and $P_\lambda(z')$ imply that

$$|w'_1 - z'_1| \leqslant 2\lambda |r_{\Omega'}(z')|, \tag{4.4.3}$$

and for each $j = 2, \dots, n$,

$$\sum_{k=2}^{m} \left(\frac{A_{jk}(z')}{2|r_{\Omega'}(z')|} \right)^{\frac{1}{k}} |w'_j - z'_j| = b_j(z', z')|w'_j - z'_j| \leqslant \lambda,$$

namely,

$$A_{jk}(z')|w'_j - z'_j|^k \leqslant 2\lambda^k |r_{\Omega'}(z')|, \tag{4.4.4}$$

for all $w' \in P_\lambda(z')$. Plugging (4.4.3) and (4.4.4) into (4.4.2), we get

$$|r_{\Omega'}(w') - r_{\Omega'}(z')| \leqslant C\lambda |r_{\Omega'}(z')|,$$

and hence $r_{\Omega'}(w') \approx r_{\Omega'}(z')$ for a sufficient small $\lambda > 0$. This means $r_{\Omega'}(w') < 0$. Thus, $P_\lambda(z') \Subset \Omega'$ for some small λ.

In [McN94b, Proposition 2.4], for convex domains of finite type, McNeal proves that if $w' \in P_\lambda(z')$ then there exist c and $\tilde{\lambda}$ such that $P_{c\lambda}(z') \subset P_{\tilde{\lambda}}(w')$. This result also holds for other domains listed in the theorem (see [McN94a, Proposition 4] and [McN03, Proposition 9], for example). Since $K_{\Omega'}(z', z') \approx \mathrm{Vol}(P_\lambda(z'))$, we have

$$K_{\Omega'}(w', w') \leqslant C K_{\Omega'}(z', z'), \quad \text{if } w' \in P_\lambda(z').$$

This completes our verification. □

The following corollary follows immediately from Theorem 4.1.3, that refines the result of McNeal [McN94a], Phong and Stein [PS77] for the L^p regularity of the Bergman projection P and also generalises the works of Zhu [Zhu06] and Zhao [Zha15] for the upper bound of the norm $\|P\|_{L^p(\Omega)\to L^p(\Omega)}$.

Corollary 4.4.2. *Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies at least one of the following settings:*

(a) Ω is a strongly pseudoconvex domain;

(b) Ω is a pseudoconvex domain of finite type and $n = 2$;

(c) Ω is a convex domain of finite type;

(d) Ω is a decoupled domain of finite type;

(e) Ω is a pseudoconvex domain of finite type whose Levi-form has only one degenerate eigenvalue or comparable eigenvalues.

Then the Bergman projection P is L^p-bounded for $1 < p < \infty$, with the upper bound

$$\|P\|_{L^p(\Omega)\to L^p(\Omega)} \leqslant C\frac{p^2}{p-1}.$$

Furthermore, the L^p-boundedness of P is sharp in the sense that there is no $\tilde{p} > p$ such that P is bounded from $L^p(\Omega)$ to $L^{\tilde{p}}(\Omega)$.

4.5 A generalised version of Schur's test

In this section, we introduce a generalised version of Schur's test that is an important tool of studying the L^p-L^q estimates for Toeplitz operators. We believe our generalised Schur's test is of some independent interest as well.

Theorem 4.5.1. *Let (X,μ), (Y,ν) be measure spaces with σ-finite, positive measures; let $1 < p \leqslant q < \infty$ and $\alpha \in \mathbb{R}$. Let $K : X \times Y \to \mathbb{C}$ and $\psi : Y \to \mathbb{C}$ be measurable functions. Assume that there exist positive measurable functions h_1, h_2 on Y and g on X such that*

$$h_1^{-1} h_2 \psi \in L^\infty(Y, d\nu)$$

and the inequalities

$$\int_Y |K(x,y)|^{\alpha p'} h_1(y)^{p'} d\nu(y) \leqslant C_1 g(x)^{p'}, \tag{4.5.1}$$

$$\int_X |K(x,y)|^{(1-\alpha)q} g(x)^q d\mu(x) \leqslant C_2 h_2(y)^q, \tag{4.5.2}$$

hold for almost every $x \in (X,\mu)$ and $y \in (Y,\nu)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and C_1, C_2 are positive constants.

Then the Toeplitz operator T_ψ associated to the kernel K and the symbol ψ defined by

$$(T_\psi u)(x) := \int_Y K(x,y)\, u(y)\, \psi(y)\, d\nu(y),$$

is bounded from $L^p(Y,\nu)$ into $L^q(X,\mu)$. Furthermore,

$$\|T_\psi\|_{L^p(Y,\nu) \to L^q(X,\mu)} \leqslant C_1^{\frac{p-1}{p}} C_2^{\frac{1}{q}} \|h_1^{-1} h_2 \psi\|_{L^\infty(Y,\nu)}.$$

Remark 4.5.2. We remark that Theorem 4.5.1 generalises the Schur's test of Zhao in [Zha15, Theorem 1], which is a special case of ours when $X = Y$, $\psi = 1$ and $h_1 = h_2$.

Before giving the proof of Theorem 4.5.1, we recall the well-known Minkowski's inequality, see [LL01, Theorem 2.4].

Theorem 4.5.3 (Minkowski's inequality). *Let (X,μ) and (Y,ν) be measure spaces with σ-finite, positive measures. Let f be a non-negative measurable function on $(X \times Y, \mu \times \nu)$ and let $1 \leqslant \eta < \infty$. Then*

$$\left(\int_X \left(\int_Y f(x,y)\, \nu(dy) \right)^\eta \mu(dx) \right)^{\frac{1}{\eta}} \leqslant \int_Y \left(\int_X f(x,y)^\eta \mu(dx) \right)^{\frac{1}{\eta}} \nu(dy).$$

Proof of Theorem 4.5.1. The proof of this theorem follows from a standard argument, as in [CM06, Zha15]. Using Hölder's inequality and (4.5.1), it follows

$$
\begin{aligned}
|(T_\psi u)(x)| &= \left| \int_Y \left(|K(x,y)|^{\alpha} h_1(y)\right) \left(|K(x,y)|^{1-\alpha} h_1(y)^{-1} |u(y)||\psi(y)|\right) d\nu(y) \right| \\
&\leqslant \left(C_1 g(x)^{p'}\right)^{\frac{1}{p'}} \left| \int_Y |K(x,y)|^{(1-\alpha)p} h_1(y)^{-p} |u(y)|^p |\psi(y)|^p d\nu(y) \right|^{\frac{1}{p}} \\
&= C_1^{\frac{1}{p'}} \left(\int_Y |K(x,y)|^{(1-\alpha)p} g(x)^p h_1(y)^{-p} |u(y)|^p |\psi(y)|^p d\nu(y) \right)^{\frac{1}{p}},
\end{aligned}
$$

and hence

$$
\|T_\psi u\|^p_{L^q(X,\mu)} \leqslant C_1^{p-1} \left(\int_X \left(\int_Y |K(x,y)|^{(1-\alpha)p} g(x)^p h_1(y)^{-p} |u(y)|^p |\psi(y)|^p d\nu(y) \right)^{\frac{q}{p}} d\mu(x) \right)^{\frac{p}{q}}.
$$

Since $1 < p \leqslant q < \infty$, we now apply Minkowski's integral inequality (Theorem 4.5.3) for $\eta = \frac{q}{p} \in [1, \infty)$. From (4.5.2), the estimate continues as

$$
\begin{aligned}
\|T_\psi u\|^p_{L^q(X,\mu)} &\leqslant C_1^{p-1} \int_Y \left(\int_X |K(x,y)|^{(1-\alpha)q} g(x)^q h_1(y)^{-q} |u(y)|^q |\psi(y)|^q d\mu(x) \right)^{\frac{p}{q}} d\nu(y) \\
&\leqslant C_1^{p-1} \int_Y \left(C_2 h_2(y)^q\right)^{\frac{p}{q}} h_1(y)^{-p} |\psi(y)|^p |u(y)|^p d\nu(y) \\
&\leqslant C_1^{p-1} C_2^{\frac{p}{q}} \int_Y h_1(y)^{-p} h_2(y)^p |\psi(y)|^p |u(y)|^p d\nu(y) \\
&\leqslant C_1^{p-1} C_2^{\frac{p}{q}} \|h_1^{-1} h_2 \psi\|^p_{L^\infty(Y,\nu)} \|u\|^p_{L^p(Y,\nu)}.
\end{aligned}
$$

This proves Theorem 4.5.1. ∎

Chapter 5

Bergman-Toeplitz operators on fat Hartogs triangles [1]

5.1 Introduction

In this chapter, we continue our study of the "gain" L^p-estimate properties of Bergman-Toeplitz operators

$$f \longrightarrow T_{K^{-\alpha}}(f) := \int_{\Omega} K_{\Omega}(\cdot, w)\, K^{-\alpha}(w, w)\, f(w)\, dV(w).$$

We obtain some L^p mapping properties of the Bergman-Toeplitz operator on fat Hartogs triangles $\Omega_k := \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1\right\}$, where $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}^+$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $A^p(\Omega)$ be the closed subspace of holomorphic functions in $L^p(\Omega)$. Given a measurable function ψ on Ω, the Bergman-Toeplitz operator with symbol ψ is defined by

$$f \longrightarrow T_{\psi}(f)(z) := \int_{\Omega} K(z, w)\, \psi(w)\, f(w)\, dV(w), \tag{5.1.1}$$

where $K(\cdot, \cdot)$ is the Bergman kernel associated to Ω. In Chapter 4 we proved that for a large class of weakly pseudoconvex smooth domains in $\mathbb{C}^n$, the Bergman-Toeplitz operator T_ψ with $\psi(z) = K^{-\alpha}(z, z)$ maps from $L^p(\Omega)$ to $A^q(\Omega)$ continuously if and only if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$, for any $1 < p \leqslant q < \infty$. As a corollary, if Ω is one of the smooth domains considered there, then its associated Bergman projection is a

[1] This chapter contains the material from the paper: T.V. Khanh, J. Liu and P.T. Thuc, Bergman-Toeplitz operators on fat Hartogs triangles, Proc. Amer. Math. Soc., https://doi.org/10.1090/proc/14218.

bounded operator from $L^p(\Omega)$ to $L^p(\Omega)$ for any $p \in (1, +\infty)$, and moreover, that is sharp in the sense that the Bergman projection is not bounded from $L^p(\Omega)$ to $A^q(\Omega)$ for any $q > p$, (see also [MS94, PS77, McN89, CD06]).

As a non-smooth case, the fat Hartogs triangle $\Omega_k \subset \mathbb{C}^2$ is defined by

$$\Omega_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1 \right\}, \qquad \text{for } k \in \mathbb{Z}^+. \tag{5.1.2}$$

The model of Hartogs triangles and their variants has recently attracted particular attention through the study of several problems in complex analysis; see e.g. [Che17b, EM16, CZ16, CS13, HZ17]. The fat Hartogs triangle Ω_k is a pseudoconvex domain but not a hyperconvex domain (see e.g. [Zwo99, CCW99] for the characterisation of pseudoconvexity and hyperconvexity of Reinhardt domains). Nevertheless, the Bergman kernel of Ω_k can be computed explicitly thanks to the work of Edholm [Edh16]. This important fact provides useful estimates on the Bergman kernel and then the L^p boundedness of the Bergman projection. In particular, Edholm and McNeal [EM16] proved that the Bergman projection associated to Ω_k is L^p-bounded if and only if $p \in (\frac{2k+2}{k+2}, \frac{2k+2}{k})$. This generalises the result in the case $k = 1$ by Chakrabarti and Zeytuncu [CZ16]. It should be interesting to add that the Bergman projections associated to the Hartogs triangle domains $\Omega_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$ with $\gamma > 0, \gamma \notin \mathbb{Q}$ are L^p-bounded if and only if $p = 2$, see [EM17].

It is reasonable to expect that for Hartogs triangles Ω_k, the Bergman projections cannot gain the L^p regularity. Moreover, it is also of particular interest to obtain a holomorphic function with higher regularity from an input function in the L^p space. Motivated by this, in this chapter we study the Bergman-Toeplitz operators $T_{K^{-\alpha}}$, where K is the Bergman kernel on the diagonal and $\alpha \in \mathbb{R}$. We shall prove that for $1 < p \leqslant q < \infty$ such that $\frac{k+2}{2k} - \frac{1}{kp} > \frac{1}{q} > \frac{k}{2k+2}$, the Bergman-Toeplitz operator $T_{K^{-\alpha}}$ is L^p-L^q bounded if and only if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$. It is natural to show that if $q \geqslant \frac{2k+2}{k}$, then $T_{K^{-\alpha}}$ cannot be bounded from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ for any $\alpha \in \mathbb{R}$. Surprisingly, for sufficiently small p, q, indeed, $\frac{1}{q} + \frac{1}{kp} \geqslant \frac{k+2}{2k}$, we shall point out that there exists $\alpha > 0$ such that $T_{K^{-\alpha}}$ is still bounded from $L^p(\Omega_k)$ to $A^q(\Omega_k)$. The precise statement of our main result is as follows:

Theorem 5.1.1. *For $k \in \mathbb{Z}^+$, let Ω_k be the Hartogs triangle domain defined by* (5.1.2) *and let $1 < p \leqslant q < \infty$. Then we have the following conclusions:*

(i) If $\frac{k}{2k+2} \geqslant \frac{1}{q}$, then for any $\alpha \in \mathbb{R}$, $T_{K^{-\alpha}}$ does not map from $L^p(\Omega_k)$ to $A^q(\Omega_k)$.

(ii) If $\frac{k+2}{2k} - \frac{1}{kp} > \frac{1}{q} > \frac{k}{2k+2}$ then $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ continuously if and only if $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$.

(iii) *If* $\frac{1}{q} \geqslant \frac{k+2}{2k} - \frac{1}{kp}$ *then* $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ *continuously if and only if* $\alpha > \frac{1}{p} - \left(\frac{k+2}{2k} - \frac{1}{kp}\right)$.

The proof of this theorem is divided into four small theorems below. In §5.2, Theorem 5.2.1 and 5.2.2, we provide the proof of part (i) and the sufficient conditions of (ii) and (iii) in Theorem 5.1.1. In §5.3, Theorem 5.3.3 and 5.3.4, we prove the necessary conditions of (ii) and (iii) in Theorem 5.1.1. The following corollary is a direct consequence of Theorem 5.1.1.

Corollary 5.1.2. *Let* P *be the Bergman projection associated to* Ω_k*. Then*

(i) P *maps from* $L^p(\Omega_k)$ *to* $A^p(\Omega_k)$ *continuously if and only if* $p \in \left(\frac{2k+2}{k+2}, \frac{2k+2}{k}\right)$*; and*

(ii) P *cannot map from* $L^p(\Omega_k)$ *to* $A^q(\Omega_k)$ *continuously if* $1 < p < q < \infty$.

We remark that Corollary 5.1.2 (i) has been proved in [EM16, Theorem 1.2].

5.2 Sufficient conditions

We first recall the basic properties of the Bergman kernel associated to Ω_k. The Bergman kernel of Ω_k can be computed explicitly as (see [EM16])

$$K(z, w) = \frac{p_k(s)\, t^2 + q_k(s)\, t + s^k p_k(s)}{k\pi^2 (1-t)^2 (t - s^k)^2}$$

for $z = (z_1, z_2), w = (w_1, w_2) \in \Omega_k$, where $s := z_1\overline{w_1}$, $t := z_2\overline{w_2}$,

$$p_k(s) := \sum_{j=1}^{k-1} j(k-j)\, s^{j-1} \text{ and } q_k(s) := \sum_{j=1}^{k} \left(j^2 + (k-j)^2 s^k\right) s^{j-1}.$$

Here, we use the convention $\sum_{j=1}^{k-1} \cdots := 0$ if $k = 1$. Since $|s^k| < |t| < 1$, the Bergman kernel has the upper bound

$$|K(z, w)| \lesssim \frac{|z_2\bar{w}_2|}{|1 - z_2\bar{w}_2|^2 \, |z_2\bar{w}_2 - (z_1\bar{w}_1)^k|^2}. \tag{5.2.1}$$

Combining the upper bound with the fact that $p_k(s) \geqslant 0$ and $q_k(s) \geqslant 1$ for $s \in \mathbb{R}^+$, we obtain the "sharp" estimate of the Bergman kernel on the diagonal

$$K(z, z) \approx \frac{|z_2|^2}{\left(1 - |z_2|^2\right)^2 \left(|z_2|^2 - |z_1|^{2k}\right)^2} \tag{5.2.2}$$

for $z = (z_1, z_2) \in \Omega_k$.

Theorem 5.1.1 (i) is covered in the following theorem.

Theorem 5.2.1. *Let $q \geqslant \frac{2k+2}{k}$. Then $T_{K^{-\alpha}}$ does not map from $L^p(\Omega_k)$ to $A^q(\Omega_k)$, for any $p > 0$ and $\alpha \in \mathbb{R}$.*

Proof. Note that if $T_{K^{-\alpha_0}}$ maps from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ for some $\alpha_0 < 0$ then $T_{K^{-\alpha}}$ maps from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ for any $\alpha \geqslant 0$. Therefore, we may assume $\alpha \geqslant 0$. In order to prove Theorem 5.2.1, we shall show that $T_{K^{-\alpha}}(\bar{z}_2) = \frac{c}{z_2}$ for some non-zero constant c. Then the conclusion follows since $\bar{z}_2 \in L^p(\Omega_k)$ for any $p > 0$, but $\frac{1}{z_2} \notin L^q(\Omega_k)$ if $q \geqslant \frac{2k+2}{k}$ (by a simple calculation).

Now, we recall the fact (see [EM16])

$$\left\{z^\beta : \beta \in \mathcal{B} := \left\{(\beta_1, \beta_2) \in \mathbb{Z}^2 : \beta_1 \geqslant 0, \beta_1 + k(\beta_2 + 1) > -1\right\}\right\}$$

forms an orthogonal basis for $A^2(\Omega_k)$. It suffices to prove $\left\langle K^{-\alpha}\bar{z}_2, z^\beta\right\rangle = 0$ for any $\beta \in \mathcal{B}$, apart from $\beta = (0, -1)$. To see this, observe that the function $\psi(z) := K^{-\alpha}(z, z)$ can be represented as $\psi(z) = \Phi(|z_1|, |z_2|)$, for a bounded function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$. Therefore

$$\begin{aligned}
\left\langle K^{-\alpha}\bar{z}_2, z^\beta\right\rangle &= \int_{\Omega_k} \Phi(|z_1|, |z_2|)\, \overline{z_2 z_1^{\beta_1} z_2^{\beta_2}} dz \\
&= \left(\int_0^{2\pi} e^{-\imath\theta_1\beta_1} d\theta_1\right)\left(\int_0^{2\pi} e^{-\imath\theta_2(\beta_2+1)} d\theta_2\right)\left(\int_U \Phi(r_1, r_2)\, r_1^{\beta_1+1} r_2^{\beta_2+2} dr_1 dr_2\right) \\
&= 0,
\end{aligned}$$

unless $\beta = (0, -1)$. Here $U := \left\{(r_1, r_2) : 0 \leqslant r_1, r_1^k < r_2 < 1\right\}$. This completes the proof of Theorem 5.2.1. □

The next theorem is the main goal of this section, in which we prove the sufficient conditions of (ii) and (iii) in Theorem 5.1.1.

Theorem 5.2.2. *Let $1 < p \leqslant q < 2 + \frac{2}{k}$. If p, q and α satisfy either*

(i) $\frac{1}{q} < \frac{k+2}{2k} - \frac{1}{kp}$ *and* $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$, *or*

(ii) $\frac{1}{q} \geqslant \frac{k+2}{2k} - \frac{1}{kp}$ *and* $\alpha > \frac{1}{p} - \left(\frac{k+2}{2k} - \frac{1}{kp}\right)$

then $T_{K^{-\alpha}}$ maps from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ continuously.

One of the fundamental tools to establish the self L^p boundedness of the Bergman projection is Schur's test lemma (see [MS94], [EM16]). We will use the generalised Schur's test, Theorem 4.5.1, to prove Theorem 5.2.2. Thus we first establish integral estimates on the Bergman kernel of the fat Hartogs triangle Ω_k.

Proposition 5.2.3. *Let a, b, c be real numbers satisfying*

$$a \geqslant 1, \quad -1 < b < 0 \quad \text{and} \quad -a + 2b + c + \frac{2}{k} > -2. \tag{5.2.3}$$

Then for any $z \in \Omega_k$,

$$\int_{\Omega_k} |K(z,w)|^a \, |r(w)|^b |w_2|^c dV(w) \quad \lesssim \quad K^{a-1}(z,z) \, |r(z)|^b |z_2|^{a-2b-2}, \tag{5.2.4}$$

where $r(z) := \left(|z_2|^2 - |z_1|^{2k}\right)\left(|z_2|^2 - 1\right)$.

Proof. Set $J(z) := \int_{\Omega_k} |K(z,w)|^a \, |r(w)|^b |w_2|^c dV(w)$. By (5.2.1), it follows

$$\begin{aligned} J(z) &\lesssim \int_{\Omega_k} \frac{|z_2\overline{w_2}|^a \left(|w_2|^2 - |w_1|^{2k}\right)^b \left(1 - |w_2|^2\right)^b |w_2|^c}{|1 - z_2\overline{w_2}|^{2a} \left|z_2\overline{w_2} - z_1^k \overline{w_1}^k\right|^{2a}} dV(w) \\ &= \int_{\mathbb{D}\setminus\{0\}} \frac{|z_2|^a \, |w_2|^{a+c} \left(1 - |w_2|^2\right)^b}{|1 - z_2\overline{w_2}|^{2a}} \left[\int_{\mathbb{D}(|w_2|^{\frac{1}{k}})} \frac{\left(|w_2|^2 - |w_1|^{2k}\right)^b}{\left|z_2\overline{w_2} - z_1^k\overline{w_1}^k\right|^{2a}} dV(w_1) \right] dV(w_2), \end{aligned}$$

where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$ and $\mathbb{D}(|w_2|^{\frac{1}{k}}) := \left\{w_1 \in \mathbb{C} : |w_1| < |w_2|^{\frac{1}{k}}\right\}$. By the change of variable $u := \frac{w_1^k}{w_2}$, the expression in the bracket [ ] can be rewritten as

$$\begin{aligned} [\cdots] &= |w_2|^{-2a+2b} \, |z_2|^{-2a} \int_{\mathbb{D}(|w_2|^{\frac{1}{k}})} \left|1 - \frac{z_1^k \overline{w_1}^k}{z_2 \overline{w_2}}\right|^{-2a} \left(1 - \left|\frac{w_1^k}{w_2}\right|^2\right)^b dV(w_1) \\ &= \frac{1}{k^2} |w_2|^{-2a+2b+\frac{2}{k}} \, |z_2|^{-2a} \int_{\mathbb{D}} \left|1 - \frac{\overline{z_1}^k}{\overline{z_2}} u\right|^{-2a} \left(1 - |u|^2\right)^b |u|^{\frac{2}{k}-2} dV(u). \end{aligned} \tag{5.2.5}$$

By Lemma 5.2.4 below, the integral term in the last line of (5.2.5) is dominated by $\left(1 - \left|\frac{z_1^k}{z_2}\right|^2\right)^{-2a+b+2}$. Thus, (5.2.5) continues as

$$[\cdots] \lesssim |w_2|^{-2a+2b+\frac{2}{k}} \, |z_2|^{-2a} \left(1 - \left|\frac{z_1^k}{z_2}\right|^2\right)^{-2a+b+2}.$$

Therefore,

$$\begin{aligned} J(z) \lesssim& |z_2|^{3a-2b-4} \left(|z_2|^2 - |z_1|^{2k}\right)^{-2a+b+2} \\ &\times \int\limits_{\mathbb{D}\setminus\{0\}} |1 - w_2\overline{z_2}|^{-2a} \left(1 - |w_2|^2\right)^b |w_2|^{-a+2b+c+\frac{2}{k}} \, dV(w_2) \\ \lesssim& |z_2|^{3a-2b-4} \left(|z_2|^2 - |z_1|^{2k}\right)^{-2a+b+2} \left(1 - |z_2|^2\right)^{-2a+b+2} \\ \lesssim& K^{a-1}(z,z) \, |r(z)|^b \, |z_2|^{a-2b-2}, \end{aligned}$$

where the second inequality follows by using Lemma 5.2.4 again since $-a + 2b + c + \frac{2}{k} > -2$; and the last one follows by (5.2.2). □

The proof of Proposition 5.2.3 is complete but we have skipped a crucial technical point which we now address.

Lemma 5.2.4. *Let a, b and c be real numbers such that*

$$a \geqslant 1, \quad -1 < b < 0 \quad \textit{and} \quad c > -2.$$

Then, for any $v \in \mathbb{D}$,

$$I_{a,b,c}(v) := \int\limits_{\mathbb{D}} |1 - u\bar{v}|^{-2a} \left(1 - |u|^2\right)^b |u|^c \, dV(u) \lesssim \left(1 - |v|^2\right)^{-2a+b+2}. \tag{5.2.6}$$

Proof. This lemma is a slight extension of [EM16, Lemma 3.2], in which the estimate

$$I_{1,b,c}(v) \lesssim (1 - |v|^2)^b \tag{5.2.7}$$

has been proved for $-1 < b < 0$ and $-2 < c \leqslant 0$. The estimate (5.2.7) can be automatically extended to the case $c > -2$. Now, if $a \geqslant 1$ then

$$|1 - u\bar{v}|^{2-2a} \leqslant (1 - |v|)^{2-2a} \lesssim (1 - |v|^2)^{2-2a};$$

and hence by (5.2.7)

$$I_{a,b,c}(v) \lesssim (1 - |v|^2)^{2-2a} I_{1,b,c}(v) \lesssim (1 - |v|^2)^{-2a+b+2},$$

provided $-1 < b < 0$ and $c > -2$. □

Now we are ready to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. Let $p' = \frac{p}{p-1}$, $0 < \beta < \min\{\frac{1}{q}, \frac{1}{p'}\}$ and $\gamma < (1+\frac{2}{k})(1-\frac{1}{p})$. Combining the choice of p', β, γ and the hypothesis $1 < p \leqslant q < 2 + \frac{2}{k}$, it is clear that the relations

$$a \geqslant 1, \quad -1 < b < 0 \quad \text{and} \quad -a + 2b + c + \frac{2}{k} > -2$$

satisfy for both choices

$$(a,b,c) = (1, -\beta p', (2\beta-\gamma)p') \quad \text{and} \quad (a,b,c) = \left(\frac{q}{p}, -\beta q, (2\beta - \frac{1}{p'})q\right).$$

Thus, by Proposition 5.2.3, the following integral estimates hold

$$\int\limits_{\Omega_k} |K(z,w)|\, |r(w)|^{-\beta p'} |w_2|^{(2\beta-\gamma)p'} dV(w) \lesssim |r(z)|^{-\beta p'} |z_2|^{2\beta p' - 1},$$

$$\int\limits_{\Omega_k} |K(z,w)|^{\frac{q}{p}} |r(z)|^{-\beta q} |z_2|^{\frac{(2\beta p'-1)q}{p'}} dV(z) \lesssim K^{\frac{q}{p}-1}(w,w)\, |r(w)|^{-\beta q} |w_2|^{\frac{q}{p}+2\beta q - 2}.$$

These estimates show that the integral estimates in Theorem 4.5.1 hold with $\alpha = \frac{1}{p'}$,

$$h_1(w) := |r(w)|^{-\beta} |w_2|^{2\beta-\gamma}, \quad g(z) := |r(z)|^{-\beta} |z_2|^{2\beta - \frac{1}{p'}}$$

and

$$h_2(w) := K^{\frac{1}{p}-\frac{1}{q}}(w,w) |r(w)|^{-\beta} |w_2|^{\frac{1}{p}+2\beta-\frac{2}{q}}.$$

In order to conclude that the Bergman-Toeplitz operator $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ continuously, we shall choose $\gamma < \left(1+\frac{2}{k}\right)\left(1-\frac{1}{p}\right)$ such that

$$\mathcal{A}(w) := h_1^{-1}(w) h_2(w) K^{-\alpha}(w,w) = K^{\frac{1}{p}-\frac{1}{q}-\alpha}(w,w) |w_2|^{\gamma+\frac{1}{p}-\frac{2}{q}}$$

is uniformly bounded for all $w \in \Omega_k$.

If $\frac{1}{q} < \frac{k+2}{2k} - \frac{1}{kp}$ and $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$, by choosing $\gamma = \frac{2}{q} - \frac{1}{p}$, we have $\mathcal{A}(w) \leqslant 1$ for all $w \in \Omega_k$. Moreover, with this choice, the requirement $\gamma < (1+\frac{2}{k})(1-\frac{1}{p})$ is equivalent to the given condition $\frac{1}{q} < \frac{k+2}{2k} - \frac{1}{kp}$. This proves the first part of Theorem 5.2.2.

If $\frac{1}{q} \geqslant \frac{k+2}{2k} - \frac{1}{kp}$ and $\alpha > \frac{1}{p} - \left(\frac{k+2}{2k} - \frac{1}{kp}\right)$ then

$$\mathcal{A}(w) \lesssim |w_2|^{-\frac{2}{p}+\frac{2}{q}+2\alpha+\gamma+\frac{1}{p}-\frac{2}{q}} = |w_2|^{2\alpha+\gamma-\frac{1}{p}}.$$

Here, we have used the fact that $\alpha > \frac{1}{p} - \left(\frac{k+2}{2k} - \frac{1}{kp}\right) \geqslant \frac{1}{p} - \frac{1}{q}$ and $K^{-1}(w,w) \lesssim |w_2|^2$ by (5.2.2). Now by choosing $\gamma = -2\alpha + \frac{1}{p}$, we have $\mathcal{A}$ is uniformly bounded. The proof is complete since the requirement $\gamma < (1+\frac{2}{k})(1-\frac{1}{p})$ is equivalent to $\alpha > \frac{1}{p} - \left(\frac{k+2}{2k} - \frac{1}{kp}\right)$.

□

5.3 Necessary conditions

To complete the proof of Theorem 5.1.1, the remaining task is to show the respective lower bounds for α. That is, if $T_{K^{-\alpha}}$ maps from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ continuously, then α must be greater than or equal to the desired values. We shall prove parts *(ii)* and *(iii)* in Theorem 5.1.1 by using two different approaches, respectively. We remark that the underlying idea of both arguments is to derive the desirable property from singular points.

In the first approach, we illustrate a technique involving the use of the pluricomplex Green function. This method may be applied to a more general context. We first recall that for a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the pluricomplex Green function with a pole $w \in \Omega$ is defined by

$$G(\cdot, w) := \sup\left\{u(\cdot) : u \in PSH^-(\Omega), \limsup_{z\to w}(u(z) - \log|z-w|) < \infty\right\}.$$

Here $PSH^-(\Omega)$ denotes the set of all negative plurisubharmonic functions in Ω. The relation between the pluricomplex Green function and the Bergman kernel has been studied by several authors, see e.g. [Her99, Bło05, CF11, Bło14a]. One of the most important facts from these results is the very weak assumption on the regularity of the domain.

The following proposition is first proved by Herbort [Her99] with the constant on the right hand side depending on the diameter of the domain, and Błocki [Bło14b] improved it to the sharp one as follows.

Proposition 5.3.1 (Herbort-Błocki)**.** *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let t be any positive number. Then for any holomorphic function f on Ω and any $w \in \Omega$,*

$$\int_{\{G(\cdot,w)<-t\}} |f(z)|^2 \, dV(z) \geqslant e^{-2nt}\frac{|f(w)|^2}{K(w,w)}.$$

The next lemma provides a relation between the Bergman kernel and the pole w on the sublevel set $\{G(\cdot, w) < -1\}$ for the Hartogs triangles Ω_k.

Lemma 5.3.2. *Let $w \in \Omega_k$,*

$$K(z,z)|z_2|^2 \approx K(w,w)|w_2|^2$$

for any $z \in \{G(\cdot, w) < -1\}$.

Proof. We first prove the following elementary fact.

Claim: *If $a, b \in \mathbb{D}$ and $\left|\frac{a-b}{1-a\bar{b}}\right| < \frac{1}{e}$ then $1-|a|^2 \approx 1-|b|^2$.*

Proof of the claim. Set $\xi = \frac{b-a}{1-a\bar{b}}$. Then $a = \frac{b-\xi}{1-\xi\bar{b}}$ and hence $\frac{1-|a|^2}{1-|b|^2} = \frac{1-|\xi|^2}{|1-\xi\bar{b}|^2}$. This implies the stated claim by the fact that

$$\frac{e-1}{e+1} < \frac{1-|\xi|}{1+|\xi|} \leqslant \frac{1-|\xi|^2}{|1-\xi\bar{b}|^2} \leqslant \frac{1+|\xi|}{1-|\xi|} < \frac{e+1}{e-1}, \quad \text{since } b \in \mathbb{D}.$$

□

We now proceed the proof of Lemma 5.3.2. Recall that, see e.g. [Kli95], for $z = (z_1, z_2)$ and $w = (w_1, w_2)$,

$$G_{\mathbb{D}\times\mathbb{D}}(z,w) = \max\left\{\log\left|\frac{z_1-w_1}{1-z_1\overline{w_1}}\right|, \log\left|\frac{z_2-w_2}{1-z_2\overline{w_2}}\right|\right\}.$$

Since the map

$$\begin{aligned} F : \Omega_k &\longrightarrow \mathbb{D}\times\mathbb{D} \\ (z_1, z_2) &\longrightarrow \left(\frac{z_1^k}{z_2}, z_2\right) \end{aligned}$$

is holomorphic, we obtain

$$G_{\mathbb{D}\times\mathbb{D}}(F(z_1,z_2), F(w_1,w_2)) \leqslant G_{\Omega_k}((z_1,z_2),(w_1,w_2))$$

for any $z, w \in \Omega_k$. Thus for any $z \in \{z \in \Omega_k : G(z,w) < -1\}$, we have

$$\left|\frac{\frac{z_1^k}{z_2} - \frac{w_1^k}{w_2}}{1 - \frac{z_1^k\overline{w_1}^k}{z_2\overline{w_2}}}\right| < \frac{1}{e} \text{ and } \left|\frac{z_2-w_2}{1-z_2\overline{w_2}}\right| < \frac{1}{e}.$$

Using the above claim, it follows

$$1-|z_2|^2 \approx 1-|w_2|^2 \quad \text{and} \quad 1-\left|\frac{z_1^k}{z_2}\right|^2 \approx 1-\left|\frac{w_1^k}{w_2}\right|^2. \tag{5.3.1}$$

Now, the desired result can be obtained by (5.3.1) and the fact

$$K(z,z)|z_2|^2 \approx \left[(1-|z_2|^2)\left(1-\left|\frac{z_1^k}{z_2}\right|^2\right)\right]^{-2}.$$

□

We now turn to the proof of the necessary condition in Theorem 5.1.1 (ii).

Theorem 5.3.3. *Let* $1 < p \leqslant q < 2 + \frac{2}{k}$ *and let* $\alpha \in \mathbb{R}$. *Assume that* $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ *continuously. Then* $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$.

Proof. First, we may assume that $\alpha < 1$, otherwise it implies the conclusion. Since $K(w,z)w_2$ is holomorphic in w, one has

$$\begin{aligned}
&\int_{\Omega_k} |K(z,w)|^2 K^{-\alpha}(w,w)|w_2|^2 dV(w) \\
&\quad = \int_{\Omega_k} K(z,w) K^{-\alpha}(w,w)\overline{w_2} K(w,z) w_2 dV(w) \\
&\quad = \int_{\Omega_k} K(z,w) K^{-\alpha}(w,w)\overline{w_2}\left(\int_{\Omega_k} K(w,\xi)K(\xi,z)\xi_2 dV(\xi)\right) dV(w) \\
&\quad = \int_{\Omega_k}\left(\int_{\Omega_k} K(w,\xi)K^{-\alpha}(w,w)K(z,w)\overline{w_2}dV(w)\right)K(\xi,z)\xi_2 dV(\xi).
\end{aligned}$$

By Hölder's inequality and the boundedness of $T_{K^{-\alpha}}$, it continues as

$$\begin{aligned}
&\int_{\Omega_k} |K(z,w)|^2 K^{-\alpha}(w,w)|w_2|^2 dV(w) \\
&\quad \leqslant \left\|\int_{\Omega_k} K(w,\cdot)K^{-\alpha}(w,w)K(z,w)\overline{w_2}dV(w)\right\|_{L^q} \|K(z,\cdot)(\cdot)_2\|_{L^{q'}} \\
&\quad \lesssim \|K(z,\cdot)(\cdot)_2\|_{L^p(\Omega_k)}\|K(z,\cdot)(\cdot)_2\|_{L^{q'}(\Omega_k)}, \qquad (5.3.2)
\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. By Proposition 5.2.3,

$$\begin{aligned}
\|K(z,\cdot)(\cdot)_2\|_{L^p(\Omega_k)}\|K(z,\cdot)(\cdot)_2\|_{L^{q'}(\Omega_k)} &\lesssim |z_2|^{2-\frac{2}{p}-\frac{2}{q'}} K^{2-\frac{1}{p}-\frac{1}{q'}}(z,z) \\
&= |z_2|^{\frac{2}{q}-\frac{2}{p}} K^{1-\frac{1}{p}+\frac{1}{q}}(z,z). \qquad (5.3.3)
\end{aligned}$$

Note that the appearance of the term w_2 above allows us to apply Proposition 5.2.3. On the other hand, the LHS of (5.3.2) satisfies

$$\int_{\Omega_k} |K(z,w)|^2 K^{-\alpha}(w,w)|w_2|^2 dw \geqslant \int_{\{G(\cdot,z)<-1\}} |K(z,w)|^2 K^{-\alpha}(w,w)|w_2|^2 dw$$

$$
\begin{aligned}
&= \int_{\{G(\cdot,z)<-1\}} |K(z,w)|^2 |w_2|^{2+2\alpha}\left(K(w,w)|w_2|^2\right)^{-\alpha} dw \\
&\gtrsim \left(K(z,z)|z_2|^2\right)^{-\alpha} \int_{\{G(\cdot,z)<-1\}} \left|K(z,w)w_2^2\right|^2 dw \\
&\gtrsim \left(K(z,z)|z_2|^2\right)^{-\alpha} K(z,z)|z_2|^4 .
\end{aligned}
$$

Here we have used Lemma 5.3.2 and Proposition 5.3.1. From this, (5.3.2) and (5.3.3), we get $\alpha \geqslant \frac{1}{p} - \frac{1}{q}$ by letting $z_2 \to 1$. This completes the proof of Theorem 5.3.3.

□

The second approach is to construct an appropriate sequence $\{f_j\}$ and establish the lower bound from the hypothesis

$$\sup_j \frac{\|T_{K^{-\alpha}}(f_j)\|_{L^q(\Omega_k)}}{\|f_j\|_{L^p(\Omega_k)}} < \infty. \tag{5.3.4}$$

This approach is standard and depends quite heavily on the intrinsic information of our domains Ω_k. Let us use this approach to prove the necessary condition in Theorem 5.1.1 (iii).

Theorem 5.3.4. *Let $1 < p \leqslant q < 2 + \frac{2}{k}$ and let $\alpha \geqslant 0$. Assume that $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ continuously. Then $\alpha > \frac{k+1}{kp} - \frac{k+2}{2k}$.*

Proof. We employ a computation similar to that in [Che17b]. Define the sequence $\{f_j\}_{j=1}^{\infty}$ by

$$f_j(z) := \begin{cases} h(|z_2|)\,\overline{z_2} & ; \quad a_{j+1} < |z_2| < 1, \\ 0 & ; \quad \text{elsewhere,} \end{cases}$$

where $a_j := \frac{1}{j^j}$ and the function $h : (0,1] \to (0,\infty)$ is defined by

$$h(x) := x^{\frac{1}{l} - 1 - \left(2+\frac{2}{k}\right)\frac{1}{p}} \text{ for } x \in (a_{l+1}, a_l]; \quad l = 1, 2, \ldots.$$

We now can easily check that

$$
\begin{aligned}
\|f_j\|^p_{L^p(\Omega_k)} &\lesssim \int_{a_{j+1}}^{1} h^p(r_2)\, r_2^{p+\frac{2}{k}+1} dr_2 \lesssim \sum_{l=1}^{\infty} \left(\int_{a_{l+1}}^{a_l} h^p(r_2)\, r_2^{p+\frac{2}{k}+1} dr_2 \right) \\
&\lesssim \sum_{l=1}^{\infty} l \left(l^{-p} - (l+1)^{-\left(1+\frac{1}{l}\right)p} \right)
\end{aligned}
$$

$$\lesssim \sum_{l=1}^{\infty} \frac{1}{l^p}.$$

Therefore

$$\sup\left\{\|f_j\|_{L^p(\Omega_k)}^p : j \in \mathbb{Z}^+\right\} < c_0 := \sum_{l=1}^{\infty} \frac{1}{l^p}. \tag{5.3.5}$$

By construction, $f_j \in L^2(\Omega_k)$, for any $j \in \mathbb{Z}^+$. Thus we can make use of the above orthogonal basis $\mathcal{B}$ of $A^2(\Omega_k)$ to obtain

$$T_{K^{-\alpha}}(f_j) = C\left(\int_{a_{j+1}}^{1} \int_{0}^{r_2^{\frac{1}{k}}} (1-r_2)^{2\alpha}\left(r_2 - r_1^k\right)^{2\alpha} r_1 r_2 h(r_2)\, dr_1 dr_2\right)\frac{1}{z_2},$$

for a constant C independent of j. Therefore

$$\|T_{K^{-\alpha}}(f_j)\|_{L^q(\Omega_k)}^q \gtrsim \sum_{l=1}^{j}\left(\int_{a_{l+1}}^{a_l} r_2^{\frac{1}{l}-1+2\left(\alpha-\frac{k+1}{kp}+\frac{k+2}{2k}\right)}(1-r_2)^{2\alpha}\, dr_2\right). \tag{5.3.6}$$

We now show that $\alpha > \frac{k+1}{kp} - \frac{k+2}{2k}$ by contradiction. Assume that $\alpha \leqslant \frac{k+1}{kp} - \frac{k+2}{2k}$. Since $(1-r_2)^{2\alpha} \geqslant \max\{1-2\alpha r_2, 2\alpha - 2\alpha r_2\}$ for any $\alpha \geqslant 0$ and $r_2 \in (0,1)$, (5.3.6) continues as

$$\|T_{K^{-\alpha}}(f_j)\|_{L^q(\Omega_k)}^q \gtrsim \sum_{l=1}^{j}\left(\int_{a_{l+1}}^{a_l} r_2^{\frac{1}{l}-1} dr_2\right) - \sum_{l=1}^{j}\left(\int_{a_{l+1}}^{a_l} r_2^{\frac{1}{l}} dr_2\right). \tag{5.3.7}$$

Note that the first sum of the RHS of (5.3.7) goes to infinity while the second sum converges as $j \to \infty$. The contradiction now follows from (5.3.4), (5.3.5) and (5.3.7) by letting $j \to \infty$. □

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