Dispersive shock waves in optical and fluid media

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Dispersive shock waves in optical and fluid media

Xin An

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Doctor of Philosophy

Supervisors:
Prof. Timothy Marchant & Prof. Noel Smyth

The University of Wollongong
School of School of Mathematics and Applied Statistics

January 7, 2019
Declaration

I, Xin An, declare that this thesis is submitted in partial fulfilment of the requirements for the conferral of the degree Doctor of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Xin An

January 7, 2019
Abstract

Dispersive shock waves (DSWs), also termed undular bores in fluid mechanics, are generated from a jump discontinuity. DSWs form due to the balance between nonlinearity and dispersion and have an oscillating wave structure where the leading and trailing edges have different velocities. DSW has been investigated intensively over the past few decades, ever since Whitham’s pioneering invention of modulation theory [127] and Gurevich and Pitaevsky’s construction of the DSW solution for the Korteweg-de Vries equation [63]. The theory was subsequently used to study DSWs governed by integrable equations. Then, based on Whitham’s and Gurevich and Pitaevsky’s research, El proposed the framework of a DSW fitting method which enables the analysis of DSWs governed by non-integrable equations [37, 41].

In this thesis, we consider the analysis of DSW in three different applications, all governed by non-integrable equations. The first is the analysis of the propagation of an optical DSW in a defocussing colloidal medium. The equations governing nonlinear light propagation in a colloidal medium consist of an NLS-type equation for the beam and an algebraic equation for the medium response. Solutions for the leading and trailing edges of the colloidal DSW are found using El’s theory.

The second is an investigation of the DSWs governed by the nonlocal Whitham equation. This equation allows the study of short wavelength effects, that led to peaked cusped waves within the DSW. The equation combines the weak nonlinearity of the KdV equation with full linear dispersion. Various dispersion relations are considered, for surface gravity waves, the intermediate long wave equation and a model dispersion relation introduced by Whitham to investigate the 120° peaked Stokes wave of highest amplitude. El’s
DSW fitting method is used to find the leading (solitary wave) and trailing (linear wave) edges. This method is found to produce results in excellent agreement with numerical solutions up until the lead solitary wave of the DSW reaches its highest amplitude. Numerical solutions show that the DSWs for the water wave and Whitham peaking kernels become modulationally unstable and evolve into multi-phase wavetrains after a critical amplitude, which is just below the DSW of maximum amplitude.

The third is the investigation of DSWs in quadratic media. A quadratic medium gives rise to a second harmonic generation, which is a nonlinear process that induces two photons with same frequency to interact with each other and generates a photon with twice the energy as before. As a second harmonic is considered, a phase locking assumption is required for the analytic solution of DSW in the quadratic system. The DSW fitting technique is again used to determine the leading and trailing edges. Excellent agreements between theory and numerical solutions are found for all three problems considered.
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Chapter 1

Introduction and background

1.1 The solitary wave and its history

A dispersive shock wave (DSW) is the dispersive resolution of a discontinuity and the equivalent of a shock wave in compressible flow, with dispersion rather than viscosity smoothing the discontinuity. An introduction to DSWs without giving an account of solitary waves would seem incomplete. It all begins in 1834, when John Scott Russell observed a soliton, due to a chance observation on the Union Canal. Later he named his new observation a great wave of translation and published a report on the phenomenon in 1844 [106]. In this work, he investigated a hump-like wave which travelled along the canal and kept its shape and form over a long distance. So he named it a wave of translation, in contrast to more typical oscillatory waves. However, it was at that time a common understanding that all waves should be oscillatory, but Russell’s observation disagreed with this. Russell deduced that a nonlinear effect must be behind this phenomenon. In fluid dynamics the wave is now called Russell’s solitary wave. In 1847, Stokes [116] pioneered the study of nonlinear dispersive waves. He found the existence of periodic wavetrains in nonlinear dispersive wave systems, as well as the dispersion relation for water waves. Subsequently, theoretical investigations of solitary waves were undertaken by Joseph Boussinesq in 1871 [13] and Lord Rayleigh in 1876 [103]. Both of their works confirmed Russell’s observations. Then, in 1895 Korteweg and de Vries [80] derived the
Korteweg de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

(1.1.1)

which is a generic model equation describing weakly nonlinear long waves on shallow water. It is of importance due to the fact that it is integrable, which means as a nonlinear partial differential equation (PDE) its analytic solution can be fully determined in many cases. There are many theoretical methods to solve the KdV equation, such as the inverse scattering transform [2] and Hirota’s bilinear method [66]. The second term in the KdV equation (1.1.1) represents the nonlinear effect and the last term gives the dispersive effect. The nonlinear effect contributes to the steeping of the wave, while the dispersion gives rise to spreading. These two term can balance each other, allowing a solitary wave to form. The equation has also appeared in Boussinesq’s works.

During the second half of the 20th century, there has been a series of developments and breakthroughs in solitary wave theory, among which Zabusky and Kruskal’s contribution deserves a special mention. In 1965, they considered the numerical interaction of solitons in a collisionless plasma [128]. They observed that when there were two solitary waves of different speeds, the faster moving wave would catch up and collide with the slower one. Interestingly, the solitary waves maintained their shapes and speeds after they interacted with each other, except that their phases experienced shifts. Then, the faster moving wave would keep moving ahead, with the slower one falling behind. The terminology soliton was firstly introduced by Zabusky and Kruskal due to the fact the solitary waves behaved like particles and underwent elastic collisions, i.e. a collision between two bodies in which the total kinetic energy of each was the same before and after the collision.

An elegant mathematical theory, the Inverse Scattering Transform (IST), was introduced to solve integrable solitary wave equations. The IST technique was invented in the context of the Cauchy problem for the KdV equation by Gardner et al. [53]. It was later shown that a similar technique can be applied to many integrable equation, such as the nonlinear Schrödinger (NLS) equation [1]. An introduction to the NLS equation and defocusing NLS term is needed here as it is an extremely important mathematical model...
CHAPTER 1. INTRODUCTION AND BACKGROUND

for many nonlinear problems. Generically, it is of the form

\[ iu_t + \frac{1}{2} u_{xx} \pm |u|^2 u = 0. \]  

(1.1.2)

where \( u \) is the complex envelope of a group of weakly interacting waves grouped around a central wavenumber. A balance between the nonlinear term and the dispersion term gives rise to the NLS soliton. If the coefficient of the nonlinear term \(|u|^2 u\) is positive, then self-focusing occurs and bright solitons arise. If the coefficient of the nonlinear term is negative, then defocusing results in dark solitons. Research on the NLS equation first arose in studying water wave stability [127]. The main current applications of the NLS equation are the propagation of light in nonlinear optical media and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps in the mean-field regime. Other application includes propagation of heat pulses in solid and quantum mechanics [30]. Various methods have been found to obtain particular solutions of the NLS equation. They are the Darboux transformation [92], the Bäcklund transformation [104], the bilinear method of Hirota [66], as well as other methods based on using a trial function to determine wave parameters [76].

1.2 A brief history of dispersive shock waves (undular bores)

Viscous bores arise when there is a balance between viscous loss and nonlinearity in nonlinear wave systems. Such bores are steady wavetrains that are dominated by viscous loss. On the other hand, when viscous effects are small, a balance between nonlinearity and dispersion occurs and DSWs, or undular bores, arise and behave as unsteady wavetrains that spread continuously. The generic manifestation of a DSW is that it consists of two edges propagating with different speeds. The two distinct speeds of propagation correspond to two limits, the harmonic (linear) wave limit where the wave amplitude is small and the solitary wave limit where the wavenumber is small. One thing worth men-
Figure 1.1: (a) A generic DSW and its inner structure. (b) Orientation and polarity of a DSW. (a) and (b) are figure 3 and figure 4 from [38].

The DSW has attracted extensive research over the past few years aided by extensive experimental observation. The study of DSWs first arose in water wave theory [12, 71]. Classic examples are tidal bores which arise in coastal regions of strong tidal flow, such as the Severn Estuary in England and the Bay of Fundy in Canada, and the tsunami that
is generated by marine earthquakes and land slips. Bores in fluids can be categorised into two types, viscous and undular bores.

Undular bores are common phenomena in nature and have been extensively documented. Figure 1.2 shows internal waves in the Sulu Sea between the Philippines and Malaysia [97]. The Sulu Sea is composed of multiple layers of water with different densities. Internal waves move along the boundary between these layers, with the topmost layer being the least dense with increasing density of deeper layers. When the lower layer comes across an obstacle, a disturbance occurs and creates a wave in the water layer, so that an internal wave arises. Internal waves can extend 25 km to 35 km in length and last for two days. The four-wave-packet waves arise every 12.42 hours during each tide. Figure 1.3 shows a satellite photograph of an atmospheric gravity wave (morning glory) moving southward off the Texas coast and out over the western Gulf of Mexico. As the
cold front moves down from the north, it pushed out the train of smooth waves to the south seen in the figure. These waves then pushed up dense air into the less dense air which was above the waves and the wavetrain was created. Under appropriate conditions, the whole evolution can produce a series of gravity waves which eventually die out after the cold front moves away. The phenomenon can retain its identity for several hours and can extend in length up to a few hundred kilometres. It was measured that the gravity wave front moves with a speed of 25 metres per second. The horizontal (crest to crest) and vertical (crest to trough) wavelengths are 13 km and 440 meters [101].

When a shallow water DSW is propagating over a smooth bed, an interesting fluid mechanics problem termed shoaling arises. It has been observed in many locations, such as the tidal bore on the river Ribble Lancashire shown in Figure 1.4. Tidal bores occur in places where the tidal range is more than 6 meters between high and low water. They can occur every day, such as the tidal bore of the Batang River in Malaysia. Other tidal bores happen during spring tides on new and full moons. In addition, the incoming tide should be flowing into a shallow and narrow body of water. A tidal bore can move rapidly and
travel a long distance up a river against the direction of the current. The breaking of a tidal bore wave occurs when the crest of the bore travels faster than the wave itself. Tidal waves can be dangerous to inexperienced swimmers and people living nearby the river, but nonetheless a magnificent phenomenon of nature.

In the context of nonlinear optical media, DSWs have been observed, simulated and researched in photorefractive crystals [42, 122], nonlinear optical fibres [21, 24] and colloidal media [4, 89]. In the context of fluid flow, there has been much research on atmosphere DSWs [19, 20, 102], DSWs in the internal tide [113], in resonant flows over topography [62, 86] and in magma flow in geophysics [82, 87, 109]. Figure 1.5 is figure 2 from [9]. It shows an experimental observation of an optical thermal DSW and its evolution in an ethanol and iodine liquid cell after 1 cm of propagation. Figure 1.5 (a) shows the input optical beam. Figure 1.5 (b) shows the initial breaking of the beam. Figure 1.5 (c) shows the quasi-steady state of the DSW 200 ms later and (d) shows the steady state of the DSW. As the heating increases, the output evolved from linear diffraction to a weak, then strong nonlinear DSW. After a relatively long time, the DSW began to move vertically and developed an asymmetry in (d) due to convection of the liquid medium.
1.2.1 Modulation theory

The KdV equation (1.1.1) and NLS equation (1.1.2) are the two classical examples of generic nonlinear dispersive wave equations. Whitham introduced a series of powerful techniques to investigate problems associated with nonlinear dispersive equations and the whole set of techniques was named Whitham modulation theory [127]. The theory provides an asymptotic method for studying slowly varying periodic waves. The so called Whitham modulation equations were originally derived using averaged conservation laws, but can also be obtained by an averaged Lagrangian approach [127]. These equations are employed to describe the key parameters of nonlinear periodic waves, such as the wave amplitude, frequency, mean height and wavelength. Subsequently, the solution was fully

Figure 1.5: Experimental evolution of dispersive shock wave after 1 cm of propagation in an ethanol + iodine liquid cell. (a) The input optical beam (b) The initial breaking of the beam. (c) The quasi-steady state of the DSW 200 ms later. (d) Steady state, with asymmetry due to convection. Figure 1.5 is figure 2 from [9].

itself.
developed in the form of a slow modulated wavetrain by Luke [84].

It is not an overstatement to say that Whitham modulation theory opened a new door to a deeper theoretical understanding of nonlinear dispersive waves. The theory has a beautiful and elegant mathematical structure, as well as being a powerful analytic tool for the investigation of dispersive waves in various nonlinear media and their stability. When the Whitham modulation equations are strictly hyperbolic, the nonlinear wavetrains are modulationally stable. Then the method of characteristics can be employed and distinct characteristic velocities can be obtained and interpreted as the nonlinear generalisation of the linear group velocity. When the Whitham modulation equations are elliptic, the initial value problem becomes ill posed and the nonlinear wavetrains are modulationally unstable. Whitham applied modulation theory to the KdV equation. Later the theory was applied to many other dispersive wave equations, such as NLS equation [38, 126, 127] and Benjamin-Ono equation [44].

As DSWs are unsteady waveforms, their analysis is more challenging than for solitary waves which are steady wave forms. The key to obtaining DSW solutions was the development of Whitham modulation theory [125–127] (related to the method of multiple scales) for analysing slowly varying, nonlinear dispersive waves. The Whitham modulation equations form a system of pdes for the slowly varying parameters of a periodic wavetrain, such as amplitude, wavenumber and mean height. For hyperbolic systems a DSW solution occurs as a simple wave solution of the modulation equations. In particular, Whitham derived the modulation equations for the KdV equation and set them in a simple Riemann invariant form [126, 127]. The DSW solution of the KdV equation was subsequently found using these modulation equations by Gurevich and Pitaevsky in [63], which physically describes an undular bore. It was then verified numerically by Fornberg and Whitham [51]. Once the connection between DSW solutions and hyperbolic Whitham modulation equations was realised, the DSW solutions of other standard (integrable) nonlinear, dispersive wave equations, such as the NLS [40], the Sine-Gordon [95] and the Gardner equations [75] were found. Kamchatnov et al. [75] developed modulation theory for the Gardner equation. The technique they employed was a reduced version
of the finite-gap integration method in Riemann invariant form. The key to finding DSW solutions from modulation equations is the ability to set them in Riemann invariant form, which is guaranteed if the underlying equation is integrable [50].

If the underlying nonlinear dispersive equation is not integrable, then the determination of the Whitham modulation equations and their Riemann invariant form, if hyperbolic, is a non-trivial problem. In these cases, the modulation equations need to be solved numerically or an approximation used to obtain the amplitude of the lead soliton. Marchant and Smyth [88] developed an approximate approach to determine the amplitude of the lead solitary wave in a DSW for various nonlinear wave equations, such as the KdV equation, the modified KdV equation, the Benjamin Ono equation, the NLS equation and the nematic equation.

A significant advance in this regard was the realisation that if a DSW is of KdV type, that is the DSW consists of solitary waves at one edge, linear waves at the other, the dispersion relation is convex and the differential equation determining the periodic wave solution is of the form $u_\theta^2 = r^2(u)P(u)$, where $\theta$ is the phase, $P(u)$ is a cubic polynomial and $r(u)$ is some smooth function which does not vanish at the roots of $P(u)$, then the Whitham modulation equations have a simplified, degenerate structure at the two edges of the DSW, the solitary and linear wave limits [37, 38, 41]. The basic assumption is that the periodic wave solution has properties similar to the elliptic function cnoidal wave solutions of the KdV (and NLS) equations. The simplified structure can then be used to determine the leading and trailing edges of the DSW from the linear dispersion relation alone without knowledge of the full Whitham modulation equations [37, 38, 41]. This method for non-integrable equations, termed dispersive shock fitting, was then used to determine the leading and trailing edges of DSWs governed by non-integrable equations arising in a number of applications, including fluids, nonlinear optics and Bose-Einstein condensates [4, 28, 38, 42, 47, 67, 82]. The shock fitting method has recently been extended to nonlinear dispersive wave equations with Benjamin-Ono type dispersion, for which the equation governing the periodic wave solution is not of the form $u_\theta^2 = r^2(u)P(u)$, but the DSW is of KdV type [44]. The latest extension of the DSW fitting method was brought about by
El et al. [25]. They successfully extended the method, so that a theoretical understanding of the structure and stability of the interior of a DSW becomes possible. A few examples were considered in order to show the efficacy of the extended theory.

In many observational measurements only the solitary wave edge of a DSW can be fully resolved [113], so the restriction of El’s method to the leading and trailing edges of a DSW is not a severe one.

1.2.2 Undular bores and resonant flow

Undular bores and resonant flow are ubiquitous phenomena in the natural world and laboratory observations of them have been realised by many researchers. Among the earliest, Thews and Landweber [118, 119] successfully found that an upstream wave can be generated by a steadily proceeding ship model in towing tanks. Subsequently, Davis and Acrivos [31] created a two layer stratified fluid with fresh water sitting on top of the salt water in order to generate internal waves in a laboratory.

In order to test if the KdV equation is a good model to describe the evolution of gravity wave of moderate amplitudes, Hammack and Segur [64] compared the predictions of the KdV equation with the experimental results of long waves on shallow water with uniform depth. The comparison showed good agreement, and therefore proved the KdV equation is a suitable model that is also relatively easy to solve. More recently, Chen [18] experimentally studied the generation of internal solitary waves over inhomogeneous topography. Later, Gourlay [57] investigated experimentally transcritical flow past a ship and they obtained numerical results of the actual flow patterns in open water or confined channels. Grimshaw [60] developed the theoretical investigation of internal solitary waves as models to describe the waves in fjords, lakes and shallow coastal seas. Ostrovsky and Stepanyants [99] compared laboratory experiments with theoretical predictions from different mathematical models for internal solitary waves, and concluded that the KdV equation gave excellent agreement with experiment results and therefore is a good model to describe the properties of solitary waves with moderate amplitude.

Grimshaw and Smyth [62] developed a theoretical study of the flow of a stratified fluid
over localised topography when the flow is near resonance. A forced KdV equation was employed to describe the phenomenon and it was investigated both analytically and numerically for different cases, such as subcritical (actual water depth is greater than the critical depth) and supercritical flow (actual water depth is less than the critical depth) and resonant and non-resonant flow. A resonant flow occurs when the basic flow speed is close to a linear long-wave phase speed for one of the long-wave modes. Marchant and Smyth [86] applied Whitham modulation theory to the extended KdV equation which includes the next higher order terms in the KdV approximation. They then derived the undular bore solution of the extended KdV equation as a simple wave solution of these modulation equations. Then, they applied Whitham modulation theory to study the resonant flow of a fluid over topography where the higher order terms are modulated. All the theoretical solutions were compared with numerical solutions, as well with experimental measurements.

Grimshaw et al. [61] derived an extended higher-order KdV equation for internal solitary waves in a density and current stratified shear flow with a free surface. Then, they determined all the coefficients of this extended KdV equation as integrals of the modal function under the framework of linear long wave theory. Lastly they considered a two-layer shear flow. More recently, internal solitary waves in the ocean were described by Apel et al. [6] by using the Boussinesq and the KdV equations. They compared the theoretical solutions of the equation with experimental measurements.

Smyth and Holloway [113] applied the extended KdV equation to describe the evolution of internal waves governed by the semi-diurnal internal tide on the Australian North West Shelf. They found the flow was dominated by nonlinearity. They also showed that dispersion gives rise to an undular bore behind this flow. Then, experimental measurements were compared with the theoretical results and good agreement was found.

In 1996, Zhang and Chwang [129] found numerical solutions for two-dimensional solitary waves generated by a moving object near critical speed in a shallow water channel. A finite difference method was employed to solve the Navier-Stokes equation for generalised Newtonian incompressible inviscid fluids. Then, the numerical solutions were
compared with experimental measurements, as well as with numerical solutions of the Boussinesq equation and the forced KdV equation. It was found that the Navier-Stokes equations gave a better approximation than the other two nonlinear wave equations, as expected.

### 1.2.3 DSWs in optical media

Ever since the initial groundbreaking experiments in optical media and on ultra-cold atoms, DSWs in optical media have attracted extensive research interest, examples being photorefractive crystals [9, 42, 122, 123], nonlinear optical fibres [49], nonlinear thermal optical media [54, 124], liquid crystals [39, 112] and colloidal media [4, 89]. One of the earliest observation of DSWs in optical media was made by Rothenberg and Grischkowsky [105]. They observed the formation of a DSW within a propagating pulse in a nonlinear optical fibre. They also found good agreement between experiment results and numerical solutions of the NLS equation. More than a decade later, Couton et al. [27] observed a DSW in iron-doped LiNbO3 samples. They also found that the initial jump condition determined the number of the solitons that are formed, as well as the DSW’s quasi-steady state regime. Wan et al. [122] found that in systems such as cold plasmas, superfluids and BECs, instead of dissipative shock waves where the structures are steady, DSWs emerged due to the fact that dispersion dominates over dissipation in these systems. Then, they demonstrated an optical experimental procedure which only allows one optical signal to control another, for researching the dynamics of DSWs. This platform enabled the researchers to better observe and understand the evolution and nonlinear interactions of DSWs. Barsi et al. [9] investigated DSWs in a thermal medium comprising an ethanol and iodine liquid cell. They also found good agreement between experimental results obtained with theoretical solutions.

The formation of a DSW by a Gaussian laser beam propagating in a thermal medium comprising Rhodamine-B filled liquid cell was investigated by Ghofraniha et al. [54]. They investigated the interplay of the non-locality of the nonlinear response and the dynamics of the DSW in a system governed by the NLS equation. Lastly, the experimental
measurements were compared with the theoretical solution, with good agreement found. In 2009, Conti et al. [26] studied the dynamics of a dark optical beam propagating in a thermal medium comprising Rodhamine-B in methanol. Optical DSW formation from a standard optical beam in a thermal defocusing media was studied experimentally by Ghofraniha et al. [55]. By employing a fluorescent medium and a high aperture microscope, they were able to determine experimental results for how non-locality and nonlinearity affect the evolution of a DSW. They then compared the measurements with numerical simulations.

A Bose-Einstein condensate (BEC) is a quantum fluid that shows similar behaviour to a compressible gas in which shock waves occur. In 2001, Dutton et al. [33] observed DSWs in a Bose-Einstein condensate by using the slow light technique. They showed the induced small density defects evolved into sound waves with large amplitude. Then, they investigated the subsequent breakdown of the superfluid and found solitons arising from the density defects, as well as the nucleation of vortices. A few years later, Simula et al. [111] employed repulsive laser potential pulses to Bose-Einstein condensates and observed DSWs. Depending on the experimental parameters, Chang et al. [17] observed DSWs experimentally in a BEC. Then, when a BEC was split, they observed the transition of a sound wave to a DSW and studied this transition by applying increasingly stronger splitting barriers.

El et al. [42] developed the DSW fitting method to investigate the dynamics of optical DSWs generated by the propagation of light beams through photorefractive media. The DSW fitting method was based on Whitham modulation theory but allowed analytical solutions to be found for non-integrable systems. El et al. [69] showed that dissipative shock wave solutions do not explain BEC DSW phenomenon. Instead, they found that small dispersion should be added to the wave dynamics and this explained the generation of a DSW. The interplay between two merging BECs was investigated by Hoefer et al. [70]. They used the NLS equation as the governing equation and an asymptotic analysis of the NLS equation in the semi-classical limit, as well as numerical simulations to show the interplay. The induced wave pattern can be explained as two rarefaction waves.
moving through a vacuum and interacting with each other, then generating modulated solitons. Lastly, they used Whitham modulation theory to analyse the soliton wave trains.

Smyth [112] investigated the resonant DSWs which arise when a light beam propagates through a nematic liquid crystal by using the DSW fitting method. The equations governing optical beams in a nematic liquid crystal are coupled equations of an NLS type equation for the optical beam and an elliptic medium response equation. The author found that besides the standard defocusing NLS-type DSW solution consisting of dark waves, a KdV-type resonant DSW also occurred, consisting of bright waves when the nematic response was highly nonlocal. Smyth also found that when the initial jump is below a certain height there was no resonant wavetrain. El and Smyth [39] found that the DSW governed by the nematic equation has positive polarity and determined the generated resonant radiation moving ahead of it. More importantly, they showed the classical shock wave velocity can be used to determine the lead soliton edge speed of the DSW and when the initial jump is under a certain value, the nematic DSW can be asymptotically described by a 5th order KdV equation.

1.2.4 DSW for magma flow

A fluid driven by viscous forces can also lead to the generation of a DSW. Spiegelman [114, 115] provided a thorough physical understanding of the equations governing magma migration. Spiegelman applied classic shock wave theory to the problem to show that viscous resistance to volume changes gives rise to the transition from simple shocks to DSWs. Elperin et al. [46] employed the KdV equation, which was a small amplitude reduction of the magma equation, to study the phenomenon of a DSW. McKenzie [93] proposed that in 1-D case the magma flow can be described by

$$f_t = [(1 - f_0 f) w]_x, \quad w_{xx} = w f^{-n} + \frac{1 - f_0 f}{1 - f_0}, \quad (1.2.1)$$

where $f$ is defined as the melt fraction, $f_0$ is the background melt fraction, $w$ is defined as the vertical velocity of the solid matrix and $n$ is determined by the material constitutive
Marchant and Smyth [87] considered the flow of magma governed by equation (1.2.2) which has periodic traveling wave solutions with one limit as a solitary wave. This solitary wave limit is called a magmon. Whitham modulation theory was employed to derive the modulation equations for the magma equation and it was found the modulation equations can either be hyperbolic or mixed hyperbolic/elliptic depending on the specific values of the wavenumber, mean height and amplitude of the modulated waves.

Lowman and Hoefer [82] investigated the case where a DSW was formed in viscously deformable media governed by a third-order nonlinear dispersive wave equation. The DSW fitting approach was used to determine the trailing and leading edge speeds, as well as the leading edge amplitude of the DSW. They also found backflow and gradient catastrophe in the Whitham modulation equations. Lastly, they compared the analytical results with numerical simulation and excellent agreement was found.

Hoefer et al. [83] investigated the interactions of large amplitude solitary waves in viscous fluid conduits. Unlike the weakly nonlinear solitary waves, strongly nonlinear solitary water wave interactions have not been paid as much attention. They applied both numerical and experimental approaches to the problem and were able to classify the interactions into three classes, purely bimodal, purely unimodal and a mixed type. Hoefer et al. [85] observed the interactions of DSWs and solitary waves in viscous fluid conduits and found that the refraction or absorption of solitary waves by DSWs, multiphase evolution and DSW merging are common, ubiquitous properties of dispersive hydrodynamics.

1.3 Colloidal optical media

The nonlinear interaction of a light beam with soft matter has been studied for decades [90, 91, 117]. A colloidal medium consists of a suspension of dielectric nanoparticles,
which migrate towards regions of high light intensity because of the optical gradient force [7, 89–91]. This leads to a change of the intensity-dependent refractive index, which induces an interplay between light and the colloidal nanoparticles. Colloidal media have various applications such as optical sensors, selective particle trapping and manipulation [29].

The equation for wave propagation in colloidal media is an NLS-type equation accompanied to a medium equation. The nonlinear term in the NLS-type equation depends on the particle interactions. In the 1-D case the equations governing the nonlinear propagation of a light beam through a colloidal media and the medium response equation are

$$i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - (\eta - \eta_0) u = 0, \quad |u|^2 = g(\eta) - g_0, \quad g(\eta) = \frac{3 - \eta}{(1 - \eta)^3} + \ln \eta.$$  \hspace{1cm} (1.3.1)

Here $u$ is the complex envelope of the electric field of the light and $\eta$ is the packing fraction of the colloidal particles, with $\eta_0$ the background fraction, $g(\eta)$ the Carnahan-Starling hard sphere formula [90, 91].

It was found that a balance between the light beam diffraction and the interplay between particle and light gives rise to spatial solitary waves. The first model of light propagation in colloids considering colloidal particle interaction was proposed by Kivshar et al [90, 91], with assuming that the colloidal suspension can be approximated by a hard-sphere gas. The Carnahan-Starling formula was used for the compressibility of the hard-sphere colloid. Then, the 1-D and 2-D colloidal equations were considered and the solitary wave propagation constant versus power curves was derived numerically. Also, bistable behaviour, arising for some parameter values, was shown. They also found significant differences in interaction behaviours for solitons from the same and different branches. In the 2-D case only two solution branches can occur one stable and the other unstable and the bistable behaviour of the 1-D colloidal solitary wave is absent.
1.4 The Whitham equation

The KdV equation (1.1.1) has been studied more than a century ever since it was derived by Korteweg and de Vries [80] in 1895 to model waves on shallow water. It can be exactly solved for a wide range of initial conditions using the IST [2, 127]. It is well known that water waves peak (and break) when their amplitude to depth ratio is large. The nonlinear shallow water equations, which neglect dispersion, show typical hyperbolic wave breaking [127]. If long wave dispersion is added to the shallow water equations, the Boussinesq and KdV equations arise [127], which possess soliton solutions. However, solutions of these equations show no breaking due to the dispersion being too strong for short waves. In addition, Stokes showed that there is a wave of maximum amplitude for steady water waves and that this wave has a sharp peak of angle 120° [116]. Again, the KdV and Boussinesq equations do not reproduce this behaviour as they possess solutions of arbitrary amplitude. The full water wave peaking is due to short wave effects, beyond the Boussinesq and KdV asymptotic approximations [127].

To understand the connection between nonlinearity, dispersion, peaking and breaking, Whitham introduced model nonlinear dispersive equations which incorporate KdV type quadratic nonlinearity and full dispersion, in particular full linear water wave dispersion [51, 127]. The original Whitham equation which combined the quadratic nonlinearity of the KdV equation with the full dispersion of gravity waves on a fluid of finite depth is

\[
\left(1.4.1\right)
\]

where \( F \) denotes the Fourier transform and the acceleration due to gravity and depth have been normalised to 1.

It was found that full water wave dispersion gives rise to a wave of greatest height, but with a cusp, not a sharp peak of 120°. To understand why the water wave kernel resulted in a wave of greatest height with a cusp and not a peak, Whitham introduced an exponential approximation to the water wave kernel which was found to give a wave of greatest height with a peak of angle 110°. Whitham also studied other dispersive
Fourier kernels which give rise to peaked waves and breaking for sufficiently asymmetric waves [110, 127]. Nonlinear dispersive equations with KdV nonlinearity and full Fourier dispersion have subsequently been termed Whitham equations and have been found useful to study peaking and breaking effects present in the full water wave equations, but based on a much simpler model equation [96]. More recent work has shown the existence of travelling and solitary wave solutions of the Whitham equation with various dispersive kernels [34, 35, 96].

An equal soliton approximation method was employed by Cadiot [16] to model DSWs for a Whitham equation. By assuming a bore consists of a train of equal amplitude solitary waves, he determined the solitary wave edge amplitude for the DSW based on mass and energy conservation equations. The dispersion relation for the Whitham equation can also be extended beyond surface water waves to internal waves in a stratified fluid [73].

1.5 Quadratic media

The widespread development and use of lasers in the early 1960s gave rise to the growth in research in nonlinear optics. Thanks to this technology, experimental physicists were able to apply a powerful coherent light beam and unravel some of the mysteries in this field. Second harmonic generation (SHG), also known as frequency doubling, was one of the nonlinear optical effects discovered at that time. SHG can be succinctly described using the following experiment. Imagine someone has an optical material with a $\chi^{(2)}$ (quadratic) nonlinear response. When a laser beam with frequency $\omega$ is incident on an optical material a laser beam with mixed frequencies $\omega$ and $2\omega$ is then detected in the scattered field.

The renaissance of interest in this field was brought about by Belashenkov et al. [10] and DeSalvo et al. [32]. Their work showed that the cascaded $\chi^{(2)}$ effect can induce quadratic materials to behave similarly to conventional Kerr materials. Thus, previously well understood nonlinear phenomena for Kerr materials, such as modulational instability, self-focusing and parametric optical solitons could also be analysed for $\chi^{(2)}$ materials. The first experimental reports for quadratic optical solitons were made by Torruellas et
al. [121] and Schiek et al. [94]. Subsequently, the observation of optical solitons by the mutual trapping of interacting waves was found by Kivshar and Buryak [14], as well as Baek et al. [108]. Later, 2-D solitary waves due to quadratic nonlinearities were investigated by Torner et al. [120]. In 2012, Trillo et al. [22] investigated wave breaking and DSW formation due to a pulse undergoing SHG in a quadratic medium. They showed the accessibility of the DSW in the limit of a large phase match and weak dispersion.

There has been a vast number of relevant equations describing optical solitary waves due to quadratic nonlinearities [15]. These equations are categorised as different types based on the so called phase-matching techniques (PMTs). The PMTs are an essential tool to give rise to efficient wave mixing and $\chi^{(2)}$ soliton formation. The simplest case of type-I SHG without walk-off between the harmonic waves is described by the normalised system [15]

$$
\frac{i}{\sigma} \frac{\partial u_1}{\partial z} + \frac{r}{2} \frac{\partial^2 u_1}{\partial x^2} + \beta u_1 - u_2 u_1^* = 0, \\
\frac{i}{\sigma} \frac{\partial u_2}{\partial z} + \frac{s}{2} \frac{\partial^2 u_2}{\partial x^2} - \alpha u_2 + \frac{u_1^2}{2} = 0,
$$

(1.5.1)

where, $\alpha = (2 + \Delta k)\sigma / \beta$ and $\Delta k = 2k_1 - k_2$,

where $u_1$ and $u_2$ are the renormalised slowly varying complex envelope of the first harmonic and second harmonic with frequencies $\omega_1$ and $\omega_2$. $\alpha$ is a normalised wave number mismatch, while $k_1$ and $k_2$ are wavenumbers of $u_1$ and $u_2$. The coefficients $r$ and $s$ are medium parameter values. In the purely spatial case, the $z$ direction is the propagation direction, and the parameter values $\sigma = 2, r = s = \pm 1$ are used, in order for the system to be modulationally stable. In the temporal case, the values of $r$ and $s$ can either be $+1$ or $-1$ and $\sigma$ can be any value [15].

1.6 Structure of thesis

This thesis consists of five chapters. Chapter 2 investigates the dynamics of an optical DSW in a defocussing colloidal medium. The equations governing nonlinear light prop-
agation in a colloidal medium consist of an NLS-type equation for the beam and an algebraic equation for the medium response. For low light intensity, these equations reduce to a perturbed higher order NLS equation. Results for the leading and trailing edge speeds and the amplitude of the solitary wave edge are found using El’s DSW fitting theory [37, 38]. The analytical and numerical results are found to be in good agreement.

Chapter 3 considers DSWs governed by the nonlocal Whitham equation in order to explore short wavelength effects that lead to peaked cusped waves within the DSW. This is done by combining the weak nonlinearity of the KdV equation with full linear dispersion relations. The dispersion relations considered are those for surface gravity waves, the intermediate long wave equation and a model dispersion relation introduced by Whitham to investigate the 120° peaked Stokes wave of highest amplitude. The dispersive shock fitting method is used to find the leading (solitary wave) and trailing (linear wave) edges of the DSW. This method is found to produce results in excellent agreement with numerical solutions up until the lead solitary wave of the DSW reaches its highest amplitude. Numerical solutions show that the DSWs for the water wave and Whitham peaking kernels become modulationally unstable and evolve into multi-phase wavetrains after a critical amplitude which is just below the DSW of maximum amplitude.

Chapter 4 studies DSWs in quadratic media. We find that in order to realise a DSW in a quadratic medium, a phase locking assumption is necessary. The DSW fitting technique is again applied to determine the leading and trailing edges of the DSW. The analytical solutions are compared with numerical solutions. Conclusions and remarks are discussed in Chapter 5. Finally, appendices and references are at the end of this thesis.
Chapter 2

Optical dispersive shock waves in defocusing colloidal media

2.1 Introduction

In this chapter the propagation of an optical DSW in the nonlinear colloidal medium will be studied. The results of this chapter appear in An et al. [4]. The equations governing optical beam propagation in a colloid consist of an NLS-type equation for the beam coupled to an algebraic equation for the concentration of the colloid particles which depends on the beam intensity [90, 91]. In the limit of low light intensity, these equations can be asymptotically reduced to a higher order NLS equation. While a colloid is normally a focusing medium, so that its refractive index increases with beam intensity, it can also be a defocusing medium [48, 117], which then supports a DSW consisting of dark solitary waves at the trailing edge and linear waves at the leading edge [7, 38, 40]. The DSW is generated by a jump initial condition in optical beam intensity. While there have been previous studies of DSW in colloids [88, 89], these have been for focusing colloids. In this case the waves of the DSW are modulationally unstable, so that the DSW structure has only a finite propagation length before becoming unstable. This is not the case for a defocusing colloidal medium. The leading and trailing edges of the colloid DSW are determined using El’s method [37, 38, 41] based on both the full colloid equations and a
higher order NLS equation, in the limit of low beam intensity. These modulation theory solutions are compared with full numerical solutions of the governing colloid equations. As well as determining the accuracy of modulation theory, these comparisons also determine the applicability of the low light intensity limit of the colloid equations.

### 2.2 Colloid equations

Let us consider the propagation of a polarised optical beam through a colloidal suspension. DSWs in nonlinear optical media are governed by NLS-type equations and, in the simplest approximation, are governed by \((1 + 1) - D\) equations \([38, 78]\). Higher dimensional \((2 + 1) - D\) DSWs governed by NLS-type equations are much more difficult to analyse and need a non-trivial azimuthal vortex structure to be stable. Indeed, \((2 + 1) - D\) DSWs, governed by not just NLS-type equations, but any nonlinear wave equation, are an open topic \([3, 77]\).

Hence, the optical beam generating the colloid DSW will be assumed to have a plane front. The \(z\) direction will then be taken to be the propagation direction, with the \(x\) direction orthogonal to this and the beam having no \(y\) dependence. The concentration of the colloid particles has a nonlinear dependence on the beam intensity. The colloid can be either a focusing medium, so that its refractive index increases with beam intensity \([45, 90]\), or defocusing, so that it decreases with intensity \([48, 117]\). In order for a stable DSW to be generated, the colloidal medium will be assumed to be defocusing. Let us denote the concentration of the colloidal particles by \(\eta\), with \(\eta_0\) the constant background concentration in the absence of the optical beam. In the slowly varying, paraxial approximation the non-dimensional equations governing the propagation of the optical beam through the colloidal suspension are then \([90, 91]\)

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - (\eta - \eta_0)u = 0, \tag{2.2.1}
\]
with the equation of state, that is the medium response equation,

\[ |u|^2 = g(\eta) - g_0, \quad g(\eta) = \frac{3 - \eta}{(1 - \eta)^3} + \ln \eta. \]  

(2.2.2)

Here \( u \) is the complex valued envelope of the electric field of the optical beam and \( g_0 = g(\eta_0) \). The Carnahan-Starling compressibility approximation has been used for the state relation \( g \). Alternative models for the compressibility alter the form of \( g \). The Carnahan-Starling approximation is valid up to the solid-fluid transition, which occurs at \( \eta = \sqrt{2\pi/9} \approx 0.496 \) in a hard-sphere fluid [65]. It should be noted that the nonlinear term in the NLS equation (2.2.1) for the optical beam has a negative coefficient, so that the equation is defocussing, in contrast to the focussing equation of previous work [90, 91]. The inequality \( \eta - \eta_0 > 0 \) was discussed and shown in their work too [91].

Hoefer [68] considered general properties of DSW solutions of generalised NLS equations of the form

\[ i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - f(|u|^2)u = 0, \]  

(2.2.3)

with details determined for a power law nonlinearity \( f \). While the colloid system (2.2.1) and (2.2.2) is in principle of this form, the nonlinearity \( f \) cannot be explicitly determined from the medium response equation (2.2.2). The colloid system then represents a further extension of the forms of nonlinear response in generalised NLS equations and the DSW solutions for such equations, in addition to the previously studied extension of a nonlocal response [39, 112].

The simplest initial condition which will result in the generation of a DSW is a step initial condition in optical intensity,

\[ u(x, 0) = \begin{cases} 
    u_-, & x < 0, \\
    u_+, & x > 0.
\end{cases} \]  

(2.2.4)

We require that \( u_- > u_+ \geq 0 \) in order for the initial condition to be breaking so that a DSW is generated, in contrast to an expansion wave for \( u_- < u_+ \). The medium equation
(2.2.2) then shows that the initial concentration of colloid particles is

\[
\eta(x, 0) = \begin{cases} 
\eta_-, & x < 0 \\
\eta_+, & x > 0 
\end{cases},
\]

(2.2.5)

where \(\eta_-\) is the concentration generated by the beam intensity \(u_-\) and \(\eta_+\) by the intensity \(u_+\). The intensity \(u\) and packing fraction are related by the state equation (2.2.2).

### 2.3 Low light intensity limit

The colloid equations (2.2.1) and (2.2.2) can be simplified in the limit in which the intensity \(|u|^2\) of the light beam is low as the medium equation (2.2.2) can then be asymptotically solved for the concentration \(\eta\) in terms of \(|u|^2\). Let us then set the concentration as

\[
\eta = \eta_0 + \eta_1|u|^2 + \eta_2|u|^4 + \ldots
\]

(2.3.1)

The low intensity limit is equivalent to \(|\eta - \eta_0| \ll 1\). The medium equation (2.2.2) can then be inverted to give

\[
\eta - \eta_0 = \frac{|u|^2}{g'(\eta_0)} - \frac{g''(\eta_0)}{2g'(\eta_0)^3} |u|^4 + \ldots,
\]

(2.3.2)

so that the electric field equation (2.2.1) becomes

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \left( g'(\eta_0) \right)^{-1} |u|^2 u + \frac{g''(\eta_0)}{2g'(\eta_0)^3} |u|^4 u = 0\]

(2.3.3)

to second order in \(|u|^2\). This is an NLS equation with a fifth order nonlinearity correction, which has been extensively studied [28, 42, 78]. It can be set in the standard form

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - |u|^2 u + \alpha |u|^4 u = 0
\]

(2.3.4)
by using the rescaled variables

\[ z = g'(\eta_0) z', \quad x = \sqrt{g'/(\eta_0)} x', \quad \alpha = \frac{g''(\eta_0)}{2g'(\eta_0)^2}, \]  

(2.3.5)

as the derivative \( g'(\eta_0) \) of the constitutive law is positive.

The linear periodic wave solution of the higher order NLS equation (2.3.4) is

\[ u = u_\infty e^{-i(u_\infty^2 - \alpha u_\infty^4) z'}. \]  

(2.3.6)

Using this wave as the background carrier wave, we then find the grey solitary wave solution of the higher order NLS equation (2.3.4) as

\[ u = B \tanh \left( \frac{\alpha}{3} \right) \left( B^3 + A^2B \right) \tanh C \theta - \frac{\alpha}{3} B^3 \tanh^3 C \theta 
+ iA - i\frac{\alpha}{3} AB^2 \tanh^2 C \theta \right] e^{-i(u_\infty^2 - \alpha u_\infty^4) z' + iv x' + \ldots}, \]  

where

\[ C = B - \alpha \left( B^3 + \frac{4}{3} A^2B \right), \quad A^2 + B^2 = u_\infty^2, \quad \theta = x' - \left( A + V - \alpha \left( \frac{2}{3} AB^2 + A^3 \right) \right) z'. \]  

(2.3.7)

\( V \) is velocity of the background level. In the limit \( \alpha = 0 \), the NLS grey soliton is obtained.

This asymptotic solitary wave solution will be used in Section 2.6 to derive the amplitude of the leading solitary wave of the DSW in the limit of low light intensity.

Figure 2.1 shows a typical DSW solution of the colloid equations (2.2.1) and (2.2.2). The parameters are \( u_- = 1.0, \ u_+ = 0.5 \) and \( \eta_0 = 0.01 \). Shown is a contour plot of the numerical simulation of (a) the electric field amplitude \( |u| \) and (b) the packing fraction \( \eta \).

It can be seen that the solution consists of four sections. The first is the original levels \( u_+ \) and \( u_- \) ahead of and behind the DSW. The second is an intermediate shelf \( u_i \) behind the actual DSW with \( u_+ < u_i < u_- \), with the third section a dispersionless level change linking the intermediate shelf \( u_i \) to the level behind \( u_- \). It should be noted that there is a small amplitude wavetrain on the level \( u_- \). This wavetrain is due to the smoothing of the discontinuity in derivative at the point the expansion wave links the level \( u_- \) [51]. The final, fourth, section is the actual DSW linking the intermediate shelf \( u_i \) to the level \( u_+ \) ahead. Each of these portions of the solution will be discussed in the following sections.
Figure 2.1: DSW solution of the colloid equations (2.2.1) and (2.2.2). The parameters are $u_- = 1.0, u_+ = 0.5$ and $\eta_0 = 0.01$. Shown are contour plots of the numerical solution of (a) the electric field amplitude $|\nu|$ and (b) the packing fraction $\eta$. 
2.4 Dispersionless Limit

As can be seen from Figure 2.1, a DSW consists of two main wave forms, the DSW itself for which dispersion balances nonlinearity, and the region away from the DSW for which dispersion can be neglected [37, 38, 41]. To understand the solution in these two regions, it is simplest to set the colloid equations (2.2.1) and (2.2.2) in the so-called hydrodynamic form via the Madelung transformation

\[ u = \sqrt{\rho} e^{i\phi}, \quad v = \phi_x. \]  

(2.4.1)

This transformation then sets the colloid equations in the form

\[ \frac{\partial \rho}{\partial z} + \frac{\partial}{\partial x} (\rho v) = 0, \]  

(2.4.2)

\[ \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} + \frac{\rho_{xx}}{4\rho} - \frac{\rho_x^2}{8\rho^3} + \frac{\partial \eta}{\partial x} = 0, \]  

(2.4.3)

where

\[ \rho = g(\eta) - g(\eta_0). \]  

(2.4.4)

The system (2.4.2) and (2.4.3) is referred to as the hydrodynamic form as it is similar to the shallow water equations [127], with the first of equation (2.4.2) that for mass conservation and the second (2.4.3) that for momentum conservation in this context. The initial condition is (2.2.4) and (2.2.5).

Away from the DSW, the solution is approximately non-dispersive as there are no significant waves, as seen from the typical solution displayed in Figure 2.1. The non-dispersive limit of the hydrodynamic equations (2.4.2) and (2.4.3) is

\[ \frac{\partial \rho}{\partial z} + \frac{\partial}{\partial x} (\rho v) = 0, \quad \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} + \frac{1}{g'(\eta)} \frac{\partial \rho}{\partial x} = 0, \quad \frac{\partial \eta}{\partial x} = \frac{1}{g'(\eta)} \frac{\partial \rho}{\partial x}. \]  

(2.4.5)
In Riemann invariant form, these equations are

\[
\begin{align*}
\frac{dx}{dz} &= V_+ = v + \sqrt{\frac{\rho}{g'(\eta)}}, \\
v - \int_{\rho_-}^{\rho} \frac{d\rho}{\sqrt{\rho g'(\eta)}} &= \text{constant on } C_-: \frac{dx}{dz} = V_- = v - \sqrt{\frac{\rho}{g'(\eta)}},
\end{align*}
\]

(2.4.6)

(2.4.7)

The dispersionless equations can be used to determine the level \(u_i\) of the intermediate shelf seen in Figure 2.1. As noted above, the levels \(u_-\) and \(u_i\) are linked by an expansion wave, which is an expansion fan on the characteristic \(C_-\) with the Riemann invariant on \(C_+\) constant in the fan. The intermediate shelf \(u_i\) terminates at the DSW, whose trailing edge has the velocity \(v_-\). This simple wave solution is

\[
(\rho,v) = \begin{cases} 
(\rho_-,0), & \frac{x}{z} < -H_- , \\
(\rho_f,M(\rho_-,\rho_f)), & -H_- \leq \frac{x}{z} \leq M(\rho_-,\rho_i) - H_i , \\
(\rho_i,M(\rho_-,\rho_i)), & M(\rho_-,\rho_i) - H_i < \frac{x}{z} < v_- .
\end{cases}
\]

(2.4.8)

The expressions for the edges of the expansion fan are involved and are

\[
\begin{align*}
H_- &= \sqrt{\frac{\rho_-}{g'(\eta_-)}}, & M(\rho_-,\rho_i) &= \int_{\rho_i}^{\rho_-} \frac{d\rho}{\sqrt{\rho g'(\eta)}} , \\
H_i &= \sqrt{\frac{\rho_i}{g'(\eta_i)}}, & M(\rho_-,\rho_f) &= \int_{\rho_f}^{\rho_-} \frac{d\rho}{\sqrt{\rho g'(\eta)}}.
\end{align*}
\]

(2.4.9)

The intensity \(\rho = \rho_f\) on the expansion fan is the solution of

\[
\rho_f = g'(\eta_f) \left[ M(\rho_-,\rho_f) - \frac{x}{z} \right]^2.
\]

(2.4.10)

This simple wave solution is not complete as the level \(u_i = \sqrt{\rho_i}\) of the intermediate shelf is not yet determined. The intermediate level \(\rho_i\) is found by the requirement that the Riemann invariant along the characteristic \(C_-\) is constant through the DSW, which links the intermediate level to the level \(u_+ = \sqrt{\rho_+}\) ahead of the DSW [37, 38, 41]. Hence the
wavenumber $v$ on the intermediate shelf is

$$v_i = M(\rho_i, \rho_+). \quad (2.4.11)$$

Matching this value of $v_i$ with the value given by the simple wave solution (2.4.8) finally gives that $\rho_i$ is determined by the solution of

$$M(\rho_-, \rho_i) = M(\rho_i, \rho_+). \quad (2.4.12)$$

This completes the solution for the colloid DSW, except for the DSW itself.

In the low light intensity limit the Riemann invariant form (2.4.6) and (2.4.7) of the non-dispersive hydrodynamic equations can be approximated by

$$2\sqrt{\rho} - \frac{2}{3} \alpha \rho^{3/2} + v = \text{constant on } C_+: \frac{dx'}{dz'} = v + \sqrt{\rho}(1 + \alpha \rho), \quad (2.4.13)$$

$$2\sqrt{\rho} - \frac{2}{3} \alpha \rho^{3/2} - v = \text{constant on } C_-: \frac{dx'}{dz'} = v - \sqrt{\rho}(1 + \alpha \rho). \quad (2.4.14)$$

The expansion fan solution (2.4.8) then becomes

$$\sqrt{\rho} = \begin{cases} 
    u_-, & Q < \frac{x}{z}, \\
    \rho_{\text{int}}, & \frac{x}{z} \leq Q \leq G, \\
    \sqrt{\rho_i}, & G < \frac{x}{z} < v_-, 
\end{cases} \quad (2.4.15)$$

with the limits of the expansion fan given by

$$Q = -\frac{u_- + \alpha u_+^3}{\sqrt{g'(\eta_0)}}, \quad G = \frac{2u_- - 3\sqrt{\rho_i} - \frac{\alpha}{3} \left(2u_+^3 + \rho_i^{3/2}\right)}{\sqrt{g'(\eta_0)}}. \quad (2.4.16)$$

The corresponding wavenumber $v$ is determined by the intensity $\rho$ given by (2.4.15) and is

$$v = 2u_- - 2\sqrt{\rho} - \frac{2\alpha}{3} \left(u_+^3 - \rho^{3/2}\right). \quad (2.4.17)$$
In the expansion fan the intensity $\rho = \rho_{\text{ref}}$ is given by the solution of

$$3 \sqrt{\rho_{\text{ref}}} + \frac{1}{3} \alpha \rho_{\text{ref}}^{3/2} = \left( 2u_- - \frac{2}{3} \alpha u_-^3 \right) - \frac{x}{\bar{z}} \frac{1}{g'(\eta_0)},$$  

(2.4.18)

which completes the non-dispersion portion of the solution in the low light intensity limit.

In the same low intensity limit, equation (2.4.12) for the intermediate level $\rho_i$ can be solved to give

$$\sqrt{\rho_i} = \sqrt{\rho_- + \sqrt{\rho_+}} + \frac{\alpha}{12} \left[ \rho_-^{3/2} + \rho_+^{3/2} - \frac{1}{4} (\sqrt{\rho_-} + \sqrt{\rho_+})^3 \right].$$  

(2.4.19)

The wavenumber $v_i$ on the intermediate shelf, given by (2.4.11), similarly becomes

$$v_i = \sqrt{\rho_-} - \sqrt{\rho_+} - \frac{\alpha}{12} \left[ 10\rho_-^{3/2} + 2\rho_+^{3/2} - \frac{3}{2} (\sqrt{\rho_-} + \sqrt{\rho_+})^3 \right].$$  

(2.4.20)

This solution for the intermediate level is also needed for the determination of the leading and trailing edges of the DSW as the trailing edge of the DSW arises from this shelf, see Figure 2.1.

### 2.5 Dam Break Solution

The simplest solution of the colloid equations (2.2.1) and (2.2.2) with the jump initial condition (2.2.4) is for $u_+ = 0$. In this limit, the intermediate shelf $u_i$ disappears and there is only the expansion wave of the previous section linking the level $u_-$ to $u_+ = 0$. The solution is then just the classical dam break solution of shallow water wave theory [127]. The expansion fan solution (2.4.8) then becomes

$$(\rho, v) = \begin{cases} 
(\rho_-, 0), & \frac{x}{\bar{z}} < -H_- \\
(\rho_f, M(\rho_-, \rho_f)), & -H_- \leq M(\rho_-, 0) \\
(0, M(\rho_-, 0)), & \frac{x}{\bar{z}} > M(\rho_-, 0) 
\end{cases}$$  

(2.5.1)
Figure 2.2: The intensity $\rho$ versus $x$ at $z = 1500$. The parameters are $\rho_- = 1.0$, $\rho_+ = 0.0$ and $\eta_0 = 0.01$. Shown are the numerical solution of the colloid equations (2.2.1) and (2.2.2) for the initial jump condition (2.2.4) and the dam break solution (2.5.1) of colloid equation and (2.5.2) of the higher order NLS equation. Initial $\rho$: (dark blue) dot-dash line; numerical solution of colloid equation: (red) solid line; modulation solution of higher order NLS: pink (dotted) line; modulation solution of colloid equation: (green) dashed line.

where, again, $\rho_f$ is the solution of (2.4.10). As the intermediate shelf has disappeared, there is no DSW as no wavetrain is needed to bring the solution down from $u_i$ to $u_+$. In the low light intensity limit, this dam break solution becomes

\[
\sqrt{\rho} = \begin{cases} 
  u_-, & Q < \frac{x}{z}, \\
  \rho_{f\text{uls}}, & Q \leq \frac{x}{z} \leq G_{\text{dam}}, \\
  0, & \frac{x}{z} > G_{\text{dam}}.
\end{cases}
\] (2.5.2)

Here $Q$ and $G_{\text{dam}}$ are given by (2.4.16) and $v$ is given by (2.4.17), with $\rho_i = 0$.

Figure 2.2 shows the intensity $\rho$ versus $x$ at $z = 1500$. The parameters are $\rho_- = 1.0$, $\rho_+ = 0.0$ and $\eta_0 = 0.01$, so that the dam break solution applies. Compared are the numerical solution of the colloid equations (2.2.1) and (2.2.2) and the dam break solutions of the higher order NLS equation (2.4.15) and the colloid equations (2.5.1). It can be seen that there is excellent agreement between the numerical solution and the dam break solution of colloid equation, while the higher order NLS solution shows clear disagreement, indicating the importance of higher order terms in the colloid equations (2.2.1) and (2.2.2). The
expansion fan itself lies in the region $-230 < x < 347$ as predicted by dam break solution
the colloid equations and $-144 < x < 288$ by the dam break solution of the higher order
NLS equation. The colloid expansion fan region is 25% longer than the higher order NLS
one. The disagreement can be explained by the size of $|\alpha|$. As $\alpha = -0.425$, neglected
terms of $O(\alpha^2)$ and higher are important. It can be seen that there is a small amplitude
dispersive wavetrain generated at the trailing edge of the expansion fan which is not ac-
counted for by the present modulation theory. This wavetrain is due to the discontinuity
in the derivative of the dam break solution at the trailing edge of the expansion fan and the
flat level $u_-$ behind the trailing edge. This small amplitude wavetrain is a general feature
of expansion fan solutions in modulation theory and acts to smooth out this discontinuity
[51].

2.6 Higher order NLS DSW

In general, when the level ahead $u_i$ is non-zero there is a DSW linking this level with an
intermediate level of height $u_i$, as seen in Figure 2.1. There is no known periodic wave
solution of the colloid equations (2.2.1) and (2.2.2), and hence no basis to calculate the
full Whitham modulation equations, from which the DSW solution could be determined.
It is then convenient to use the method of El [37, 38, 41] to determine the leading and
trailing edges of this DSW. This will be done in Section 2.7. However, due to the highly
nonlinear constitutive relation (2.2.2), the exact solution of El’s equations cannot be found
and they have to be solved numerically. In this section, El’s method will be used for the
higher order NLS equation (2.3.4) as the leading and trailing edge equations can be solved
exactly in this low intensity limit. It should be noted that El’s method has been applied
to similar higher order NLS equations in previous work [28, 42]. The higher order NLS
equation (2.3.3) differs from that used in this previous work only in the scaling of the cubic
and quintic nonlinear terms due to the colloid constitutive relation (2.2.2). The present
DSW solution for the leading and trailing edges for the higher order NLS equation (2.3.3)
is then a rescaled version of that of this previous work.

El [37, 38, 41] showed that the leading and trailing edges of a DSW can be determined
from the linear dispersion relation for the relevant equation. In hydrodynamic form the higher order NLS equation (2.3.4) is

$$\frac{\partial \rho}{\partial z'} + \frac{\partial}{\partial x'} \left( \rho v \right) = 0, \quad \frac{\partial v}{\partial z'} + v \frac{\partial v}{\partial x'} + (1 - 2 \alpha \rho) \frac{\partial \rho}{\partial x'} - \frac{\partial}{\partial x'} \left( \frac{\rho x'}{4 \rho} - \frac{\rho^2}{8 \rho^2} \right) = 0. \quad (2.6.1)$$

We then seek a linear travelling wave solution

$$\rho = \bar{\rho} + \rho_1 e^{i(kx' - \omega z')}, \quad v = \bar{v} + v_1 e^{i(kx' - \omega z')}, \quad (2.6.2)$$

of these equations, where $|\rho_1| \ll \bar{\rho}$ and $|v_1| \ll |\bar{v}|$. $\bar{\rho}$ and $\bar{v}$ are defined as the background levels of $\rho$ and $v$. This gives the dispersion relation

$$\omega = \bar{v} k + k \sqrt{\bar{\rho} (1 - 2 \alpha \bar{\rho}) + \frac{1}{4} k^2} = \bar{v} k + k \sqrt{\bar{\rho} + \frac{1}{4} k^2 - \alpha \bar{\rho}^2 k \left( \bar{\rho} + \frac{1}{4} k^2 \right)^{-1/2}} + \ldots \quad (2.6.3)$$

The determination of the leading and trailing edges of a DSW results from a matching of these edges with the dispersionless solution away from the DSW.

### 2.6.1 Linear wave edge of DSW

El [37, 38, 41] gives that the linear wave edge of the DSW, the leading edge, is determined by the differential equation

$$\frac{dk}{d \bar{\rho}} = \frac{\partial \omega}{\partial \bar{\rho}} \left( V_+ - \frac{\partial \omega}{\partial k} \right)^{-1}, \quad (2.6.4)$$

where $V_+$ is the velocity of the forward propagating characteristic (2.4.6). At the edges of the DSW, $\bar{v}$ and $\bar{\rho}$ are related due to the coincidence of two characteristics of the full modulation equations [37, 38, 41]. This relation is determined by the DSW jump condition which requires that the Riemann invariant on the $C_-$ characteristic is the same at both ends of the DSW. This gives

$$\bar{v} = 2 \sqrt{\bar{\rho}} - \frac{2}{3} \alpha \rho^{3/2} - 2 \sqrt{\rho_+} + \frac{2}{3} \alpha \rho_+^{3/2}, \quad (2.6.5)$$
as \( v = 0 \) and \( \rho = \rho_+ \) ahead of the DSW.

The equation (2.6.4) for the linear edge of the DSW can now be solved on using the dispersion relation (2.6.3) with \( \bar{\nu} \) determined by (2.6.5). Substituting the dispersion relation (2.6.3) and the mean level (2.6.5) into the differential equation (2.6.4) gives

\[
\frac{dk}{d\bar{\rho}} = \frac{1}{2} \sqrt{\frac{1}{2} \frac{(1 - 4\alpha\bar{\rho}) k}{W^{1/2} - W + \frac{k^2}{4}}},
\]

(2.6.6)

where

\[
W = \bar{\rho} \left( 1 - 2\alpha\bar{\rho} \right) + \frac{k^2}{4},
\]

(2.6.7)

The change of variable

\[
\gamma = \sqrt{1 - \frac{k^2}{4(\bar{\rho} - 2\alpha\bar{\rho}^2)}},
\]

(2.6.8)

is now used to simplify this equation for the leading edge, so that it becomes

\[
\frac{d\gamma}{d\bar{\rho}} = - \frac{1 + \gamma}{2\bar{\rho}} + \alpha \left[ \frac{4(1 + \gamma)\gamma}{2\gamma + 1} - (1 + \gamma) \right].
\]

(2.6.9)

The leading, linear edge of the DSW can now be determined by solving this equation with the boundary condition which links it to the trailing, solitary wave edge. This boundary condition is \( k = 0 \) when \( \bar{\rho} = \sqrt{\rho_i} \), with \( \rho_i \) given by (2.4.19), as a solitary wave has zero wavenumber \([37, 38, 41]\). It can then be found that

\[
\gamma = 2 \sqrt{\frac{\rho_i}{\bar{\rho}}} - 1 + \alpha \left[ 6 \sqrt{\bar{\rho} \rho_i} + 32 \rho_i - 38 \rho_i^{3/2} \right] + 128 \rho_i^{3/2} \log \left( 3 \sqrt{\rho_i} \right).
\]

(2.6.10)

The wavenumber \( k_+ \) at the leading, linear edge of the DSW is hence given by (2.6.10) and (2.6.8) with \( \bar{\rho} = \rho_+ \). Furthermore, the position of this edge of the DSW is given by the linear group velocity, which is

\[
c_g = \frac{\partial \omega}{\partial k} = \frac{\rho_+ + \frac{1}{4}k_+^2}{\sqrt{\rho_+ + \frac{1}{4}k_+^2}} - \frac{\alpha \rho_+^3}{(\rho_+ + \frac{1}{4}k_+^2)^{3/2}}.
\]

(2.6.11)
2.6.2 Solitary wave edge of DSW

The trailing, solitary wave, edge of the DSW can be determined in a similar fashion. This edge is determined in a similar fashion to the linear wave edge, but with a “conjugate” wavenumber \( \tilde{k} \) and “conjugate” frequency \( \tilde{\omega} \) related by the “conjugate” dispersion relation

\[
\tilde{\omega} = -i\omega(i\tilde{k}, \tilde{\rho}, \tilde{v}),
\]

where \( \omega \) is given by the linear dispersion relation (2.6.3). Thus, the “conjugate” dispersion relation is

\[
\tilde{\omega} = \tilde{v}\tilde{k} + \tilde{k}\sqrt{\tilde{\rho}(1 - 2\alpha\tilde{\rho}) - \frac{1}{4}\tilde{k}^2} = \tilde{v}\tilde{k} + \tilde{k}\sqrt{\tilde{\rho} - \frac{1}{4}\tilde{k}^2 - \alpha\tilde{\rho}\tilde{k}\left(\tilde{\rho} - \frac{1}{4}\tilde{k}^2\right)^{-1/2}} + \ldots
\]

(2.6.12)

As for the leading edge, the solitary wave edge of the DSW is determined by [37, 38, 41]

\[
\frac{d\tilde{k}}{d\tilde{\rho}} = \frac{\partial \tilde{\omega}}{\partial \tilde{\rho}} \left(V - \frac{\partial \tilde{\omega}}{\partial \tilde{k}}\right)^{-1}. \quad (2.6.13)
\]

Again, the Riemann invariant condition (2.6.5) is used to relate \( \tilde{v} \) and \( \tilde{\rho} \) at the trailing edge of the DSW as, again, two of the modulation equations coincide at this edge. Using the conjugate dispersion relation (2.6.12) and the mean level (2.6.5), the trailing edge equation (2.6.13) becomes

\[
\frac{d\tilde{k}}{d\tilde{\rho}} = \frac{1}{2} \frac{(1 - 4\alpha\tilde{\rho})\tilde{k}}{\sqrt{\tilde{\rho}(1 + \alpha\tilde{\rho})}V^{1/2} - V + \frac{\tilde{k}^2}{4}}, \quad (2.6.14)
\]

with

\[
V = \tilde{\rho}(1 - 2\alpha\tilde{\rho}) - \frac{\tilde{k}^2}{4}. \quad (2.6.15)
\]

Again, the change of variable

\[
\tilde{\gamma} = \sqrt{1 - \frac{\tilde{k}^2}{4(\tilde{\rho} - 2\alpha\tilde{\rho}^2)}} \quad (2.6.16)
\]

is now used to simplify this equation, which becomes

\[
\frac{d\tilde{\gamma}}{d\tilde{\rho}} = -\frac{1 + \tilde{\gamma}}{2\tilde{\rho}} + \alpha \left[\frac{4(1 + \tilde{\gamma})\tilde{\gamma}}{2\tilde{\gamma} + 1} - (1 + \tilde{\gamma})\right]. \quad (2.6.17)
\]
This equation is solved with the boundary condition linking the trailing edge with the linear, leading edge of the DSW, so that $\tilde{k} = 0$ at $\tilde{\rho} = \rho_+$ [37, 38, 41]. The trailing edge solution is then found to be

$$\gamma = 2 \sqrt{\frac{\rho_+}{\tilde{\rho}}} - 1 + \alpha \left( 6 \sqrt{\rho_+} + 32 \rho_+ - 38 \frac{\rho_+^{3/2}}{\sqrt{\rho}} + 128 \frac{\rho_+^{3/2}}{\sqrt{\rho}} \log \frac{4 \sqrt{\rho_+} - \sqrt{\rho}}{3 \sqrt{\rho_+}} \right).$$

(2.6.18)

The conjugate wavenumber $\tilde{k}_i$ at the trailing edge of the DSW is finally given by (2.6.18) and (2.6.16) with $\tilde{\rho} = \rho_i$ and $\rho_i$ given by (2.4.19).

The position of the trailing, solitary wave, edge of the DSW is determined by the solitary wave velocity

$$v_- = \frac{\tilde{\omega}}{\tilde{k}} = v_i + \sqrt{\rho_i - \frac{1}{4} k_i^2} - \frac{\alpha \rho_i^2}{\sqrt{\rho_i - \frac{1}{4} k_i^2}}.$$

(2.6.19)

Finally, the amplitude of the trailing edge solitary wave of the DSW can be deduced from the grey solitary wave solution (2.3.7). Let us define the amplitude of the trailing edge solitary wave as

$$A_s = u_i - A,$$

(2.6.20)

which is the difference between the intermediate level $u_i$ and the minimum of the solitary wave $|u|$. The velocity (2.3.7) of the (grey) solitary wave is given by the amplitude/velocity relation

$$v_- = A + v_i - \alpha \left( \frac{2}{3} AB^2 + A^3 \right).$$

(2.6.21)

On noting that $A^2 + B^2 = u_i^2$ and $|\alpha| \ll 1$, this equation can be solved to give

$$A = v_- - v_i + \frac{1}{3} \alpha (v_- - v_i) \left[ 2u_i^2 + (v_- - v_i)^2 \right].$$

(2.6.22)

This amplitude expression is similar to the equivalent results (92) in [42] and (66) in [28], which is an asymptotic solution. The difference between these expressions and the present ones is that the present results are derived based on the general grey soliton solution (2.3.7) of the higher order NLS equation for the colloid response (2.2.2). It should be noted that the leading and trailing edge velocities (2.6.11) and (2.6.19), respectively, are
in the scaled \((x', z')\) coordinates and need to be transformed back to the coordinates \((x, z)\) via the transformations (2.3.5) to be compatible with the colloid equations (2.2.1) and (2.2.2).

## 2.7 Full colloid DSW

In the previous section the leading and trailing edges of the DSW solution of the higher order NLS equation (2.3.3) were determined by using the theory of El [37, 38, 41], which is an asymptotic approximation to the edges of the DSW of the full colloid equations (2.2.1) and (2.2.2) in the limit of low light intensity. This leads to explicit solutions for the leading and trailing edges, detailed in Sections 2.6.1 and 2.6.2. The same method will now be used to determine the leading and trailing edges of the DSW solution of the full colloid equations (2.2.1) and (2.2.2). However, due to the complexity of the medium equation (2.2.2) the differential equations for the leading and trailing edges cannot be solved explicitly and need to be solved numerically.

The dispersion relation for the full colloid equations can be found by substituting the linear wave solution

\[
\rho = \bar{\rho} + \rho_1 e^{i(kx - \omega z)}, \quad v = \bar{v} + v_1 e^{i(kx - \omega z)},
\]

where \(|\rho_1| \ll \bar{\rho}\) and \(|v_1| \ll |\bar{v}|\), into the hydrodynamic equations (2.4.3). This gives

\[
\omega = k\bar{v} + k\sqrt{\frac{\bar{\rho}}{g'(\bar{\eta})} + \frac{1}{4} k^2}, \quad \text{with} \quad g(\bar{\eta}) - g_0 = \bar{\rho}.
\]

Again, at the trailing and leading edges of the DSW \(\bar{v}\) is related to \(\bar{\rho}\) through the condition that the Riemann invariant on the characteristic \(C_- \) (2.4.7) is the same at the two edges of the DSW. We then have

\[
\bar{v} = \int_{\bar{\rho}}^{\bar{\rho}'} \frac{d\rho}{\sqrt{\rho g'(\eta)}}.
\]
CHAPTER 2. THE COLLOID EQUATION

2.7.1 Linear wave edge of DSW

The leading, linear wave, and trailing, solitary wave, edges of the DSW solution of the full colloid equations (2.2.1) and (2.2.2) can be determined in a similar fashion as was done in Section 2.6 for the DSW solution of the higher order NLS equation (2.3.3).

The leading, linear wave, edge of the DSW is determined by the differential equation (2.6.4). On using the full dispersion relation (2.7.2) and expression (2.7.3) for $\bar{v}$, this equation becomes

$$\frac{dk}{d\bar{\rho}} = \frac{k}{\sqrt{\bar{\rho}g'(\bar{\eta})}} \left[ \sqrt{\bar{\rho}g'(\bar{\eta})} + \frac{1}{4} k^2 + \frac{1}{2} k \left( \frac{1}{g'(\bar{\eta})} - \frac{\bar{\rho}g''(\bar{\eta})}{g'(\bar{\eta})} \right) \right] \sqrt{\bar{\rho}g'(\bar{\rho})} + \frac{1}{4} k^2 - \frac{\bar{\rho}g'(\bar{\eta})}{g'(\bar{\eta})} - \frac{1}{2} k^2. \quad (2.7.4)$$

As for the higher order NLS DSW of Section 2.6 this equation is simplified using the change of variable

$$\gamma(\bar{\rho}) = \sqrt{1 + \frac{1}{4} k^2 \frac{g'(\bar{\eta})}{\bar{\rho}}}. \quad (2.7.5)$$

The leading, linear edge of the DSW is then determined by

$$\frac{d\gamma}{d\bar{\rho}} = -\frac{(1 + \gamma) \left( \frac{2}{g'(\bar{\eta})} + (2\gamma - 1) \left( \frac{\bar{\rho}g''(\bar{\eta})}{g'(\bar{\eta})} \right)' \right)}{2(2\gamma + 1) \frac{\bar{\rho}g'(\bar{\eta})}{g'(\bar{\eta})}}. \quad (2.7.6)$$

This equation is solved with the condition $k(\rho_i) = 0$ which connects the linear edge of the DSW to the solitary wave edge [37, 38, 41], so that $\gamma(\rho_i) = 1$. With this solution the wavenumber at the leading edge of the DSW is

$$k = k(\rho_+) = 2 \sqrt{\frac{\rho_+}{g'(\eta_+)}} \sqrt{\gamma^2(\rho_+) - 1}. \quad (2.7.7)$$

The position of the leading edge of the DSW is therefore finally given by the group velocity

$$c_g = \frac{\partial \omega}{\partial k} = \sqrt{\frac{\rho_+}{g'(\eta_+)}} \left( 2\gamma(\rho_+) - \frac{1}{\gamma(\rho_+)} \right). \quad (2.7.8)$$

This leading edge position will be compared with numerical solutions of the colloid equations (2.2.1) and (2.2.2) in Section 2.9.
2.7.2 Solitary wave edge of DSW

The trailing, solitary wave, edge of the bore is determined in a similar fashion. The equation governing the trailing edge of the DSW is (2.6.13). As for the higher order NLS equation DSW, in general the “conjugate” dispersion relation is given by \( \tilde{\omega} = -i\omega(i\tilde{k}, \tilde{\rho}, \tilde{v}) \), so that the linear dispersion relation (2.7.2) gives for the colloid DSW

\[
\tilde{\omega} = \tilde{k}\tilde{v} + \tilde{k} \sqrt{\frac{\tilde{\rho}}{g'(\tilde{\eta})}} - \frac{1}{4} \tilde{k}^2, \quad \text{with} \quad g(\tilde{\eta}) - g_0 = \tilde{\rho}. \tag{2.7.9}
\]

With these conjugate variables and the relation (2.7.3) between \( \tilde{v} \) and \( \tilde{\rho} \) at the trailing edge of the DSW, equation (2.6.13) gives that the trailing edge is determined by

\[
\frac{d\tilde{k}}{d\tilde{\rho}} = \frac{\tilde{k}}{\sqrt{\tilde{\rho} g'(\tilde{\eta})}} \sqrt{\frac{\tilde{\rho}}{g'(\tilde{\eta})} - \frac{1}{4} \tilde{k}^2 + \frac{1}{2} \left( \frac{1}{g'(\tilde{\eta})} - \frac{\dot{\tilde{\rho}} g''(\tilde{\eta})}{g'(\tilde{\eta})} \right)}.
\tag{2.7.10}
\]

As for the linear edge, this equation can be simplified by the change of variable

\[
\tilde{\gamma}(\tilde{\rho}) = \sqrt{1 - \frac{1}{4} \tilde{k}^2 \tilde{g}'(\tilde{\eta})}, \tag{2.7.11}
\]

so that the final equation for the trailing, solitary wave edge is

\[
\frac{d\tilde{\gamma}}{d\tilde{\rho}} = -\frac{(1 + \tilde{\gamma}) \left( \frac{2}{g'(\tilde{\eta})} + (2\tilde{\gamma} - 1)(\frac{\tilde{\rho}}{g'(\tilde{\eta})})' \right)}{2(2\tilde{\gamma} + 1)(\frac{\tilde{\rho}}{g'(\tilde{\eta})})}, \tag{2.7.12}
\]

This equation is solved with the condition \( \tilde{k}(\rho_+) = 0 \) [37, 38, 41] which connects the trailing edge to the leading edge, giving \( \tilde{\gamma}(\rho_+) = 1 \). With this solution, the velocity of the trailing edge of the DSW is

\[
v_- = \frac{\tilde{\omega}}{\tilde{k}} = \int_{\rho_+}^{\rho_i} \frac{d\rho}{\rho \sqrt{g'(\eta)}} + \tilde{\gamma}(\rho_i) \sqrt{\frac{\rho_i}{g'(\eta)}} , \tag{2.7.13}
\]

on using expression (2.7.3) for \( \tilde{v} \).

The differential equations (2.7.6) and (2.7.12) for the leading and trailing edges of the
DSW, respectively, were solved numerically. The solution for the trailing edge of the DSW is not complete yet as, while (2.7.13) gives the velocity of the trailing edge, the amplitude of the solitary wave at the trailing edge has not been determined. The colloid equations (2.2.1) and (2.2.2) have no known exact solitary wave solutions. However, the exact amplitude-velocity relation for the grey colloid solitary wave can be found without knowledge of this solution as (see Appendix A)

\[
V_s = \frac{\sqrt{2} |f_m| \sqrt{\sigma u_\infty^2 - \sigma |f_m|^2 + F(\eta_m, \eta_\infty)}}{u_\infty^2 - |f_m|^2},
\]  

(2.7.14)

with \( |f_m| \) the minimum value of \(|u|\) and \(u_\infty\) the background carrier wave amplitude. The amplitude of the grey solitary wave is then

\[
A_s = u_\infty - |f_m|,
\]  

(2.7.15)

the difference between the background level \(u_\infty\) and the minimum of \(|u|\). We now recognise \(V_s\) as the difference between leading solitary wave velocity \(v_-\) from (2.7.13) and \(v_i\) from (2.4.11). The minimum \( |f_m| \) is found by applying Newton’s method to (2.7.14), resulting in the amplitude \(A_s\). Hoefer [68] obtained general results for DSW solutions for NLS equations with a general nonlinearity \(f(|u|^2)u\). The electric field equation (2.2.1) has a nonlinearity of this form, but without an explicit expression for \(f\) as the nonlinearity \((\eta - \eta_0)u\) is determined implicitly from the medium response (2.2.2). Hoefer found that in this general case the structure of the simple wave solution of the Whitham modulation equations for the DSW can break down in that the characteristic velocity of the simple wave ceases to be monotonic, which results in the formation of a multi-phase wavetrain [38, 82, 127]. In terms of the present work, this can occur if

\[
V_+ = \frac{\partial \omega}{\partial k} \quad \text{or} \quad V_+ = \frac{\partial \tilde{\omega}}{\partial \tilde{k}}
\]  

(2.7.16)

in equations (2.6.4) and (2.6.13) for the leading and trailing edges of the DSW. It can be seen from the specific equations (2.7.6) and (2.7.12) for the leading and trailing edges that
such a breakdown of the simple wave solution cannot occur for the colloid equations.

\section*{2.8 Vacuum Point}

As the jump height $u_i - u_+$ increases, the amplitudes of the waves in the DSW grow until there is a jump height for which the trailing solitary wave of the DSW has its minimum at $|u| = |f_m| = 0$. This point is referred to as the vacuum point \cite{40}. At the vacuum point the trailing grey solitary wave of the DSW becomes dark as $|f_m| = 0$ and the amplitude-velocity (2.7.14) gives $V_s = 0$. Hence, the velocity of the trailing edge of the DSW is $v_- = v_i$ and the trailing edge conjugate dispersion relation (2.7.9) gives

$$\frac{\rho_i}{g'(\eta_i)} = \frac{1}{4} k^2. \tag{2.8.1}$$

This condition cannot be solved exactly to determine the vacuum point. However, it can be solved from expression (2.7.11) for the conjugate wavenumber $\tilde{k}$, with $\tilde{\gamma}$ determined by numerically solving the trailing edge equation (2.7.12).

The vacuum point condition (2.8.1) can be asymptotically solved in the low power
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Figure 2.4: The intermediate level \( \rho_i \) versus \( \rho_+ \). The parameter values are \( \rho_- = 1.0 \) and \( \eta_0 = 0.01 \). Shown are the NLS equation modulation solution (2.4.19) with \( \alpha = 0 \): (light blue) line with \( \star \); higher order NLS equation modulation solution (2.4.19): (red) line with \( \bullet \); modulation solution of full colloid equations (2.4.12): (green) line with \( \times \); numerical solution of colloid equations: (black) line.

Limit. The higher order NLS equation conjugate dispersion relation (2.6.12) gives the onset of the vacuum point as

\[
\rho_i - 2\alpha \rho_i^2 = \frac{1}{4} \tilde{k}^2. \tag{2.8.2}
\]

Using the conjugate wavenumber \( \tilde{k} \) determined by (2.6.18) and (2.6.16), the vacuum point then first occurs when \( \rho_+ \) and \( \rho_i \) satisfy

\[
2 \sqrt{\frac{\rho_+}{\rho_i}} + \alpha \left( 6 \sqrt{\rho_i \rho_+} + 32 \rho_+ - 38 \frac{\rho_+^{3/2}}{\sqrt{\rho_i}} + 128 \frac{\rho_+^{3/2}}{3 \sqrt{\rho_i}} \log \frac{4 \sqrt{\rho_+ - \sqrt{\rho_i}}}{3 \sqrt{\rho_+}} \right) = 1. \tag{2.8.3}
\]

2.9 Comparison with numerical solutions

In this section, numerical solutions of the colloid equations (2.2.1) and (2.2.2) with the jump initial condition (2.2.4) will be compared with the modulation solutions of the NLS equation (higher order NLS equation with \( \alpha = 0 \)), the higher order NLS equation and the full colloid equations. The numerical solutions of the colloid equations (2.2.1) and (2.2.2) were obtained using the numerical scheme of [89]. The electric field equation (2.2.1) was solved using a hybrid method Runge-Kutta finite difference method, see Appendix B.1.
Figure 2.5: The trailing solitary wave amplitude $A_s$ versus $\rho_+$. The parameter values are $\rho_- = 1.0$ and $\eta_0 = 0.01$. Shown are the full numerical solution of the colloid equations (2.2.1) and (2.2.2) and modulation theory. NLS modulation theory (2.6.20) with $\alpha = 0$: (light blue) line with $\star$; higher order NLS modulation theory (2.6.20): (red) line with $\bullet$; full modulation theory (2.7.15): (green) line with $\times$; numerical solution: black line.

Figure 2.3 shows the optical intensity $\rho$ versus $x$ at $z = 1500$. The parameter values are $\rho_- = 1.0$, $\rho_+ = 0.5$ and $\eta_0 = 0.01$. Compared are the non-dispersive modulation solution of the colloid equations (2.2.1) and (2.2.2) and the higher order NLS equation (2.3.3), as well as the numerical solution of the colloid equations. The expansion fan lies in the region $-227 < x \leq -120$ as predicted by the modulation solution of the colloid equation and $-144 < x \leq -84$ by the modulation solution of the higher order NLS equation. The colloid expansion fan region is 44% longer than the higher order NLS one. Also, the intermediate level lies in $-120 \leq x < 160$ as given by the modulation solution of the colloid equations and $-84 \leq x < 155$ as given by the higher order NLS equation modulation solution. The colloid intermediate level region is 15% longer than the higher order NLS one. As for the dam break solution, there is again excellent agreement between the modulation theory solution (2.4.8) and the numerical solution, but the higher order NLS solution (2.4.15) shows significant differences due to higher order terms being important. There is again a backward propagating small amplitude wavetrain on the initial level $\rho_-$, which is again generated by the smoothing due to dispersion of the discontinuity in slope of the modulation simple wave solution where it joins the level $\rho_-$. As $\rho_+ \neq 0$, a DSW is generated ahead of the intermediate level $\rho_i$. 
Figure 2.4 shows the intermediate level $\rho_i$ versus $\rho_+$. The parameter values are $\rho_- = 1.0$ and $\eta_0 = 0.01$. Shown are numerical solutions and the full modulation solution (2.4.12) of the colloid equations (2.2.1) and (2.2.2), the higher NLS equation modulation theory solution (2.4.19) and the NLS equation modulation theory solution (2.4.19) with $\alpha = 0$. It can be seen that both the full non-dispersive solution and the higher order NLS equation solution are in excellent agreement with the full numerical solution. As $\rho_+ \to 1$, all the modulation theory solutions converge to $\rho_i = 1$ as in this limit there is no initial jump and the full solution is $\rho = 1$. As $\rho_+$ decreases, the intermediate level decreases from the level behind and the differences between the modulation theory and numerical solutions increase. It should be noted that a vacuum point arises for $\rho_+ < 0.1$ and the modulation theory solutions are not valid below this level. Unfortunately, to continue the modulation theory solution below the vacuum point requires a knowledge of the full modulation equations [40]. It is important to note that the intermediate level as given by modulation theory for the NLS equation is in excellent agreement with the numerical values, except in the limit of large jumps $\rho_- - \rho_+$ towards the vacuum point, with the maximum error 8.5%. Higher order NLS equation modulation theory gives values for the intermediate level $\rho_i$ which are almost identical with those of full modulation theory and with numerical results, which is expected given the accurate predictions of NLS equation modulation theory. Modulation theory for the higher order NLS equation gives intermediate levels which differ by at most 2.8% from numerical values.

Figure 2.5 shows the amplitude $A_s$ of the trailing solitary wave versus $\rho_+$. The parameter values are $\rho_- = 1.0$ and $\eta_0 = 0.01$. Shown are the numerical solution and the modulation theory amplitude (2.7.15) for the full colloid equations, the amplitude (2.6.20) for the higher order NLS equation and the amplitude (2.6.20) for the NLS equation (when $\alpha = 0$). It can be seen that the full modulation theory gives amplitudes in excellent agreement with numerical solutions for the full range of jump heights $\rho_- - \rho_+$. As for the intermediate level both full modulation theory and modulation theory for the higher order NLS equation give amplitudes in near perfect agreement with numerical results. The full colloid equation modulation amplitude differs by at most 0.5% and the higher order NLS
Figure 2.6: The trailing solitary wave amplitude $A_s$ versus the background packing fraction $\eta_0$. The parameter values are $\rho_- = 1.0$ and $\rho_+ = 0.5$. Shown are full numerical solutions of the colloid equations (2.2.1) and (2.2.2) and modulation theory. NLS equation modulation theory (2.6.20) with $\alpha = 0$: (light blue) line with $\star$; higher order NLS equation modulation theory (2.6.20): (red) line with $\bullet$; full modulation theory (2.7.15): (green) line with $\times$; numerical solution: (black) line.

equation modulation amplitude differs by at most 2% from the numerical values. Again, as for the intermediate level, modulation theory for the NLS equation gives amplitudes in good agreement with numerical solutions, except for large jump heights with $\rho_+$ near the vacuum point, at which point the modulation amplitude differs by 7% from the numerical value. The numerical and modulation amplitudes converge to $A_s = 0$ as $\rho_+ \to 1$ as the jump vanishes in this limit.

Figure 2.6 shows the amplitude $A_s$ of the trailing solitary wave of the DSW versus the background packing fraction $\eta_0$. The parameter values are $\rho_- = 1.0$ and $\rho_+ = 0.5$. Shown are amplitudes from numerical solutions of the colloid equations (2.2.1) and (2.2.2), the full modulation theory amplitude (2.7.15), the NLS equation modulation theory amplitude (2.6.20) with $\alpha = 0$ and the higher order NLS equation amplitude (2.6.20). The full modulation theory is, again, in excellent agreement with the numerical results, with an overall difference less than 0.5%. In contrast to the comparisons of Figures 2.4 and 2.5, the amplitudes as given by the modulation theories for the NLS and higher order NLS equations show significant differences from the numerical values. For high background packing fractions the full modulation theory and the NLS and higher order NLS equation
Figure 2.7: The trailing solitary wave velocity \( v_- \) versus \( \rho_+ \). The parameter values are \( \rho_- = 1.0 \) and \( \eta_0 = 0.01 \). Shown are: NLS modulation theory (2.6.19) with \( \alpha = 0 \): (light blue) line with \( \ast \); higher order NLS modulation theory (2.6.19): (red) line with \( \bullet \); full modulation theory (2.7.13): (green) line with \( \times \); numerical solution of colloid equations (2.2.1) and (2.2.2): (black) line.

Modulation theories give similar results, but the NLS and higher order NLS modulation theories show significant disagreement with the numerical results as the background packing fraction decreases. The NLS and higher order NLS equation modulation theories differ by 2.9% and 2.0% from numerical results in this limit.

Figure 2.7 shows the trailing solitary wave velocity \( v_- \) versus \( \rho_+ \). The parameter values are \( \rho_- = 1.0 \) and \( \eta_0 = 0.01 \). Compared are the numerical velocity of the trailing solitary wave of the DSW and the modulation theory values, the full modulation theory value (2.7.13), the higher order NLS modulation theory value (2.6.19) and the NLS modulation theory value (2.6.19) with \( \alpha = 0 \). These comparisons are shown down to \( \rho_+ = 0.1 \), at which value the vacuum point first appears and the modulation theories of Sections 2.6 and 2.7 cease to be valid. As for the previous comparisons, the full modulation theory velocity is in excellent agreement with the numerical velocity, with a difference less than 0.3%. The higher order NLS equation modulation theory gives a velocity in good agreement with the numerical value, but showing differences from it, unlike the intermediate level and amplitude comparisons of Figures 2.4 and 2.5. As for the amplitude comparison of Figure 2.6 modulation theory for the NLS equation does not yield a good prediction for the trailing solitary wave velocity, with differences ranging from 19% to 30%, showing
the importance of higher order effects in determining the trailing solitary wave velocity.

Figure 2.8 shows a contour plot of the DSW portion of a numerical solution of the colloid equations (2.2.1) and (2.2.2). To investigate the accuracy of the modulation theory prediction of the leading edge of the DSW by the linear group velocity $c_g$, this figure also has superimposed the leading edge prediction of the modulation theories of Sections 2.6.1 and 2.7.1. The parameter values used are $\rho_- = 1.0$, $\rho_+ = 0.5$, and $\eta_0 = 0.01$. NLS modulation theory (2.6.11) with $\alpha = 0$: (light blue) line with $\star$; higher order NLS modulation theory (2.6.11): (red) line with yellow $\diamond$; full modulation theory (2.7.8): (green) line with $\times$.

Figure 2.8: Contour plot of the DSW portion of the numerical solution of the colloid equations (2.2.1) and (2.2.2). Superimposed are the positions of the leading, linear edge of the DSW as given by the group velocity $c_g$ of modulation theory. The parameter values are $\rho_- = 1.0$, $\rho_+ = 0.5$, and $\eta_0 = 0.01$. NLS modulation theory (2.6.11) with $\alpha = 0$: (light blue) line with $\star$; higher order NLS modulation theory (2.6.11): (red) line with yellow $\diamond$; full modulation theory (2.7.8): (green) line with $\times$. The full modulation theory front position is given by the group velocity (2.7.8) and the higher order NLS equation modulation theory front position is given by the group velocity (2.6.11). The front position as given by the NLS equation modulation theory is the higher order NLS result (2.6.11) with $\alpha = 0$. This comparison is more complicated than the similar comparison of the trailing, solitary wave edge of Figure 2.7 as numerical solutions show no distinct leading edge, in contrast to the predictions of modulation theory, with an extended train of waves of decreasing amplitude bringing the level back to $\rho_+$ [38, 51]. Nevertheless, the front position comparison of Figure 2.8 shows that the full modulation theory gives a front position which encompasses the majority of the larger amplitude waves of the DSW. As for the trailing edge comparison of Figure 2.7, the higher order NLS equation modulation theory gives a good approximation to the leading
Figure 2.9: Vacuum point versus background packing fraction $\eta_0$ for $\rho_- = 1.0$. Shown are the NLS modulation theory prediction (2.8.3) with $\alpha = 0$: (light blue) line with *; higher order NLS modulation theory prediction (2.8.3): (red) line with ●; full modulation theory prediction (2.8.1): (green) line with ×; numerical solution of colloid equations (2.2.1) and (2.2.2): (black) line.

edge position, while NLS equation modulation theory is not in agreement with numerical solutions. This again indicates that the higher order terms in the constitutive law (2.2.2) are important and that the response of the medium to the optical beam cannot be approximated by Kerr’s Law. It should be noted for the comparisons of Figure 2.8 that, in general, modulation theory does not do as well in predicting the linear edge position of a DSW than the solitary wave edge [38, 51, 58, 59, 81]. In terms of asymptotic theory there is an additional layer of linear waves between the initial level and the linear edge of the Whitham modulation theory solution [58, 59, 81].

Figure 2.9 shows a comparison for the vacuum point versus the background packing fraction $\eta_0$ for $\rho_- = 1.0$. This comparison is for the value of $\eta_+$ at which the vacuum point first occurs, that is the value of $\rho_+$ at which $|u|$ first vanishes within the DSW, as discussed in Section 2.8. The full modulation theory gives that this vacuum point is given implicitly by (2.8.1), while higher order NLS equation modulation theory gives that it is the solution of (2.8.3), with the NLS equation modulation theory value given by this expression with $\alpha = 0$. The NLS equation modulation theory gives a constant vacuum point independent of $\eta_0$, in contrast to the other modulation theories. The full modulation theory gives near perfect agreement for the vacuum point for the full range
of background packing fractions $\eta_0$, with the overall difference from the numerical value less than 0.1%. The higher order NLS modulation theory gives reasonable agreement, with increasing disagreement as the background packing fraction decreases, similar to the amplitude comparison of Figure 2.6. The difference grows to 2.8% at $\eta_0 = 0.01$. This is due to the higher order NLS equation (2.3.3) being valid in the limit $|\eta - \eta_0| \ll 1$, which becomes less valid as $\eta_0$ decreases. As for the other DSW properties discussed above, the NLS equation modulation theory does not give good agreement for the vacuum point, again highlighting the importance of the higher order terms in the constitutive law (2.2.2) for the colloid.

2.10 Conclusions

The DSW solution for optical beam propagation in a defocussing colloidal solution has been derived. As the full Whitham modulation theory [125–127] is not available, due to the underlying colloid equations (2.2.1) and (2.2.2) not being integrable [50], modulation equations for the leading and trailing edges of the DSW were derived using the method of El [37, 38, 41]. This method enables the derivation of differential equations for these leading and trailing edges without knowledge of the full modulation equations by using the degenerate nature of these modulation equations at the edges of the DSW for cases in which the DSW is of KdV type. It was found that in the limit of low beam power, or small deviations of the colloid concentration from the background value, the full colloid equations can be approximated by a higher order NLS equation. El’s method was also used to derive the leading and trailing edges of DSW for this higher order NLS equation, as has been done in previous work [28, 42].

The modulation theory for the full colloid equations was found to give near perfect predictions for the amplitude and velocity of the trailing solitary wave of the DSW and the velocity of the linear, leading edge of the DSW. It was also found that the modulation theory for the higher order NLS equation gave reasonable predictions for the leading and trailing edges of the DSW, but not for other properties, such as the vacuum point. This highlights that the full constitutive law (2.2.2) is not always necessary and that Tay-
lor expansions of it for small packing fractions can lead to adequate approximations to
the full constitutive law. The constitutive law (2.2.2) is one of many derived under var-
ious approximations for colloidal media. It is an open question as to whether these full
constitutive laws are needed and whether low power Taylor expansions provide adequate
approximations. However, the present work highlights the power of El’s method to de-
rive the leading and trailing edges of a KdV-like DSW, particularly for nonlinear wave
equations which are not integrable.
Chapter 3

Dispersive shock waves governed by the Whitham equation and their stability

3.1 Introduction

This chapter studies the DSW solution of the Whitham equation for three different linear dispersion relations, for surface water waves on a fluid of finite depth [127], for the intermediate long wave equation for a stratified fluid [73] and a model kernel introduced by Whitham to model wave peaking as for the Stokes wave of greatest amplitude [127]. The results of this chapter appear in An et al. [5]. The shock fitting method [37, 38, 41] is used to obtain the leading, solitary wave edge and the trailing, linear wave edge of the DSWs. It is found that the shock fitting method gives results in excellent agreement with numerical solutions for all three dispersion relations. A particular emphasis of the study is the behaviour of the DSW as the leading solitary wave approaches the wave of greatest amplitude. Numerical solutions show that the DSWs for the water wave and peaking kernels then become unstable just below this maximum and evolve into a two phase wavetrain. This cannot be captured with the dispersive shock fitting method as it assumes a single phase wavetrain. The standard shock fitting method assumes that the DSW has the standard KdV type structure, for which certain admissibility criteria need to be satisfied which relate to the genuine nonlinearity and hyperbolicity of the Whitham
modulation equations [38, 82]. In the shock fitting method, failure of the admissibility conditions is shown by the non-monotonicity of the trailing edge group velocity and the leading edge solitary wave velocity as the levels ahead and behind vary. In particular, loss of hyperbolicity indicates that the bore becomes modulationally unstable, which in numerical solutions typically results in a multi-phase wavetrain appearing in the DSW, which is what is found here for the Whitham equations for the water wave and peaking kernels. However, while it is found that the admissibility condition for genuine nonlinearity fails at the trailing edge as the initial jump height increases, the breakdown of the DSW is not due to this. It is found from numerical solutions that the instability is generated at the leading, solitary wave edge and propagates back through the DSW. This instability is due to a Benjamin-Feir instability [11] of the periodic wave solution of the Whitham equation as the wavenumber decreases [11, 127], which was detailed by Sanford et al [107]. It is found that this Benjamin-Feir instability arises just after the breakdown of genuine nonlinearity and dominates it.

3.2 Whitham equations and dispersion relations

3.2.1 Intermediate long waves

The KdV equation models the propagation of weakly nonlinear long waves at the interface of a two layer fluid in the limit in which the wavelength is much larger than the total depth of the fluids. However, when the lower layer is relatively thick compared with the wavelength, the waves are governed by the intermediate long wave equation [73], which is integrable [79]. This equation was introduced as a model for weakly nonlinear waves much longer than a pycnocline thickness in a stratified fluid of finite total depth. The intermediate long wave equation with quadratic nonlinearity is

\[ u_t + 2uu_x + i \int_{-\infty}^{\infty} G(x - \xi) u_{\xi\xi}(\xi, t) d\xi = 0, \quad G = \mathcal{F}^{-1}\left\{ \coth k - \frac{1}{k} \right\}, \quad (3.2.1) \]
where $\mathcal{F}$ denotes the Fourier transform. This equation has the exact soliton solution [100]

$$u = \beta + \frac{r \sin r}{\cosh(r(x - V_s t)) + \cos r}, \quad V_s = 2\beta + 1 - r \cot r. \quad (3.2.2)$$

The amplitude of the soliton is then $A_s = r \sin r / (1 + \cos r)$, with its velocity $V_s$ implicitly determined from this through the parameter $r$. In principle, as the intermediate long wave equation is integrable, its Whitham modulation equations can be determined and set in Riemann invariant form. In the present work, the intermediate long wave equation will be used as an example equation which can be expressed as a Whitham equation, but which does not have a wave of maximum height and which possesses DSWs that are stable.

### 3.2.2 Water waves

The original Whitham equation [51, 127] combined the quadratic nonlinearity of the KdV equation with the full dispersion of gravity waves on a fluid of finite depth. This equation is

$$u_t + 2uu_x - u_x + \int_{-\infty}^{\infty} K(x - \xi)u_\xi(\xi, t) d\xi = 0, \quad K = \mathcal{F}^{-1} \left\{ \sqrt{\tanh k} \right\}. \quad (3.2.3)$$

where the acceleration due to gravity and depth have been normalised to 1. In the limit of long waves $k \to 0$, the dispersion relation for this equation reduces to $\omega = \sqrt{\tanh k}/k = k - \frac{1}{6}k^3 + O(k^5)$. This then gives the KdV dispersive term, as required. As noted in the Introduction, this water wave equation gives a wave of greatest height with a cusp [127], rather than the Stokes limiting wave with a peak of angle $120^\circ$ [116].

The dispersive shock fitting method provides the velocity of the leading solitary wave of a DSW. To find its corresponding amplitude, the amplitude-velocity relation for a solitary wave needs to be known, but this cannot be found analytically for the water wave Whitham equation (3.2.3). Hence, its amplitude-velocity relation must be found numerically in order to obtain comparisons with numerical solutions for the leading wave amplitude. In this regard, we seek a travelling wave solution $u = f(\theta) = f(x - Vt)$. The water
Figure 3.1: Numerical DSW solution of the Whitham equation for water waves (3.2.3). The parameters are $u_- = 0.1$ and $u_+ = 0.0$. Shown are (a) the surface profile $u$ versus $x$ at $t = 2000$, (b) contour evolution plot of $u$ in the $(x,t)$ plane.

The wave equation (3.2.3) becomes

$$-V f + f^2 + \int_{-\infty}^{\infty} K(\theta - \xi) f(\xi) \, d\xi = C = -V \beta + \beta^2 + \beta \int_{-\infty}^{\infty} K(\theta - \xi) \, d\xi$$

(3.2.4)

on integrating and assuming that $f \to \beta$ as $|\theta| \to \infty$. It is noted that the boundary value can be taken as $\beta = 0$ as the Whitham equation has the invariance that if the solitary wave solution $f \to \beta$ as $\theta \to \pm \infty$, the transformation $f \to f - \beta$ results in $V \to V + 2\beta$. The steady wave solution equation (3.2.4) was solved numerically using the spectral method of Ehrnström and Kalisch [34, 35, 96], from which numerical amplitude-velocity results can be obtained. The work of Ehrnström and Kalisch [34, 35] shows that solitary wave
solutions can be obtained by calculating periodic wave solutions and then increasing their wavelength until a solitary wave results.

### 3.2.3 Peaking model

As noted in the Introduction, the Whitham equation (3.2.3) does not give the peaked highest wave of Stokes [116], but gives a cusped wave of maximum amplitude instead. To further study the type of dispersive kernel which generates a peaked wave of maximum amplitude, Whitham [110, 127] introduced the nonlocal equation

\[
  u_t + 2uu_x - u_x + \int_{-\infty}^{\infty} L(x - \xi)u_{\xi}(\xi, t) \, d\xi = 0, \quad L = \mu e^{-\nu|x|}, \quad \mu = \pi/4, \quad \nu = \pi/2.
\] (3.2.5)

Whitham showed that this equation has a peaked highest wave solution of peak angle 110°. As the Whitham equation (3.2.5) is just a model, it is not expected that it would reproduce the 120° Stokes wave. An additional reason for the choice of the dispersive kernel in the Whitham equation (3.2.5) is that it is the Green’s function for the operator \(d^2/dx^2 - \nu^2\), so that it is the solution of

\[
  \frac{d^2L}{dx^2} - \nu^2 L = -\nu^2 \delta(x).
\] (3.2.6)

This enables the solitary wave solution of of the Whitham equation (3.2.5) to be determined exactly.

The simplest initial condition which will result in the generation of a DSW is the step initial condition

\[
  u(x, 0) = \begin{cases} 
    u_-, & x < 0, \\
    u_+, & x > 0 
  \end{cases}
\] (3.2.7)

with \(u_- > u_+\). The DSW solutions of the Whitham equations (3.2.1)–(3.2.5) are of similar appearance and only differ in the details, such as leading wave amplitude and leading and trailing edge velocities, for a given jump \(u_- - u_+\). Figure 3.1 shows a DSW solution for the surface water wave Whitham equation (3.2.3) for the level ahead \(u_+ = 0\) and the level behind \(u_- = 0.1\). These initial values mean that the waves of the DSW are far from the
maximum amplitude wave. The DSW has a similar appearance to the KdV DSW [51], which is discussed in Section 3.3(a). The amplitude of the lead solitary wave is 0.23, which is slightly higher than the KdV DSW lead wave amplitude of $2(u_- - u_+) = 0.2$.

### 3.3 DSWs for the Whitham equation

As discussed in the Introduction, El [37, 38, 41] developed the dispersive shock fitting method to determine the details of the leading and trailing edges of a DSW of KdV type. The importance of this method is that it is applicable even when the Whitham modulation equations for the governing nonlinear dispersive wave equation are not known, provided that the modulation equations are strictly hyperbolic, genuinely nonlinear and have solitary and linear wave limits. The shock fitting method is based on the deduction that the leading and trailing edges of a DSW can be determined solely from the non-dispersive form of the equation applying outside of the DSW region and the linear dispersion relation for the equation. The non-dispersive form of all the Whitham equations (3.2.1)–(3.2.5) applying outside the DSW are

$$\frac{\partial \bar{u}}{\partial t} + 2\bar{u} \frac{\partial \bar{u}}{\partial x} = 0.$$  \hspace{1cm} (3.3.1)

The notation $\bar{u}$ is used for the non-dispersive region as the solution in the non-dispersive region matches with the mean of $u$ at the edges of the DSW [37, 38, 41]. The characteristic velocity of the non-dispersive equation is then $V(\bar{u}) = 2\bar{u}$.

El [37, 38, 41] showed that matching between the non-dispersive region behind the DSW and the trailing edge of the DSW gives that the wavenumber $k$ at the trailing edge of the DSW is determined by

$$\frac{dk}{d\bar{u}} = \frac{\partial \omega}{\partial \bar{u}} \frac{1}{V(\bar{u}) - \frac{\partial \omega}{\partial k}},$$  \hspace{1cm} (3.3.2)

with $\omega = \omega(\bar{u}, k)$ the linear dispersion relation and $\bar{u}$ the mean of $u$ in the DSW. The boundary condition for this differential equation is $k(u_+) = 0$, which links the trailing edge to the leading, solitary wave edge of the DSW where the wavenumber vanishes. The
position of the trailing edge of the DSW is then determined from the group velocity

\[ s_- = \frac{\partial \omega(u_-, k_-)}{\partial k} \]  

(3.3.3)

at the trailing edge.

The leading, solitary wave edge of the DSW is determined in a similar fashion, but in terms of “conjugate” variables. The “conjugate frequency” \( \tilde{\omega} \) is given in terms of the “conjugate wavenumber” \( \tilde{k} \) by \( \tilde{\omega} = -i\omega(\tilde{u}, i\tilde{k}) \). Then, as for the linear wave edge, the leading, solitary wave edge of the DSW is determined by

\[ \frac{d\tilde{k}}{d\tilde{u}} = \frac{\frac{\partial \tilde{\omega}}{\partial \tilde{u}}}{V(\tilde{u}) - \frac{\partial \tilde{\omega}}{\partial \tilde{k}}} \]  

(3.3.4)

with the boundary condition \( \tilde{k}(u_-) = 0 \) to link to the trailing, linear edge. The position of the leading edge of the DSW is thus given by its velocity \( s_+ \), with

\[ s_+ = \frac{\tilde{\omega}(u_+, \tilde{k}_+)}{\tilde{k}_+}. \]  

(3.3.5)

Modulation theory gives that this is the same as the solitary wave velocity \( V_s \), so that \( s_+ = V_s \). The reason for the use of these complex “conjugate” variables is due to the underlying assumption of the dispersive shock fitting method that the periodic wave solution is given by \( u_0^2 = r^2(u)P(u) \), where \( P(u) \) is a cubic polynomial and \( r(u) \) is some smooth function which does not vanish at the roots of \( P(u) \). This then connects the periodic wave solution to elliptic function type behaviour, for which a solitary wave solution in the real direction is connected to a “conjugate” (linear) periodic wave in the imaginary direction.

In Section 3.5 the details of the leading and trailing edges of the DSWs for the Whitham equations (3.2.1)–(3.2.5) will be compared with their equivalents for the KdV equation, which is (3.2.3) with \( K = 1 + \frac{1}{6}\partial^2/\partial x^2 \). The velocity \( s_- \) of the trailing edge and the amplitude \( A_s \) and velocity \( s_+ = V_s \) of the leading edge of the KdV DSW are [51, 63]

\[ s_- = 4u_+ - 2u_-, \quad s_+ = V_s = \frac{2}{3}u_+ + \frac{4}{3}u_-, \quad A_s = \frac{3}{2}s_+ - 3u_+ = 2(u_+ - u_-). \]  

(3.3.6)
With these preliminaries, the leading and trailing edges of the DSWs for the Whitham equations (3.2.1)–(3.2.5) can be determined.

### 3.3.1 Intermediate long wave DSW

The linear dispersion relation for the intermediate long wave equation (3.2.1) is

\[ \omega = 2\bar{u}k - k^2 \coth k + k. \]  

(3.3.7)

The trailing, linear wave edge of the DSW is then determined by the differential equation (3.3.2) with the boundary condition \( k(u_+) = 0 \) to match with the leading edge. Solving this equation gives

\[ \frac{1}{2} \ln \frac{e^{2k} - 1}{k} + \frac{k}{e^{2k} - 1} = \bar{u} - u_+ + \ln \sqrt{2} + \frac{1}{2}. \]  

(3.3.8)

Then at the trailing, linear edge \( k = k_- \) and \( \bar{u} = u_- \). The velocity \( s_- \) of the trailing edge is the linear group velocity

\[ s_- = \frac{\partial \omega}{\partial k} = 2u_- + k_\coth^2 k_- - 2k_- \coth k_- - k_-^2 + 1 \]  

(3.3.9)

with \( k_- \) the solution of (3.3.8) with \( k = k_- \) and \( \bar{u} = u_- \).

The leading, solitary wave edge is determined in a similar fashion from the conjugate equation (3.3.4) with the condition \( \tilde{k}(u_-) = 0 \) linking to the trailing edge. Solving this differential equation gives that the conjugate wavenumber \( \tilde{k} = \tilde{k}_+ \) at the leading edge is the solution of

\[ \tilde{k}_+ \cot \tilde{k}_+ + \ln \frac{\sin \tilde{k}_+}{\tilde{k}_+} = -2(u_- - u_+) + 1. \]  

(3.3.10)

With this solution, the velocity of the leading edge of the DSW is the solitary wave velocity

\[ s_+ = V_s = \frac{\tilde{\omega}(u_+, \tilde{k}_+)}{\tilde{k}_+} = 2u_+ - \tilde{k}_+ \cot (\tilde{k}_+) + 1. \]  

(3.3.11)

The amplitude \( A_s \) of the leading edge of the DSW is then determined from the intermedi-
ate long wave soliton solution (3.2.2).

### 3.3.2 Water wave DSW

The linear dispersion relation for the Whitham equation (3.2.3) with water wave dispersion is

\[
\omega = 2\bar{u}k + \sqrt{k\tanh k - k}.
\]  

(3.3.12)

The differential equation (3.3.2) for the DSW trailing edge cannot be solved analytically with this dispersion relation, so it will be solved numerically. The velocity of the trailing edge of the DSW is then the group velocity

\[
s_{-} = \frac{\partial \omega(u_{-}, k_{-})}{\partial k} = 2u_{-} + \frac{1}{2} \frac{\tanh (k_{-}) + k_{-} \text{sech}^2 (k_{-})}{\sqrt{k_{-} \tanh (k_{-})}} - 1.
\]  

(3.3.13)

The solitary wave edge of the DSW is determined in a similar fashion using the conjugate equation (3.3.4). As for the linear edge, this equation cannot be solved analytically and its solution is found numerically with the condition \( \tilde{k}(u_{-}) = 0 \) to connect the solitary wave edge of the DSW to the linear wave edge. The velocity of the leading edge of the DSW is then the solitary wave velocity

\[
s_{+} = V_s = \frac{\tilde{\omega}(u_{+}, \tilde{k}_{+})}{\tilde{k}_{+}} = 2u_{+} + \sqrt{\frac{\tan (\tilde{k}_{+})}{\tilde{k}_{+}}} - 1.
\]  

(3.3.14)

As there is no known solitary wave solution of the Whitham equation (3.2.3), its amplitude-velocity relation will be determined from numerical solitary wave solutions, as outlined in Section 3.2(b).

The jump height \( \Delta = u_{-} - u_{+} \) at which the lead wave of the water wave DSW first forms a cusp can be obtained from numerical solutions of the solitary wave equation (3.2.4). For given \( u_{-} \) and \( u_{+} \) the velocity \( V = s_{+} \) is obtained from the shock fitting expression (3.3.14). This velocity is then used in the solitary wave equation (3.2.4). The jump height is increased until the numerical scheme ceases to converge. In this manner it is found that the DSW reaches maximum amplitude and forms a cusp when \( \Delta = 0.162 \).
3.3.3 Peaking DSW

The final DSW to be determined is that for the model Whitham equation (3.2.5) which has a peaked wave of maximum amplitude. The linear dispersion relation for this equation is

$$\omega = 2\pi k + k \left( \frac{2\mu \nu}{k^2 + \nu^2} - 1 \right). \quad (3.3.15)$$

As for the intermediate long wave example of Section 33.3.1, the trailing edge equation (3.3.2) is solved using this dispersion relation and the matching condition $k(u_+) = 0$ at the leading edge. It is then found that the wavenumber at the trailing edge is the solution of

$$\frac{\mu \ln (k_-^2 + \nu^2)}{\nu} - \frac{2\mu \nu}{k_-^2 + \nu^2} = 2(u_- - u_+) + \ln \left(\frac{\pi}{2}\right) - 1,$$  

$$\mu \ln (k_-^2 + \nu^2) - \frac{2\mu \nu}{k_-^2 + \nu^2} = 2(u_- - u_+) + \ln \left(\frac{\pi}{2}\right) - 1, \quad (3.3.16)$$

on setting $k = k_-$ and $\bar{u} = u_-$ in the solution. The velocity of the linear wave edge is then the group velocity

$$s_- = \frac{\partial \omega(u_-, k_-)}{\partial k} = 2u_- + \frac{2\mu \nu}{k_-^2 + \nu^2} - 1 - \frac{4k_-^2 \mu \nu}{(k_-^2 + \nu^2)^2}. \quad (3.3.17)$$

The leading, solitary wave edge of the DSW is determined in a similar fashion. Solving the conjugate equation (3.3.2) gives that the conjugate wavenumber $\tilde{k}_+$ at the leading edge is the solution of

$$\frac{1}{4} \ln \left( \frac{\nu^2 - \tilde{k}_+^2}{\nu^2} \right) + \frac{1}{2} \left( \frac{2\mu \nu}{\tilde{k}_+^2 - \nu^2} + 1 \right) = u_+ - u_-.$$

The velocity of the leading edge of the DSW is thus the solitary wave velocity

$$s_+ = V_s = \frac{\tilde{\omega}_s(u_+, \tilde{k}_+)}{\tilde{k}_+} = 2u_+ + \frac{2\mu \nu}{\nu^2 - \tilde{k}_+^2} - 1. \quad (3.3.19)$$

For this peaked wave case the amplitude-velocity relation can be determined analytically as the dispersive kernel of the Whitham equation (3.2.5) is the solution of the ordinary differential equation (3.2.6).

We seek a solitary wave solution of the Whitham equation (3.2.5) of the form $u =$
$f(x-Vt) = f(\theta)$. Substituting this travelling wave form into the equation and integrating once gives

$$-Vf + f^2 - f + \int_{-\infty}^{\infty} L(\theta - \xi) f(\xi) \, d\xi - C_1 = 0,$$

(3.3.20)

where $C_1$ is a constant of integration. We now use that $L$ is the solution of (3.2.6), so that

$$(\partial_\theta^2 - \nu^2) \left[(V + 1)f - f^2\right] + \nu^2 (f - C_1) = 0.$$

(3.3.21)

Multiplying (3.3.21) by $\partial_\theta ((V + 1)f - f^2)$ and integrating with respect to $\theta$ gives

$$[[V + 1 - 2f]f' - \nu^2 \left[[V + 1)f - f^2\right]^2 - (V + 1)f^2 + \frac{4}{3}f^3\right]
+ 2\nu^2 C_1 \left[f^2 - (V + 1)f\right] = C_2,$$

(3.3.22)

where $C_2$ is another constant of integration. The constants of integration can be found on assuming that the solitary wave is on a constant background, so that $f \to \beta$ and $f' \to 0$ as $\theta \to \pm \infty$. Thus, we obtain

$$C_1 = -V \beta + \beta^2, \quad C_2 = \nu^2 \left\{[(V + 1)\beta - \beta^2]^2 + (V + 1)\beta^2 - \frac{4}{3}\beta^3\right\}.$$

(3.3.23)

At the solitary wave maximum, $f = f_m$ and $f' = 0$. We then obtain from the differential equation (3.3.22) and the expressions (3.3.23) for the constants of integration that the solitary wave velocity is the solution of the quadratic

$$(V + 1)^2 - (1 + 2f_m + 2\beta)(V + 1) + (f_m + \beta)^2 + \frac{4}{3}f_m + \frac{2}{3}\beta = 0.$$

(3.3.24)

The positive square root of this quadratic is chosen to give the correct velocity as $f_m \to \beta$, so that

$$V + 1 = \frac{1}{2} \left\{1 + 2f_m + 2\beta + \sqrt{1 - \frac{4}{3} f_m^2 - \beta^2}\right\}.$$

(3.3.25)

As shown by Whitham [127], equation (3.3.22) governing the periodic wave solution gives that peaking occurs when $V + 1 = 2f_p$, where $f_p$ denotes the peak height. By setting
3.4 DSW breakdown

As the height $\Delta = u_- - u_+$ of the initial jump (3.2.7) grows, the amplitude of the lead solitary waves of the DSWs increase for the Whitham equations with the water wave and peaking kernels until they reach a maximum. The lead wave of the water wave DSW first has a cusp at $\Delta \approx 0.162$. This results in the leading edge of the DSW becoming flat as the maximum wave is approached as the jump height increases. The resulting DSW appears similar to a DSW propagating in water of decreasing depth [43]. In this case, the shoaling of the DSW causes it to eject a train of solitary waves ahead of it, due to mass conservation. While the DSWs have similar appearance, the cause is different here as the approach to a maximum wave causes the levelling of the leading edge of DSWs in the present work.

Numerical results show that when $\Delta = 0.152$, the water wave DSW becomes unstable.
For higher jump heights, the lead solitary wave of the DSW develops an instability, which propagates back through it, resulting in the DSW becoming a multiphase wavetrain. The development of this instability is illustrated in Figures 3.2 and 3.3. The numerical DSW solution shown in Figure 3.2(a) is just below the threshold, with $\Delta = 0.15$. The DSW has the standard KdV form, with some flattening of the leading wave amplitude due to these waves approaching the limiting wave form. An unstable DSW just above the threshold, with $\Delta = 0.157$, is illustrated in Figure 3.2(b) with the generation of a multi-phase wavetrain is clearly seen, with the instability arising at around $t = 1500$. The detailed evolution of the instability is shown in Figure 3.3 for $\Delta = 0.157$. It can be seen from this figure that the instability first occurs at the leading edge of the DSW at around $t = 2500$ and propagates back through it. It will now be shown that this behaviour is due to the modulational instability of the periodic wave solution of the Whitham equation. The Whitham equation with the peaking kernel (3.2.5) shows similar instability behaviour, with the threshold
jump height $\Delta = 0.234$ for the onset of the instability, again slightly lower than the critical jump height $\Delta = 0.239$.

### 3.4.1 Admissibility conditions

The DSW fitting method is based on the DSW being of KdV type. To ensure this, various admissibility conditions need to be satisfied [38, 82]. These relate to the genuine nonlinearity and hyperbolicity of the Whitham modulation equations governing the DSW. The breakdown of these conditions can lead to linear degeneracy and modulational instability due to zero dispersion. In detail, we require that

$$
\frac{d}{du_-} s_-(u_-) \neq 0, \quad \frac{d}{du_+} s_-(u_+) \neq 0, \quad \frac{d}{du_-} s_+(u_-) \neq 0, \quad \frac{d}{du_+} s_+(u_+) \neq 0 \quad (3.4.1)
$$

for the shock fitting method to be valid. The first and third criteria imply that the Whitham modulation equations form a genuinely nonlinear system at the linear and solitary wave edges of the DSW, respectively. The breakdown of either of these two conditions means that a centred simple wave solution of the modulation equations, corresponding to a DSW, is not possible. The second and fourth criteria mean that the Whitham modulation equations are strictly hyperbolic at the linear and solitary wave edges of the DSW, respectively. The breakdown of this condition at the linear wave edge means that there is zero dispersion and the DSW is not modulationally stable. A gradient catastrophe occurs, resulting in compression and implosion of the DSW. The numerical solutions displayed in Figures 3.2(b) and 3.3 seem to show modulational instability due to zero dispersion. However, the breakdown of the DSW is more involved than this.

The validity of the admissibility conditions (3.4.1) can be checked using the leading and trailing edge DSW solutions of Section 3.3. Differentiating the trailing edge velocity (3.3.9) and leading edge velocity $s_+$ (3.3.11) with respect to $u_-$ and $u_+$ for the intermediate long wave Whitham equation (3.2.1), it is found that the admissibility conditions are always satisfied, so that the DSW solution of Section 3.4 is valid for any jump height $\Delta = u_- - u_+$. Numerical solutions of the intermediate long wave equation (3.2.1) do not
show any instability of the DSW, as is expected as the intermediate long wave equation is integrable.

For the case of the water wave DSW, the derivative $ds_-/du_-$ vanishes when $k = 1.29$, at which wavenumber the jump height is $\Delta = 0.148$. The other derivatives in the admissibility conditions (3.4.1) do not vanish. The modulation equations then have a breakdown of genuine nonlinearity at the trailing edge for sufficiently large initial jumps. As stated above, the lead wave of the DSW first reaches the maximum amplitude for a jump height $\Delta = 0.162$, so the loss of genuine nonlinearity occurs just below this maximum wave amplitude. However, the numerical solutions displayed in Figures 3.2(b) and 3.3 are more indicative of modulation instability, with the generation of a multiphase wavetrain [38]. The resolution of this is found from the stability of the nonlinear periodic wavetrain for the Whitham equation with the water wave kernel [107]. This work found that the nonlinear periodic wave solution undergoes Benjamin-Feir instability when its amplitude is sufficiently large. Rescaling their results to the present form (3.2.3) of the Whitham equation, this critical amplitude is 0.39, which compares favourably with the numerical lead wave amplitude 0.439 for the jump height $\Delta = 0.151$ just before the DSW becomes unstable. In addition, the linear trailing edge also approaches instability at this jump height. Shock fitting gives the wavenumber $k_- = 1.31$ at the trailing edge for $\Delta = 0.152$, which is very close to the stability boundary for Benjamin-Feir instability for (weakly nonlinear) Stokes water waves on water of finite depth $k = 1.36$ [127]. For instance, the wavenumber at the instability around $x = 500$ in Figure 3.2(b) is centred around 0.962, which again is just in the Benjamin-Feir instability region for weakly nonlinear water waves. The water wave DSW then becomes unstable due to Benjamin-Feir instability of the underlying periodic wavetrain solution at its two edges. The jump height for the onset of numerical instability is just above that for the loss of genuine nonlinearity, so that the Benjamin-Feir instability dominates the effects of loss of genuine nonlinearity in the numerical solutions.

The DSW for the Whitham equation with the peaking kernel, (3.2.5), shows similar instability behaviour as for the water wave Whitham equation. This is not unexpected as the peaking kernel was introduced as an approximation to the water wave kernel so
that the limiting wave had a peak, rather than a cusp, which matches the limiting Stokes water wave [127]. Again, the derivative $ds_-/du_-$ vanishes, this time for the wavenumber $k = 1.070$ for the jump height $\Delta = 0.254$, so that there is a loss of genuine nonlinearity. In addition, $ds_-/du_+ = 0$ when $k = 2.721$, for which $\Delta = 0.721$. However, the leading wave of the DSW peaks at the jump height $\Delta = 0.239$. Hence, in contrast to the water wave kernel, the breaking of the admissibility conditions (3.4.1) is not relevant for the peaking kernel. However, similar to the water wave kernel, the leading wave of the DSW becomes unstable at the jump height $\Delta = 0.234$ for which the lead solitary wave has amplitude 0.66, again due to modulational instability of the underlying periodic wave. The instability then propagates through the DSW, generating a multiphase wavetrain. The solution in this case is similar to those displayed in Figures 3.2(b) and 3.3.

3.5 Comparisons with numerical solutions

Numerical solutions of the Whitham equations (3.2.1), (3.2.3) and (3.2.5) with the jump initial condition (3.2.7) will be compared with the DSW modulation solutions of the three Whitham equation variants and also KdV theory. The numerical solutions were obtained using a hybrid spectral method [34], see Appendix B.2.

Figure 3.4 shows comparisons for the lead solitary wave velocity $s_+$, lead solitary wave amplitude $A_s$ and trailing edge velocity $s_-$ versus the level ahead $u_+$ for the intermediate long wave equation (3.2.1). Shown are results from full numerical solutions and the modulation theory of Section 3.3 for the amplitude and velocity of the leading, solitary wave edge of the DSW and the group velocity of the trailing, linear wave edge. In addition, the figure shows the equivalent results for the KdV DSW (3.3.6), as the intermediate long wave equation reduces to the KdV equation in the limit of small amplitudes. The level behind $u_- = 1$ is kept fixed as the level ahead $u_+$ is varied. Figure 3.4(a) shows this comparison for the velocity $s_+$ of the lead solitary wave of the DSW. It can be seen that dispersive shock fitting gives leading edge velocities in near perfect agreement with numerical values, with a difference of less than 1%. As expected, the leading edge velocity of the equivalent KdV DSW converges to the Whitham equation value as the jump height
Figure 3.4: Comparisons between dispersive shock fitting solutions and numerical solutions for intermediate long wave equation (3.2.1). (a) front velocity $s_+$, (b) front amplitude $A_s$, (c) trailing edge velocity $s_-$. Here $u_+ = 1.0$. Numerical solution: (green) +; modulation solution (3.3.11): (purple) □; modulation theory (3.3.6) for KdV equation: (blue) △.
Figure 3.5: Comparisons between dispersive shock fitting solutions and numerical solutions for the water wave equation (3.2.3). (a) and (b) front velocity $s_+$, (c) and (d) front amplitude $A_s$, (e) and (f) trailing edge velocity $s_-$. For (a), (c) and (e) $u_+ = 0$. For (b), (d) and (f) $u_- = 0.2$. Numerical solution: (purple) $+$; shock fitting solution (3.3.14) and (3.2.4): (green) $\Box$; DSW modulation solution (3.3.6) for KdV equation: (blue) $\triangle$. 
$u_- - u_+$ decreases and the wave amplitudes become small. In the KdV limit the solitary wave amplitude has the exact value $2(u_- - u_+)$, which is twice the jump height. However, the KdV velocity differs from the numerical velocity by less than 5% over the entire jump range. Similar conclusions can be made for the leading solitary wave amplitude $A_s$ comparison shown in Figure 3.4(b). The numerical amplitude and shock fitting amplitude are in excellent agreement and the KdV modulation theory amplitude converges to these in the limit of a small initial jump. The modulation theory results for the intermediate long wave equation differ by less than 1% from the numerical values, while the KdV modulation theory differs by up to 9%.

The final comparison of Figure 3.4 is that of Figure 3.4(c) for the velocity $s_-$ of the trailing, linear edge of the DSW. The agreement between the dispersive shock fitting results and the numerical values, while still very good, is slightly worse than for the leading edge, but with the largest difference still less than 8%. In addition, the KdV trailing edge velocity converges to the Whitham equation value as the jump height decreases, as expected. The major result is that the discrepancy for the KdV trailing edge velocity is much larger than for the leading edge amplitude and velocity, with the difference up to 50%, showing the importance of higher order dispersion on the linear group velocity. In addition, the agreement of the dispersive shock fitting velocity, while very good, is slightly worse than for the leading edge amplitude and velocity. The reason for the poorer agreement between the shock fitting and numerical trailing edge velocities is that the trailing edge position is less certain than the leading edge one as there is no sharp rear edge to the numerical DSW, as can be seen from Figure 3.1. There is no distinct trailing edge, but a long tail of waves of nearly equal amplitude. Modulation theory predicts that the amplitudes of the waves in the rear of the DSW decrease linearly [51, 63, 127]. The numerical trailing edge position can then be estimated by linear extrapolation of the rear crests of the DSW down to the level $u_-$ behind [36]. The crests chosen are the distinct, moderate amplitude waves of decreasing amplitude before the long train nearly uniform, small amplitude waves seen in Figure 3.1 (a). This process is still arbitrary to some extent, which explains the increased difference between the shock fitting and numerical values.
Figure 3.6: Comparisons between dispersive shock fitting solutions and numerical solutions for peaking equation (3.2.5). (a) and (b) front velocity $s_+$, (c) and (d) front amplitude $A_s$, (e) and (f) trailing edge velocity $s_-$. For (a), (c) and (e) $u_+ = 0.0$. For (b), (d) and (f) $u_- = 0.2$. Numerical solution: (purple) $+$; shock fitting theory (3.3.19) and (3.3.25): (green) $\Box$; DSW modulation solution (3.3.6) for KdV equation: (blue) $\triangle$. 
Figure 3.5 shows comparisons for the lead solitary wave velocity $s_+$, lead solitary wave amplitude $A_s$ and the trailing edge velocity $s_-$ versus $u_-$ and $u_+$ for the Whitham equation with the water wave kernel (3.2.3). This figure has additional comparisons for which the level behind $u_-$ varies, due to the breakdown of the admissibility conditions and instability discussed above. The numerical results in this figure are presented beyond the instability threshold at $\Delta = 0.151$. The DSW leading and trailing edges reach an approximate steady state before instability develops, so doing this provides a meaningful comparison. The agreement between the numerical and shock fitting values for the lead solitary wave amplitude and the velocities of the lead solitary wave and trailing, linear edge of the DSW are very good, as for the intermediate long wave equation, except for larger jumps. While the lead wave and trailing wave velocities are in good agreement for the larger jump heights, the lead wave amplitude shows increasing disagreement. Figures 3.5(c) and (d) were calculated from the numerical solitary wave amplitude-velocity relation based on equation (3.2.4) governing a solitary wave. However, the difference is no more than 13%. In detail, the differences between modulation theory and the numerical values of the lead wave and trailing wave velocities and the lead wave amplitude are no more than 3%, 7% and 9%, respectively. Again, the KdV DSW solution shows increasing disagreement as the jump height grows, with the differences with the numerical results increasing to 8%, 58% and 27% for the lead wave amplitude and velocity and the trailing edge velocity, respectively. The reason for the difference between shock fitting and numerical results can be understood from the evolution of the DSW as the maximum amplitude is approached, shown in Figure 3.2(a). The numerical lead wave amplitude shown in Figure 3.5(c) levels off as the maximum amplitude is approached, but the shock fitting amplitude shows rapid growth. Dispersive shock fitting gives accurate predictions for the leading and trailing velocities of the DSW, even up to instability. On the other hand, while KdV modulation theory for the lead wave velocity is quite close to the numerical and shock fitting values, even up to instability, this is not the case for the lead wave amplitude and trailing edge velocity. In particular, the KdV trailing edge velocity is far from the numerical values. The KdV velocity linearly decreases, while the numerical and shock fitting values have
a slowing rate of decrease. The higher wave frequencies which are incorporated in the Whitham equation, but not asymptotically incorporated in the KdV equation, then have a major role in the DSW evolution.

Figure 3.6 shows comparisons for the lead solitary wave velocity $s_+$, lead solitary wave amplitude $A_s$ and the trailing edge velocity $s_-$ versus the level behind $u_-$ (the level ahead is fixed at $u_+ = 0.0$) and the level ahead $u_+$ (the level behind is fixed at $u_- = 0.2$) for the peaking wave Whitham equation (3.2.5). In general, the agreement between the numerical solution and the dispersive shock fitting results for the lead solitary wave amplitude and velocity and the trailing edge group velocity is excellent up until the DSW becomes unstable for large jump heights, at which point the numerical values terminate. The dispersive shock fitting results continue past this point, but these cease to be relevant. The peaking condition of Whitham [127], see Section 3.3(c), gives that the DSW peaks at $u_- = 0.239$. This value was obtained by assuming a zero background depth in equation (3.3.22), as the initial condition has $u_+ = 0$ for Figure 3.6. The peak wave is determined by $f' \to +\infty$ at the peak. As for the lead wave velocity for the water wave Whitham equation (3.2.3), shown in Figures 3.5(a) and (b), the KdV modulation theory lead wave velocity shown in Figures 3.6(a) and (b) is again in quite good agreement with the numerical values. However, this is not true for the lead wave amplitude shown in Figures 3.6(c) and (d) and the trailing edge velocity shown in Figures 3.6(e) and (f). This is emphasised for the trailing edge velocity as the Whitham equation velocity levels off, while the KdV velocity linearly decreases. Overall, the comparisons of Figure 3.6 are similar to those for the water wave Whitham equation of Figure 3.5, which is expected as Whitham introduced the model equation (3.2.5) to mimic the properties of the full water wave equation (3.2.3), but to be analytically more tractable, and to produce a highest wave with a sharp peaked crest, as for the full water wave equations, rather than a cusp, as for (3.2.3) [127]. The differences between the modulation theory results and the numerical values for the lead wave velocity, trailing edge velocity and lead wave amplitude are no more than 2%, 3% and 2%, respectively. On the other hand, the same results for the KdV DSW modulation solution differ from the numerical values by up to 8%, 48% and 28% for the lead
wave velocity, the trailing edge velocity and the lead wave amplitude.

3.6 Conclusions

The dispersive shock wave fitting method [37, 38, 41] has been used to find the properties of the leading and trailing edges of DSWs governed by the Whitham equation [127] with various dispersion relations (kernels). The Whitham equation is a model equation used to incorporate short wave effects not present in long wavelength, weakly nonlinear expansions of the water wave equations, such as the KdV equation. This is done by incorporating the weak nonlinearity of the KdV equation with the full water wave dispersion via a nonlocal Fourier integral term with a kernel based on the full dispersion relation. It is found that the shock fitting method again gives results in excellent agreement with full numerical solutions for the leading and trailing edges of a DSW. The results further emphasise the importance of higher frequencies in the evolution of water waves as their amplitudes increase, leading to peaking and other effects. These lead to the amplitudes of the waves in the DSW, particularly at the leading edge, to asymptote to a constant, in sharp contrast to a KdV DSW. The DSW then develops a series of equal amplitude waves at its leading edge. As the level of the initial jump $\Delta = u_- - u_+$ rises, the DSW for the water wave and peaking Whitham equations becomes unstable due to a Benjamin-Feir instability of the underlying periodic wavetrain. This manifests itself by the lead solitary wave becoming unstable. The instability then propagates back through the DSW, generating a multiphase wavetrain. For these equations, the DSWs also violate some of the admissibility conditions for the shock fitting method to be valid. However, these violations either occur for large jump heights which are not relevant, or occur for jump heights just below the instability threshold, so that they do not manifest themselves.
Chapter 4

Dispersive shock waves in quadratic media

4.1 Introduction

In this chapter the propagation of optical DSWs in a quadratic medium will be studied. The system governing the DSWs consists of two coupled NLS-type equations, one for the fundamental mode and one for the second harmonic. The DSWs are generated by jump initial conditions in the optical beam intensity. It is found that in order to obtain analytical solutions for the DSWs in quadratic media, a phase locking assumption is needed which is related to the form of the second harmonic. The leading and trailing edges of the DSW and the solitary wave amplitude are determined using El’s DSW fitting method [37, 38, 41]. These modulation theory solutions are compared with full numerical solutions of the governing quadratic media equations.

4.2 Quadratic system

Let us consider the propagation of a coherent optical beam through a quadratic ($\chi^{(2)}$) medium instead of the conventional Kerr ($\chi^{(3)}$) medium. A nonlinear effect termed SHG was discovered [52], and can be explained as a nonlinear optical process in which two optical beams with the same frequency interact with a nonlinear material. These two
beams’ energy is combined, so that both beams $u_1$ and $u_2$ exist together and a new optical
beam with twice the initial frequency and half of the initial wavelength is created. The
simplest case of SHG in a 1-D medium is of the form, after renormalisation,

\[
\begin{align*}
    i \frac{\partial u_1}{\partial z} &+ \frac{r}{2} \frac{\partial^2 u_1}{\partial x^2} + \beta u_1 - u_2 u_1^* = 0, \\
    \sigma i \frac{\partial u_2}{\partial z} &+ \frac{s}{2} \frac{\partial^2 u_2}{\partial x^2} - \alpha u_2 + \frac{u_1^2}{2} = 0,
\end{align*}
\]

(4.2.1)

where $\alpha = \frac{(2 + \Delta k)\sigma}{\beta}$, and $\Delta k = 2k_1 - k_2$.

Here $u_1$ and $u_2$ are the slowly varying complex envelopes of the first and second harmonics
with frequencies $\omega_1$ and $\omega_2$, $\alpha$ is a wave number mismatch, while $k_1$ and $k_2$ are the
Corresponding wavenumbers of $u_1$ and $u_2$. In the purely spatial case, the $z$ direction is
the propagation direction, and the medium parameter values are $\sigma = 2$, $r = s = \pm 1$ in
order for the system to be modulationally stable [15]. In this chapter, we only consider
the defocusing case for which $r = s = 1$. The system then becomes

\[
\begin{align*}
    i \frac{\partial u_1}{\partial z} &+ \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} + \beta u_1 - u_2 u_1^* = 0, \\
    2i \frac{\partial u_2}{\partial z} &+ \frac{1}{2} \frac{\partial^2 u_2}{\partial x^2} - \alpha u_2 + \frac{u_1^2}{2} = 0.
\end{align*}
\]

(4.2.2)

Let us briefly consider the limit of large $\alpha$, which corresponds to large positive values of
the mismatch $\Delta k$. In this case, the second equation of the system (4.2.2) can be approx-
imately reduced to the form $u_2 \approx u_1^2/(2\alpha)$. The substitution of this expression into the
first equation of system (4.2.2) results in the NLS equation for the first harmonic

\[
\begin{align*}
    i \frac{\partial u_1}{\partial z} &+ \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} - \frac{1}{2\alpha} |u_1|^2 u_1 = 0.
\end{align*}
\]

(4.2.3)

Equation (4.2.3) possesses exact dark soliton solutions. We call the limit of large $\alpha$ the
cascading limit. In this limit the cascadet $\chi^{(2)}$ effects are Kerr-like and the second har-
monic component $u_2$ is much weaker than the first harmonic $u_1$.

The simplest initial condition which will result in the generation of a DSW is a step
initial condition in optical intensity. Then, the initial condition for the first harmonic can be taken as

\[ u_1(x,0) = \begin{cases} 
  u_{1-} e^{ik_1 x}, & x < 0, \\
  u_{1+} e^{ik_1 x}, & x > 0.
\end{cases} \tag{4.2.4} \]

The second harmonic can not be defined independently of the first harmonic, but needs to be consistent with the quadratic system (4.2.2). Continuous wave solutions have the form \( u_1 = f_{1\infty} e^{i \theta_1} \) and \( u_2 = f_{2\infty} e^{i \theta_2} \), where \( \theta_1 = k_1 x - \omega_1 z \) and \( \theta_2 = k_2 x - \omega_2 z \). When substituted into the quadratic system (4.2.2) we find

\[
\begin{align*}
  f_{1\infty} \omega_1 - \beta f_{1\infty} - \frac{1}{2} f_{1\infty} k_1^2 + f_{2\infty} e^{i(\theta_2 - 2 \theta_1)} &= 0, \tag{4.2.5} \\
  4 f_{2\infty} \omega_1 - 2 f_{2\infty} k_1^2 - \alpha f_{2\infty} + \frac{1}{2} f_{1\infty} e^{i(2 \theta_1 - \theta_2)} &= 0.
\end{align*}
\]

We then have the following results

\[
\theta_2 = 2 \theta_1, \quad \omega_1 = f_{2\infty} + \frac{1}{2} k_1^2 - \beta, \quad 4 f_{2\infty}^2 - (\alpha + 4 \beta) f_{2\infty} + \frac{1}{2} f_{1\infty}^2 = 0. \tag{4.2.6}
\]

The initial condition for \( u_2 \) is found by solving the quadratic equation (4.2.6) and is

\[ u_2(x,0) = \begin{cases} 
  \frac{(\alpha + 4 \beta) - \sqrt{(\alpha + 4 \beta)^2 - 8 u_{1-}^2}}{8} e^{2k_1 x}, & x < 0, \\
  \frac{(\alpha + 4 \beta) + \sqrt{(\alpha + 4 \beta)^2 - 8 u_{1-}^2}}{8} e^{2k_1 x}, & x > 0.
\end{cases} \tag{4.2.7} \]

As \( u_2 \) is a second harmonic, the wavenumber is \( 2k_1 \). Note the \( \alpha + 4 \beta > 2 \sqrt{2} u_{1-} \) to ensure modulational stability of the continuous wave solution. If not, modulational instability will arise.

### 4.3 Dispersionless limit

Figure 4.1 shows a DSW solution of the quadratic system (4.2.2). The parameter values are \( u_{1-} = 1.0, u_{1+} = 0.5, \alpha = 10.0 \) and \( \beta = 0.5 \). Shown are a contour plot of the numerical solution (a) of the first harmonic electric field amplitude \( |u_1| \) and (b) of the second
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Figure 4.1: DSW solution of the quadratic system (4.2.2). The parameters are $u_{1-} = 1.0$, $u_{1+} = 0.5$, $\alpha = 10.0$ and $\beta = 0.5$. Shown are contour plots of the numerical solutions of (a) first harmonic electric field amplitude $|u_1|$ and (b) second harmonic electric field amplitude $|u_2|$ of the quadratic system (4.2.2).
harmonic electric field amplitude $|u_2|$. The solutions for $|u_1|$ and $|u_2|$ have a similar structure. It can be seen that the numerical solution consists of four parts. The first part is the far fields consisting of the initial levels far in front of and behind the DSW. The second part directly behind the DSW is the intermediate shelf, while the third part is an expansion fan linking the intermediate shelf to the uniform level behind the DSW. The fourth is the DSW itself and it links the intermediate shelf to the uniform level ahead of the DSW. Behind the DSW, there are small amplitude wave trains caused by the smoothing of the discontinuity in the derivative where the expansion wave connects the level behind.

As seen from Figure 4.1, away from the DSW, the dispersive effects are small. In this region, the Madelung transformation can be applied to the quadratic system (4.2.2) to obtain the hydrodynamic form. Let

$$u_1 = \sqrt{\rho_1}e^{i\phi_1}, \quad \phi_{1x} = v_1, \quad u_2 = \sqrt{\rho_2}e^{i\phi_2}, \quad \phi_{2x} = v_2. \quad (4.3.1)$$

This transformation then sets the quadratic system in the form

$$\frac{\partial \rho_1}{\partial z} + \frac{\partial}{\partial x} (\rho_1 v_1) - 2\rho_1 \sqrt{\rho_2} \sin (\phi_2 - 2\phi_1) = 0, \quad (4.3.2)$$
$$v_{1x} - \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\rho_{1xx}}{\rho_1} - \frac{1}{8} \frac{(\rho_{1x})^2}{\rho_1^2} \right) + v_1 v_{1x} + \frac{1}{2} \frac{\rho_{2x}}{\sqrt{\rho_2}} \cos (\phi_2 - 2\phi_1) - \sqrt{\rho_2} v_2 \sin (\phi_2 - 2\phi_1) + 2\sqrt{\rho_2} v_1 \sin (\phi_2 - 2\phi_1) = 0,$$
$$2\frac{\partial \rho_2}{\partial z} + \frac{\partial}{\partial x} (\rho_2 v_2) - \rho_1 \sqrt{\rho_2} \sin (\phi_2 - 2\phi_1) = 0,$$
$$2v_{2x} - \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\rho_{2xx}}{\rho_2} - \frac{1}{8} \frac{(\rho_{2x})^2}{\rho_2^2} \right) + v_2 v_{2x} + \frac{1}{4} \left( -2\rho_{2}^{-1/2} \rho_{1x} + \rho_1 \rho_{2}^{-3/2} \rho_{2x} \right) \cos (\phi_2 - 2\phi_1) + \frac{1}{2} \rho_1 \rho_2^{-1/2} (v_2 - 2v_1) \sin (\phi_2 - 2\phi_1) = 0.$$
assumption $\phi_2 = 2\phi_1$ is needed. Using the assumption gives

\[
\frac{\partial \rho_1}{\partial z} + \frac{\partial}{\partial x} (\rho_1 v_1) = 0, \quad \frac{2}{\partial z} (\rho_2 v_2) = 0,
\]

\[
\frac{\partial v_1}{\partial t} - \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{1}{\rho_1} - \frac{1}{8} \frac{\rho_1}{\rho_2^2} \right) + v_1 v_1 - \frac{1}{2} \frac{\rho_2}{\sqrt{\rho_2}} = 0,
\]

\[
2v_2 - \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\rho_2}{\rho_2^2} - \frac{1}{8} \frac{\rho_2}{\rho_2^2} \right) + v_2 v_2 + \frac{1}{4} \left[ -2 \rho_2^{-1/2} \rho_1 + \rho_1 \rho_2^{-3/2} \rho_2 \right] = 0.
\]

The system (4.3.3) is named the hydrodynamic form due to its similarity to the shallow water equations [127], with the first and third equations representing mass conservation and the second and fourth equations momentum conservation. As seen from Figure 4.1, away from the DSW, the solution is approximately non-dispersive as there are no significant waves. Therefore, by dropping higher order terms of (4.3.3) which describe the dispersion, the non-dispersive limit of the hydrodynamic equations system (4.3.3) is

\[
\frac{\partial \rho_1}{\partial z} + \frac{\partial}{\partial x} (\rho_1 v_1) = 0, \quad v_1 + v_1 v_1 + \frac{1}{2} \frac{\rho_2}{\sqrt{\rho_2}} = 0,
\]

\[
2v_2 - \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\rho_2}{\rho_2^2} - \frac{1}{8} \frac{\rho_2}{\rho_2^2} \right) + v_2 v_2 + \frac{1}{4} \left[ -2 \rho_2^{-1/2} \rho_1 + \rho_1 \rho_2^{-3/2} \rho_2 \right] = 0.
\]

The consistency of the dispersionless hydrodynamic form (4.3.4) together with phase locking assumption $v_2 = 2v_1$ give a relation between $\rho_1$ and $\rho_2$ [23], which is

\[
\rho_1 = 8 \left( \alpha + \beta \right) \sqrt{\rho_2} - 8\rho_2, \quad \text{or} \quad \sqrt{\rho_2} = \alpha + 4\beta - \sqrt{\left( \alpha + \beta \right)^2 - 8\rho_1}.
\]

(4.3.5)

Equations (4.3.5) and (4.2.7) have a similar form and (4.2.7) is a special case of (4.3.5). Then, in order to find the characteristics, we need to obtain the eigenvalues of the dispersionless hydrodynamic form (4.3.4). The two eigenvalues with physical meaning are chosen. They are

\[
\lambda_{1,2} = v_1 \pm \sqrt{\frac{\rho_1 - \sqrt{\rho_1^2 - 64\rho_1 \rho_2}}{32 \rho_2^1/2}}.
\]

(4.3.6)

On the non-dispersive section of the DSW, $\rho_1$ and $\rho_2$ satisfy the condition (4.3.5). Sub-
stitute the condition (4.3.5) into equation (4.3.6) gives the characteristics

$$\lambda_{1,2} = v_1 \pm H_{\rho_1}, \quad (4.3.7)$$

where, 

$$H_{\rho_1} = \frac{\rho_1 - \sqrt{\rho_1^2 + 2\rho_1\Lambda\sqrt{\Lambda^2 - 8\rho_1} - 2\Lambda^2\rho_1}}{4\left[\Lambda - \sqrt{\Lambda^2 - 8\rho_1}\right]}, \quad \text{and} \quad \Lambda = \alpha + 4\beta.$$ 

In Riemann invariant form, the non-dispersive equations (4.3.4) are

$$v_1 + M(\rho_1, \rho_{1+}) = \text{constant on } C_+: \quad \frac{dx}{dz} = V_+ = v_1 + H_{\rho_1}, \quad (4.3.8)$$
$$v_1 - M(\rho_1, \rho_{1+}) = \text{constant on } C_-: \quad \frac{dx}{dz} = V_- = v_1 - H_{\rho_1}, \quad (4.3.9)$$

where

$$M(\rho_1, \rho_{1+}) = \int_{\rho_{1+}}^{\rho_{1-}} \frac{H_{\rho_1}}{\rho_1} d\rho_1. \quad (4.3.10)$$

$M(\rho_{1-}, \rho_{1+})$ cannot be obtained in an analytical form, so Simpson’s rule is used to calculate it. The non-dispersive equations can be used to determine the level $u_{1i}$ of the intermediate shelf seen in Figure 4.1 (a). As noted above, the levels $u_{1-}$ and $u_{1i}$ are linked by an expansion wave, which is an expansion fan on the characteristic $C_-$ with the Riemann invariant on $C_+$ constant in the fan. The intermediate shelf $u_{1i}$ terminates at the DSW, whose trailing edge has the velocity $v_{1-}$. This simple wave solution is

$$(\rho_1, v_1) = \begin{cases} (\rho_{1-}, 0), & \frac{x}{z} < -H_{\rho_{1-}}, \\ (\rho_{f}, M(\rho_{1-}, \rho_{f})), & -H_{\rho_{1-}} \leq \frac{x}{z} \leq M(\rho_{1-}, \rho_{1i}) - H_{\rho_{1}}, \quad (4.3.11) \\ (\rho_{1i}, M(\rho_{1-}, \rho_{1i})), & M(\rho_{1-}, \rho_{1i}) - H_{\rho_{1}} < \frac{x}{z} < v_{1-}. \end{cases}$$

The intensity $\rho_1 = \rho_{f}$ on the expansion fan is the solution of

$$\rho_{1f} = \frac{\rho_1}{H_{\rho_1}^2} \left[M(\rho_{1-}, \rho_{1f}) - \frac{x}{z}\right]^2. \quad (4.3.12)$$

This simple wave solution is not complete as the level $\rho_{1i}$ of the intermediate shelf is not yet determined. The intermediate level $\rho_i$ is found by the requirement that the Riemann invariant along the characteristic $C_-$ is constant through the DSW, which links the intermediate level to the level $\rho_{1+}$ ahead of the DSW [38, 41]. Hence, the wavenumber $v_1$ on
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Figure 4.2: Electric field amplitude $|u_1|$ versus $x$ at $z = 1500$. The parameter values are $u_{1-} = 1.0$, $u_{1+} = 0.0$, $\alpha = 10$ and $\beta = 0.5$. Shown are the numerical solution of the quadratic system (4.2.2), the dam break solution (4.4.1) and (4.3.12) of the quadratic system and (4.5.1) of the cascading limit NLS equation. Numerical solution: purple (solid) line; modulation solution of the quadratic system: green (solid) line; modulation solution of the cascading limit NLS equation: blue (solid) line.

the intermediate shelf is

$$v_{1i} = M(\rho_{1-}, \rho_{1+}).$$

(4.3.13)

Matching this value of $v_{1i}$ with the value given by the simple wave solution finally gives that $\rho_{1i}$ is determined by

$$M(\rho_{1-}, \rho_{1+}) = 2M(\rho_{1-}, \rho_{1i}) = 2M(\rho_{1i}, \rho_{1+}).$$

(4.3.14)

The value of $\rho_{1i}$ needs to be determined numerically. After obtaining the value of $\rho_{1i}$, the solution of the non-dispersive part of the DSW in quadratic system is complete.

4.4 Dam break solution

The simplest solution of the quadratic equation (4.2.2) with the jump conditions (4.2.4) and (4.2.7) is for $u_{1+} = 0$. In this limit, the intermediate shelf $u_{1i}$ disappears and there is only the expansion wave of the previous section linking the level $u_{1-}$ to $u_{1+} = 0$. The solution is then just the classical dam break solution of shallow water wave theory. The
expansion fan solution then becomes

\[
\begin{align*}
(p_1, v_1) &= \begin{cases} 
(p_{1-}, 0), & \frac{x}{z} < -H_{p_{1-}}, \\
(p_{1f}, M(p_{1-}, p_{1f})), & -H_{p_{1-}} \leq \frac{x}{z} \leq M(p_{1-}, 0), \\
(0, M(p_{1-}, 0)), & M(p_{1-}, 0) < \frac{x}{z},
\end{cases}
\end{align*}
\]

(4.4.1)

Figure 4.2 shows the electric field amplitude $|u_1|$ versus $x$ at $z = 1500$. The parameters are $u_{1-} = 1.0$, $u_{1+} = 0.0$, $\alpha = 10.0$ and $\beta = 0.5$. Shown are the numerical solution of the quadratic system (4.2.2), the dam break solution (4.4.1) and (4.3.12) of the quadratic system and (4.5.1) of the cascading limit NLS equation. For these parameters no DSW occurs, just a dam break solution. It can be seen that there is excellent agreement between the numerical solution of the first harmonic of the quadratic system and the dam break solution, while the cascading limit NLS solution shows disagreement, indicating the importance of higher order terms. The expansion fan itself lies between the region $-311 < x < 615$ as predicted by the full dam break solution, and $-335 < x < 670$ by the dam break solution of the cascading limit NLS equation. The NLS expansion fan region is 8% longer than the quadratic media one. The disagreement can be explained by the size of the normalised mismatch wave number $\alpha$. As $\alpha = 10$, neglected terms of $O(\alpha^{-2})$ are relatively important. The difference between the cascading limit NLS equation and the numerical solution of the quadratic system increases for small values of $\alpha$. This is illustrated in Figure 4.10. It can be seen that there is a small amplitude dispersive wave train behind the trailing edge of the expansion fan that can not be explained by the present modulation theory. This wave train arises as there is a discontinuity in the derivative at the trailing edge of the expansion fan and the flat level $u_-$ behind it. The small amplitude wave train is a common characteristic of expansion fan solutions in modulation theory. It works as a smoothing effect for this discontinuity [51].
4.5 DSWs in the quadratic system

When the level ahead $u_{1,i}$ is non-zero a DSW will be formed to connect this level with the intermediate level $u_{1,i}$, which is illustrated in Figures 4.1 (a) and (b). If the governing equations are integrable, then it is possible to derive the full Whitham modulation equations and a simple wave solution for the DSW. However, this is not the case here. Therefore, a new approach is needed and has been provided by El [37, 38, 41]. This approach is to determine the leading and trailing edge of the DSW without needing the full Whitham modulation equations. He showed that the linear dispersion relation for the relevant equation can determine the leading and trailing edges of a DSW.

In section 4.7 the details of the non-dispersive part and the leading and trailing edges of DSWs for the quadratic system (4.2.2) will be compared with theoretical results for the cascading limit NLS equation (4.2.3). The intermediate level $\rho_i$, the expansion fan $\rho_{\text{f cascading}}$, the velocity of the trailing edge $v_-$, the amplitude $A_s$ and velocity of the leading edge $c_g$ of the defocusing NLS DSW are:

$$\sqrt{\rho_i} = \frac{\sqrt{\rho_-} + \sqrt{\rho_+}}{2}, \quad \rho_{\text{f cascading}} = \frac{1}{3} \left( 2\sqrt{\rho_1} - \sqrt{2\alpha} \zeta \right),$$

$$c_g = \frac{2\sqrt{\rho_-} - \rho_+}{\sqrt{2\alpha}}, \quad v_- = \frac{\sqrt{\rho_-} + \sqrt{\rho_+}}{2\sqrt{2\alpha}}, \quad A_s = \sqrt{\rho_-} - \sqrt{\rho_+}.$$

The dispersion relation for the quadratic system (4.2.2) can be found by substituting the linear wave solution

$$\rho_1 = \bar{\rho}_1 + \tilde{\rho}_1 e^{i(kx - \omega z)}, \quad \rho_2 = \bar{\rho}_2 + \tilde{\rho}_2 e^{i(2k - \omega z)}, \quad \rho_1 \ll |\bar{\rho}_1|, \quad |\tilde{\rho}_1| \ll \sqrt{\rho}_1, \quad |\rho_2| \ll \bar{\rho}_2\text{ and } |\tilde{\rho}_2| \ll \sqrt{\rho}_2,$$

where $|\bar{\rho}_1| < |\tilde{\rho}_1|$, $|\tilde{\rho}_2| < |\bar{\rho}_2|$ and $|\tilde{\rho}_2| < |\bar{\rho}_2|$, into the hydrodynamic form
(4.3.3). We obtain
\[
\begin{pmatrix}
-\omega + \bar{v}_1 k & \bar{p}_1 k & 0 & 0 \\
\frac{k^3}{4\bar{p}_1} & -\omega + \bar{v}_1 k & \frac{k}{\sqrt{\bar{p}_2}} & 0 \\
0 & 0 & -2\omega + \bar{v}_2 k & \bar{p}_2 k \\
-\frac{k}{4\sqrt{\bar{p}_2}} & 0 & \frac{k^3}{\bar{p}_2} + \frac{\bar{p}_1}{4\bar{p}_2^{3/2}} k & -2\omega + \bar{v}_2 k
\end{pmatrix}
\begin{pmatrix}
\bar{p}_1 \\
\bar{v}_1 \\
\bar{p}_2 \\
\bar{v}_2
\end{pmatrix} = 0. 
\tag{4.5.3}
\]

Throughout the DSW, the phase locking assumption \(\bar{v}_2 = 2\bar{v}_1\) needs to be satisfied. The dispersion relation for right propagating waves for these linearised equations is then
\[
\omega = \bar{v}_1 k + k \sqrt{\frac{k^2}{4} + \frac{\bar{p}_1 + \sqrt{\bar{p}_1^2 - 64\bar{p}_1\bar{p}_2}}{32\bar{p}_2^{1/2}}}. 
\tag{4.5.4}
\]

Then, since \(\bar{p}_1\) and \(\bar{p}_2\) represent the values of \(p_1\) and \(p_2\) in the non-dispersive section of the DSW, satisfying equation (4.3.5) gives a linearised equation
\[
\omega = \bar{v}_1 k + k \sqrt{\frac{k^2}{4} + H_{\bar{p}_1}}, \quad \bar{v}_1 = M(\bar{p}_1, \rho_1^+). 
\tag{4.5.5}
\]

The leading and trailing edges of a DSW are determined from a matching of these edges with the dispersionless solution away from the DSW.

### 4.5.1 Linear wave edge

According to El’s theory [37, 38, 41], the linear wave edge of the DSW is determined by the following differential equation
\[
\frac{dk}{d\bar{p}_1} = -\frac{\partial \omega}{\partial \bar{p}_1} \left( V_+ - \frac{\partial \omega}{\partial k} \right)^{-1}, 
\tag{4.5.6}
\]
when
\[
\frac{\partial \omega}{\partial \bar{p}_1} = k \frac{H_{\bar{p}_1}}{\bar{p}_1} + k \frac{H_{\bar{p}_1} \frac{\partial H_{\bar{p}_1}}{\partial \bar{p}_1}}{\sqrt{\frac{k^2}{4} + H_{\bar{p}_1}^2}}, \quad \frac{\partial \omega}{\partial k} = v_1 + \sqrt{\frac{k^2}{4} + H_{\bar{p}_1}^2} + \frac{k^2}{4\sqrt{\frac{k^2}{4} + H_{\bar{p}_1}^2}},
\]
\[
\frac{dk}{d\bar{p}_1} = \frac{k \sqrt{\frac{k^2}{4} + H_{\bar{p}_1}^2} H_{\bar{p}_1} + kH_{\bar{p}_1} \frac{\partial H_{\bar{p}_1}}{\partial \bar{p}_1}}{H_{\bar{p}_1} \sqrt{\frac{k^2}{4} + H_{\bar{p}_1}^2} - \left( \frac{k^2}{4} + H_{\bar{p}_1}^2 \right) - k^2/4}.
\]
$V_+ = \tilde{v}_1 + H\rho_1$ is defined as the forward moving characteristic. The leading edge equation (4.5.6) is simplified using the change of variable

$$\gamma(\bar{\rho}_1) = \sqrt{1 + \frac{k^2}{4H^2 \bar{\rho}_1}}.$$  (4.5.7)

Then,

$$\frac{dk}{d\bar{\rho}_1} = \frac{2\sqrt{\gamma^2 - 1}}{(1 - \gamma)(1 + 2\gamma)} \left( \frac{H\bar{\rho}_1}{\bar{\rho}_1} \gamma + \frac{\partial H\bar{\rho}_1}{\partial \bar{\rho}_1} \right).$$  (4.5.8)

After this transformation, the differential equation is

$$\frac{d\gamma}{d\bar{\rho}_1} = \frac{1 + \gamma}{\bar{H}_{\rho_1}} \left[ (1 - \gamma) \frac{\partial H\bar{\rho}_1}{\partial \bar{\rho}_1} - \frac{1}{1 + 2\gamma} \left( H\bar{\rho}_1 \gamma + \frac{\partial H\bar{\rho}_1}{\partial \bar{\rho}_1} \right) \right].$$  (4.5.9)

This equation is solved with the condition $k(\rho_1) = 0$ which connects the linear edge of the DSW to the solitary wave edge, so that $\gamma(\rho_1) = 1$. An analytic solution for the speeds of the leading and trailing edges cannot be obtained. Therefore, a numerical solution is necessary. With this solution the wavenumber at the leading edge of the DSW is

$$k = k(\rho_{1+}) = 2\sqrt{\gamma(\rho_{1+})^2 - 1H\rho_{1+}}.$$  (4.5.10)

The position of the leading wave edge of the DSW is therefore finally given by the group velocity

$$c_g = \frac{\partial \omega}{\partial k} = H\rho_{1+} \left( 2\gamma(\rho_{1+}) - \frac{1}{\gamma(\rho_{1+})} \right).$$  (4.5.11)

### 4.5.2 Solitary wave edge

The trailing, solitary wave, edge of the DSW is determined in a similar fashion. The equation governing the trailing edge of the DSW is [38, 41]

$$\frac{d\tilde{k}}{d\bar{\rho}_1} = \frac{\partial \tilde{\omega}}{\partial \bar{\rho}_1} \left( V_+ - \frac{\partial \tilde{\omega}}{\partial \tilde{k}} \right)^{-1}.$$  (4.5.12)
In general the “conjugate” dispersion relation is given by \( \tilde{\omega} = -i \omega (i \tilde{k}, \tilde{\rho}_1, \tilde{v}) \), so that the linear dispersion relation (4.5.5) gives for the quadratic DSW

\[
\tilde{\omega} = \tilde{v}_1 \tilde{k} + \tilde{k} \sqrt{-\frac{\tilde{k}^2}{4} + H_{\tilde{\rho}_1}^2},
\]

when

\[
\frac{\partial \tilde{\omega}}{\partial \tilde{\rho}_1} = -H_{\tilde{\rho}_1} \frac{\partial H_{\tilde{\rho}_1}}{\partial \tilde{\rho}_1}, \quad \frac{\partial \tilde{\omega}}{\partial \tilde{k}} = \tilde{v}_1 + \sqrt{-\frac{\tilde{k}^2}{4} + H_{\tilde{\rho}_1}^2} - \frac{\tilde{k}^2}{4\sqrt{-\frac{\tilde{k}^2}{4} + H_{\tilde{\rho}_1}^2}},
\]

\[
\frac{d \tilde{k}}{d \tilde{\rho}_1} = \frac{\tilde{k} H_{\tilde{\rho}_1}}{H_{\tilde{\rho}_1} \sqrt{-\frac{\tilde{k}^2}{4} + H_{\tilde{\rho}_1}^2} - \left(-\frac{\tilde{k}^2}{4} + H_{\tilde{\rho}_1}^2\right) + \frac{\tilde{k}^2}{4}}.
\]

This differential equation is simplified using the change of variable

\[
\tilde{\gamma}(\tilde{\rho}_1) = \sqrt{1 - \frac{\tilde{k}^2}{4H_{\tilde{\rho}_1}^2}}.
\]

Then,

\[
\frac{d \tilde{k}}{d \tilde{\rho}_1} = \frac{2 \sqrt{1 - \gamma^2}}{(1 - \gamma)(1 + 2\gamma)} \left[ H_{\tilde{\rho}_1} \tilde{\gamma} + \frac{\partial H_{\tilde{\rho}_1}}{\partial \tilde{\rho}_1} \right].
\]

After the transformation, the differential equation is of the form

\[
\frac{d \tilde{\gamma}}{d \tilde{\rho}_1} = \frac{1 + \tilde{\gamma}}{H_{\tilde{\rho}_1} \tilde{\gamma} \left[ (1 - \tilde{\gamma}) \frac{\partial H_{\tilde{\rho}_1}}{\partial \tilde{\rho}_1} - \frac{1}{1 + 2\tilde{\gamma}} \left( H_{\tilde{\rho}_1} \tilde{\gamma} + \frac{\partial H_{\tilde{\rho}_1}}{\partial \tilde{\rho}_1} \right) \right]}.
\]

This equation is solved numerically with the condition \( \tilde{k}(\tilde{\rho}_1) = 0 \) which connects the trailing edge to the leading edge, giving \( \tilde{\gamma}(\tilde{\rho}_1) = 1 \). With this solution, the velocity of the trailing edge of the DSW is

\[
v_- = \frac{\bar{\omega}}{\tilde{k}} = \int_{\tilde{\rho}_1}^{\tilde{\rho}_{1+}} \frac{H_{\tilde{\rho}_1}}{\tilde{\rho}_1} d\tilde{\rho}_1 + H_{\tilde{\rho}_1} \tilde{\gamma}(\tilde{\rho}_{1+}),
\]

on using expression (4.5.5).

We also wish to determine the trailing solitary wave amplitude of the DSW, so that we use the amplitude-velocity relation for a solitary wave. We look for a solitary wave solution by substituting the ansatz \( u_1 = f_1 e^{i \psi_1 e^{i \xi_1 z}} \) and \( u_2 = f_2 e^{i \psi_1 e^{i \xi_1 z}} \) where \( f_1 = \)
Figure 4.3: First and second harmonic trailing solitary wave edge amplitude $A_s$ and $A_{s2}$ versus $V_s$. The parameter values are $\alpha = 10$, $\beta = 0.5$, $f_{1\infty} = 0.7507$ and $f_{2\infty} = 0.0237$. Shown are (a) the numerical solution of equation (4.5.18) and (b) the numerical solution of equation (4.5.18).

We solve the pair of 2nd order ODEs (4.5.18) and (4.2.6) numerically and obtain the lead solitary wave amplitude $A_s$ as a function of $V_s$, which is defined as the difference between the trailing wave speed $v_-$ from (4.5.17) and the $v_{1i}$ from (4.3.13).

It is shown in both Figures 4.3 (a) and (b) that when $V_s = 0$ the trailing solitary wave amplitude is equal to the intermediate level and vacuum points arise. This will be fully
Figure 4.4: Electric field amplitude $|u_1|$ versus $x$ at $z = 1500$. The parameter values are $u_{1_\perp} = 1.0$, $u_{1_\parallel} = 0.5$, $\alpha = 10$ and $\beta = 0.5$. Shown are the numerical solution of the quadratic system (4.2.2), the simple wave solution (4.3.11) and intermediate level (4.3.12) of the quadratic system, as well as the solution (4.5.1) of the cascading limit NLS equation. Numerical solution: purple (solid) line; modulation solution of the quadratic system: green (solid) line; modulation solution of the cascading limit NLS equation: blue (solid) line.

explained in Section 4.6. Also, there is a near linear relation between first harmonic trailing solitary wave amplitude $A_s$ and $V_s$ and a nonlinear relation between second harmonic trailing solitary wave amplitude $A_{s2}$ versus $V_s$.

4.6 Vacuum point

As the jump height $u_{1_\parallel} - u_1$ grows, the amplitudes of the waves in the DSW increase until there is a jump height for which the trailing solitary wave of the DSW has its minimum at $A_s = u_{1_\parallel}$. This is named the vacuum point and the dark trailing solitary wave of the DSW occurs, with the amplitude-velocity relation (4.5.18) giving $V_s = 0$. Hence, the trailing wave edge speed of the DSW is $v_{1_\parallel} = v_{1_\parallel}$ and the conjugate dispersion relation for the trailing edge (4.5.13) gives

$$4H_{\tilde{p}_1}^2 = \tilde{k}^2.$$  

(4.6.1)

The vacuum point cannot be determined analytically. However, it can be obtained from expression (4.6.1) for the conjugate wavenumber $\tilde{k}$, with $\tilde{\gamma}$ determined by numerically
Figure 4.5: The trailing solitary wave amplitude $A_s$ versus $u_{1+}$. The parameter values are $\alpha = 10$, $\beta = 0.5$, and $u_{1-} = 1.0$. Shown are: numerical solution of quadratic system: (purple) line with $+$; modulation theory (4.5.18) of full quadratic system: (green) line with $\times$; the cascading limit NLS modulation theory (4.5.1): (blue) line with $\star$.

solving the trailing edge equation (4.5.16). The vacuum point condition (4.6.1) can be applied to the cascading limit NLS equation, which gives the condition analytically

$$3\sqrt{p_{1+}} = \sqrt{p_{1-}}.$$  \hspace{1cm}(4.6.2)

4.7 Comparison with numerical solutions

In this section, numerical solutions of the quadratic system (4.2.2) with the jump initial conditions (4.2.4) and (4.2.7) will be compared with the modulation theory solutions of the cascading limit NLS equation and the shock fitting solution of the full quadratic system. The numerical solutions of the quadratic system were obtained using a hybrid Runge-Kutta and finite difference method, see Appendix B.3 for details.

Figure 4.4 shows electric field amplitude $|u_1|$ versus $x$ at $z = 1500$. The parameter values are $u_{1-} = 1.0$, $u_{1+} = 0.5$, $\alpha = 10$ and $\beta = 0.5$. Shown are the numerical solution of the quadratic system (4.2.2), the simple wave solution (4.3.11) and intermediate level (4.3.12) of the quadratic system, as well as the solution (4.5.1) of the cascading limit NLS equation. The expansion fan lies in the region $-311 < x \leq -78$ as predicted by the modulation theory solution of the full quadratic system and $-336 < x \leq -84$ by the
Figure 4.6: The intermediate value $u_1$ versus $u_{1+}$. The parameter values are $\alpha = 10$, $\beta = 0.5$, and $u_{1-} = 1.0$. Shown are: numerical solution of quadratic system: (purple) line with $+$; modulation theory (4.3.14) of full quadratic system: (green) line with $\times$; the cascading limit NLS modulation theory (4.5.1): (blue) line with $\ast$.

The NLS expansion fan region is 8% longer than the quadratic media one and shifted to the left. Also, the intermediate level lies in the region $-77 \leq x < 231$ as given by the modulation theory solution of the quadratic system and $-84 \leq x < 252$ as given by the cascading limit NLS equation modulation theory solution. The NLS intermediate level region is 9% longer than the quadratic media one and is shifted to the left. Similarly to the dam break solution, excellent agreement between the modulation theory solution (4.3.11) and the numerical solution of intermediate level is found, but the cascading limit NLS solution (4.5.1) shows larger disagreement due to higher order terms being important. A backward moving small amplitude wave train on the initial level $u_{1-}$ is again shown. It is caused by the smoothing effect of dispersion to the discontinuity in slope of the modulation simple wave solution where it joins the level $u_{1-}$. As $u_{1+} \neq 0$, a DSW is generated ahead of the intermediate level $u_{1-}$.

Figure 4.5 shows the amplitude $A_s$ of the trailing solitary wave versus $u_{1+}$. The parameter values are $u_{1-} = 1.0$, $\alpha = 10.0$ and $\beta = 0.5$. Shown are the numerical solution of the quadratic system, the modulation theory amplitude (4.5.18) for the quadratic system (4.2.2) and the amplitude (4.5.1) for the cascading limit NLS equation. It can be seen
Figure 4.7: The trailing solitary wave velocity $v_-$ versus $u_{1-}$. The parameter values are $\alpha = 10$, $\beta = 0.5$, and $u_{1-} = 1.0$. Shown are: numerical solution of quadratic system: (purple) line with $+$; modulation theory (4.5.17) of full quadratic system: (green) line with $\times$; the cascading limit NLS modulation theory (4.5.1): (blue) line with $\ast$.

that the full modulation theory gives amplitudes in excellent agreement with numerical solutions for the full range of jump heights $u_{1-} - u_{1+}$. As for the intermediate level, both modulation theory for the quadratic system and modulation theory for the cascading limit NLS equation give amplitudes in near perfect agreement with numerical results for the quadratic system. The quadratic system modulation amplitude differs by at most 0.5% and the cascading limit NLS equation modulation amplitude differs by at most 1% from the numerical values. The numerical and modulation amplitudes converge to $A_s = 0$ as $u_{1+} \to 1$ as the jump vanishes in this limit.

Figure 4.6 shows the intermediate level $u_{1i}$ versus $u_{1+}$. The parameter values are $u_{1-} = 1.0$, $\alpha = 10.0$ and $\beta = 0.5$. Shown are numerical solutions of the quadratic system (4.2.2), the full modulation solution (4.3.14) of the quadratic system (4.2.2) and the cascading limit NLS equation modulation theory solution (4.5.1). It can be seen that both the full non-dispersive solution and the cascading limit NLS equation solution are in excellent agreement with the full numerical solution. As $u_{1+} \to 1$, all the modulation theory solutions converge to $u_{1i} = 1$ because there is no initial jump in this limit and the full solution is $u_1 = 1$. As $u_{1+}$ decreases, the intermediate level decreases and the disagreement between the modulation theory and numerical solutions grow slightly. It is worth noting that
a vacuum point occurs for \( u_{1+} < 0.4 \) and the modulation theory solutions cease to be valid below this level. Unfortunately, it is not possible to continue the theoretical investigation below the vacuum point, since it requires a knowledge of the full modulation equations [40]. The cascading limit NLS modulation theory gives predictions for the intermediate level \( u_{1i} \) which are in almost perfect agreement with those of the full modulation theory of the quadratic system and with the numerical results for the full quadratic system. Modulation theory for the cascading limit NLS equation gives intermediate levels which differ by at most 0.5% from numerical values, while the quadratic system modulation solution differs by 0.1% from numerical values.

Figure 4.7 shows the trailing solitary wave velocity \( v_- \) versus \( u_{1+} \). The parameter values are \( \alpha = 10, \beta = 0.5 \) and \( u_{1-} = 1.0 \). Shown are the numerical solution of the quadratic system, modulation theory (4.5.17) for the full quadratic system and the cascading limit NLS modulation theory (4.5.1). These comparisons are shown down to \( u_{1+} = 0.4 \), at which value the vacuum point first appears. Similarly to the previous comparisons, the full modulation theory velocity for the quadratic system is in excellent agreement with the numerical velocity of the quadratic system, with a difference less than 0.5%. An obvious disagreement is seen when the full numerical solution of the quadratic system
and the cascading limit NLS modulation theory velocity are compared. The differences between them range up to 8%, showing the significance of higher order terms’ effect in determining the trailing solitary wave velocity.

Figure 4.8 shows the leading linear wave velocity $c_g$ versus $u_{1+}$. The parameter values are $\alpha = 10$, $\beta = 0.5$ and $u_{1-} = 1.0$. Shown are numerical solutions of the quadratic system, modulation theory (4.5.11) for the full quadratic system and the cascading limit NLS modulation theory (4.5.1). The estimation of the numerical leading linear wave speed is still arbitrary to some extent. It is measured by linear extrapolation of the crests of the DSW down to the level $u_{1+}$ ahead. The crests chosen are the distinct, moderate amplitude waves of decreasing amplitude before the long wave train of nearly uniform small amplitude waves seen in Figure 4.1. Similarly to the previous comparisons, the full modulation theory linear wave velocity of the quadratic system is in excellent agreement with the numerical velocity of the quadratic system, with a difference of less than 1%. A small, but noticeable, difference of 3% is found when the full numerical solution of the quadratic system and the cascading limit equation modulation theory velocity are compared.

Figure 4.9 shows the vacuum point versus wave number mismatch $\alpha$. The parameter values are $u_{1-} = 1.0$, $u_{1+} = 0.5$, and $\beta = 0.5$. Shown are: numerical solution of quadratic system: (purple) line with $+$; modulation theory (4.6.1) of full quadratic system : (green) line with $\times$; the cascading limit NLS modulation theory (4.6.2): (blue) line with $\ast$. 

\[ \text{Figure 4.9: Vacuum point versus wave number mismatch } \alpha. \text{ The parameter values are } u_{1-} = 1.0, u_{1+} = 0.5, \text{ and } \beta = 0.5. \text{ Shown are: numerical solution of quadratic system: (purple) line with } +; \text{ modulation theory (4.6.1) of full quadratic system: (green) line with } \times; \text{ the cascading limit NLS modulation theory (4.6.2): (blue) line with } \ast. \]
Figure 4.10: Comparisons between the DSW fitting solutions and numerical solutions for the quadratic system (4.2.2). (a) trailing wave amplitude $A_s$, (b) trailing edge speed $V_-$, (c) intermediate level $u_1^i$, (d) front (linear wave edge) speed $c_g$. The parameters are $u_- = 1.0$, $u_+ = 0.5$, and $\beta = 0.5$. Shown are: numerical solutions: (purple) line with +; modulation theory of full quadratic system: (green) line with $\times$; the cascading limit NLS modulation theory: (blue) line with *.

ter values are $u_1^- = 1.0$, $u_1^+ = 0.5$ and $\beta = 0.5$. Shown are numerical solutions of the quadratic system, modulation theory (4.6.1) for the full quadratic system and the cascading limit NLS modulation theory (4.6.2). This comparison is for the value of $\alpha$ at which the vacuum point first occurs, that is the value of $u_1^+$ at which $|u_1|$ first vanishes within the DSW, as discussed in Section 4.6. The full modulation theory gives the vacuum point implicitly by (4.6.1), while for the cascading limit NLS equation modulation theory, it is given by (4.6.2). The cascading limit NLS equation modulation theory gives a constant vacuum point independent of $\alpha$. All three solutions show near perfect agreement and a slightly better agreement is found between the quadratic system modulation solution and the numerical solution for the vacuum point for the quadratic system, with a difference
less than 0.1%.

Figure 4.10 shows comparisons between the DSW fitting solutions and the numerical solutions of the quadratic system (4.2.2) for (a) the trailing wave amplitude $A_s$, (b) the trailing edge speed $V_-$, (c) the intermediate level $u_{1i}$, and (d) the front (linear wave edge) speed $c_g$. The parameter values are $u_-=1.0$, $u_+=0.5$ and $\beta=0.5$. Shown are numerical solutions, the modulation theory of the full quadratic system and the cascading limit NLS modulation theory. In general, the agreement between the numerical solution of the quadratic system and the DSW fitting method results for the trailing solitary wave amplitude and the trailing solitary wave speed, the intermediate level and the lead linear wave speed is excellent, even when the wave number mismatch $\alpha$ is small. On the other hand, the disagreement between the DSW fitting solutions of the cascading limit NLS equation and the full numerical solution of the quadratic system increases as $\alpha$ decreases.

For the trailing solitary wave amplitude $A_s$, there is excellent agreement between the numerical solution and the DSW fitting method solution for the quadratic system (4.5.18), the difference being less than 0.3%. The difference between the numerical solution and the DSW fitting method solution (4.5.1) of the cascading limit NLS equation can be as large as 2.7% when $\alpha = 2$. The trailing solitary wave speed $v_-$ shows excellent agreement between the numerical solution and the DSW fitting method solution (4.5.17) for the quadratic system, the difference being less than 1%. A much larger difference between the numerical solution and the DSW fitting method solution (4.5.1) occurs for the cascading limit NLS equation, up to 47% when $\alpha = 2$. For the intermediate level $u_{1i}$, the comparison between the numerical solution of the quadratic system and the analytical solution for the quadratic system shows excellent agreement, with less than 0.1% difference, while the disagreement with the cascading limit NLS solution grows rapidly as $\alpha$ decreases with the disagreement up to 1.3%. For the leading linear wave speed $c_g$, the difference between the numerical solution of quadratic system and the DSW fitting technique is less than 1.7% overall. However, the difference between the numerical solution of the quadratic system and the DSW fitting solution for the cascading limit ranges up to 24%. Hence, the DSW fitting solution for the quadratic system is needed, especially
when $\alpha$ is small.

### 4.8 Conclusion

The DSW solution for optical beam propagation in a defocussing quadratic medium has been derived. These equations form a coupled system of two NLS-type equations. A phase locking assumption is needed to give rise to DSWs in this medium. As the quadratic system (4.2.2) is non-integrable, the modulation equations for the leading and trailing edges of the DSW were obtained by the DSW fitting technique. This powerful technique enables the derivation of the leading and trailing wave speeds without knowing the full details of the Whitham modulation equations. It was found that the cascading limit NLS equation provides a good approximation when the wave number mismatch $\alpha$ is large. Full analytical solutions can be found in this case due to its integrability. However, when $\alpha$ is small, the NLS equation is less accurate and the DSW shock fitting solutions for the quadratic system (4.2.2) needs to be used.
Chapter 5

Conclusion

This thesis examines the DSWs generated by three different nonlinear wave equations describing a colloidal medium, peaking and breaking waves and quadratic media. As these equations are not integrable, the full Whitham modulation theory can not be applied to these model equations. Hence, El’s shock fitting technique was employed to derive solutions describing the leading and trailing edges of the DSW.

The DSW fitting method is used throughout this research. This powerful technique is based on Whitham’s modulation theory and Gurevich Pitaevsky’s pioneering ideas, and then adapted by El in order to describe non-integrable systems. It has been used in numerous applications and found to be a near perfect tool to determine the dynamics of the DSW by derivation of differential equations for its leading and trailing edges. The reason that it enables the derivation of these edges without knowing the full detail of the Whitham modulation theory is because the method uses the degenerate nature of these modulation equations at the edges of the DSW for cases in which the DSW is of KdV type.

In solving the DSWs in colloid media, it was found that in limit of low beam power, a higher-order NLS equation can be used to approximate the full colloid equation. The DSW fitting method for the full colloid equations was shown to give excellent predictions for the amplitude and velocities of the trailing and leading edges of the DSW. It was also found that the DSW fitting method gave reasonable predictions except near the vacuum point. This suggested that Taylor expansions of the full constitutive law for small packing
fractions can lead to reasonable approximations without using the complicated full case.

The DSWs governed by the Whitham equation was investigated. The Whitham equation is a model equation used to incorporate short wave effects not present in the KdV equation. The DSW fitting technique were used to understand the dynamics of the leading and trailing edges of the DSW. The stability of the Whitham equation was also considered and the threshold to instability was determined numerically. It was found the importance of higher frequencies in the evolution of water waves as their amplitudes increase, leading to peaking and other effects. The admissibility conditions were checked to prove the validity of the shock fitting method. Violations occurred but do not manifest themselves because these violations either occur for large jump heights which are not relevant, or occur for jump heights just below the instability threshold.

In determining the DSWs in quadratic media, it was found a phase locking assumption is necessary to determine the analytical solutions of DSWs. This helped simplify the problem, and the DSW fitting technique was employed again to determine the leading and trailing edges of the DSWs. It was also found that when the wavenumber mismatch $\alpha$ is large, the cascading limit NLS equation provided good predictions to the quadratic system. However, when $\alpha$ is small, the disagreement increased, and a modulation theory solution for the quadratic system becomes necessary.

In summary, we have investigated, both analytically and numerically, the DSWs in optical and fluid media and demonstrated El’s DSW fitting technique gives good estimation of the dynamics of a DSW for these applications. It is hoped that the theoretical and numerical treatment of DSWs in this work will motivate more future experimental or theoretical studies of DSWs in other media.

A few extensions of this work can be suggested here. The first one is to analyse the optical DSW in colloidal media with a more general constitutive relation. For example, a series form for the compressibility state relation $g(\eta)$ could be considered, see [7]. This would allow DSW for colloids with experimentally determined material properties (which determine the series coefficients) to be analysed.

The study of DSWs in nematic crystal with competing nonlinearities also becomes an
option, since Jung et al. [74] observed the formation of two-humped spatial solitons. They analysed and showed competing effects between the focusing (re-orientational) and thermal (defocusing) nonlinearities can induce two-humped solitons. It would be interesting and insightful to explore how a DSW would behave for competing nonlinearities.

A newly extended DSW fitting method brought about by El et al. [25] allows us to consider the interior structure of a DSW. Another possible extension of my work will be applying this theoretical tool to the Whitham equation and other equation with modulational instability arising in the interior of the DSW. The stability of the Whitham equation is already considered but mainly numerically, and a theoretical understanding of the phenomenon will be helpful.

It is also possible to study DSWs governed by 2-D nematic equations. The nematic equations reduce to the Kadomtsev-Petviashvilli (KP) equations in the small jump limit. The DSWs in the KP equations have been investigated by R.S. Johnson [72]. He showed that with a proper ansatz, the KP and 2-D Benjamin-Ono (2DBO) equation can be reduced to cylindrical KdV equation and cylindrical BO equation. Later, Ablowitz et al. [3] employed the Whitham modulation equations to describe evolution of the DSWs in these two equations. Thus, it would be interesting to see if a similar technique as used here, can be used for a 2-D nematic equation to determine properties of the DSW.
Bibliography


Appendix A

Amplitude-velocity relation for the grey colloidal solitary wave

In this appendix, the amplitude-velocity relation for a grey colloidal solitary wave is derived. Let us seek a grey solitary wave solution of the colloid equations (2.2.1) and (2.2.2) of the form

\[ u = f(\theta)e^{-i\sigma z}, \quad \eta = \eta(x-Vz), \quad \theta = x-Vz, \]  

(A.0.1)

where \( V \) is the velocity and \( \sigma \) is the propagation constant. The carrier wave giving the level as \( \theta \to \pm \infty \) is

\[ u = u_{\infty}e^{-i\sigma z}, \quad \sigma = \eta_{\infty} - \eta_0, \quad g(\eta_{\infty}) - g_0 = u_{\infty}^2. \]  

(A.0.2)

Substituting the grey solitary wave form (A.0.1) into the colloid equations (2.2.1) and (2.2.2) gives

\[ \sigma f - iV f' + \frac{1}{2} f'' - (\eta - \eta_0)f = 0, \quad |f|^2 = g(\eta) - g_0. \]  

(A.0.3)

We split this equation into real and imaginary parts with \( f = f_r + if_i \), giving

\[ \sigma f_r + V f'_i + \frac{1}{2} f''_r - (\eta - \eta_0)f_r = 0, \]  

(A.0.4)

\[ \sigma f_i - V f'_r + \frac{1}{2} f''_i - (\eta - \eta_0)f_i = 0 \]  

(A.0.5)
and \( \sigma \) given by (A.0.2). Multiplying the real part (A.0.4) by \( f'_r \) and the imaginary part (A.0.5) by \( f'_i \), adding and then integrating once gives

\[
\frac{1}{2} \sigma (f_r^2 + f_i^2) + \frac{1}{4} (f_r'^2 + f_i'^2) - \int (\eta - \eta_0) (f_r f'_r + f_i f'_i) \, d\theta = D, \tag{A.0.6}
\]

where \( D \) is a constant of integration. Differentiating the constitutive law (A.0.3) enables the integral to be rearranged as an integral in \( \eta \), so that (A.0.6) can be determined explicitly as

\[
\sigma (f_r^2 + f_i^2) + \frac{1}{2} (f_r'^2 + f_i'^2) - F(\eta, \eta_\infty) = D, \tag{A.0.7}
\]

with

\[
F(\eta, \eta_\infty) = \left[ \frac{1}{(1-\eta)^3} - \frac{1}{(1-\eta)^2} - \frac{1}{1-\eta} + \eta_0 \frac{\eta}{2} - \eta_0 \frac{\eta}{2} g(\eta) \right] \eta_\infty. \tag{A.0.8}
\]

The constant of integration \( D \) can be found by taking \( \theta \to \infty \), resulting in \( D = \sigma u_\infty^2 \), since \(|f|^2 \to u_\infty^2 \) as \( \theta \to \infty \).

Similarly, multiplying the real part (A.0.4) by \( f_i \) and the imaginary part (A.0.5) by \( f_r \), subtracting and integrating gives

\[
f_i f'_r - f_r f'_i = (u_\infty^2 - f_r^2 - f_i^2) V. \tag{A.0.9}
\]

Let us now consider the minimum of the grey solitary wave profile \(|u|\) and denote values at this minimum by an \( m \) subscript. At the minimum for \(|u|^2\), we have that

\[
f_{rm} f'_{rm} + f_{im} f'_{im} = 0. \tag{A.0.10}
\]

Hence, the imaginary part expression (A.0.9) gives

\[
V = \frac{|f_m|^2}{f_{im} (u_\infty^2 - |f_m|^2)} f'_{rm}. \tag{A.0.11}
\]
Similarly, using the minimum relation (A.0.10) in the real part expression (A.0.8) we have

\[
\frac{|f_m|^2}{f_{rm}^2} (f'_{rm})^2 = 2D - 2\sigma |f_m|^2 + 2F(\eta_m, \eta_\infty),
\]

(A.0.12)

where \(F(\eta_m, \eta_\infty)\) is similarly defined as \(F(\eta, \eta_\infty)\). Combining (A.0.11) and (A.0.12) we can deduce the amplitude-velocity relation for a grey colloid solitary wave as

\[
V = \sqrt{2|f_m| \sqrt{\sigma u_\infty^2 - \sigma |f_m|^2 + F(\eta_m, \eta_\infty)}}
\]

\[
\frac{u_\infty^2 - |f_m|^2}{u_\infty^2 - |f_m|^2}.
\]

(A.0.13)

We note that this amplitude-velocity relation has been derived independently of the solution for the grey solitary wave.
Appendix B

The numerical schemes

B.1 The colloid equation

The numerical method for the colloid equation (2.2.1) and (2.2.2) consists of two parts. In the main part, a hybrid method is employed, the solutions are obtained by using a centred finite-differences to the spatial coordinate $x$, and the Runge-Kutta 4th order method to the time-like coordinate $z$. The second part handles the numerical solutions of the medium response equation (2.2.2) which are obtained by Newton’s method. We choose the scheme for its high accuracy and stability. We discretise the solution as

$$ u_{m,n} = u(z_m = m\Delta z, x_n = n\Delta x), $$

$$ \eta_{m,n} = \eta(z_m = m\Delta z, x_n = n\Delta x), \quad n = 1, ..., N. \quad (B.1.1) $$

Then, by discretising the terms involving $x$-derivative, we can treat the colloid equation (2.2.1) and (2.2.2) similarly as an ODE

$$ u_{mz} = f(u_{m,n}) = \frac{i}{2\Delta x^2} (u_{m,n+1} - 2u_{m,n} + u_{m,n-1}) $$

$$ + \frac{i}{2} (u_{m,n+1} + u_{m,n-1}) (\eta_{m,n} - \eta_0), \quad \text{where} \quad |u_{m,n}|^2 = g(\eta_{m,n}) - g_0. \quad (B.1.3) $$
The fourth-order Runge-Kutta then gives the solution at $z_{m+1}$ as

$$u_{m+1,n} = u_{m,n} + \frac{1}{6} (a_{m,n} + 2b_{m,n} + 2c_{m,n} + d_{m,n}),$$

where

$$a_{m,n} = \Delta z f(u_{m,n}), \quad b_{m,n} = \Delta z f(u_{m,n} + \frac{a_{m,n}}{2}),$$
$$c_{m,n} = \Delta z f(u_{m,n} + \frac{b_{m,n}}{2}), \quad d_{m,n} = \Delta z f(u_{m,n} + c_{m,n})$$  \hspace{1cm} (B.1.4)

The packing fraction $\eta$ is defined as an implicit function of $u$. Thus for every $z$-step of the Runge-Kutta method (B.2.4), an explicit expression for the change of $\eta$ due to the change of $u$ has to be obtained. This explicit form of the small change is

$$\delta \eta = \frac{1}{g'(\eta)} (u \delta u^* + u^* \delta u),$$ \hspace{1cm} (B.1.5)

and it is necessary to calculate the expressions for $b$, $c$, and $d$ at each $z$-step in the Runge-Kutta method. Then, after each $u_{m+1,n}$ is calculated, the corresponding $\eta_{m+1,n}$ needs to be calculated by using Newton’s iteration technique to the equation (2.2.2). The boundary conditions are chosen to be the value of $u$ and $\eta$ when $x \to \infty$. The accuracy of the numerical method at each $z$-step is $O(\Delta z^4, \Delta x^2)$.

### B.2 The Whitham equation

The numerical solutions of the Whitham equation (3.2.3) were obtained using Pseudo-spectral method in the spatial coordinate $x$ and a 4th order Runge-Kutta method to the time coordinate $t$. The spatial period is normalised to $[0, 2\pi]$ for convenience. The, the interval is discretised into $2N$ equidistant points, and the space is $\Delta x = \pi/N$. The discretised function $u(x,t)$ can be transformed to the discrete Fourier space by

$$\hat{u}(k,t) = \mathcal{F}\{u\} = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} u(j\Delta x, t) e^{-\pi jk/N},$$ \hspace{1cm} (B.2.1)
where \( k = 0, \pm 1, \ldots, \pm N \). The inversion formula is defined as
\[
    u(j \Delta x, t) = \mathcal{F}^{-1}\{\hat{u}\} = \frac{1}{\sqrt{2N}} \sum_k \hat{u}(k, t)e^{\pi jk/N},
\] (B.2.2)
where only one half of the contributions at \( k = \pm N \) are included in the sum over \( k \). These transformation can be efficiently performed by the fast Fourier transform (FFT) scheme.

Then, the Whitham equation can be approximated by
\[
    u_t = f(u(j \Delta x, t)) = \mathcal{F}^{-1}\{-i(k)\mathcal{F}\{(u(j \Delta x, t)^2)\}\} - \mathcal{F}^{-1}\{(ik)\mathcal{F}\{K\} \mathcal{F}\{u(j \Delta x, t)\}\}
\] (B.2.3)

The fourth-order Runge-Kutta then gives the solution at \( t + \Delta t \) as
\[
    u(j \Delta x, t + \Delta t) = u(j \Delta x, t) + \frac{1}{6} \left( a(j \Delta x, t) + 2b(j \Delta x, t) + 2c(j \Delta x, t) + d(j \Delta x, t) \right),
\] where
\[
    a(j \Delta x, t) = \Delta t f(u(j \Delta x, t)),
\]
\[
    b(j \Delta x, t) = \Delta t f(u(j \Delta x, t) + \frac{a(j \Delta x, t)}{2}),
\]
\[
    c(j \Delta x, t) = \Delta t f(u(j \Delta x, t) + \frac{b(j \Delta x, t)}{2}),
\]
\[
    d(j \Delta x, t) = \Delta t f(u(j \Delta x, t) + c(j \Delta x, t)).
\] (B.2.4)

The boundary conditions are chosen to be the value of \( u \) when \( x \to \infty \). To obtain accurate results for initial jumps producing DSWs whose leading edge wave is near the wave of maximum height the time step was halved and the number of Fourier modes doubled until there was no change in the solution to the accuracy reported here. For the numerical calculations, the time step used was \( \Delta t = 0.025 \) and the number of points was \( N = 2^{18} \). Also, the initial condition (3.2.7) was smoothed using the hyperbolic tangent,
\[
    u(x, 0) = \frac{u_+ - u_-}{2.0} \tanh\left( \frac{x}{w} \right) + \frac{u_+ + u_-}{2.0}, \quad \text{where,} \quad w = 140. \] (B.2.5)
B.3 The quadratic media

The numerical solutions of the quadratic system (4.2.2) were obtained by using centred finite differences in the spatial coordinate $x$ and a fourth-order Runge-Kutta method for the time-like, propagation direction $z$. Let us use the notation

$$u_{1,m,n} = u_1(z_m = m\Delta z, x_n = n\Delta x),$$
$$u_{2,m,n} = u_2(z_m = m\Delta z, x_n = n\Delta x),$$

where $n = 1, \ldots, N$, $m = 1, 2, \ldots$, for the numerical solution. The quadratic system can be written in the form of a pair of ODEs by discretising the $x$-derivatives using centred differences to obtain

$$u_{1,c} = f(u_{1,m,n}) = \frac{i}{2\Delta x^2} (u_{1,m,n+1} + u_{1,m,n-1} - 2u_{1,m,n})$$
$$+ \frac{i\beta}{2} (u_{1,m,n+1} + u_{1,m,n-1}) + i\frac{2}{2} u_{1,m,n} (u_{1,m,n+1} + u_{1,m,n-1})$$
$$u_{2,c} = g(u_{2,m,n}) = \frac{i}{4\Delta x^2} (u_{2,m,n+1} + u_{2,m,n-1} - 2u_{2,m,n})$$
$$- \frac{i\alpha}{2} u_{2,m,n} + \frac{i}{16} (u_{1,m,n+1} + u_{1,m,n-1})^2$$

The fourth-order Runge-Kutta method then gives the solution at $z_{m+1}$ as

$$u_{1,m+1,n} = u_{1,m,n} + \frac{1}{6} (a_{1,m,n} + 2b_{1,m,n} + 2c_{1,m,n} + d_{1,m,n}),$$
$$u_{2,m+1,n} = u_{2,m,n} + \frac{1}{6} (a_{2,m,n} + 2b_{2,m,n} + 2c_{2,m,n} + d_{2,m,n}),$$

where

$$a_{1,m,n} = \Delta z f(u_{1,m,n}, u_{2,m,n}), \quad b_{1,m,n} = \Delta z f(u_{1,m,n} + \frac{a_{1,m,n}}{2}, u_{2,m,n} + \frac{a_{2,m,n}}{2}),$$
$$c_{1,m,n} = \Delta z f(u_{1,m,n} + \frac{b_{1,m,n}}{2}, u_{2,m,n} + \frac{b_{2,m,n}}{2}), \quad d_{1,m,n} = \Delta z f(u_{1,m,n} + c_{1,m,n}, u_{2,m,n} + c_{2,m,n}),$$
$$a_{2,m,n} = \Delta z g(u_{1,m,n}, u_{2,m,n}), \quad b_{2,m,n} = \Delta z g(u_{1,m,n} + \frac{a_{1,m,n}}{2}, u_{2,m,n} + \frac{a_{2,m,n}}{2}),$$
$$c_{2,m,n} = \Delta z g(u_{1,m,n} + \frac{b_{1,m,n}}{2}, u_{2,m,n} + \frac{b_{2,m,n}}{2}), \quad d_{2,m,n} = \Delta z g(u_{1,m,n} + c_{1,m,n}, u_{2,m,n} + c_{2,m,n}).$$
The boundary conditions are chosen to be the value of $v$ and $w$ when $x \rightarrow \infty$. The accuracy of the numerical method at each $z$-step is $O(\Delta z^4, \Delta x^2)$. 