2006

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Zhu, Song-Ping: A New Analytical-Approximation Formula for the Optimal Exercise Boundary of American Put Options 2006.

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Keywords

American Put Options, Laplace, Transform, Moving Boundary Value Problems

Disciplines

Physical Sciences and Mathematics

Publication Details

This article was originally published as Zhu, SP, A New Analytical-Approximation Formula for the Optimal Exercise Boundary of American Put Options, International Journal of Theoretical and Applied Finance, 9(7), 2006, 1141-1177. Copyright World Scientific Publishing Company. Original journal available [here](#).

A New Analytical-Approximation Formula for the Optimal Exercise Boundary of American Put Options

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Abstract

In this paper, a new analytical formula as an approximation to the value of American put options and their optimal exercise boundary is presented. A transform is first introduced to better deal with the terminal condition and, most importantly, the optimal exercise price which is an unknown moving boundary and the key reason that valuing American options is much harder than valuing its European counterparts. The pseudo-steady-state approximation is then used in the performance of the Laplace transform, to convert the systems of partial differential equations to systems of ordinary differential equations in the Laplace space. A simple and elegant formula is found for the optimal exercise boundary as well as the option price of the American put with constant interest rate and volatility. Other hedge parameters as the derivatives of this solution are also presented.

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1 Introduction

Financial derivatives, such as options, warrants and swaps are widely used as risk management tools in today's financial markets. Among them, options are still the most popularly used derivative products to hedge a portfolio of assets. Since most options traded on organized exchanges are of American style (i.e., they can be exercised at any time before the expiry date of the option), it is extremely important to correctly price these American-style options. Unfortunately, pricing American options is much harder than pricing their European counterparts, and the search for

accurate and efficient methods to value American options as well as the corresponding Greeks has been pursued by many researchers in the past two decades.

For American options, the essential difficulty lies in the problem that they are allowed to be exercised at any time before the expiration day, and mathematically, such an early exercise right purchased by the holder of the option has changed the problem into a so-called free boundary value problem, since the optimal exercise price prior to the expiration of the option is now time-dependent and is part of the solution. As a result of the unknown boundary being part of the solution of the problem, the valuation of American options becomes a highly nonlinear problem like any other free boundary value problems (e.g., Stefan problems of melting ice (Hill (1987))). This is very different from the valuation of European options as the latter is a linear problem if the well-known Black-Scholes equation (1973) is solved. Therefore, it is really this nonlinear feature that has hindered the search for an analytical solution. In the past two decades, it was widely believed that an exact formula for the optimal exercise price for the valuation of American options is very difficult or even impossible (see Huang *et al.* (1996), Wu and Kwok (1997), Ju (1998), Longstaff and Schwartz (2001)), and research efforts have been focused on the development of other approaches that can be employed to evaluate American options approximately.

The valuation problem of American put options can be traced back to McKean (1965) and Merton (1973) who first suggested that the valuation of American options should be treated as a free boundary value problem. Karatzas (1988) argued that the free boundary value problem should be treated in the frame of stochastic processes. All these early works have drawn considerable research interests in this area. In the literature, there have been two types of approximate approaches, numerical methods and analytical approximations, for the valuation of American options. Each type has its own advantages and limitations.

Of all numerical methods, there are two subcategories, those with which the Black-Scholes equation is directly solved with both time and stock price being discretized and those based on the risk-neutral valuation at each time step. The former subcategory typically includes the finite difference method (Brennan and Schwartz (1977), Schwartz (1977), Wu and Kwok (1997)), the finite element method (Allegritto *et al.* (2001)) and the radial basis function method (Hon and Mao (1997)). On the other hand, the latter subcategory typically includes the binomial method (Cox *et al.* (1979)), the Monte Carlo simulation method (Grant *et al.* (1996)) and the least squares method (Longstaff and Schwartz (2001)). Many of these methods still require intensive computation before a solution of reasonable accuracy can be obtained and in some cases, such as the explicit finite-difference scheme, the method may not even converge, as pointed out by Huang *et al.* (1996). For American options, the solution near maturity can be of singular behavior. Naturally, it is difficult for most numerical methods to calculate the option price accurately in the

neighborhood of maturity.

Analytical approximations, as another alternative, were also intensively sought in the past two decades. Typical methods in this category include the compound-option approximation method (Geske and Johnson (1984)), the quadratic approximation method (MacMillan (1986), Barone-Adesi and Whaley (1987)), the interpolation method (Johnson (1983)), the capped option approximation (Broadie and Detemple (1996)), the randomization approach (Carr (1998)) and the integral-equation method (Kim (1990), Jacka (1991), Carr *et al.* (1992), Huang *et al.* (1996), Ju (1998)). It must be pointed out that all these analytical approximation methods still require a certain degree of computation at the end. However, unlike the numerical methods mentioned earlier, various approximations are made in order to reduce the intensity of the final numerical computation, a feature that distinguishes these methods from those in the first category.

For the optimal exercise boundary near the expiration time, analytical approximations based on asymptotic expansions were worked out by Kuske and Keller (1998) and Stamicar *et al.* (1999). Chen and Chadam (2000) provided some convincing mathematical arguments to justify the asymptotic behavior of the optimal exercise boundary near expiry proposed by Stamicar *et al.* (1999). But all these approximation formulae are supposed to be valid for options with rather short life time.

Recently, based on an intuitive argument that the time derivative of the American put option at the optimal exercise price must vanish at all time, Bunch and Johnson (2000) derived a nonlinear algebraic equation, to which the optimal exercise price must satisfy. It was a remarkably good attempt to find an analytical formula for American put options with some very simple analytical approximations derived for different range of τ values (τ is the time to the expiration of the option). In their simple formulae, the optimal exercise price is written in terms of an unknown number α , which needs to be approximated and solved from a nonlinear algebraic equation iteratively. As will be demonstrated later, the formula they proposed to approximate α for a large τ value appears to be correct whereas the one used when τ is not too large appears to be of a problem. In fact, Basso *et al.* (2002) has already reported some problems using Bunch and Johnson's formulae.

In this paper, a new approximation formula for the optimal exercise boundary of American put options with constant interest rate and volatility is presented. The formula is found through solving the Black-Scholes (1973) equation in the Laplace space, with the utilization of the pseudo-steady state approximation during the Laplace transform. Being written explicitly in terms of all given inputs of the problem, such as interest rate, volatility and time to expiration, the new formula for the optimal exercise boundary is simple. It also requires no iteration at all. The option price, as well as the Greeks, for the simplest American put with constant interest rate and volatility can be written as a by-product in the solution process of

finding the optimal exercise boundary.

The paper is organized into four sections. In Section 2, a detailed description of the newly-found approximate formula for the optimal exercise boundary is provided, together with a formula for the option price. In Section 3, three examples are given, and our conclusions are stated in Section 4. Any mathematical derivations that are not immediately needed in the main body of the paper and yet are important for readers who may be interested in the details of derivation are left in the appendices.

2 The New Analytical-Approximation Solution

Since one can easily show that, without dividends, American call options would be equivalent to their European counterparts, i.e, it is always optimal to hold an American call to maturity when there are no dividend payments to the underlying asset, we shall thus concentrate on solving the Black-Scholes equation for an American put option with constant interest rate and volatility but no dividend payments to the underlying asset. This section is subdivided into two subsections, in the first of which an analytical-approximation formula for the optimal exercise price of American put options is presented and discussed, and in the second of which, the corresponding formula for the price of the American put is provided.

2.1 The Optimal Exercise Price of the American Put

Let $V(S, t)$ denote the value of an American put option, with S being the price of the underlying asset and t being the current time. With six main assumptions, Black and Scholes (1973) showed that the value of a call or put option V can be modeled by the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (1)$$

where r is the risk-free interest rate and σ is the volatility of the underlying asset price. Eq. (1) is widely referred to as the Black-Scholes equation. In this paper, r and σ are assumed to be constant.

Eq. (1) is the governing equation for any financial derivative in the Black-Scholes world. To model a specific type of options, Eq. (1) needs to be solved together with a set of appropriate boundary conditions. For both European and American put options, there is a far-field boundary condition

$$\lim_{S \rightarrow \infty} V(S, t) = 0, \quad (2)$$

which simply states that a put option becomes worthless when the price of the underlying asset becomes very large. On the other hand, unlike European options,

there is a critical asset price, $S_f(t)$, below or equal to which it is optimal to exercise the American put option. The optimal nature can be understood by the arbitrage opportunity to make a risk-free profit if the option is not exercised when the stock price is equal to or less than this critical asset price, which is usually referred to, in the literature, as the optimal exercise price (Wilmott *et al.* (1995)), the critical stock price (Bunch and Johnson (2000)), or optimal exercise boundary (Wu and Kwok (1997)), among some other less frequently used names. In this paper, we shall mainly refer to it as the optimal exercise price, but occasionally use the other two names as well. It is the existence of the optimal exercise price that has made the process of finding an exact formula for the valuation of American options much more difficult than that of finding an exact formula for the valuation of European options.

With the presence of the optimal exercise price, it can be shown (see Wilmott *et al.* (1995)) that the boundary conditions at the optimal exercise price $S = S_f(t)$ are

$$\begin{cases} V(S_f(t), t) = X - S_f(t), \\ \frac{\partial V}{\partial S}(S_f(t), t) = -1, \quad \text{or } \frac{\partial V}{\partial S} \text{ be continuous on } S = S_f(t), \end{cases} \quad (3)$$

in which X is the strike price of the option. The first equation in (3) simply states that the option price should be nothing but the intrinsic value of the option when the optimal exercise price is reached and the second one states that the option price is “smoothly” connected to the pay-off function at $S = S_f(t)$. From a mathematical point of view, this constitutes a so-called free-boundary value problem, in which the boundary location itself is part of the solution of the problem. Although the governing differential equation itself is linear in terms of the unknown function V , it is the unknown boundary that has made this type of problem highly nonlinear. The nonlinearity of the problem is clearly manifested once a Landau transform is used to convert the moving boundary problem to a fixed boundary value problem as demonstrated by Wu and Kwok (1997); the product term of the unknown functions $\frac{1}{S_f} \frac{dS_f}{dt} \frac{\partial v}{\partial s}$, which now appears in the partial differential equation, gives a good measure of the strength of the nonlinearity.

The fact that the value of a put option must be equal to its payoff function sets up the terminal condition

$$V(S, T) = \max\{X - S, 0\}, \quad (4)$$

where T is the expiration time of the option. Eqs. (1)-(4) constitute a differential system, the solution of which will give rise to the value of the American option at any time t before the expiration time T and at any price S .

To solve this system effectively, we shall first non-dimensionalize all variables by

introducing dimensionless variables

$$V' = \frac{V}{X}, \quad S' = \frac{S}{X}, \quad \tau' = \tau \cdot \frac{\sigma^2}{2} = (T - t) \cdot \frac{\sigma^2}{2}.$$

With all primes dropped from now on, the dimensionless system reads as

$$\left\{ \begin{array}{l} -\frac{\partial V}{\partial \tau} + S^2 \frac{\partial^2 V}{\partial S^2} + \gamma S \frac{\partial V}{\partial S} - \gamma V = 0, \\ V(S_f(\tau), \tau) = 1 - S_f(\tau), \\ \frac{\partial V}{\partial S}(S_f(\tau), \tau) = -1, \\ \lim_{S \rightarrow \infty} V(S, \tau) = 0, \\ V(S, 0) = \max\{1 - S, 0\}, \end{array} \right. \quad (5)$$

in which $\gamma \equiv \frac{2r}{\sigma^2}$ can be viewed as an interest rate relative to the volatility of the underlying asset price. The nondimensional differential system (5) shows that the solution will be a two-parameter family. That is, the solution of the system depends only on two parameters; one is the relative interest rate, γ , and the other one is the dimensionless total time, $\tau_{exp} = T \cdot \frac{\sigma^2}{2}$, from the initial time $t = 0$ to the expiration time T of the option. It should also be noticed that due to the introduction of the time to expiration τ as the difference between the expiration time T and the current time t , terminal condition (4) has become an initial condition in (5).

If we define a new function $U(S, \tau)$ as

$$U = \begin{cases} V + S - 1, & \text{if } S_f \leq S < 1, \\ V, & \text{if } S \geq 1, \end{cases} \quad (6)$$

differential system (5) can be written as the following two sets of equations and boundary conditions

$$\left\{ \begin{array}{l} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \gamma U = \gamma, \\ U(S_f(\tau), \tau) = 0, \\ \frac{\partial U}{\partial S}(S_f(\tau), \tau) = 0, \\ U(S, 0) = 0, \end{array} \right. \quad \text{if } S_f \leq S < 1 \quad (7)$$

$$\begin{cases} -\frac{\partial U}{\partial \tau} + S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \gamma U = 0, \\ \lim_{S \rightarrow \infty} U(S, \tau) = 0, \\ U(S, 0) = 0. \end{cases} \quad \text{if } S \geq 1 \quad (8)$$

One should notice that the initial condition in (7) and (8) now becomes a much easier form to deal with than that in (5). The boundary conditions at the moving boundary $S = S_f(\tau)$ also become homogeneous (at the expenses that the differential equation in (7) has now become non-homogeneous), which will considerably facilitate the solution procedure.

To guarantee V being a C^1 function of S , the continuity of the unknown function $V(S, \tau)$ and its derivative are demanded on the boundary $S = 1$, which results in the following interfacial matching conditions

$$\lim_{S \rightarrow 1^-} U = \lim_{S \rightarrow 1^+} U, \quad (9)$$

$$\lim_{S \rightarrow 1^-} \frac{\partial U}{\partial S} = \lim_{S \rightarrow 1^+} \frac{\partial U}{\partial S} + 1, \quad (10)$$

where 1^- indicates S approaching 1 from the left and 1^+ indicates S approaching 1 from the right.

Now, we perform the Laplace transform on systems (7)-(10). For the option price $U(S, \tau)$ and the optimal exercise price $S_f(\tau)$, we can certainly show that all three conditions for the exitance of the Laplace transform (cf. Hildebrand (1976)) are satisfied, and we shall denote all variables in the Laplace space with bars. For example,

$$\mathcal{L}U(S, \tau) = \int_0^\infty e^{-p\tau} U(S, \tau) d\tau = \bar{U}(S, p), \quad \mathcal{L}S_f(\tau) = \int_0^\infty e^{-p\tau} S_f(\tau) d\tau = \bar{S}_f(p).$$

Under the Laplace transform, systems (7)-(10) become the following ordinary differential equation systems, respectively, in terms of parameter p after the initial conditions have been substituted in

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2 \bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma \bar{U} = \frac{\gamma}{p}, \\ \bar{U}(p\bar{S}_f, p) = 0, \\ \frac{d\bar{U}}{dS}(p\bar{S}_f, p) = 0, \end{cases} \quad (11)$$

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2 \bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma \bar{U} = 0, \\ \lim_{S \rightarrow \infty} \bar{U}(S, p) = 0, \end{cases} \quad (12)$$

$$\begin{cases} \bar{U}(1^-, p) = \bar{U}(1^+, p), \\ \frac{d\bar{U}}{dS}(1^-, p) = \frac{d\bar{U}}{dS}(1^+, p) + \frac{1}{p}, \end{cases} \quad (13)$$

One should notice that the derivation of the differential equations in (11) and (12) under the Laplace transform, the interfacial matching conditions in (13) and the far-field boundary condition in (12) is straightforward. However, treatment of the two nonlinear moving boundary conditions in (7) requires an approximation based on the pseudo-steady-state approximation used in the heat transfer for Stefan problems.

For the classical Stefan (1889) problem, the Boltzmann similarity solution technique can be used to find the exact solution for a one-dimensional problem (see Gupta (2003)). However, based on a pseudo-steady-state approximation, approximate solutions can be worked out with amazingly accuracy, especially when the ratio of the specific latent heat to the heat capacity is small (see Fulford and Broadbridge (2002)). Although under a valid pseudo-steady-state approximation, the interface, at which the phase change takes place, is supposed to move slowly in comparison with the heat conduction, the results obtained based on the pseudo-steady-state approximation agree with the exact solution even near the initial time, when the interface moves with a large speed. This has motivated me to apply the pseudo-steady-state approximation to the current problem to derive an approximation solution because from the previously published numerical results, we know the behavior of $S_f(t)$ is very much like that of the moving boundary $S(t)$ in the classical Stefan problem.

Strictly speaking, when the Laplace transform is performed on the boundary conditions defined on the moving boundary $S_f(\tau)$, S in

$$\mathcal{L}U(S, \tau) = \int_0^\infty e^{-p\tau} U(S, \tau) d\tau,$$

should be replaced by $S_f(\tau)$ and the result is a function of p only. Based on the pseudo-steady-state approximation, if we assume that the optimal exercise boundary moves slowly in comparison with the “diffusion” of the option price, S can still be held as a constant during the Laplace transform and will then be replaced by the Laplace transform performed on the interfacial condition $S = S_f(\tau)$ with S being held as a constant as well (i.e., $\mathcal{L}S = \mathcal{L}S_f(\tau) \implies \frac{S}{p} = \bar{S}_f$). That is, we argue that the moving boundary condition $U(S_f(\tau), \tau) = 0$ in the original time space can be approximated by the boundary condition $\bar{U}(S, p) = 0$, with $S = p\bar{S}_f$ in the Laplace space. Similarly, we have the same argument for the 2nd moving boundary condition in (11). Of course, like the pseudo-steady-state approximation used for the classical Stefan problem, the assumption that $S_f(\tau)$ is nearly a constant function during the Laplace transform does not necessarily result in a boundary that is “slowly moving”. In the classical Stefan problem, the approximate solution based on the

pseudo-steady-state approximation has a infinite speed at $t = 0$, just like that of the exact solution of the Stefan problem (see Fulford and Broadbridge (2002)). For the similar reason, we shall expect that the result of using approximation technique will result in an approximation with a reasonably high accuracy. The verification of the accuracy of this approximation will be performed after the approximate solution is obtained.

The solution of differential systems (11)-(13) is of the form

$$\bar{U} = \begin{cases} D_1 S^{q_1} + D_2 S^{q_2} - \frac{\gamma}{p(p+\gamma)}, & \text{if } S_f \leq S < 1, \\ D_3 S^{q_1} + D_4 S^{q_2}, & \text{if } S \geq 1, \end{cases} \quad (14)$$

where q_1 and q_2 are roots of the characteristic equation of the homogeneous part of the corresponding equation

$$q_{1,2} = \frac{1-\gamma}{2} \pm \sqrt{\left(\frac{1-\gamma}{2}\right)^2 + (p+\gamma)}, \quad (15)$$

and D_1, D_2, D_3 and D_4 are four arbitrary complex constants to be determined in order to satisfy all boundary conditions. To facilitate the derivation, Eq. (15) can be written in different forms as

$$q_{1,2} = b \pm \sqrt{b^2 + (p+\gamma)} = b \pm \sqrt{p+a^2}, \quad (16)$$

where $a = \frac{1+\gamma}{2}$ and $b = \frac{1-\gamma}{2}$. It should be noticed that when γ varies in the domain $(0, \infty)$, a is always positive but b can be either positive or negative. In fact, b varies in the range $(\frac{1}{2}, -\infty)$. Furthermore, a and b are related as

$$a + b = 1, \quad a - b = \gamma, \quad a^2 = b^2 + \gamma.$$

As shown in Appendix A, if we choose a proper contour for the Laplace inverse transform, it can be shown that the real part of q_1 is always positive and the real part of q_2 is always negative. Therefore, D_3 has to be set to zero in order to satisfy the far-field boundary condition (2).

The satisfaction of the remaining boundary conditions as well as the interface conditions in (11)-(13) leads to a set of algebraic equations,

$$\begin{cases} D_1(p\bar{S}_f)^{q_1} + D_2(p\bar{S}_f)^{q_2} = \frac{\gamma}{p(p+\gamma)}, \\ D_1 q_1 (p\bar{S}_f)^{q_1-1} + D_2 q_2 (p\bar{S}_f)^{q_2-1} = 0, \\ D_1(1)^{q_1} + D_2(1)^{q_2} - \frac{\gamma}{p(p+\gamma)} = D_4(1)^{q_2}, \\ D_1 q_1 (1)^{q_1} + D_2 q_2 (1)^{q_2} = D_4 q_2 (1)^{q_2} + \frac{1}{p}, \end{cases} \quad (17)$$

the solution of which leads to a simple and yet elegant formula for the optimal exercise price in the Laplace space

$$\bar{S}_f = \frac{1}{p} \left[\frac{\gamma q_2}{\gamma q_2 - (p + \gamma)} \right]^{\frac{1}{q_1}}, \quad (18)$$

as well as three coefficients from which the option price $U(S, \tau)$ will be determined (they are shown in the next subsection).

Although Eq. (18) is remarkably simple, it is unfortunately still in terms of the Laplace parameter p . In order to obtain an analytical formula for the optimal exercise price, one still needs to carry out the Laplace inverse transform, a formidable process that often prevents this great technique being widely used to solve partial differential equations. However, the significance of Eq. (18) should never be underestimated, even though it is still in terms of the Laplace parameter p . As many researchers have pointed out, finding the optimal exercise price is the key to solve the American option problem. Once the optimal exercise price is found, the problem becomes a fixed boundary value problem like a European option problem; either numerical solutions or analytical solutions for this type of problem can be readily found.

By definition, the inversion of Eq. (18) should lead to the optimal exercise price $S_f(\tau)$ in the time domain. However, for the inverse Laplace transform

$$\begin{aligned} S_f(\tau) &= \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{e^{p\tau}}{p} \cdot \left[\frac{\gamma q_2}{\gamma q_2 - (p + \gamma)} \right]^{\frac{1}{q_1}} dp \\ &= \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{e^{p\tau}}{p} \cdot \exp \left\{ \frac{-\log \left[1 - \frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})} \right]}{b + \sqrt{p+a^2}} \right\} dp, \end{aligned} \quad (19)$$

to exist, we need to show (cf. Brown and Churchill (1996)) that $\bar{S}_f(p)$ is analytic to the right of the straight line $\text{Re}(p) = \mu$, where μ is an appropriately chosen positive number. In our problem here, other than a branch cut that ends at $p = -a^2$, $p = 0$ is the only simple pole in $\bar{S}_f(p)$. To ensure that $\text{Re}(q_1) > 0$ and $\text{Re}(q_2) < 0$, we can show that it is sufficient to choose any μ such that

$$\mu > 0. \quad (20)$$

Together with a set of other proofs, the proof of this requirement as a sufficient condition to guarantee the satisfaction of the far-field condition (2) is shown in Appendix A.

To actually invert (19) is not simple; the integrand appears to have two branch cuts, one from $p = -\infty$ to $p = -a^2$ and another one is associated with the logarithm

function. I first tried to use some published numerical Laplace inverse subroutines because it was believed that an analytical inversion was almost impossible. Then, after many failed attempts of using various numerical inversion methods (e.g., Papoulis (1957) and Stehfest (1970)), including those that appear to be very robust in inverting integrand that exhibits oscillatory behavior (Valsa and Brancik (1998)), I realized that none of these numerical inversion schemes was good enough to invert a complex function with branch cuts; the numerical inversion of (19) reached to a dead end.

This project was halted until I realized that there was no need to deal with both branch cuts, after a proper conformal mapping had been introduced. The branch cut of the logarithm function will never be reached. Details of the proof have been left in Appendix B for interested readers.

Figure 1: An illustration sketch of the contour used to evaluate the Laplace inverse transform.

In order to evaluate (19), we can now construct a closed contour as illustrated in Fig. 1. C_1 is a straight line placed at $\text{Re}(p) = \mu$ with μ being any positive number. C_2 and C_6 are two parts of a large circle with a radius R eventually approaching infinity. C_4 on the other hand is a circle of infinitesimally small radius and centered at the point $p = -a^2$. C_3 is a straight line connecting the end of C_2 and the beginning of C_4 , and is placed slightly above the negative real axis, whereas

C_5 is placed slightly below the negative real axis, connecting the end of C_4 and the beginning of C_6 . The direction of the integration is counter-clockwise as marked by the arrows in Fig. 1.

According to Cauchy's residue theorem (Brown and Churchill (1996)), we have

$$\begin{aligned} \sum_{j=1}^6 \int_{C_j} e^{p\tau} \bar{S}_f(p) dp &= 2\pi i \sum_{k=1}^n \operatorname{Res}_{p=p_k} \left\{ e^{p\tau} \bar{S}_f(p) \right\} \\ &= 2\pi i \sum_{k=1}^n \operatorname{Res}_{p=p_k} \left\{ \frac{e^{p\tau}}{p} \exp \left\{ \frac{-\log \left[1 - \frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})} \right]}{b + \sqrt{p+a^2}} \right\} \right\}, \end{aligned} \quad (21)$$

in which $i = \sqrt{-1}$, n is the total number of singular points inside the closed contour $C_1 - C_6$, and $\operatorname{Res}_{p=p_k} \{\cdot\}$ stands for taking the residue of the complex function included inside of the curly brackets at $p = p_k$. Of course, the integral corresponding to $j = 1$ on the left hand side of (21) is the integral we need to perform the inverse Laplace transform on $\bar{S}_f$ as defined in Eq. (19).

One can easily show that as $|p| \rightarrow \infty$, $|e^{p\tau} \bar{S}_f(p)| \rightarrow 0$. Therefore, according to the Jordan Lemma (cf. Brown and Churchill (1996)), the integrals on C_2 and C_6 vanish as $R \rightarrow 0$. It can also be easily shown that the integral on C_4 vanishes as the radius of C_4 approaches zero, since $p = -a^2$ is not a simple pole of the integrand $e^{p\tau} \bar{S}_f(p)$.

There is only one isolated simple pole of $e^{p\tau} \bar{S}_f(p)$ at $p = 0$. The residue of $e^{p\tau} \bar{S}_f(p)$ at this point can be readily evaluated as,

$$\operatorname{Res}_{p=0} [e^{p\tau} \bar{S}_f(p)] = \frac{\gamma}{1 + \gamma}, \quad (22)$$

which turns out to be the perpetual optimal exercise price shown by Samuelson (1965). In comparison to his derivation, here we have amazingly reached the same conclusion naturally as the residue of the integrand in (19).

With the proof left in Appendix B that there are no more cuts other than that shown in Fig. 1, the only non-trivial integrals that one needs to evaluate are on the straight lines C_3 and C_5 . On C_3 , $p + a^2 = \rho e^{i\pi}$ and $\sqrt{p + a^2} = \sqrt{\rho} e^{i\frac{\pi}{2}} = i\sqrt{\rho}$. On the other hand, on C_5 , $p + a^2 = \rho e^{-i\pi}$ and $\sqrt{p + a^2} = \sqrt{\rho} e^{-i\frac{\pi}{2}} = -i\sqrt{\rho}$. This results in the cancelation of the real part of integrands of the integrals on C_3 and C_5 , which become

$$I_3 + I_5 = 2ie^{-a^2\tau} \int_0^\infty \frac{e^{-\rho\tau}}{\rho + a^2} \cdot \operatorname{Im} \left\{ \exp \left[\frac{-\log(1 + \frac{b-i\sqrt{\rho}}{\gamma})}{b - i\sqrt{\rho}} \right] \right\} d\rho. \quad (23)$$

In (23), $\operatorname{Im} \{\cdot\}$ stands for taking the imaginary part of complex function inside of the curly brackets.

After dividing both sides of Eq. (23) by $2\pi i$ and substituting the result into Eq. (19) and Eq. (21), an analytical formula is obtained for the optimal exercise price S_f

$$S_f(\tau) = \frac{\gamma}{1+\gamma} + \frac{e^{-a^2\tau}}{\pi} \int_0^\infty \frac{e^{-\tau\rho}}{a^2+\rho} e^{-f_1(\rho)} \sin[f_2(\rho)] d\rho, \quad (24)$$

where

$$f_1(\rho) = \frac{1}{b^2+\rho} \left[b \ln \left(\frac{\sqrt{a^2+\rho}}{\gamma} \right) + \sqrt{\rho} \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right], \quad (25)$$

$$f_2(\rho) = \frac{1}{b^2+\rho} \left[\sqrt{\rho} \ln \left(\frac{\sqrt{a^2+\rho}}{\gamma} \right) - b \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right]. \quad (26)$$

Eq. (23) shows that the optimal exercise price has two components. The first one is the so called perpetual optimal exercise price, which does not depend on the time to expiration. The 2nd term, which is always of positive value because the integrand is always a positive function of ρ , depends on the remaining time to the expiration of the option. As will be shown in the examples, the contribution of the time-dependent term always asymptotically approaches zero when τ is very large or when the option is far from expiration, leaving the optimal exercise price being virtually equal to the perpetual optimal exercise price, $P_{cri} = \frac{\gamma}{1+\gamma}$.

Financially, it is natural to expect that part of the optimal exercise price should be the perpetual optimal exercise price, which is the optimal exercise price for options with infinite lifespan. However, it is amazing that Eq. (24) mathematically demonstrates that the optimal exercise price for an American put with a finite lifespan can be elegantly split into two parts, the first of which represents the perpetual optimal exercise price of the corresponding option without maturity and the second part of which represents an early exercise premium reflected in the underlying asset price due to the fact that the option under consideration is actually of a finite lifespan. The closer to maturity, the higher this premium is going to be.

One should also notice the analogy between this formula for the optimal exercise price of American options and that of the option price of European options. Both of these formulae are in terms of an integral defined in a semi-infinite range and both of them can only be evaluated numerically. For the price of European options, the integral happens to be the cumulative distribution function of the standard normal distribution, which can be written in terms of error functions. (See Wilmott *et al.* (1995)) Here, our formula cannot be written in terms of error functions. However, it is also an integral defined on the semi-infinite interval $[0, \infty)$ with an exponentially decay integrand.

Since there has been so much research done in the valuation of American options, this paper could have just stopped here because once the optimal exercise price is found, the price of American options can be treated as a problem of finding that of European-like options, which is entirely a linear problem due to the fixed boundary conditions in conjunction with the linear differential operator in the Black-Scholes equation. In the literature, many numerical solution approaches or approximate solution approaches hinge on the successful prediction of optimal exercise price S_f . Once the optimal exercise price is found, the remaining task of determining the option price and other hedge parameters can be easily fulfilled. For example, Huang *et al.* (1996) showed that important hedge parameters such as Δ , Γ , Θ , $Vega$ and Rho can be written in a closed form in terms of the optimal exercise price S_f . However, they had to calculate S_f recursively, which can be computationally intensive if the time to expiration is long. The formula presented here, on the other hand, does not suffer from this problem; the amount of computational work in order to evaluate the integral in (24) is virtually nothing in comparison with solving an integral equation in which an unknown function inside of an integral sign needs to be found. For a nonlinear problem like evaluating the price of American options, the integral equation is usually nonlinear and some kind of iteration needs to be designed to solve it numerically. To evaluate the integral in (24), on the other hand, requires no iteration at all. The computational time is virtually the same for any τ value. The only exceptional case is when $\tau = 0$, where the integral requires a little longer time to calculate as discussed below. In fact, the value of the integral is a function of τ and γ only, and hence this integral as a function of τ can be precalculated and tabulated for each value of γ as a parameter (similar to the tabulated error function used in calculating the price of European options). Then, there is no need to repeatedly evaluate this integral for different American put options. This may make the valuation of American options much faster in trading practice.

The numerical evaluation of the integral in (24) is not difficult at all; although the range of integration extends to infinity, the integrand vanishes to zero at infinity as well. In fact, for all non-zero τ , the integrand approaches zero exponentially because of the factor $e^{-\tau\rho}$. The worst scenario is at the expiration time when $\tau = 0$, where the integrand approaches zero at a rate of $O(\frac{1}{\rho})$. This actually turns out to be a good feature since at the expiration time we know that the optimal exercise price of an American option should be exactly equal to the strike price X of the option (see Huang *et al.* (1996)), or an arbitrage opportunity would exist otherwise, we can thus use this feature to set up a lower bound of accuracy (or an upper bound for the numerical error) in the numerical evaluation of the integral. In other words, we can determine the level of accuracy when we evaluate the integral with $\tau = 0$ by comparing the result with the exact solution of the unity optimal exercise price in nondimensional case, we can then be assured that when $\tau > 0$, the accuracy can only be better.

Here, let's use a sample case discussed in Wu and Kwok (1997) and Carr and Faguet (1994) to illustrate the level of accuracy in the computation of the optimal exercise price at $\tau = 0$. The parameters used by them are

- Strike price $X = \$100$,
- Risk-free interest rate $r = 0.1$,
- Volatility $\sigma = 0.3$,
- Time to expiration $T = 1$ (year).

In terms of the dimensionless variables, the two parameters involved are $\gamma = 2.2222$ and $\tau_{exp} = 0.045$.

There are many choices for the computation of the integral involved in Eq. (24). For example, one could try to make a variable change so that a standard IMSL subroutine could be called to do the calculation. However, with many high level and sophisticated programs available these days, I decided to adopt the simplest approach that requires minimal programming effort. The built-in numerical integration procedure in Maple is amazingly powerful in handling integrals of this type. Calculated with Maple 6, the results of S_f in terms of the upper limit R are tabulated in Table 1.

Table 1: Optimal exercise price at the expiration time

R	1×10^6	1×10^8	1×10^{10}	1×10^{13}
$S_f(0)$	0.9954771159	0.9994008655	0.9999254241	0.9999969643

Clearly, with the upper limit being 1×10^6 , the value of S_f is already correct to the 2nd decimal place and within 1% of the exact solution of 1. When the upper limit was increased to 1×10^8 in Maple, the result computed by the formula Eq. (24) became correct to the 3rd decimal place. The author intended to stop the calculation here as the convergence had been clearly demonstrated. The only reason that the calculation continued until the upper limit became 10 trillion was because of Jonathan Zhu (author's 10 year-old son)'s curiosity. He was wondering what the largest number Maple could handle without having any idea of what an upper limit was and what his dad was calculating. However, his curiosity actually helped to show that in Maple, an accuracy up to the fifth decimal place, in the worst scenario when $\tau = 0$, can be achieved. Most importantly, this prompted the author to examine if a simple variable transformation described below could actually lower the upper limit because "10 trillion" appears to be a huge number already to the author, perhaps not so big to a 10-year old kid as the imagination at that age can be so unlimited!

With a simple transform, $\rho = \zeta^2$, the integral in Eq. (24) can be changed to

$$S_f(\tau) = \frac{\gamma}{1+\gamma} + \frac{2e^{-a^2\tau}}{\pi} \int_0^\infty \frac{\zeta e^{-\tau\zeta^2}}{a^2 + \zeta^2} e^{-f_1^*(\zeta)} \sin[f_2^*(\zeta)] d\zeta, \quad (27)$$

where

$$f_1^*(\zeta) = \frac{1}{b^2 + \zeta^2} \left[b \ln \left(\frac{\sqrt{a^2 + \zeta^2}}{\gamma} \right) + \zeta \tan^{-1} \left(\frac{\zeta}{a} \right) \right], \quad (28)$$

$$f_2^*(\zeta) = \frac{1}{b^2 + \zeta^2} \left[\zeta \ln \left(\frac{\sqrt{a^2 + \zeta^2}}{\gamma} \right) - b \tan^{-1} \left(\frac{\zeta}{a} \right) \right]. \quad (29)$$

This simple change of variable results in a much smaller upper limit in order to achieve the same level of accuracy. For example, for the case discussed here, when the upper limit was set to 10^5 in Eq. (27) instead of 10^{10} in Eq. (24), the $S_f(0)$ ends up in the same value of 0.9999254241. Actually, with the powerful integration routines built in Maple, such a change of variable is not so crucial in the calculation of S_f , other than to have somewhat increased the computational efficiency in the evaluation of the involved integral defined on $[0, \infty)$. It is, however, absolutely necessary later when the option price is computed as the integrand would otherwise have a singularity at $\rho = 0$.

Figure 2: Optimal exercise prices for the case in Example 1

For the remaining τ values, the computation in Maple was just as easy as it was for the case when $\tau = 0$. An interesting phenomenon observed in the calculation for

nonzero τ is that the upper limit cannot be as large as one wishes once the exponential factor $e^{-\tau\rho}$ has made the integrand vanish much faster. Generally speaking, the larger the τ is, the faster the integrand approaches zero and the smaller the upper limit needs to be in order to achieve a certain degree of accuracy. The results presented in Fig. 2 for the optimal exercise price of this example are calculated with an upper limit of 10^6 .

An attempt was made to compare the results calculated by the present analytical formula with the results presented in Wu and Kwok (1997) in their Figure 1. An overall-good agreement can be observed in Fig. 2, it is clearly noticeable that the optimal exercise prices obtained with the front-fixing finite difference method proposed by Wu and Kwok (1997) are slightly higher than those obtained with the newly-developed formula Eq. (24). At the expiration time, $t = T$, the optimal exercise price in Wu and Kwok (1997) is $B(T) = \$76.25$ whereas it is $S_f(T) = \$75.49$ in our calculation based on the formula (24). The difference between these two solutions at $t = T$ is less than 1%.

In Fig. 2, the value of perpetual optimal exercise price $\frac{\gamma}{1+\gamma} \cdot X = \68.97 is plotted too to graphically show the level towards which the optimal exercise price asymptotically approaches when τ becomes large.

It should also be noticed that the newly-developed formula appears to work for large as well as small τ as demonstrated in Fig. 2. As τ becomes large, one can see, from Fig. 2, that S_f gets closer to the perpetual optimal exercise price and becomes more and more of a constant. This is indeed in line with the pseudo-steady-state approximation we have adopted in deriving formula (24). However, for small τ values, the variation of $S_f(\tau)$ appears to be at odds with the pseudo-steady state approximation. Since it is well known that the optimal exercise price is not differentiable near expiry (see Barles *et al.* (1995)), one naturally may question the validity of the pseudo-steady state approximation used here. To understand why the approximate solution also works well beyond the pseudo-steady state, one needs to compare the current situation with the one when the pseudo-steady state approximation is applied to solve the classical Stefan problem. If one has noticed how the pseudo-steady state approximation actually renders accurate solutions for the classical Stefan problem (see Fulford and Broadbridge (2002)) for large as well as small time, the fact that the current solution still works even when the interface $S_f(\tau)$ moves quite fast near $\tau = 0$ is not surprising at all. In fact, this phenomenon is quite common in applied mathematics. For example, a perturbation solution is meant to be valid when some parameters are small. But, quite often, a solution derived based on the assumption of some small parameters still works well when these parameters are actually of much larger values than they are supposed to be in the original assumption (see van Dyke (1964)). Another plausible explanation is that van Moerbeke (1976) demonstrated the connection between early exercise boundary and a Stefan-type free boundary problem for the heat equation; it is thus

not surprising at all that the pseudo-steady-state approximation works so well for this early exercise boundary problem as it did for the classic Stefan problem.

2.2 The Option Price of the American Put

Once $\bar{S}_f(p)$ is found, D_1 , D_2 and D_4 can be easily found from (17) and written in terms of the Laplace parameter p as

$$\begin{aligned} D_1 &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{q_2}{q_2 - q_1} \cdot \frac{1}{(p\bar{S}_f)^{q_1}} \\ &= -\frac{\gamma}{p(p+\gamma)} \cdot \frac{b - \sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right], \end{aligned} \quad (30)$$

$$\begin{aligned} D_2 &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{q_1}{q_1 - q_2} \cdot \frac{1}{(p\bar{S}_f)^{q_2}} \\ &= \frac{\gamma}{p(p+\gamma)} \cdot \frac{b + \sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right]^{\frac{q_2}{q_1}}, \end{aligned} \quad (31)$$

$$\begin{aligned} D_4 &= D_1 + D_2 - \frac{\gamma}{p(p+\gamma)} \\ &= \frac{\gamma}{p(p+\gamma)} \cdot \left\{ -\frac{b - \sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right] \right. \\ &\quad \left. + \frac{b + \sqrt{p+a^2}}{2\sqrt{p+a^2}} \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right]^{\frac{q_2}{q_1}} - 1 \right\}. \end{aligned} \quad (32)$$

Consequently, $U(S, \tau)$ can be written as

$$U(S, \tau) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_1(p) dp, \quad (33)$$

for $S_f(t) \leq S \leq 1$, and

$$U(S, \tau) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \frac{\gamma e^{p\tau}}{p(p+\gamma)} F_2(p) dp, \quad (34)$$

for $S > 1$. In Eq. (33) and Eq. (34), $F_1(p)$ and $F_2(p)$ are obtained by substituting Eq. (30)-Eq. (32) into Eq. (14) and can be written as

$$\begin{aligned} F_1(p) &= \frac{1}{2} \left(1 - \frac{b}{\sqrt{p+a^2}} \right) \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right] \cdot S^{q_1} \\ &\quad + \frac{1}{2} \left(1 + \frac{b}{\sqrt{p+a^2}} \right) \cdot \left[1 - \frac{p+\gamma}{\gamma(b - \sqrt{p+a^2})} \right]^{q_2/q_1} \cdot S^{q_2} - 1, \end{aligned} \quad (35)$$

and

$$F_2(p) = \left\{ \frac{1}{2} \left(1 - \frac{b}{\sqrt{p+a^2}} \right) \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})} \right] + \frac{1}{2} \left(1 + \frac{b}{\sqrt{p+a^2}} \right) \cdot \left[1 - \frac{p+\gamma}{\gamma(b-\sqrt{p+a^2})} \right]^{q_2/q_1} - 1 \right\} \cdot S^{q_2}. \quad (36)$$

If we use the same technique employed to invert $\bar{S}_f(p)$ shown in the previous subsection, we can find an analytical formula for the price of American options in the time domain as the inverse Laplace transform of $\bar{U}(S, p)$. After the function $U(S, \tau)$ is converted back to $V(S, \tau)$, we obtain

$$V(S, \tau) = \frac{1}{1+\gamma} \left[\frac{\gamma}{(1+\gamma)S} \right]^\gamma + \frac{\gamma}{2\pi} S^b e^{-a^2\tau} \int_0^\infty \frac{e^{-\rho\tau}}{(a^2+\rho)(b^2+\rho)} \cdot \left\{ e^A \left[\frac{b}{\sqrt{\rho}} \cos(\Phi + \sqrt{\rho} \ln S) + \sin(\Phi + \sqrt{\rho} \ln S) \right] - \sin(\sqrt{\rho} \ln S) - \left(\frac{bc}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{\gamma} \right) \cos(\sqrt{\rho} \ln S) \right\} d\rho, \quad (37)$$

where

$$A(\rho) = \frac{1}{b^2+\rho} \left[(b^2-\rho) \ln \left(\frac{\sqrt{a^2+\rho}}{\gamma} \right) + 2b\sqrt{\rho} \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right], \quad (38)$$

$$\Phi(\rho) = \frac{1}{b^2+\rho} \left[2b\sqrt{\rho} \ln \left(\frac{\sqrt{a^2+\rho}}{\gamma} \right) - (b^2-\rho) \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right]. \quad (39)$$

One should notice that the final solution shown in Eq. (37) is actually written in one form for both $S_f(t) \leq S < 1$ and $S \geq 1$. (See the detailed derivation in Appendix C) This is actually quite amazing as we started our solution procedure by only demanding that the unknown function $U(S, \tau)$ and its partial derivative with respect to S being continuous across the interfacial boundary $S = 1$. The fact that an American put option price can be written in a uniform function rather than a function of two branches shows that our solution is in fact infinitely differentiable everywhere in the domain $[0, \tau_{exp}] \times [S_f(\tau), \infty)$, which is truly remarkable as this has made the differentiability of the newly-found solution better than expected!

It may appear that the 2nd term in Eq. (37) will have a convergence problem when $S \rightarrow \infty$, if $b > 0$. A careful examination, however, reveals that this is actually not the case (cf. in Appendix A). An example is also presented in the next section (Example 3) to numerically show that, although the option price may approach zero much slower than in the case where $b < 0$, satisfying the far-field boundary condition of a put option is never a problem. The fundamental reason that the 2nd term in Eq. (37) still approaches zero despite the fact that $S^b \rightarrow \infty$ when $S \rightarrow \infty$

for the case $b > 0$ is that b is not the only factor that influences the behavior of the value of a put option for a large stock price. As shown in Eq. (A.6) of Appendix A, when the absolute value of the negative exponent $-\sqrt{b^2 + q_{2i}^2 + (p_{2r} + \gamma)}$ exceeds the value of b for an appropriately chosen contour of integration to perform the inverse Laplace transform, the behavior of the solution for large S values is governed by a net negative exponent of S , which results in the satisfaction of the far-field boundary condition (2). In Eq. (37), the effect of the negative exponent $-\sqrt{b^2 + q_{2i}^2 + (p_{2r} + \gamma)}$ is built inside of the integral sign, leaving the factor S^b outside of the integral sign to give a false impression about the behavior of the put option value $V(S, \tau)$ for a large stock price S .

From Eq. (37), we can easily show that there is a perpetual price limit for American put options for each fixed stock price S . This is the term resulting from evaluating the residue of the integrand of Eq. (37), similar to the case when the perpetual optimal exercise price is part of the optimal exercise price S_f . As $\tau \rightarrow \infty$, the terms under the integral sign in Eq. (37) vanish and we obtain

$$P_{opt}(\gamma, S) = \lim_{\tau \rightarrow \infty} V(\gamma, S, \tau) = \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)S} \right]^\gamma. \quad (40)$$

However, unlike the perpetual optimal exercise price associated with the optimal exercise price, this limit is an upper limit for a given stock price since the value of the second term (the integral term) in Eq. (37) is always negative whereas the value of the integral term in Eq. (24) is always positive. As S increases, the value of the integral term becomes more negative and eventually its absolute value becomes equal to P_{opt} , resulting in the zero option value at the large end of the S axis.

To facilitate the discussion on $P_{opt}(\gamma, S)$, we can rewrite Eq. (40) as

$$P_{opt}(\gamma, S) = G_1(\gamma)G_2(\gamma, S), \quad (41)$$

where the first function $G_1(\gamma) = \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)} \right]^\gamma$ is a function of γ only and the second one, $G_2(\gamma, S) = S^{-\gamma}$, depends on both γ and S .

As depicted in Fig. 3, it can be easily shown that $G_1(\gamma)$ is monotonically decreasing from 1 to 0, whereas the perpetual optimal exercise price associated with S_f , $P_{cri} = \frac{\gamma}{1 + \gamma}$, is a monotonically increasing function of γ from 0 to 1. The upper limit of the option price for a fixed stock price at any time depends also on the stock price S , as dictated by the second function in Eq. (41), which exponentially decays when γ increases if $S > 1$ but exponentially grows when γ increases if $S < 1$. Therefore, for $S > 1$, function G_2 simply makes the upper limit smaller after it is multiplied to G_1 . On the other hand, for $S < 1$, we shall expect that the upper limit decreases first and then passes a minimum point before increases to infinity as $\gamma \rightarrow \infty$. The minimum point at which the upper limit reaches the minimum value

Figure 3: Graphs of $P_{cri}(\gamma)$, $G_1(\gamma)$ and $P_{out}(\gamma)$

can be easily shown as

$$\gamma_{min} = \frac{S}{1 - S}. \quad (42)$$

The third curve in Fig. 3 shows the variation of $P_{opt}(\gamma, S)$ with S being set to 0.8. Clearly, P_{opt} has reached minimum when $\gamma_{min} = 4$, as calculated from Eq. (42). Since the upper limit of the option price given in Eq. (40) is a value that an American option can never pass beyond in its lifespan, γ_{min} is the relative risk-free interest rate, at which this upper limit is the minimum for the case of the stock price being less than the strike price. The reason that such a minimum relative risk-free interest rate does not exist for the case $S \geq 1$ is because when the stock price is higher than the strike price, a put option is “out of money” already and an increase in the risk-free interest rate would only worsen the option price. One should also notice that there is no problem with S being in the denominator in Eq. (40) since the smallest S value for American options is S_f , which is always greater than zero as long as $\gamma \neq 0$.

To actually compute the integral involved in Eq. (37), one should remove the singularity at $\rho = 0$ first. This can be easily achieved by letting $\rho = \zeta^2$. Eq. (37)

thus becomes

$$\boxed{
\begin{aligned}
V(S, \tau) = & \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)S} \right]^\gamma + \frac{\gamma}{\pi} S^b e^{-a^2 \tau} \int_0^\infty \frac{e^{-\zeta^2 \tau}}{(a^2 + \zeta^2)(b^2 + \zeta^2)} \\
& \cdot \left\{ e^{A^*} \left[b \cos(\Phi^* + \zeta \ln S) + \zeta \sin(\Phi^* + \zeta \ln S) \right] \right. \\
& \left. - \zeta \sin(\zeta \ln S) - (bc + \frac{\zeta^2}{\gamma}) \cos(\zeta \ln S) \right\} d\zeta,
\end{aligned}
} \tag{43}$$

where

$$A^*(\zeta) = \frac{1}{b^2 + \zeta^2} \left[(b^2 - \zeta^2) \ln \left(\frac{\sqrt{a^2 + \zeta^2}}{\gamma} \right) + 2b\zeta \tan^{-1} \left(\frac{\zeta}{a} \right) \right], \tag{44}$$

$$\Phi^*(\zeta) = \frac{1}{b^2 + \zeta^2} \left[2b\zeta \ln \left(\frac{\sqrt{a^2 + \zeta^2}}{\gamma} \right) - (b^2 - \zeta^2) \tan^{-1} \left(\frac{\zeta}{a} \right) \right]. \tag{45}$$

Eq. (43) can be readily evaluated and the results from several examples are discussed in the next section.

3 Examples and Discussions

As reviewed in the Introduction, there have been several approximation solutions for the valuation of American put options and their optimal exercise price. It would be interesting to compare the current formula with some of those previously published numerical or approximate solutions. Therefore, three examples are presented in this section for the purpose of validation. To help readers who may not be used to discussing financial problems with dimensionless quantities, all results are converted back to dimensional quantities in this section before they are graphed and presented.

3.1 Example 1

This is a sample case discussed in Wu and Kwok (1997) and Carr and Faguet (1994), and all parameters have already been listed in Section 2.1.

The optimal exercise price as a function of time to expiration has already been shown in Fig. 2. Clearly, it is a monotonically decreasing function of $T - t$ or a monotonically increasing function of t . When the time approaches the expiration time T of the option, the optimal exercise price sharply rises towards the strike price $X = \$100$. At $t = T$, $S_f(T) = X$ as we expected. Fig. 2 also exhibits that the rate of change of S_f is much larger near the expiration time than when the option contract is far away from the expiration.

With the newly-developed formula, we can now graph option value for a fixed stock price as a function of time to expiration. Shown in Fig. 4 is the option value,

Figure 4: Option price as a function of time at the fixed stock price $S = \$100$

calculated from Eq. (43), as a function of time to expiration with S being fixed to \$100. As expected, the option value decreases as the time to expiration approaches zero and it is equal to zero at the expiration time. In Fig. 4, the perpetual upper limit of American put option value with the stock price $S = \$100$ is also plotted. The interpretation of this perpetual price is quite simple; at any time, the American put option can never be worth more than \$13.59 in this case.

Of course, we can also graph the option value vs. the stock price at a fixed time. Depicted in Fig. 5 are the option prices $V(S, t)$ as a function of S at four instants, $\tau = T - t = 1$ (year), $\tau = 0.66$ (years), $\tau = 0.44$ (years), and $\tau = 0.22$ (years), respectively. Clearly, the option price is a decreasing function of stock price. As it gets closer to the expiration of the option, the option price becomes very close to the payoff function $\max\{X - S, 0\}$. In fact, when $t = T = 1$ (year), the option price is just the S axis starting from $S = \$100$, since $S_f(0) = \$100$ implies that $V(S, t) = 0$ for all $S \geq \$100$.

The stars on Fig. 5 show the value of $P_{opt}(\gamma, S)$ in Eq. (40). Clearly, for each stock price S , this is the upper bound for the option price discussed in Sec. 2.2. The absolute value of the second term that involves the integral in Eq. (43) (this value itself is always negative) is the difference of this upper bound and the option price. As S becomes large, the absolute value of the second term approaches $P_{opt}(\gamma, S)$, resulting in the option price approaching zero.

Theoretically, S needs to become infinite before a put option becomes worthless. But, from our newly-developed formula, one can observe from Fig. 5 that if, at any

Figure 5: Option prices for the case in Example 1

time, the stock price becomes about 1.6 times larger than the strike price, the option price becomes almost worthless for the interest rate given in this example.

3.2 Example 2

This is the same example used in Bunch and Johnson (2000). The dimensional parameters are

- Strike price $X = \$40$,
- Risk-free interest rate $r = 0.0488$,
- Volatility $\sigma = 0.3$,
- Time to expiration $T = 1$ (year),

and the two dimensionless parameters can be easily calculated as

- Relative risk-free interest rate $\gamma = 1.084$,
- Dimensionless time to expiration $\tau_{exp} = 0.045$.

Fig. 6 shows the comparison of the optimal exercise price produced by the current formula with those produced by using Eqs. (23) and (29) in Bunch and Johnson (2000), respectively. Overall, these results agree well. Bunch and Johnson

Figure 6: Optimal exercise prices for the case in Example 2

(2000) believed that the results produced by Eq. (23) “ought to have smaller errors”, Fig. 6 clearly suggests otherwise. However, it can be observed, from Fig. 6, that the results produced with their approximation formula (29) agree better with those produced by the current formula. The reason might be because their Eq. (29) is for a small τ value and the maximum τ value for this case is only 0.045, which should fall into this category. On the other hand, their Eq. (23) was supposed to be used for large τ values if it is used in conjunction with Eq. (A9) or for intermediate τ values if it is used in conjunction with Eq. (A10). It is not clear which equation they used to produce the data in their Figure 1. If it is the former, it is obviously not correct as the τ values in this problem are by no means large. On the other hand, if it is the latter, it would not be surprising at all that Eq. (23) in conjunction with Eq. (A10) produced larger error than Eq. (29) did. This is because Eq. (A10) was mistakenly derived by equating an expression that is valid for a large τ value with another one that is valid for any τ value; the result should be valid for a large τ value only!

One should also notice that in contrast to the very smooth and monotonically decreasing curves produced by the current analytical formulae for both the optimal exercise price and the option price, curves produced by some numerical solutions do not appear to be smooth and truly monotonic (e.g., Grant *et al.* (1996) and Hon and Mao (1997)). This could be due to the total number spatial grids limited by a particular numerical approach. For example, when radial basis functions are used (see Hon and Mao (1997)), a great difficulty would be encountered in the inversion

of the final solution matrix when the grid-spacing refinement is taken beyond a certain point. This problem also exists in the finite-difference approach (see Tavella and Randall (2000)); localized oscillations have been observed when grid spacing is refined beyond a point.

The option price as a function of S and t can be easily calculated as well, using the new formula. All graphs for the option price value as a function of stock price are similar to Fig. 5 and Fig. 7 and are thus not presented here.

3.3 Example 3

In both of the previous examples, the relative risk-free interest rates γ are all greater than one, resulting in a negative b value. When the risk-free interest rate r is small enough or the volatility of the underlying asset price σ is large enough, the relative risk-free interest rate will be less than unity, resulting in a positive b . Although we have proved in Appendix B that the far-field boundary condition (2) is automatically satisfied by the newly-found solution, as long as we choose the straight line C_1 (cf. Fig. 1) such that it locates to the right of the origin on the P plane, we may still desire to verify this proof through an example. In this example, we let

- Strike price $X = \$100$,
- Risk-free interest rate $r = 0.001$,
- Volatility $\sigma = 0.3$,
- Time to expiration $T = 10$ (years),

and the corresponding dimensionless parameters are

- Relative risk-free interest rate $\gamma = .0222$,
- Dimensionless time to expiration $\tau_{exp} = 0.45$.

For this set of parameters, one should notice that $b = .4889$, which is very close to the limiting value of 0.5. This extreme case should well serve the purpose of this example.

Graphed in Fig. 7 is the option price at four instants, $T - t = 10$ (years), $T - t = 5$ (years), $T - t = 1$ (year) and $T - t = 0.22$ (years), respectively. Clearly, even with this rather extreme case, the option price for a smaller time to expiration becomes worthless when the stock price reaches about twice of the strike price, like the case discussed in Example 1. On the other hand, when there is plenty of time left before an American put option expires, the underlying asset price has to be very large before the option becomes worthless. In this particular example, the option price is less than \$1 when the stock price has reached 10 times of the strike price

Figure 7: Option prices for the case in Example 3

X as shown in Fig 8. Therefore, a positive b may slow down the approaching zero of the option price when S becomes large, the far-field condition (2) is nevertheless satisfied.

One should also notice that the price upper limit marked by stars in Fig. 7 is closer to unity in comparison with those shown in Fig. 5. This is in agreement with the limit $P_{out}(\gamma, S)$ when γ approaches zero. When γ is very close to zero, the line formed by the stars should be close to a constant line of $P_{out} = 1$.

Since the expiration time in the previous two examples are still in the category of short-term options, we want verify that the newly-found analytical solution is not restricted by the lifespan of an option in any way. Furthermore, there are solutions that appear to be correct for short-term options but exhibit problems when the lifespan of an option is long. For example, in Barone-Adesi and Elliot's (1991) approximate solution, it is in error because optimal exercise price becomes non-monotonic and does not asymptotically approach the perpetual optimal exercise price in case of a long maturity time. Thereby, a long-term put option with an expiration time of 10 years is chosen in this example to demonstrate that there is no problem for the optimal exercise price to be calculated when τ is large. In fact, for large τ values, the evaluation of the integral in Eq. (24) may even take slightly less computational time; a smaller upper limit is needed to achieve a desired level of accuracy. Depicted in Fig. 9 is the variation of the optimal exercise price as a function of the time to expiration up to 10 years. When T is further increased, the contribution from the second term in Eq. (24) would eventually become insignificant

Figure 8: Option price when the time to expiration of the option is 10 years

to that of the first term; the optimal exercise price is nothing but the perpetual optimal exercise price when there is still a long time left before the expiration of an American put option.

4 Conclusions

In this paper, an analytical approximation formula for the optimal exercise price of the simplest American put with constant interest rate and volatility is presented. The formula is obtained by solving the well-known Black-Scholes equation in the Laplace space with an approximation made in the Laplace transform of the moving boundary conditions. It is shown that the optimal exercise price, which is the key difficulty in the valuation of American options, can be expressed as the perpetual optimal exercise price plus an early exercise premium that monotonically decreases with the remaining time to the expiration of the option. An analytical formula for the price of American put options is also found as a time-independent perpetual upper limit (for put options) less an early exercise cost (a negative premium in this case) that can be written in a simple integration similar to the cumulative distribution function of the standard normal distribution in the valuation of European options.

Three examples are presented to compare the newly developed analytical approximation solution with some previously published numerical solutions or approximate solutions. The validation process through these examples has demonstrated that the newly-developed formula for the optimal exercise boundary gives reasonably accu-

Figure 9: Optimal exercise price for the case in Example 3

rate results for large as well as small time to expiration.

It is envisaged that the proposed approach can be extended to find an approximation formula for American (both call and put) options with continuous dividend payments, which can find applications in foreign exchange options and index options. The research for such an extension is undertaken at the moment and the results will be published in a forthcoming paper.

Acknowledgements

The author wishes to gratefully acknowledge the financial support from the City University of Hong Kong for a visiting professorship during April and May of 2001; the fundamental concept presented in this paper was conceived there and some preliminary work that leads to the final completion of this paper was carried out there as well. He is indebted to some valuable help from Dr. X. Lu and Mr. Jonathan Zhu in drawing Figs. 1, 10 and 11. Without their support, these figures could have never been drawn so nicely. He would also like to thank the departmental administrative assistants, Ms. C. Silveri and Ms. S. Denny, who have kindly typed most of the mathematical equations presented in this paper.

Some very constructive remarks from an anonymous referee are also gratefully acknowledged as they have helped the author to improve the paper.

Appendix A: To show that the far-field boundary condition is satisfied if μ is chosen to be greater than zero

To find the asymptotic behavior of the solution (14) when $S \rightarrow \infty$, we need to analyze the real and imaginary part of $q_{1,2}$ in Eq. (16) when p varies along the straight line C_1 (cf. Fig. 1), on which the inverse Laplace transform is performed. $q_{1,2}$ are the roots of the characteristic equations corresponding to the homogeneous part of the ordinary differential equations in (11) and (12). The two characteristic equations are identical because the homogeneous part of the ordinary differential equation in (11) and that in (12) are the same. Thus, only one characteristic equation needs to be dealt with and it is

$$q^2 + (\gamma - 1)q - (p + \gamma) = 0. \quad (\text{A.1})$$

Let p and q be written in terms of their real and imaginary parts respectively as

$$p = p_r + ip_i, \quad (\text{A.2})$$

$$q = q_r + iq_i. \quad (\text{A.3})$$

Substituting Eqs. (A.2) and (A.3) into the characteristic equation Eq. (A.1), we obtain

$$q_r^2 - q_i^2 + 2q_r q_i i + (\gamma - 1)q_r + i(\gamma - 1)q_i - (p_r + \gamma) - ip_i = 0. \quad (\text{A.4})$$

Eq. (A.4) is equivalent to the following two equations

$$\begin{cases} q_r^2 + (\gamma - 1)q_r - q_i^2 - (p_r + \gamma) = 0, \\ [2q_r + (\gamma - 1)]q_i = p_i, \end{cases} \quad (\text{A.5})$$

from which we can solve for q_r and q_i as

$$\begin{aligned} q_{1r,2r} &= b \pm \sqrt{b^2 + q_i^2 + (p_r + \gamma)}, \\ q_{1i,2i} &= \pm \frac{p_i}{2\sqrt{b^2 + q_i^2 + (p_r + \gamma)}}, \end{aligned} \quad (\text{A.6})$$

with $b = \frac{1-\gamma}{2}$.

Clearly, if C_1 is chosen such that

$$\mu = p_r > -\gamma, \quad (\text{A.7})$$

we can ensure that $q_{1r} > 0$ and $q_{2r} < 0$. Therefore, to satisfy the far-field boundary condition (2), we must demand $D_3 = 0$, if C_1 is placed to the right of the point $p_r = -\gamma$.

Since $q_2 < 0$, when $S \rightarrow \infty$,

$$\bar{U} = D_4 S^{q_2} = D_4 S^{q_{2r} + i q_{2i}} \rightarrow 0, \quad (\text{A.8})$$

no matter what the b value is.

Since $p = 0$ is a simple pole of the integrands in Eqs. (19), (33) and (34), we want to make sure that on the right side of the straight line C_1 (cf. Fig. 1) there is no singularity. Therefore, we want to choose μ such that $p_r = \mu > 0$. Since $\gamma > 0$, $\mu > 0$ implies the satisfaction of Eq. (A.7), our final sufficient condition to ensure the satisfaction of the far-field boundary condition (2) is to place the straight C_1 anywhere to the right of the origin (i.e., $p_r = \mu > 0$).

Appendix B: To show that $e^{\tau p} \bar{S}_t(p)$ only has one branch cut located between $p = -a^2$ and $p = -\infty$ and a simple pole at $p = 0$

The simple pole at $p = 0$ is obvious. The branch cut on a part of the negative real axis on the P plane is obvious too because of the function $\sqrt{p + a^2}$. So all we need to do is to focus on showing that there are no other singularities on the P plane.

Figure 10: An illustration sketch of the P plane

Consider a conformal mapping

$$w = 1 - \frac{p + \gamma}{\gamma(b - \sqrt{p + a^2})}, \quad (\text{B.1})$$

with W being the argument of the logarithmic function in $\bar{S}_f(p)$ (cf. Eq. (19)), which is the only other possible source of singularity. W can be also written as

$$\begin{aligned} w &= 1 - \frac{(p + \gamma)(b + \sqrt{p + a^2})}{\gamma[b^2 - (p + a^2)]} = 1 - \frac{(p + \gamma)(b + \sqrt{p + a^2})}{\gamma(-\gamma - p)} \\ &= 1 + \frac{b + \sqrt{p + a^2}}{\gamma} = 1 + \frac{b + \sqrt{(p + \gamma) + b^2}}{\gamma}. \end{aligned} \quad (\text{B.2})$$

Now, on C_3 (see Fig 1), $p + a^2 = pe^{i\pi}$. The corresponding line on the W plane is

$$w = 1 + \frac{b + \sqrt{\rho}e^{i\frac{\pi}{2}}}{\gamma} = \left(1 + \frac{b}{\gamma}\right) + i\frac{\sqrt{\rho}}{\gamma}, \quad (\text{B.3})$$

which is a half straight line located at $\text{Re}(w) = 1 + \frac{b}{\gamma}$ as shown in Fig. 11. On the

Figure 11: An illustration sketch of the W plane

other hand, on C_5 , $p + a^2 = pe^{-i\pi}$. Correspondingly,

$$w = 1 + \frac{b + \sqrt{\rho}e^{i(-\frac{\pi}{2})}}{\gamma} = 1 + \frac{b}{\gamma} - i\frac{\sqrt{\rho}}{\gamma} \quad (\text{B.4})$$

represents another half straight line located at $\text{Re}(w) = 1 + \frac{b}{\gamma}$ too, but on the lower half of the W plane.

Therefore it is clear now that the entire P plane with a branch cut between $p = -a^2$ and $p = -\infty$ shown in Fig. 10 is mapped onto the right side of the straight line $\text{Re}(w) = 1 + \frac{b}{\gamma}$ as illustrated on Fig. 11. Since

$$1 + \frac{b}{\gamma} = 1 + \frac{1}{\gamma} \left(\frac{1 - \gamma}{2} \right) = \frac{1}{2\gamma} + 1 - \frac{1}{2} = \frac{1}{2} \left(\frac{1}{\gamma} + 1 \right) > 0, \quad (\text{B.5})$$

the branch cut of the logarithmic function, which is defined on the negative real axis of the W plane, will never be reached.

Appendix C: Derivation of Analytical formula Eq. (37) for the price of American options

The inverse Laplace transform of Eqs. (33) and (34) is performed in a similar way to that of Eq. (19) when we inverted $\bar{S}_f(p)$ to obtain $S_f(\tau)$ in Sec 2.1. It can be easily shown again that the only non-trivial integrals that we need to evaluate are on the straight lines C_3 and C_5 shown in Fig. 1. The cancelation of the real part of the integrand of the integrals on C_3 and C_5 again leads to

$$U(S, \tau) = \begin{cases} Q_1(\gamma, S) + Q_2(\gamma, S) \\ + \frac{\gamma}{2\pi} S^b e^{-a^2 \tau} \int_0^\infty \frac{e^{-\rho \tau}}{(a^2 + \rho)(b^2 + \rho)} (F_{1aim} + F_{1bim}) d\rho, & \text{if } S_f \leq S < 1, \\ Q_3(\gamma, S) + Q_4(\gamma, S) \\ + \frac{\gamma}{2\pi} S^b e^{-a^2 \tau} \int_0^\infty \frac{e^{-\rho \tau}}{(a^2 + \rho)(b^2 + \rho)} (F_{2aim} + F_{2bim}) d\rho, & \text{if } S \geq 1, \end{cases} \quad (\text{C.1})$$

where

$$F_{1aim} = \left[e^A \sin \Phi - \frac{\sqrt{\rho}}{\gamma} \right] \cos(\sqrt{\rho} \ln S) \\ + \left[e^A \cos \Phi - c \right] \sin(\sqrt{\rho} \ln S), \quad (\text{C.2})$$

$$F_{1bim} = \frac{b}{\sqrt{\rho}} \left\{ \left[e^A \cos \Phi - c \right] \cos(\sqrt{\rho} \ln S) \right. \\ \left. - \left[e^A \sin \Phi - \frac{\sqrt{\rho}}{\gamma} \right] \sin(\sqrt{\rho} \ln S) \right\}, \quad (\text{C.3})$$

$$F_{2aim} = [e^A \cos \Phi + c - 2] \sin(\sqrt{\rho} \ln S) \\ + [e^A \sin \Phi - \sqrt{\rho}/\gamma] \cos(\sqrt{\rho} \ln S), \quad (\text{C.4})$$

$$F_{2bim} = \frac{b}{\sqrt{\rho}} \left\{ \left[e^A \cos \Phi - c \right] \cos(\sqrt{\rho} \ln S) \right. \\ \left. - \left[e^A \sin \Phi + \frac{\sqrt{\rho}}{\gamma} \right] \sin(\sqrt{\rho} \ln S) \right\}, \quad (\text{C.5})$$

with

$$A = \frac{1}{b^2 + \rho} \left[(b^2 - \rho) \ln \left(\frac{\sqrt{a^2 + \rho}}{\gamma} \right) + 2b\sqrt{\rho} \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right], \quad (\text{C.6})$$

$$\Phi = \frac{1}{b^2 + \rho} \left[2b\sqrt{\rho} \ln \left(\frac{\sqrt{a^2 + \rho}}{\gamma} \right) - (b^2 - \rho) \tan^{-1} \left(\frac{\sqrt{\rho}}{a} \right) \right]. \quad (\text{C.7})$$

$$c = 1 + \frac{b}{\gamma} \quad (\text{C.8})$$

In Eq. (C.1), $Q_1(\gamma, S)$ and $Q_2(\gamma, S)$ are the residues of $F_1(p)$ at $p = 0$ and $p = -\gamma$, respectively, and $Q_3(\gamma, S)$ and $Q_4(\gamma, S)$ are the residues of $F_2(p)$ at $p = 0$ and $p = -\gamma$, respectively. They are generally a function of γ and S . The evaluation of $Q_1(\gamma, S)$ and $Q_3(\gamma, S)$ is simple and the results are

$$Q_1 = (S - 1) + \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)S} \right]^\gamma, \quad (\text{C.9})$$

$$Q_3 = \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)S} \right]^\gamma. \quad (\text{C.10})$$

The evaluation of $Q_2(\gamma, S)$ and $Q_4(\gamma, S)$, on the other hand, is not trivial and is left in Appendix D.

The addition of F_{1aim} and F_{1bim} in Eq. (C.1) can be further simplified and combined into a simple form

$$\begin{aligned} F_{1im} = F_{1aim} + F_{1bim} &= e^A \left[\frac{b}{\sqrt{\rho}} \cos(\Phi + \sqrt{\rho} \ln S) + \sin(\Phi + \sqrt{\rho} \ln S) \right] \\ &\quad - \sin(\sqrt{\rho} \ln S) - \left(\frac{bc}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{\gamma} \right) \cos(\sqrt{\rho} \ln S). \end{aligned} \quad (\text{C.11})$$

Similarly, if $F_{2aim} + F_{2bim}$ is simplified and combined into F_{2im} , one can now easily show that $F_{2im} = F_{1im}$, after making use of Eq. (C.8). That is, the value of an American put option can be expressed in a single formula as

$$\begin{aligned} V(S, \tau) &= \frac{1}{1 + \gamma} \left[\frac{\gamma}{(1 + \gamma)S} \right]^\gamma + \frac{\gamma}{2\pi} S^b e^{-a^2 \tau} \int_0^\infty \frac{e^{-\rho \tau}}{(a^2 + \rho)(b^2 + \rho)} \\ &\quad \cdot \left\{ e^A \left[\frac{b}{\sqrt{\rho}} \cos(\Phi + \sqrt{\rho} \ln S) + \sin(\Phi + \sqrt{\rho} \ln S) \right] \right. \\ &\quad \left. - \sin(\sqrt{\rho} \ln S) - \left(\frac{bc}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{\gamma} \right) \cos(\sqrt{\rho} \ln S) \right\} d\rho, \end{aligned} \quad (\text{C.12})$$

after $U(S, \tau)$ being converted back to the $V(S, \tau)$ according to Eq. (6).

Appendix D: Evaluation of Q_2 and Q_4

The evaluation of Q_2 and Q_4 involves some limiting processes. Depending on the value of b (or γ), we shall evaluate some expressions used in $F_1(p)$ and $F_2(p)$ first.

It is easy to evaluate the limits of the following expressions with $p \rightarrow -\gamma$:

$$q_1(-\gamma) = \left[b + \sqrt{p + a^2} \right]_{p \rightarrow -\gamma} = \left[b + \sqrt{(p + \gamma) + b^2} \right]_{p \rightarrow -\gamma} = \begin{cases} 2b, & \text{if } b \geq 0 \text{ or } \gamma \leq 1, \\ 0, & \text{if } b < 0 \text{ or } \gamma > 1; \end{cases} \quad (\text{D.1})$$

$$q_2(-\gamma) = \left[b - \sqrt{p + a^2} \right]_{p \rightarrow -\gamma} = \left[b - \sqrt{(p + \gamma) + b^2} \right]_{p \rightarrow -\gamma} = \begin{cases} 0, & \text{if } b \geq 0 \text{ or } \gamma \leq 1, \\ 2b, & \text{if } b < 0 \text{ or } \gamma > 1; \end{cases} \quad (\text{D.2})$$

$$\frac{q_2(-\gamma)}{q_1(-\gamma)} = \left[\frac{b - \sqrt{(p + \gamma) + b^2}}{b + \sqrt{(p + \gamma) + b^2}} \right]_{p \rightarrow -\gamma} = \begin{cases} 0, & \text{if } b > 0 \text{ or } \gamma < 1, \\ -1, & \text{if } b = 0 \text{ or } \gamma = 1, \\ \infty, & \text{if } b < 0 \text{ or } \gamma > 1; \end{cases} \quad (\text{D.3})$$

$$\frac{q_2(-\gamma)}{q_2(-\gamma) - q_1(-\gamma)} = \frac{1}{1 - \frac{q_1(-\gamma)}{q_2(-\gamma)}} = \left[\frac{1}{2} \left(1 - \frac{b}{\sqrt{(p + \gamma) + b^2}} \right) \right]_{p \rightarrow -\gamma} = \begin{cases} 0, & \text{if } b > 0 \text{ or } \gamma < 1, \\ \frac{1}{2}, & \text{if } b = 0 \text{ or } \gamma = 1, \\ 1, & \text{if } b < 0 \text{ or } \gamma > 1; \end{cases} \quad (\text{D.4})$$

$$\frac{q_1(-\gamma)}{q_1(-\gamma) - q_2(-\gamma)} = \frac{1}{1 - \frac{q_2(-\gamma)}{q_1(-\gamma)}} = \left[\frac{1}{2} \left(1 + \frac{b}{\sqrt{(p + \gamma) + b^2}} \right) \right]_{p \rightarrow -\gamma} = \begin{cases} 1, & \text{if } b > 0 \text{ or } \gamma < 1, \\ \frac{1}{2}, & \text{if } b = 0 \text{ or } \gamma = 1, \\ 0, & \text{if } b < 0 \text{ or } \gamma > 1; \end{cases} \quad (\text{D.5})$$

Then, utilizing the rationalization process shown in Eq. (B.2) already, one can obtain

$$\left[1 - \frac{p + \gamma}{\gamma(b - \sqrt{p + a^2})} \right]_{p \rightarrow -\gamma} = \begin{cases} \frac{1}{\gamma}, & \text{if } b \geq 0 \text{ or } \gamma \leq 1, \\ 1, & \text{if } b < 0 \text{ or } \gamma > 1. \end{cases} \quad (\text{D.6})$$

One should notice that in Eq. (D.4), the limiting process of $p \rightarrow -\gamma$ is taken with γ being fixed to the given value, since γ is a parameter as far as evaluating the residue of the corresponding complex integrand is concerned. This is especially important for the case when $\gamma = 1$. A completely different limit value would have been reached, had one let p approach $-\gamma$ first and then set $b = 0$ afterward.

Now, with the aid of Eqs. (D.1)- (D.6), $Q_2(\gamma, S)$ can be easily evaluated as

$$\begin{aligned} Q_2(\gamma, S) &= \text{Res}_{p = -\gamma} \left[\frac{\gamma e^{p\tau}}{p(p + \gamma)} F_1(p) \right] \\ &= \begin{cases} -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[0 \cdot \frac{1}{\gamma} \cdot S^{2b} + 1 \cdot \left(\frac{1}{\gamma} \right)^0 \cdot S^0 - 1 \right] = 0, & \text{if } b > 0 \text{ or } \gamma < 1, \\ -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[\frac{1}{2} \cdot 1 \cdot S^0 + \frac{1}{2} \cdot (1)^{-1} \cdot S^0 - 1 \right] = 0, & \text{if } b = 0 \text{ or } \gamma = 1, \\ -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[1 \cdot 1 \cdot S^0 + 0 \cdot (1)^\infty \cdot S^{2b} - 1 \right] = 0, & \text{if } b < 0 \text{ or } \gamma > 1. \end{cases} \end{aligned} \quad (\text{D.7})$$

Similarly, $Q_4(\gamma, S)$ can be calculated as

$$\begin{aligned}
Q_4(\gamma, S) &= \text{Res}_{p=-\gamma} \left[\frac{\gamma e^{p\tau}}{p(p+\gamma)} F_2(p) \right] \\
&= \begin{cases} -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[0 \cdot \frac{1}{\gamma} + 1 \cdot \left(\frac{1}{\gamma}\right)^0 - 1 \right] \cdot S^0 = 0, & \text{if } b > 0 \text{ or } \gamma < 1, \\ -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1)^{-1} - 1 \right] \cdot S^0 = 0, & \text{if } b = 0 \text{ or } \gamma = 1, \\ -\frac{\gamma}{\gamma} e^{-\gamma\tau} \cdot \left[1 \cdot 1 + 0 \cdot (1)^\infty - 1 \right] \cdot S^{2b} = 0, & \text{if } b < 0 \text{ or } \gamma > 1. \end{cases}
\end{aligned} \tag{D.8}$$

Appendix E: Hedge Parameters

In this Appendix, some hedge parameters are listed in dimensionless form. Converting back to dimensional form is straightforward.

Among all five hedge parameters, Θ is the easiest one to be calculated. Using Eq. (43), we obtain

$$\Theta = \frac{\partial V}{\partial \tau} = -\frac{\gamma}{\pi} S^b e^{-a^2 \tau} \int_0^\infty \frac{e^{-\zeta^2 \tau}}{(b^2 + \zeta^2)} \cdot Y_1 d\zeta, \tag{E.1}$$

where

$$\begin{aligned}
Y_1 &= e^{A^*} \cdot [b \cos(\Phi^* + \zeta \ln S) + \zeta \sin(\Phi^* + \zeta \ln S)] \\
&\quad - \zeta \sin(\zeta \ln S) - \left(bc + \frac{\zeta^2}{\gamma}\right) \cos(\zeta \ln S).
\end{aligned}$$

One should notice that this is always positive as the value of the integrand is always negative. In other words, at a fixed stock price S , the option value increases with τ .

Δ is also easy to compute. The result is:

$$\Delta = \frac{\partial V}{\partial S} = - \left[\frac{\gamma}{(1+\gamma)S} \right]^{\gamma+1} + \frac{\gamma}{\pi} S^{b-1} e^{-a^2 \tau} \int_0^\infty \frac{e^{-\zeta^2 \tau}}{(a^2 + \zeta^2)(b^2 + \zeta^2)} \cdot (bY_1 + \zeta Y_2) d\zeta, \tag{E.2}$$

where

$$\begin{aligned}
Y_2 &= e^{A^*} \cdot [-b \sin(\Phi^* + \zeta \ln S) + \zeta \cos(\Phi^* + \zeta \ln S)] \\
&\quad - \left[\zeta \cos(\zeta \ln S) - \left(bc + \frac{\zeta^2}{\gamma}\right) \sin(\zeta \ln S) \right].
\end{aligned}$$

The calculation of Γ can be based on that of Δ after taking the derivative with respect to S one more time, which results in

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{(1+\gamma)^2}{\gamma} \cdot \left[\frac{\gamma}{(1+\gamma)S} \right]^{\gamma+2}$$

$$+\frac{\gamma}{\pi}S^{b-2}e^{-a^2\tau}\int_0^\infty\frac{e^{-\zeta^2\tau}}{(a^2+\zeta^2)(b^2+\zeta^2)}\cdot[(b^2-b-\zeta^2)Y_1-\gamma\zeta Y_2]d\zeta. \quad (\text{E.3})$$

The calculation of *Vega* and *Rho* will all involve the derivative of V with respect to γ first. Then, using the chain rule, *Vega* is obtained by multiplying $\frac{\partial V}{\partial \gamma}$ to $\frac{\partial \gamma}{\partial \sigma}$ and *Rho* is obtained by multiplying $\frac{\partial V}{\partial \gamma}$ to $\frac{\partial \gamma}{\partial r}$. Because a , b , A^* and Φ^* are all functions of γ , the partial derivative of $\frac{\partial V}{\partial \gamma}$ is very length and cumbersome, although it can indeed be done in Maple quite easily. Hence, the expression of $\frac{\partial V}{\partial \gamma}$ is not included here.

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