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Function spaces and multiplier operators

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Abstract

Let G denote a locally compact Hausdorff abelian group. Then a bounded linear operator T from L^2(G) into L^2(G) is a bounded multiplier operator if, under the Fourier transform on L^2(G), for each function f in L^2(G), T(f) changes into a bounded function U times the Fourier transform of f. Then U is called the multiplier of T. An unbounded multiplier operator has a similar definition, but its domain is a dense subspace of L^2(G) and the multiplier function need not be bounded. For example, differentiation on the first order Sobolev subspace of L^2(R) is an unbounded multiplier operator with multiplier mapping x into ix, and the Laplace operator on the second order Sobolev subspace of L^2(R^2) is an unbounded multiplier operator with multiplier mapping (x,y) into –x^2-y^2. Now in 1972 (J. Functional Analysis 11, pp.407-424), G. Meisters and W. Schmidt had effectively characterized the range of the differentiation operator on the first order Sobolev subspace of L^2(-pi,pi) as a space of first order differences. Corresponding results for the real line, the Laplace operator in n dimensions, and other differential operators were obtained by the first named author (see Springer Lecture Notes in Mathematics, vol. 1586, for example). The present paper extends earlier results so that they apply to general bounded or unbounded multiplier operators on the space L^2(G) and on certain spaces of abstract distributions on G. Descriptions of the ranges of such operators are obtained, and these ranges are either Banach or Hilbert spaces in weighted L^p or L^2 norms under the Fourier transform, and the operators become isometries from their domains onto these spaces. These spaces, that is the ranges of the operators, are described in terms of finite differences involving pseudomeasures on the group.

Keywords

Fourier transform, differences, difference spaces, Hilbert space, Banach space, Sobolev space, abstract distributions, unbounded operators, multiplier operators, multipliers, range, factorisation, differential operator, pseudomeasure

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Function spaces and multiplier operators

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1 Introduction

Let G denote a locally compact Hausdorff abelian group. If $f \in L^2(G)$ the Fourier transform of f is denoted by $f^\wedge$ or $\widehat{f}$. Then a bounded linear operator T from $L^2(G)$ into $L^2(G)$ is called a *bounded multiplier operator* if there is a function $\Upsilon \in L^\infty(\widehat{G})$ such that

$$T(f)^\wedge = \Upsilon \widehat{f}, \quad \text{for all } f \in L^2(G).$$

In this case the function Υ is called the *multiplier* of the multiplier operator T. Let δ_x denote the Dirac measure at a point $x \in G$ and let $*$ denote the usual convolution of functions, measures and distributions on G when such a convolution is defined. Then a bounded linear operator T on $L^2(G)$ is said to *commute with translations* if

$$T(\delta_x * f) = \delta_x * T(f), \quad \text{for all } x \in G \text{ and } f \in L^2(G).$$

It is a standard result in multiplier theory (see [2, vol. II, p.216] or [5, Theorem 4.1.1], for example), that a bounded linear operator T on $L^2(G)$ is a multiplier operator if and only if T commutes with translations.

One aim of the present work is to extend the above characterization of bounded multiplier operators to *unbounded* multiplier operators (see definitions below). However, the main aim is to describe the ranges of both bounded and unbounded multiplier operators on $L^2(G)$ in terms of what are called *difference spaces* or *generalized difference spaces* of $L^2(G)$. For example, if the Haar measure on G is denoted by μ_G, if $\widehat{G}$ denotes the dual group of G, and if T is a multiplier operator on $L^2(G)$ with multiplier Υ, it is shown in Theorem 5.2 that there is a family of pseudomeasures $\{\nu_a : a \in \mathbb{R}\}$ on G with the following properties:

(1) $\nu_a * \nu_b = \nu_{a+b}$, for all $a, b \in \mathbb{R}$;

(2) the range $\mathcal{R}(T)$ of T is the vector subspace of $L^2(G)$ finitely spanned by $\{f - \nu_a * f : a \in \mathbb{R}$ and $f \in L^2(G)\}$, and this space is a Hilbert space in the norm $|||\cdot|||$ given by

$$|||f||| = \int_{\widehat{G}} |\widehat{f}|^2 (1 + |\Upsilon|^{-2})\, d\mu_G, \quad \text{for all } f \in \mathcal{R}(T); \quad \text{and}$$

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(3) for each $f \in \mathcal{R}(T)$, there are numbers $a_1, a_2, a_3 \in \mathbb{R}$ and functions $f_1, f_2, f_3 \in L^2(G)$ such that $f = \sum_{j=1}^{3}(f_j - \nu_{a_j} * f_j)$.

The vector subspace of $L^2(G)$ in (2), which is finitely spanned by the vectors of the form $f - \nu_a * f$ with $a \in \mathbb{R}$ and $f \in L^2(G)$, is an example of a difference space and such spaces have been studied in various contexts (see [6, 7, 8, 9, 10] where further references may be found). However, note that this previous work has used measures only, not general pseudomeasures. Typically, for $L^2(G)$, a difference space becomes a Hilbert space in a norm stronger than the L^2-norm.

The overall aim of this work is to extend the above type of description of the ranges of bounded multiplier operators so as to describe the ranges of a class of *unbounded* multiplier operators on certain spaces of abstract distributions on G. The general programme has been influenced by the work of the first author in [8] and [9, Chapter III], which contain new descriptions of the ranges of the differentiation operator, the Laplace operator, the one-dimensional wave operator and more general partial differentiation operators. However, the work of [8, 9] on these lines is restricted to $\mathbb{R}^n$, while the setting here is on any locally compact Hausdorff abelian group G. Characterizations of the ranges of multiplier operators on spaces of abstract distributions on such groups are given. Whereas the multiplier operators of [8, 9] are one-to-one and so have multipliers which are almost everywhere non-zero on the dual group, the multiplier operators considered here are not necessarily one-to-one. Among the operators considered here are all bounded multiplier operators on $L^2(G)$ and also a wide family of unbounded multiplier operators on this space.

2 The spaces $\mathcal{F}^p(G)$ and bounded operators

Let G denote a locally compact Hausdorff abelian group. Its dual group will be denoted by $\widehat{G}$. Haar measures μ_G and $\mu_{\widehat{G}}$ on G and $\widehat{G}$ respectively are taken to be normalized so that the Fourier Inversion Theorem holds [11, p.22]. Let $L^p(G)$, for $1 \le p < \infty$, denote the usual Lebesgue spaces of all (equivalence classes of) complex valued Borel measurable functions on G with respect to μ_G. However, in order to allow for groups which are not countable unions of compact subsets, the definition we use for $L^\infty(G)$ is that of Cohn [1]. Thus, a complex valued Borel measurable function f on G is said to be *essentially bounded* if there is a number $M > 0$ such that $|f|^{-1}([M, \infty))$ is locally μ_G-null. In this case, $||f||_\infty$ is defined to be the infimum of all such numbers M. Then, $L^\infty(G)$ is defined to be the space of all (equivalence classes of) essentially bounded, Borel measurable functions, and $||f||_\infty$ is well defined for all f in $L^\infty(G)$ (see [1, pp.98-104] for full details). Let $M(G)$ denote the space of all complex valued regular Borel measures on G of bounded total variation. Then $M(G)$ is a Banach space in the total variation norm. The Fourier transform of $\nu \in M(G)$ is given by

$$\widehat{\nu}(\gamma) \;=\; \int_G \overline{\gamma}\, d\nu, \quad \text{for all } \gamma \in \widehat{G}.$$

If $\nu \in M(G)$, then $\widehat{\nu}$ is a bounded, uniformly continuous function on $\widehat{G}$; while if $\nu \in L^1(G)$, then $\widehat{\nu}$ is in $C_0(\widehat{G})$, the space consisting of all the complex-valued continuous functions on $\widehat{G}$ which vanish at infinity.

Let V be the space of all functions f in $L^1(G)$ whose Fourier transforms vanish outside some compact subset of $\widehat{G}$, this set generally depending on f. Thus, V consists precisely of those f in $L^1(G)$ whose Fourier transforms have compact support in the Fourier algebra $A(\widehat{G})$ which is the set of all Fourier transforms of all functions in $L^1(G)$ [11, p.9]. Then V is a vector subspace of $L^1(G)$, and the set of Fourier transforms of functions in V forms a dense subspace, in the uniform norm $||\cdot||_\infty$, of $C_0(\widehat{G})$, [9, p.11]. Also, as $\widehat{f} \in L^1(\widehat{G}) \cap L^2(\widehat{G})$ for all $f \in V$, it follows from the Fourier Inversion Theorem that $V \subseteq L^1(G) \cap L^2(G) \cap C_0(G)$.

Now let $1 \leq p < \infty$ be given. A norm $|||\cdot|||_p$ may be defined on V by letting

$$|||f|||_p \;=\; ||\widehat{f}||_p, \quad \text{for all } f \in V.$$

The Banach space completion of V in this norm is denoted by $\mathcal{F}^p(G)$, the norm on this space being also denoted by $|||\cdot|||_p$. The definitions of V and $|||\cdot|||_p$ mean that the Fourier transform is an isometry from V in the norm $|||\cdot|||_p$ onto its range $\widehat{V} = \{\widehat{f} : f \in V\} \subseteq L^p(\widehat{G})$ in the norm $||\cdot||_p$. Also, as $\widehat{V} \subseteq L^p(G)$, and as $\widehat{V}$ is dense in $C_0(\widehat{G})$, it follows that $\widehat{V}$ is dense in $L^p(\widehat{G})$. Thus the Fourier transform extends from V to all of $\mathcal{F}^p(G)$ and becomes a linear isometry from $\mathcal{F}^p(G)$ onto $L^p(\widehat{G})$. The Fourier transform of a vector $f \in \mathcal{F}^p(G)$ is also denoted by $\widehat{f}$. If $f \in \mathcal{F}^p(G)$ and $\mu \in M(G)$, then $\widehat{f} \in L^p(\widehat{G})$ and $\widehat{\mu} \in L^\infty(\widehat{G})$, so that $\widehat{\mu}\widehat{f} \in L^p(\widehat{G})$. Hence there is a unique element in $\mathcal{F}^p(G)$, which is denoted by $\mu * f$, such that $(\mu * f)^\wedge = \widehat{\mu}\widehat{f}$.

Now make the definition

$$\mathcal{W}_p(G) = \Big\{ \; f : f \in \mathcal{F}^p(G) \text{ and } \widehat{f} \in L^1(\widehat{G}) \; \Big\}.$$

We prove two propositions, which enable us to identify $\mathcal{W}_p(G)$ as a subspace of $C_0(G)$ in a natural way.

2.1 Proposition. *Let $1 \leq p < \infty$. If $f \in \mathcal{W}_p(G)$, put*

$$|||f|||'_p = ||\widehat{f}||_1 + ||\widehat{f}||_p.$$

Then $|||\cdot|||'_p$ is a norm on $\mathcal{W}_p(G)$ and V is a $|||\cdot|||'_p$-dense vector subspace of $\mathcal{W}_p(G)$.

Proof. As $\widehat{f} \in L^1(\widehat{G}) \cap L^p(\widehat{G})$ when $f \in V$, we see that $V \subseteq \mathcal{W}_p(G)$. Now, let $\varepsilon > 0$. Then, as $\widehat{f} \in L^1(\widehat{G}) \cap L^p(\widehat{G})$, there is a compact subset K of $\widehat{G}$ such that

$$||\widehat{f}(1 - \chi_K)||_1 < \varepsilon \quad \text{and} \quad ||\widehat{f}(1 - \chi_K)||_p < \varepsilon. \tag{2.1}$$

Let U be a relatively compact open subset of $\widehat{G}$ such that $K \subseteq U$. By an obvious adaptation of Lemma I.2.1(b) in [9] (where the assumption that the set K_2 is compact is stronger than required), we see that there is a function $g \in V$ such that

$$0 \leq \widehat{g} \leq 1, \quad \widehat{g}(K) = 1, \quad \text{and} \quad \widehat{g}(U^c) = \{0\}.$$

Then, as $|\widehat{f}(1 - \widehat{g})| \leq |\widehat{f}(1 - \chi_K)|$, it follows from (2.1) that

$$||\widehat{f}(1 - \widehat{g})||_p \leq ||\widehat{f}(1 - \chi_K)||_p < \varepsilon, \tag{2.2}$$

and that

$$||\widehat{f}(1 - \widehat{g})||_1 \leq ||\widehat{f}(1 - \chi_K)||_1 < \varepsilon. \tag{2.3}$$

Also, since $\widehat{V}$ is dense in $L^p(\widehat{G})$, there is a function $h \in V$ such that

$$||\widehat{f} - \widehat{h}||_p < \frac{\varepsilon}{\mu_{\widehat{G}}(U)^{1/p'} + 1}, \quad \text{where } p' = p/(p-1). \tag{2.4}$$

Using (2.2) and (2.4) we now have

$$\begin{aligned} ||\widehat{f} - (g*h)^\wedge||_p &= ||\widehat{f} - \widehat{g}\,\widehat{h}||_p, \\ &\leq ||\widehat{f} - \widehat{f}\,\widehat{g}||_p + ||\widehat{f}\,\widehat{g} - \widehat{g}\,\widehat{h}||_p, \\ &= ||\widehat{f}(1-\widehat{g})||_p + ||\widehat{g}(\widehat{f} - \widehat{h})||_p, \\ &< \varepsilon + \frac{\varepsilon}{\mu_{\widehat{G}}(U)^{1/p'} + 1}, \\ &< 2\varepsilon. \end{aligned}$$

Similarly, but this time using (2.3), (2.4) and Hölder's inequality,

$$\begin{aligned} ||\widehat{f} - (g*h)^\wedge||_1 &= ||\widehat{f} - \widehat{g}\,\widehat{h}||_1, \\ &\leq ||\widehat{f} - \widehat{f}\,\widehat{g}||_1 + ||\widehat{f}\,\widehat{g} - \widehat{g}\widehat{h}||_1, \\ &\leq ||\widehat{f}(1-\widehat{g})||_1 + ||\widehat{g}(\widehat{f} - \widehat{g})||_1, \\ &\leq \varepsilon + ||\widehat{g}||_{p'} \cdot ||\widehat{f} - \widehat{h}||_p, \\ &\leq \varepsilon + \mu_{\widehat{G}}(U)^{1/p'} \frac{\varepsilon}{\mu_{\widehat{G}}(U)^{1/p'} + 1}, \\ &< 2\varepsilon. \end{aligned}$$

Put $v = g * h$. Then v belongs to V and satisfies

$$||\widehat{f} - \widehat{v}||_1 + ||\widehat{f} - \widehat{v}||_p < 4\varepsilon.$$

This means that V is $|||\cdot|||_p'$-dense in $\mathcal{W}_p(G)$. □

2.2 Proposition. *Let $1 \leq p < \infty$. If $f \in \mathcal{W}_p(G)$, put*

$$S(f)(x) = \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(x)\, d\mu_{\widehat{G}}(\gamma), \quad \text{for all } x \in G.$$

Then S is a continuous, one-to-one linear mapping from $\mathcal{W}_p(G)$ (in the norm $|||\cdot|||_p'$) into $C_0(G)$ (in the norm $||\cdot||_\infty$), and the restriction of S to V is the identity mapping.

Proof. The function $S(f)$ is defined and is in $C_0(G)$ for all $f \in \mathcal{W}_p(G)$ because in this case $\widehat{f} \in L^1(\widehat{G})$. The map S is clearly linear. Also, S is one-to-one, because if $S(f) = 0$ then $\int_{\widehat{G}} \widehat{f}(\gamma)\gamma(x)\, d\mu_{\widehat{G}}(\gamma) = 0$ for all $x \in G$, and it follows from [11, p.17] that $\widehat{f} = 0$ and so $f = 0$. When $f \in V$, the Fourier Inversion Theorem gives $S(f) = f$ (see [5, p.252] or [11, Section 1.5]), so that the restriction of S to V is the identity. Also,

$$||S(f)||_\infty = \sup_{x \in G} \Big| \int_{\widehat{G}} \widehat{f}(\gamma)\gamma(x)\, d\mu_{\widehat{G}}(\gamma) \Big| \leq \int_{\widehat{G}} |\widehat{f}(\gamma)|\, d\mu_{\widehat{G}}(\gamma) = ||\widehat{f}||_1 \leq |||f|||_p'.$$

Thus, S is continuous from $\mathcal{W}_p(G)$ into $C_0(G)$ as described in the statement in the Proposition. □

Propositions 2.1 and 2.2 together show that we are justified in identifying an element f of $\mathcal{W}_p(G)$ with the element $S(f)$ of $C_0(G)$ (this identification is in fact equality when $f \in V$). Under this identification we can write $\mathcal{W}_p(G) \subseteq C_0(G)$.

The space $\mathcal{F}^\infty(G)$ is defined by letting $\mathcal{F}^\infty(G)$ be the dual space $\mathcal{F}^1(G)^*$ of $\mathcal{F}^1(G)$. By analogy with the spaces $\mathcal{F}^p(G)$, $1 \leq p < \infty$, the concept of the Fourier transform will be defined on $\mathcal{F}^\infty(G)$ so that the Fourier transform becomes a linear isometry from $\mathcal{F}^\infty(G)$ onto $L^\infty(\widehat{G})$. Just for the moment, let the Fourier transform from $\mathcal{F}^1(G)$ onto $L^1(\widehat{G})$ be denoted by F. Then the adjoint F_{adj} of F maps $L^1(\widehat{G})^*$ isometrically onto $\mathcal{F}^1(G)^*$. Since $\widehat{G}$ is a locally compact group, it follows from [1, Theorem 9.4.8] that we can identify $L^1(\widehat{G})^*$ with $L^\infty(\widehat{G})$. Consequently, $F_{\text{adj}} : L^\infty(\widehat{G}) \longrightarrow \mathcal{F}^\infty(G)$ is a surjective isometry. Then, for every $f \in \mathcal{F}^\infty(G)$, make the definition that the Fourier transform $F'(f)$ of f is

$$F'(f) \;=\; (F_{\text{adj}})^{-1}\,(f).$$

With this definition, it is clear that $F' : \mathcal{F}^\infty(G) \longrightarrow L^\infty(\widehat{G})$ also is a surjective isometry, but we need to check that the definition is consistent with the definition of the Fourier transform which is already there for the space $M(G)$, in the following sense: there is a natural way in which $M(G)$ may be regarded as a subspace of $\mathcal{F}^\infty(G)$, and the definition of the Fourier transform given on $\mathcal{F}^\infty(G)$, when restricted to $M(G)$, coincides with the ordinary definition of the Fourier transform. Since $V \subseteq M(G)$ and since V is dense in the space $\mathcal{F}^p(G)$ for $1 \leq p < \infty$, it will then follow that the definitions of the Fourier transform for any vector which is in at least two of the spaces $M(G)$ and $\mathcal{F}^p(G)$ for $1 \leq p < \infty$ are consistent.

Now, if $\nu \in M(G)$, then there is an associated element θ_ν of $\mathcal{F}^\infty(G) = \mathcal{F}^1(G)^*$ given by

$$\langle \theta_\nu, h\rangle = \int_{\widehat{G}} \widehat{\nu}\,\widehat{h}\,d\mu_{\widehat{G}}, \quad \text{for all } h \in \mathcal{F}^1(G). \tag{2.5}$$

Since θ_ν is in $\mathcal{F}^\infty(G)$, there is a function $\psi \in L(\widehat{G})$ such that $F_{\text{adj}}(\psi) = \theta_\nu$. Then, for every $h \in \mathcal{F}^1(G)$,

$$\langle \theta_\nu, h\rangle \;=\; \langle F_{\text{adj}}(\psi), h\rangle \;=\; \int_{\widehat{G}} \psi F(h)\,d\mu_{\widehat{G}} \;=\; \int_{\widehat{G}} \psi\widehat{h}\,d\mu_{\widehat{G}}.$$

Comparing this with (2.5), we deduce that $\psi = \widehat{\nu}$. That is,

$$F'(\theta_\nu) = (F_{\text{adj}})^{-1}(\theta_\nu) = \psi = \widehat{\nu}. \tag{2.6}$$

Any $\nu \in M(G)$ may be identified with the element θ_ν of $\mathcal{F}^\infty(G)$. That is, we put "$\nu = \theta_\nu$". In this way, $M(G)$ may be regarded as a subspace of $\mathcal{F}^\infty(G)$, and (2.6) can then be seen as saying that the two definitions of the Fourier transform on $M(G)$, namely the usual one and the restriction of F' to the subspace $M(G)$ of $\mathcal{F}^\infty(G)$, are the same and so consistent. The notation $\widehat{f}$ will be used for the Fourier transform of an element $f \in \mathcal{F}^\infty(G)$. More generally, the Fourier transform of any object f is denoted by $\widehat{f}$, and the inverse Fourier transform of f is denoted by $f^\vee$. In a similar way as for the discussion above, it is not hard to see that if $1 \leq p \leq \infty$, then for each $\nu \in \mathcal{F}^\infty(G)$ and $f \in \mathcal{F}^p(G)$, their convolution $\nu * f$ may be defined as an element of $\mathcal{F}^p(G)$ by making the definition that $\nu * f = (\widehat{\nu}\widehat{f})^\vee$, in which case $(\nu * f)^\wedge = \widehat{\nu}\,\widehat{f}$.

Analogously to the definition of $\mathcal{W}_p(G)$ for $1 \leq p < \infty$, define the space $\mathcal{W}_\infty(G)$ by putting

$$\mathcal{W}_\infty(G) = \Big\{\; f : f \in \mathcal{F}^\infty(G) \;\text{ and }\; \widehat{f} \in L^1(\widehat{G}) \;\Big\}.$$

Then it is clear that $\mathcal{W}_\infty(G) \subseteq \mathcal{W}_1(G)$ and since $\mathcal{W}_1(G)$ has been proved to be contained in $C_0(G)$, we have $\mathcal{W}_\infty(G) \subseteq C_0(G)$.

A more detailed discussion of the construction of the spaces $\mathcal{F}^p(G)$, for $1 \le p < \infty$, and their properties may be found in [9, Chapter I, section 2] and, for the case of a compact group G, in [9]. Note that Plancherel's Theorem [11, p.26], which asserts that $L^2(G)$ and $L^2(\widehat{G})$ are isometrically isomorphic under the Fourier transform, means that the space $\mathcal{F}^2(G)$ may be identified with $L^2(G)$. The spaces $\mathcal{F}^p(G)$, $1 \le p \le \infty$, are called *spaces of abstract distributions* on G. The space $\mathcal{F}^\infty(G)$ is known as the space of *pseudomeasures* on G. Each measure in $M(G)$ can be regarded as a pseudomeasure, as explained above, so that measures are indeed pseudomeasures. Note also that the spaces $\mathcal{F}^p(G)$ have been used in harmonic analysis on the real line and elsewhere (see [2, 4, 5], for example) and that the space of pseudomeasures also has been widely used in harmonic analysis [3, 7, 8]. It is known that if $1 < p < 2$, then $\mathcal{F}^p(G) \subseteq L^{p'}(G)$ with $p' = p/(p-1)$ (see [3, Theorem (31.20), Definition (31.21)], for example).

Now, for future use, some results are presented concerning the spaces $\mathcal{F}^p(G)$, $\mathcal{W}_p(G)$ and V. Note that if $x \in G$, then $\delta_x \in M(G)$ denotes the Dirac measure at x.

2.3 Lemma. *Let $1 \le p < \infty$ and let $p' = p/(p-1)$ with $p' = \infty$ when $p = 1$. Then the following statements hold.*

(1) *The dual space of $\mathcal{F}^p(G)$ is $\mathcal{F}^{p'}(G)$, and the duality between them is given by*

$$\langle f, g\rangle = \int_{\widehat{G}} \widehat{f}\, \widehat{g}\, d\mu_{\widehat{G}}, \quad \text{for all } f \in \mathcal{F}^p(G) \text{ and } g \in \mathcal{F}^{p'}(G).$$

(2) *$V * \mathcal{F}^p(G) \subseteq \mathcal{W}_p(G)$ and $V * \mathcal{F}^\infty(G) \subseteq \mathcal{W}_\infty(G)$.*

(3) *If $h \in V$ and $f \in \mathcal{F}^p(G)$, then $h * f$ is the function on G given by*

$$(h * f)(x) = \langle h, \delta_{x^{-1}} * f\rangle = \langle \delta_{x^{-1}} * h, f\rangle, \quad \text{for all } x \in G.$$

Proof. (1) This is immediate from the definition of the spaces $\mathcal{F}^p(G)$, and duality between L^p and $L^{p'}$ (when $p = 1$, see [1, Theorem 9.4.8], for example).

(2) If $h \in V$ and $f \in \mathcal{F}^p(G)$ for some $p \in [1, \infty]$, then $(h * f)^\wedge = \widehat{h}\, \widehat{f} \in L^1(G)$, so that $h * f \in \mathcal{W}_p(G)$.

(3) If $x \in G$, then

$$\begin{aligned}(h * f)(x) &= \int_{\widehat{G}} (h * f)^\wedge(\gamma)\, \gamma(x)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} \widehat{h}(\gamma)\widehat{f}(\gamma)\gamma(x)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} \widehat{h}(\gamma)\, (\delta_{x^{-1}} * f)^\wedge(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \langle h, \delta_{x^{-1}} * f\rangle.\end{aligned}$$

Also, proceeding again as above,

$$(h * f)(x) = \int_{\widehat{G}} \widehat{h}\,(\gamma)\widehat{f}(\gamma)\gamma(x)\, d\mu_{\widehat{G}}(\gamma) = \int_{\widehat{G}} (\delta_{x^{-1}} * h)^\wedge(\gamma)\, \widehat{f}(\gamma)\, d\mu_{\widehat{G}}(\gamma) = \langle \delta_{x^{-1}} * h, f\rangle,$$

which completes the proof of (3). □

Let $1 \le p < \infty$. A bounded linear operator $T : \mathcal{F}^p(G) \longrightarrow \mathcal{F}^p(G)$ is said to *commute with translations* if $T(\delta_x * f) = \delta_x * T(f)$, for all $x \in G$ and all $f \in \mathcal{F}^p(G)$. The adjoint of a linear operator T is denoted by T_{adj}.

2.4 Lemma. *Let $1 \le p < \infty$, let $T : \mathcal{F}^p(G) \longrightarrow \mathcal{F}^p(G)$ be a bounded linear operator which commutes with translations, and let f, g, $h \in V$. Then the following statements hold.*

(1) *$g * T(f)$ and $f * T_{\rm adj}(g)$ are functions which are equal to each other on G, and which belong to $\mathcal{W}_p(G) \cap \mathcal{W}_{p'}(G)$.*

(2) *$\langle h, T(f) * g\rangle = \langle h, T(f * g)\rangle$.*

(3) *$T(f)^\wedge\, \widehat{g} = \widehat{f}\, T(g)^\wedge$.*

Proof. (1) That $g * T(f)$ and $f * T_{\rm adj}(g)$ are functions on G comes from Lemma 2.3(3). Now for any $x \in G$, using Lemma 2.3(3) gives,

$$\begin{aligned}(g * T(f))(x) &= \langle g, \delta_{x^{-1}} * T(f)\rangle,\\ &= \langle g, T(\delta_{x^{-1}} * f)\rangle,\\ &= \langle T_{\rm adj}(g), \delta_{x^{-1}} * f\rangle,\\ &= \langle \delta_{x^{-1}} * f, T_{\rm adj}(g)\rangle,\\ &= (f * T_{\rm adj}(g))(x),\end{aligned}$$

so that $g * T(f) = f * T_{\rm adj}(g)$. Also, it follows from Lemma 2.3(2) that $g * T(f)$ is in $\mathcal{W}_p(G)$ and that $f * T_{\rm adj}(g)$ is in $\mathcal{W}_{p'}(G)$. Hence,

$$g * T(f) = f * T_{\rm adj}(g) \in \mathcal{W}_p(G) \cap \mathcal{W}_{p'}(G),$$

and (1) has been proved.

(2) By (1) we have

$$\begin{aligned}\langle h, g * T(f)\rangle &= \int_{\widehat{G}} \widehat{h}(\gamma)\, T(f)^\wedge(\gamma)\, \widehat{g}(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} \widehat{g}(\gamma)\, (h * T(f))^\wedge(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} \widehat{g}(\gamma)\, (f * T_{\rm adj}(h))^\wedge(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} \widehat{g}(\gamma)\widehat{f}(\gamma)\, (T_{\rm adj}(h))^\wedge(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \int_{\widehat{G}} (f * g)^\wedge(\gamma)\, (T_{\rm adj}(h))^\wedge(\gamma)\, d\mu_{\widehat{G}}(\gamma),\\ &= \langle T_{\rm adj}(h), f * g\rangle,\\ &= \langle h, T(f * g)\rangle.\end{aligned}$$

(3) Since V is dense in $\mathcal{F}^{p'}(G)$ when $1 < p < \infty$, and weak*-dense in $\mathcal{F}^{p'}(G)$ when $p = 1$, and since $\mathcal{F}^{p'}(G)$ is the dual of $\mathcal{F}^p(G)$, it follows from (2) and symmetry that $g * T(f) = T(f * g) = f * T(g)$. Taking Fourier transforms gives $T(f)^\wedge\, \widehat{g} = \widehat{f}\, T(g)^\wedge$, as required. □

Let $1 \le p < \infty$. A bounded linear operator on $\mathcal{F}^p(G)$ is said to be a *multiplier operator* if there is a function $\Upsilon \in L(\widehat{G})$ such that $T(f)^\wedge = \Upsilon \widehat{f}$ for all $f \in \mathcal{F}^p(G)$. Whereas it is a standard result of multiplier theory that a bounded operator on $L^2(G)$ commutes with translations if and only if it is a multiplier operator (see [5 Theorem 4.1.1] or [12, Theorem I.3.18], for example), it is important for us to note that the corresponding result is also true for the spaces $\mathcal{F}^p(G)$, $1 \le p < \infty$.

2.5 Theorem. *Let $1 \leq p < \infty$ and let T be a bounded linear operator on $\mathcal{F}^p(G)$. Then T commutes with translations if and only if T is a multiplier operator. In such a case, the multiplier Υ of T is in $L^\infty(\widehat{G})$ and the operator norm $||T||$ of T equals $||\Upsilon||_\infty$.*

Proof. Assume that T commutes with translations. Let $\{U_\alpha\}_{\alpha \in A}$ be a family of relatively compact, non-void open subsets of $\widehat{G}$ whose union is $\widehat{G}$. By [9, Lemma I.2.1], for each $\alpha \in A$, there is a function $f_\alpha \in V$ such that $\widehat{f_\alpha}(U_\alpha) = \{1\}$. Now observe that by Lemma 2.4(3), if $\alpha, \beta \in A$ are such that $U_\alpha \cap U_\beta \neq \emptyset$, then $T(f_\alpha)^\wedge = T(f_\beta)^\wedge$ on $U_\alpha \cap U_\beta$. Thus, a function $\psi : \widehat{G} \longrightarrow \mathbb{C}$ may be defined by

$$\psi(g) = T(f_\alpha)^\wedge(g), \quad \text{if } g \in \widehat{G} \text{ and } g \in U_\alpha.$$

If $f \in V$ and $g \in \widehat{G}$ with $g \in U_\alpha$, then it follows from Lemma 2.4(3) that

$$T(f)^\wedge(g) = T(f)^\wedge(g)\, \widehat{f_\alpha}(g) = T(f_\alpha)^\wedge(g)\, \widehat{f}(g) = \psi(g)\widehat{f}(g).$$

That is,

$$T(f)^\wedge = \psi \widehat{f}, \quad \text{for all } f \in V. \tag{2.7}$$

Now it appears that ψ may not be Borel measurable and so need not be a multiplier for T, but we shall show that there is a function $\Upsilon \in L^\infty(G)$ such that $\{\gamma : \gamma \in \widehat{G} \text{ and } \Upsilon(\gamma) \neq \psi(\gamma)\}$ is a locally $\mu_{\widehat{G}}$-null subset of $\widehat{G}$. To this end, let K be a compact subset of $\widehat{G}$ and observe that by [9, Lemma I.2.1] there is a function $f \in V$ such that $\widehat{f}(K) = \{1\}$. Then we have

$$\psi \chi_K = \psi \chi_K \widehat{f} = \chi_K T(f)^\wedge,$$

and since $\chi_K T(f)^\wedge$ is Borel measurable so is $\psi \chi_K$. Since this is true for all compact subsets K of $\widehat{G}$, it is easy to see that ψf is Borel measurable for any function $f \in L^1(\widehat{G})$. Hence, the linear functional

$$f \longmapsto \int_{\widehat{G}} \psi f \, d\mu_{\widehat{G}}$$

from $L^1(\widehat{G})$ into $\mathbb{C}$ is in the dual space of $L^1(\widehat{G})$. However, because $\widehat{G}$ is a locally compact group, the dual space of $L^1(\widehat{G})$ is $L^\infty(\widehat{G})$ by [1, Theorem 9.4.8], and it follows that there is a function $\Upsilon \in L^\infty(\widehat{G})$ such that

$$\int_{\widehat{G}} \Upsilon f \, d\mu_{\widehat{G}} = \int_{\widehat{G}} \psi f \, d\mu_{\widehat{G}}, \quad \text{for all } f \in L^1(\widehat{G}).$$

From this we see that the equality $\Upsilon \chi_K = \psi \chi_K$ holds $\mu_{\widehat{G}}$-almost everywhere for any compact subset K of $\widehat{G}$, so that $\Upsilon = \psi$ locally $\mu_{\widehat{G}}$-almost everywhere. Then from (2.7) we have

$$T(f)^\wedge = \Upsilon \widehat{f}, \quad \text{for all } f \in V. \tag{2.8}$$

Since T is bounded on $\mathcal{F}^p(G)$ and since V is dense in $\mathcal{F}^p(G)$, the equality (2.8) in fact holds for all $f \in \mathcal{F}^p(G)$. That is, Υ is the multiplier for T and belongs to $L^\infty(\widehat{G})$.

Moreover, from (2.8), for any $f \in \mathcal{F}^p(G)$,

$$\begin{aligned} |||T(f)|||_p &= ||T(f)^\wedge||_p, \\ &= ||\Upsilon \widehat{f}||_p, \\ &\leq ||\Upsilon||_\infty \cdot ||\widehat{f}||_p, \end{aligned}$$

and we deduce that $||T|| \leq ||\Upsilon||_\infty$.

To prove that $||\Upsilon||_\infty \leq ||T||$, assume the contrary, that is, $||T|| < ||\Upsilon||_\infty$. Then there would be a positive number ε and a compact subset L of $\widehat{G}$ such that

$$\mu_{\widehat{G}}(L) > 0, \quad \text{and} \quad ||T|| + \varepsilon < |\Upsilon(\gamma)|, \text{ for all } \gamma \in L.$$

This would imply that

$$(||T|| + \varepsilon)\, \mu_{\widehat{G}}(L)^{1/p} < ||\Upsilon \chi_L||_p. \tag{2.9}$$

By the regularity of $\mu_{\widehat{G}}$, choose an open subset U of $\widehat{G}$ which contains L and satisfies

$$||T||\, \mu_{\widehat{G}}(U)^{1/p} < (||T|| + \varepsilon)\, \mu_{\widehat{G}}(L)^{1/p}. \tag{2.10}$$

By an evident modification of [9, Lemma I.2.1] (where it is not necessary to assume that K_2 is compact), there is a function $f \in V$ such that

$$0 \leq \widehat{f} \leq 1, \quad \widehat{f}(L) = \{1\}, \quad \text{and} \quad \widehat{f}(U^c) = \{0\}.$$

Since $\Upsilon \chi_L = \Upsilon \widehat{f} \chi_L = T(f)^\wedge \chi_L$, it follows from (2.10) that

$$\begin{aligned} ||\Upsilon \chi_L||_p &\leq ||T(f)^\wedge||_p, \\ &\leq ||T|| \cdot ||\widehat{f}||_p, \\ &\leq ||T||\, \mu_{\widehat{G}}(U)^{1/p}, \\ &< (||T|| + \varepsilon)\, \mu_{\widehat{G}}(L)^{1/p}, \end{aligned}$$

which would be a contradiction of (2.9). Thus, $||\Upsilon||_\infty \leq ||T||$, and hence we have $||\Upsilon||_\infty = ||T||$.

Conversely, let T be a multiplier operator with multiplier Υ. We show that T must commute with translations. If $f \in \mathcal{F}^p(G)$ and $x \in G$, then

$$\begin{aligned} T(\delta_x * f)^\wedge(g) &= \Upsilon(g)\overline{\gamma(x)}\widehat{f}(\gamma), \\ &= \widehat{\delta}_x(\gamma)\, T(f)^\wedge(g), \\ &= (\delta_x * T(f))^\wedge(g), \end{aligned}$$

for every $g \in \widehat{G}$ and we deduce that $T(\delta_x * f) = \delta_x * T(f)$. Thus the multiplier operator T commutes with translations. □

3 Unbounded multiplier operators

Let G be a locally compact Hausdorff abelian group with dual group $\widehat{G}$. and let $1 \leq p < \infty$. A Borel measurable function $\theta : \widehat{G} \longrightarrow [0, \infty)$ is said to be *essentially bounded away from* 0 on $\widehat{G}$ if there is a positive number ε such that $\theta(g) \geq \varepsilon$ for $\mu_{\widehat{G}}$-almost all $g \in \widehat{G}$. Given such a function θ, let $L^p(\widehat{G}, \theta d\mu_{\widehat{G}})$ denote the space of all complex valued Borel measurable functions ψ on $\widehat{G}$ such that $\int_{\widehat{G}} |\psi|^p\, \theta\, d\mu_{\widehat{G}} < \infty$. Let

$$\mathcal{F}^{p,\theta}(G) \ = \Big\{ f : f \in \mathcal{F}^p(G) \text{ and } \widehat{f} \in L^p(\widehat{G}, \theta d\mu_{\widehat{G}}) \Big\}.$$

Then define a norm $|||\cdot|||_{p,\theta}$ on $\mathcal{F}^{p,\theta}(G)$ by

$$|||f|||_{p,\theta} \;=\; \left(\int_{\widehat{G}} |\widehat{f}|^p \,\theta \, d\mu_{\widehat{G}}\right)^{1/p}, \quad \text{for all } \; f \in \mathcal{F}^{p,\theta}(G)$$

A vector subspace W of the space $\mathcal{F}^p(G)$ is said to be *translation invariant* if $\delta_x * f \in W$ for all $f \in W$ and all $x \in G$.

3.1 Lemma. *Let $1 \leq p < \infty$ and let $\theta : \widehat{G} \longrightarrow [0, \infty)$ be a Borel measurable function which is essentially bounded away from 0 on $\widehat{G}$. Then $\mathcal{F}^{p,\theta}(G)$ is a translation invariant vector subspace of $\mathcal{F}^p(G)$ which is a Banach space in the norm $|||\cdot|||_{p,\theta}$.*

Proof. The fact that $\mathcal{F}^{p,\theta}(G)$ is a Banach space in the given norm follows immediately from [9, Lemma III.2.1]. Now let $f \in \mathcal{F}^{p,\theta}(G)$ and let $x \in G$. Then,

$$\int_{\widehat{G}} |(\delta_x * f)^\wedge(\gamma)|^p \,\theta(\gamma)\, d\mu_{\widehat{G}}(\gamma) = \int_{\widehat{G}} |\widehat{f}(\gamma)|^p \,\theta(\gamma)\, d\mu_{\widehat{G}}(\gamma) < \infty,$$

where the fact that $|\gamma(x)| = 1$ has been used. This shows that $\delta_x * f \in \mathcal{F}^{p,\theta}(G)$ and it follows that $\mathcal{F}^{p,\theta}(G)$ is translation invariant. □

Now let θ_1, $\theta_2 : G \longrightarrow [0, \infty)$ be two Borel measurable functions on G which are essentially bounded away from 0. Then, analogously to the bounded multiplier case, a linear operator $T : \mathcal{F}^{p,\theta_1}(G) \longrightarrow \mathcal{F}^{p,\theta_2}(G)$ is said to be a *multiplier operator* if there is a Borel measurable function $\Upsilon : \widehat{G} \longrightarrow \mathbb{C}$ such that

$$T(f)^\wedge = \Upsilon \widehat{f}, \quad \text{for all } \; f \in \mathcal{F}^{p,\theta_1}(G).$$

In this case, Υ is called the *multiplier* of T. When $\theta_1 = \theta_2 = 1$, this definition is the same as the one for bounded multiplier operators, but generally these more general multiplier operators are not defined on the whole of $\mathcal{F}^p(G)$ and may be called *unbounded.*

3.2 Theorem. *Let G be a locally compact Hausdorff abelian group, let $1 \leq p < \infty$ and for each $j \in \{1, 2\}$ let $\theta_j : \widehat{G} \longrightarrow [0, \infty)$ be a Borel measurable function which is essentially bounded away from 0 on $\widehat{G}$. Let $T : \mathcal{F}^{p,\theta_1}(G) \longrightarrow \mathcal{F}^{p,\theta_2}(G)$ be a linear map. Then the following conditions are equivalent.*

(1) *T is bounded and $T(\delta_x * f) = \delta_x * T(f)$, for all $x \in G$ and $f \in \mathcal{F}^{p,\theta_1}(G)$.*

(2) *There is a Borel measurable function $\Upsilon : \widehat{G} \longrightarrow \mathbb{C}$ such that*

$$\Upsilon \left(\frac{\theta_2}{\theta_1}\right)^{1/p} \in L^\infty(\widehat{G}), \quad \textit{and} \quad (Tf)^\wedge = \Upsilon \widehat{f}, \quad \textit{for all } \; f \in \mathcal{F}^{p,\theta_1}(G).$$

When (1) *and* (2) *hold, the inequality* $||T|| \leq ||\Upsilon \left(\frac{\theta_2}{\theta_1}\right)^{1/p} ||_\infty$ *is valid.*

Proof. Assume that (1) holds. Then, for $j \in \{1, 2\}$, let $S_j : \mathcal{F}^p(G) :\longrightarrow \mathcal{F}^{p,\theta_j}(G)$ denote the operator defined by

$$S_j(f) \;=\; \left((\theta_j)^{-1/p}\, \widehat{f}\,\right)^\vee, \quad \text{for all } \; f \in \mathcal{F}^p(G).$$

The map S_j is a linear isometry from $\mathcal{F}^p(G)$ onto $\mathcal{F}^{p,\theta_j}(G)$ whose inverse is given by

$$S_j^{-1}(g) \;=\; \left((\theta_j)^{1/p}\, \widehat{g}\right)^\vee, \quad \text{for all } \; g \in \mathcal{F}^{p,\theta_j}(G).$$

Also, S_j commutes with translations. For, if $a \in G$ and $f \in \mathcal{F}^p(G)$, then

$$\begin{aligned}
[S_j(\delta_a * f)]^\wedge &= (\theta_j)^{-1/p}\,(\delta_a * f)^\wedge\,,\\
&= (\theta_j)^{-1/p}\,\widehat{\delta}_a \widehat{f},\\
&= \widehat{\delta}_a\, S_j(f)^\wedge,\\
&= (\delta_a * S_j(f))^\wedge,
\end{aligned}$$

so that, on taking inverse Fourier transforms, $S_j(\delta_a * f) = \delta_a * S_j(f)$.

Now the operator $S = S_2^{-1} \circ T \circ S_1 : \mathcal{F}^p(G) \longrightarrow \mathcal{F}^p(G)$ is bounded and as S_1, S_2 and T all commute with translations, so does S. By Theorem 2.3, there is a function $\psi \in L^\infty(\widehat{G})$ such that

$$S(f)^\wedge \;=\; \psi \widehat{f}, \quad \text{for all } \; f \in \mathcal{F}^p(G).$$

Then, for all $g \in \mathcal{F}^{p,\theta_1}(G)$,

$$\begin{aligned}
T(g)^\wedge &= [S_2 \circ S \circ S_1^{-1}(g)]^\wedge,\\
&= (\theta_2)^{-1/p}\,[S \circ S_1^{-1}(g)]^\wedge,\\
&= (\theta_2)^{-1/p}\,\psi\,[S_1^{-1}(g)]^\wedge,\\
&= (\theta_2)^{-1/p}\,\psi\,(\theta_1)^{1/p}\,\widehat{g},\\
&= \Upsilon \widehat{g},
\end{aligned}$$

where $\Upsilon = (\theta_2)^{-1/p}\,(\theta_1)^{1/p}\,\psi$. It follows that $\Upsilon\,(\theta_1)^{-1/p}\,(\theta_2)^{1/p} = \psi \in L^\infty(\widehat{G})$, so that (2) is established.

Conversely, let (2) hold. Then, since T maps $\mathcal{F}^{p,\theta_1}(G)$ into $\mathcal{F}^{p,\theta_2}(G)$, we have

$$\begin{aligned}
\int_{\widehat{G}} |T(f)^\wedge|^p\,\theta_2\,d\mu_{\widehat{G}} &= \int_{\widehat{G}} |\widehat{f}|^p\,|\Upsilon|^p\,\theta_1 \frac{\theta_2}{\theta_1}\,d\mu_{\widehat{G}},\\
&\leq \; ||\Upsilon\,(\theta_2)^{1/p}\,(\theta_1)^{-1/p}||_\infty^p \cdot |||f|||_{p,\theta_1}^p,
\end{aligned}$$

for all $f \in \mathcal{F}^{p,\theta_1}(G)$. Hence T is bounded from $\mathcal{F}^{p,\theta_1}(G)$ into $\mathcal{F}^{p,\theta_2}(G)$. □

4 Difference spaces and pseudomeasures

A main aim of this work is to describe the ranges of multiplier operators. As explained in Theorem 2.5, such operators may be characterized as the ones which commute with translations. The ranges of these operators are described in terms of *difference spaces*. As these spaces are not well known, it may be helpful to give some initial idea of how they are defined and of their significance in this work. Let G be a locally compact Hausdorff abelian group. Let $n \in \mathbb{N}$ and let $|\cdot|$ denote the usual Euclidean norm on the space $\mathbb{R}^n$. Let $\phi : \widehat{G} \longrightarrow \mathbb{R}^n$ be a given Borel measurable function and let $s > 0$. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ be a given bounded, Borel measurable function and let $y \in \mathbb{R}^n$ be given. Then, there is a pseudomeasure $\mu_{\alpha,\phi,y} \in \mathcal{F}^\infty(G)$ such that

$$\widehat{\mu}_{\alpha,\phi,y}(\gamma) = 1 - \alpha(\langle \phi(\gamma), y \rangle), \quad \text{for all } \; \gamma \in \widehat{G}.$$

The vector space of all functions in $L^2(G)$ which are finite sums of functions of the form

$$f - \mu_{\alpha,\phi,y} * f,$$

for some $f \in L^2(G)$ and some $y \in R^n$, is a special type of difference subspace of $L^2(G)$. The expression $f - \mu_{\alpha,\phi,y} * f$ is a type of "finite difference". Under suitable conditions on α, it will be shown that a function f in $L^2(G)$ belongs to this difference space if and only if

$$\int_{\widehat{G}} \left(1 + |\phi(\gamma)|^{-s}\right) |\widehat{f}(\gamma)|^2 \, d\mu_{\widehat{G}}(\gamma) < \infty. \tag{4.1}$$

Then, for this special case, it will be shown in the following section that this space is the range of any weighted multiplier operator whose multiplier is of the form ψ^s, where ψ is any $\mathbb{C}$-valued, Borel measurable function on $\widehat{G}$ such that $|\phi| = |\psi|$. Note that equation (4.1) shows that this difference space is the space $\mathcal{F}^{p,1+|\phi|^{-s}}(G)$, spaces of which type were considered in Section 3. In fact, a general difference space may be regarded as a special type of a space $\mathcal{F}^{p,\theta}(G)$; namely, one for which there is an alternative characterization due to the fact that its elements are precisely those vectors in $\mathcal{F}^p(G)$ which are finite sums of certain "difference expressions" formed from vectors in $\mathcal{F}^p(G)$ and convolution operations.

Motivated by the above comments, and by the work in [9], in the present section is developed a more general theory of difference spaces which is designed to assist in characterizing the range of any of the multiplier operators of the type described by Theorem 3.2. The setting is more general than in [9], in that the results are for any locally compact Hausdorff abelian group rather than simply $\mathbb{R}^n$ and in that a more general class of multiplier operators is also considered than in [9].

The usual inner product of vectors x and y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$. If W is a vector subspace of $\mathbb{R}^n$ and $c > 0$, then let $B_W(c) = \{x : x \in W \text{ and } |x| < c\}$. A vector space W is said to be a subspace of a Euclidean space if there is $n \in \mathbb{N}$ such that $W \subseteq \mathbb{R}^n$. If J is a finite interval of natural numbers and A_j is a set for each $j \in J$, then $\prod_{j\in J} A_j$ denotes the Cartesian product of the sets A_j, starting with the least element $j \in J$. Similarly, in this case, if $x_j \in A_j$ for each $j \in J$, then $\prod_{j\in J} x_j$ denotes the element $(x_j)_{j\in J}$ of $\prod_{j\in J} A_j$. The Lebesgue measure on a vector subspace W of a Euclidean space is denoted by μ_W. When $W = \mathbb{R}^n$ we write $\mu_n = \mu_{\mathbb{R}^n}$. If $m \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are measures, then $\prod_{j=1}^m \lambda_j$ will denote their product.

4.1 Theorem. *Let $\mathbb{R}_+$ denote the set of all non-negative real numbers. Let G be a locally compact Hausdorff abelian group with dual group $\widehat{G}$. Let $m \in \mathbb{N}$ and let $g : \mathbb{R}^m \longrightarrow \mathbb{R}_+$ be a given locally integrable function. Define a function $\mathcal{G} : \mathbb{R}_+^m \longrightarrow \mathbb{R}_+$ by*

$$\mathcal{G}(z_1, z_2, \ldots, z_m) = \int_{\prod_{k=1}^m (-z_k, z_k)} g \, d\mu_m, \quad \textit{for all } (z_1, z_2, \ldots, z_m) \in \mathbb{R}_+^m.$$

For each $k \in \{1, 2, \ldots, m\}$ let W_k be a non-zero vector subspace of some Euclidean space, and let $\phi_k : \widehat{G} \longrightarrow W_k$ be a given Borel measurable function. Also, let $c_1, c_2, \ldots, c_m > 0$ be given constants, and let $f : \widehat{G} \longrightarrow \mathbb{R}_+$ be a given Borel measurable function such that

$$f(\gamma) = 0 \quad \textit{for } \mu_{\widehat{G}}\textit{-almost all } \gamma \in \bigcup_{k=1}^m \phi_k^{-1}(\{0\}).$$

Suppose that

$$\int_{\widehat{G}} f(\gamma)\ \mathcal{G}\left(c_1|\phi_1(\gamma)|,\ c_2|\phi_2(\gamma)|, \ldots,\ c_m|\phi_m(\gamma)|\right) \left(\prod_{k=1}^{m} |\phi_k(\gamma)|^{-1}\right) d\mu_{\widehat{G}}(\gamma) < \infty,$$

Then, for $\left(\prod_{k=1}^{m} \mu_{W_k}\right)$*-almost all* $y = (y_1, y_2, \ldots, y_m) \in \prod_{k=1}^{m} B_{W_k}(c_k)$,

$$\int_{\widehat{G}} f(\gamma)\, g(\langle y_1, \phi_1(\gamma)\rangle, \langle y_2, \phi_2(\gamma)\rangle, \ldots, \langle y_m, \phi_m(\gamma)\rangle)\, d\mu_{\widehat{G}}(\gamma) < \infty.$$

Proof. If $\phi_k = 0$ for some $k \in \{1, 2, \ldots, m\}$, then $f(\gamma) = 0$ for $\mu_{\widehat{G}}$-almost all $\gamma \in \widehat{G}$ and the result is clearly true. So we may assume that $\phi_k \neq 0$ for every $k \in \{1, 2, \ldots, m\}$, which implies that $W_k \neq \{0\}$ for every $k \in \{1, 2, \ldots, m\}$. Let $Z_k = \phi_k^{-1}(\{0\})$ for each $k \in \{1, 2, \ldots, m\}$, and let $Z = \cup_{k=1}^{m} Z_k$. The assumption is that $f(\gamma) = 0$ for $\mu_{\widehat{G}}$-almost all $\gamma \in Z$. For each $k \in \{1, 2, \ldots, m\}$ take a non-zero vector $z_k \in W_k$, and let $\phi'_k : \widehat{G} \longrightarrow W_k$ be the function given by

$$\phi'_k(\gamma) = \begin{cases} \phi_k(\gamma), & \text{if } \gamma \notin Z_k, \\ z_k, & \text{if } \gamma \in Z_k, \end{cases}$$

so that $\phi'_k(g) \neq 0$ for every $g \in \widehat{G}$.

Now,

$$\begin{aligned}
&\int_{\widehat{G}} f(\gamma)\ \mathcal{G}\left(c_1|\phi_1(\gamma)|,\ c_2|\phi_2(\gamma)|, \ldots,\ c_m|\phi_m(\gamma)|\right) \left(\prod_{k=1}^{m} |\phi_k(\gamma)|^{-1}\right) d\mu_{\widehat{G}}(\gamma) < \infty \\
\Longrightarrow &\int_{\widehat{G}\cap Z^c} f(\gamma)\ \mathcal{G}\left(c_1|\phi_1(\gamma)|,\ c_2|\phi_2(\gamma)|, \ldots,\ d_m|\phi_m(\gamma)|\right) \left(\prod_{k=1}^{m} |\phi_k(\gamma)|^{-1}\right) d\mu_{\widehat{G}}(\gamma) < \infty \\
\Longrightarrow &\int_{\widehat{G}\cap Z^c} f(\gamma)\ \mathcal{G}\left(c_1|\phi'_1(\gamma)|,\ c_2|\phi'_2(\gamma)|, \ldots,\ c_m|\phi'_m(\gamma)|\right) \left(\prod_{k=1}^{m} |\phi'_k(\gamma)|^{-1}\right) d\mu_{\widehat{G}}(\gamma) < \infty \\
\Longrightarrow &\int_{\widehat{G}} f(\gamma)\ \mathcal{G}\left(c_1|\phi'_1(\gamma)|,\ c_2|\phi'_2(\gamma)|, \ldots,\ c_m|\phi'_m(\gamma)|\right) \left(\prod_{k=1}^{m} |\phi'_k(\gamma)|^{-1}\right) d\mu_{\widehat{G}}(\gamma) < \infty,
\end{aligned}$$

since $f(\gamma) = 0$ for $\mu_{\widehat{G}}$-almost all $\gamma \in Z$. It then follows that for $\left(\prod_{k=1}^{m} \mu_{W_k}\right)$-almost all $y = (y_1, y_2, \ldots, y_m) \in \prod_{k=1}^{m} B_{W_k}(c_k)$,

$$\int_{\widehat{G}} f(\gamma)\ g\big(\langle \phi'_1(\gamma), y_1\rangle, \langle \phi'_2(\gamma), y_2\rangle, \ldots, \langle \phi'_m(\gamma), y_m\rangle\big)\, d\mu_{\widehat{G}}(\gamma) < \infty.$$

In fact, Theorem II.2.2 in [9] applies to deduce this, by replacing the vector space W there with the locally compact Hausdorff abelian group $\widehat{G}$, by replacing the functions P_{V_k} there with the functions ϕ'_k here, and replacing the constants d_k there with c_k here. However, since $f(\gamma) = 0$ for $\mu_{\widehat{G}}$-almost all $\gamma \in Z$ and since $\phi_k(\gamma) = \phi'_k(\gamma)$ for all $\gamma \in Z^c$, we see that for $\left(\prod_{k=1}^{m} \mu_{W_k}\right)$-almost all $y = (y_1, y_2, \ldots, y_m) \in \prod_{k=1}^{m} B_{W_k}(c_k)$,

$$\int_{\widehat{G}} f(\gamma)\ g\big(\langle \phi_1(\gamma), y_1\rangle\rangle, \langle \phi_2(\gamma), y_2\rangle, \ldots, \langle \phi_m(\gamma), y_m\rangle\big)\, d\mu_{\widehat{G}}(\gamma) < \infty,$$

as required. □

STANDING ASSUMPTIONS FOR THE RESULTS OF THIS SECTION.

Now we proceed to the situation and notations necessary to describe and prove the main results of this section. There will be a standing notation that G is a locally compact Hausdorff abelian group with identity element e and dual group $\widehat{G}$. Let $n,\, q \in \mathbb{N}$ and let $s_1, s_2, \ldots, s_q$ be q strictly positive real numbers. Let $J_1, J_2, \ldots, J_q$ be q disjoint, finite, and consecutive subintervals of $\mathbb{N}$ with $1 \in J_1$. Assume that for each $j = 1, 2, \ldots, q$ the number of elements in J_j is m_j and let $m = m_1 + m_2 + \cdots + m_q$. Then by the assumptions on the J_j, it follows that

$$\{1, 2, \ldots, m\} = \bigcup_{j=1}^{q} J_j. \tag{4.2}$$

Also, for each $k \in \cup_{j=1}^{q} J_j$, let α_k be a complex valued, bounded continuous function on $\mathbb{R}$. It is assumed that there is a constant $K > 0$ such that for each $j \in \{1, 2, \ldots, q\}$

$$\int_{(-z,z)^{m_j}} \frac{d\mu_{m_j}\left(\prod_{k \in J_j} x_k\right)}{\sum\limits_{k \in J_j} |\alpha_k(x_k)|^p} \le \begin{cases} K z^{m_j}, & \text{if } z \ge 1, \\ K z^{m_j - p s_j}, & \text{if } 0 \le z < 1. \end{cases} \tag{4.3}$$

It is further assumed that there is a constant $c > 0$ such that given $j \in \{1, 2, \ldots, q\}$

$$|\alpha_k(x)| \le c|x|^{s_j}, \text{ for all } x \in \mathbb{R} \text{ and for all } k \in J_j. \tag{4.4}$$

Incidentally, note that (4.3) and (4.4) imply that

$$\int_{(-1,1)^{m_j}} \frac{d\mu_{m_j}(x_1, x_2, \ldots, x_{m_j})}{\sum_{k=1}^{m_j} |x_k|^{p s_j}} < \infty,$$

which is the case only when $m_j > p s_j$. Thus, the assumptions (4.1), (4.2) and (4.3) imply that $m_j > p s_j$ for all $j = \{1, 2, \ldots, q\}$.

If $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ is any bounded Borel measurable function, if $\phi : \widehat{G} \longrightarrow W$ is a Borel measurable function from $\widehat{G}$ into a vector subspace W of some Euclidean space, and if $y \in W$, then the function from $\widehat{G}$ into $\mathbb{C}$ given by $\gamma \longmapsto \alpha(\langle \phi(\gamma), y \rangle)$ is in $L^\infty(\widehat{G})$, and so there is a pseudomeasure $\mu_{\alpha,\phi,y}$ in $\mathcal{F}^\infty(G)$ such that

$$\widehat{\mu}_{\alpha,\phi,y}(\gamma) = 1 - \alpha(\langle \phi(\gamma), y \rangle), \quad \text{for all } \gamma \in \widehat{G}.$$

Note that if $y = 0$ then $\widehat{\mu}_{\alpha,\phi,y}(x) = 1 - \alpha(0)$ so that $\mu_{\alpha,\phi,0} = (1 - \alpha(0))\delta_e$. In particular, if $\alpha(0) = 0$, then $\mu_{\alpha,\phi,0} = \delta_e$. We assume that for each $j \in \{1, 2, \ldots, q\}$, V_j is a vector subspace of some Euclidean space and that $\phi_j : \widehat{G} \longrightarrow V_j$ is a given Borel measurable function. If $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{F}^\infty(G)$, then the element $\mu_1 * \mu_2 * \cdots * \mu_n$ of $\mathcal{F}^\infty(G)$, obtained by an n-fold convolution of pseudomeasures, may sometimes be denoted by $\prod_{j=1}^{n} * \, \mu_j$.

The above sets out the assumptions and notations for this section. Before proceeding, we give an indication of the overall picture. The abstract distributions f in a typical difference space considered here can be written in the form

$$f = \sum_{(k_1, k_2, \ldots, k_q) \in \prod_{j=1}^{q} J_j} \left(\prod_{j=1}^{q} * \, (\delta_e - \mu_{\alpha_{k_j}, \phi_j, y_{k_j}}) \right) * f_{k_1 k_2 \ldots k_q}, \tag{4.5}$$

for some $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} V_j^{m_j}$ and for some choice of $f_{k_1 k_2 \ldots k_q} \in \mathcal{F}^p(G)$, this latter choice being made for each $(k_1, k_2, \ldots, k_q) \in \prod_{j=1}^{q} J_j$. It is not at all obvious that under the assumed conditions the abstract distributions which can be written in this way form a vector space, but it is in fact the case. Now the following result gives a description of when an equation like (4.5) is possible, and this is in terms of the finiteness of a singular integral such as occurs in the conclusion of Theorem 4.1. Thus, the significance of Theorem 4.1 lies in the fact that it gives a sufficient condition for an abstract distribution to be expressible in the form (4.5), and thus tells us when a given abstract distribution is in a particular difference space.

4.2 Theorem. *Let $1 \le p < \infty$ and let $f \in \mathcal{F}^p(G)$. Let α_k and ϕ_j be functions given as above, and let $(y_1, y_2, \dots, y_m) \in \prod_{j=1}^q V_j^{m_j}$ also be given. Then, for each $(k_1, k_2, \dots, k_q) \in \prod_{j=1}^{q} J_j$, there is a function $f_{k_1 k_2 \dots k_q} \in \mathcal{F}^p(G)$ such that (4.5) holds if and only if*

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod_{j=1}^{q} \left(\sum_{k \in J_j} |1 - \widehat{\mu}_{\alpha_k, \phi_j, y_k}(\gamma)|^p \right)} < \infty.$$

Proof. Essentially, this is proved in [9, Proposition I.4.3]. Note that the proof given there is in the case where the $\mu_{\alpha_j, \phi_j, k_{k_j}}$ are measures rather than pseudomeasures (the latter being quite possible in the present context), but the proof is exactly the same once allowance is made for this. □

The following basic result characterizes the difference spaces as spaces of the type $\mathcal{F}^{p,\theta}(G)$ which were considered in Section 3. We adopt the conventions that $x/\infty = 0$ when $x > 0$ and that $1/0 = \infty$. Moreover, when A is the empty set, any product $\prod_{j \in A}$ is understood to be 1.

4.3 Theorem. *Let $1 \le p < \infty$ and let $q \in \mathbb{N}$. Let $s_j \in (0, \infty)$ and let $\psi_j : \widehat{G} \longrightarrow \mathbb{C}$ be a given Borel measurable function for each $j \in \{1, 2, \dots, q\}$. Suppose that $\Psi : \widehat{G} \longrightarrow [0, \infty]$ is the function given by*

$$\Psi \;=\; \sum_{A \subseteq \{1,2,\dots,q\}} \left(\prod_{j \in A} |\psi_j|^{-p s_j} \right). \tag{4.6}$$

For each $j \in \{1, 2, \dots, q\}$, let V_j be a non-zero vector subspace of some Euclidean space and let $\phi_j : \widehat{G} \longrightarrow V_j$ be a given Borel measurable function such that

$$|\psi_j(\gamma)| = |\phi_j(\gamma)|, \quad \text{for } \mu_{\widehat{G}}\text{-almost all } \gamma \in \widehat{G}.$$

When $j \in \{1, 2, \dots, m\}$, the m_j-fold product $\prod_{j=1}^q \mu_{V_j}$ (on the product space $V_j^{m_j}$) of the single measure μ_{V_j} is denoted by $\mu_{V_j}^{m_j}$. Then the following conditions are equivalent for a given abstract distribution $f \in \mathcal{F}^p(G)$.

(1) *$\widehat{f} \in L^p(\widehat{G}, \Psi d\mu_{\widehat{G}})$.*

(2) *For $\left(\prod_{j=1}^q \mu_{V_j}^{m_j}\right)$-almost all $(y_1, y_2, \dots, y_m) \in \prod_{j=1}^q V_j^{m_j}$, it holds that for all $(k_1, k_2, \dots, k_q) \in \prod_{j=1}^q J_j$ there is a vector $f_{k_1 k_2 \dots k_q} \in \mathcal{F}^p(G)$ such that*

$$f = \sum_{(k_1, k_2, \dots, k_q) \in \prod_{j=1}^{q} J_j} \left(\prod_{j=1}^{q} {}^{*} \left(\delta_e - \mu_{\alpha_{k_j}, \phi_j, y_{k_j}} \right) \right) * f_{k_1 k_2 \dots k_q}. \tag{4.7}$$

(3) There is a vector $(y_1, y_2, \dots, y_m) \in \prod_{j=1}^q V_j^{m_j}$ such that for each $(k_1, k_2, \dots, k_q) \in \prod_{j=1}^q J_j$ there is a vector $f_{k_1 k_2 \dots k_q} \in \mathcal{F}^p(G)$ so that (4.7) is satisfied.

Proof. Let $d_1, d_2, \dots, d_q > 0$ be given numbers. Let $k \in \{1, 2, \dots, m\}$. Then by (4.2) there is a unique $j \in \{1, 2, \dots, q\}$ such that $k \in J_j$. Let $W_k = V_j$ when $k \in J_j$ with $j \in \{1, 2, \dots, q\}$. Hence, a function $\eta_k : \widehat{G} \longrightarrow V_j$ may be defined by putting $\eta_k = \phi_j$ if $k \in J_j$, and a number c_k may be defined by putting $c_k = d_j$ if $k \in J_j$.

Define a function $g : \mathbb{R}^m \longrightarrow [0, \infty]$ by putting

$$g(x_1, x_2, \dots, x_m) \;=\; \frac{1}{\prod_{j=1}^{q} \left(\sum_{k \in J_j} |\alpha_k(x_k)|^p \right)},$$

for all $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$. Thus, if $\mathcal{G} : \mathbb{R}^m \longrightarrow [0, \infty]$ denotes the function given by

$$\mathcal{G}(z_1, z_2, \dots, z_m) = \int_{\prod_{j=1}^m (-z_j, z_j)} g \, d\mu_m, \quad \text{for all } (z_1, z_2, \dots, z_m) \in \mathbb{R}_+^m,$$

and if it is noted that $\mu_m = \prod_{j=1}^q \mu_{m_j}$, then it follows that

$$\begin{aligned} \mathcal{G}\big(c_1|\eta_1(\gamma)|, \dots, c_m|\eta_m(\gamma)|\big) &= \int_{\prod_{k=1}^m (-c_k|\eta_k(\gamma)|, c_k|\eta_k(\gamma)|)} g \, d\mu_m, \\ &= \int_{\prod_{j=1}^q (-d_j|\phi_j(g)|, d_j|\phi_j(g)|)^{m_j}} g \, d\mu_m, \\ &= \prod_{j=1}^q \left(\int_{(-d_j|\phi_j(\gamma)|, d_j|\phi_j(\gamma)|)^{m_j}} \frac{d\mu_{m_j}\left(\prod_{k \in J_j} x_k\right)}{\sum_{k \in J_j} |\alpha_k(x_k)|^p} \right). \end{aligned} \tag{4.8}$$

Now let $A \subseteq \{1, 2, \dots, q\}$, and make the definition that

$$Z(A; d_1, d_2, \dots, d_q) = \Big\{ \gamma : \gamma \in \widehat{G} \text{ and } d_j|\phi_j(\gamma)| \geq 1 \text{ for all } j \in A; \text{ and } d_j|\phi_j(g)| < 1 \text{ for all } j \notin A \Big\}.$$

Let $g \in Z(A; d_1, d_2, \dots, d_q)$. Then (4.3) and (4.8) give

$$\begin{aligned} \mathcal{G}\big(c_1|\eta_1(\gamma)|, c_2|\eta_2(\gamma)|, \dots, c_m|\eta_m(\gamma)|\big) &\leq K \left(\prod_{j \in A} \Big(d_j|\phi_j(\gamma)|\Big)^{m_j} \right) \left(\prod_{j \notin A} \Big(d_j|\phi_j(\gamma)|\Big)^{m_j - ps_j} \right), \\ &\leq L(d_1, d_2, \dots, d_q) \left(\prod_{j=1}^q |\phi_j(\gamma)|^{m_j} \right) \left(\prod_{j \notin A} |\phi_j(\gamma)|^{-ps_j} \right), \end{aligned} \tag{4.9}$$

where K is the constant in (4.3), and $L(d_1, d_2, \dots, d_q)$ is a constant which depends upon $d_1, d_2, \dots, d_q$ but does not depend upon A. Since $\prod_{k=1}^m |\eta_k(\gamma)| \;=\; \prod_{j=1}^q |\phi_j(\gamma)|$, it follows from (4.9) that

$$\begin{aligned} &\mathcal{G}\big(c_1|\eta_1(\gamma)|, c_2|\eta_2(\gamma)|, \dots, c_m|\eta_m(\gamma)\big) \left(\prod_{k=1}^m |\eta_k(\gamma)|^{-1} \right) \\ &\qquad \leq L(d_1, d_2, \dots, d_q) \left(\prod_{j \notin A} |\phi_j(\gamma)|^{-ps_j} \right), \\ &\qquad = L(d_1, d_2, \dots, d_q) \left(\prod_{j \notin A} |\psi_j(\gamma)|^{-ps_j} \right). \end{aligned} \tag{4.10}$$

From the definition of $Z(A; d_1, d_2, \ldots, d_q)$, it is clear that

$$\widehat{G} = \bigcup \left\{ Z(A; d_1, d_2, \ldots, d_q) : A \subseteq \{1, 2, \ldots, q\} \right\},$$

and that this union is disjoint. Since $L(d_1, d_2, \ldots, d_q)$ is independent of the subset A of $\{1, 2, \ldots, q\}$, it follows from (4.6) and (4.10) that, for all $\gamma \in \widehat{G}$,

$$\mathcal{G}\big(c_1|\eta_1(\gamma)|, c_2|\eta_2(\gamma)|, \ldots, c_m|\eta_m(\gamma)|\big) \left(\prod_{k=1}^{m} |\eta_k(\gamma)|^{-1} \right) \leq L(d_1, d_2, \ldots, d_q)\Psi(\gamma). \tag{4.11}$$

Now we prove that (1) implies (2). That is, we are assuming that $\widehat{f} \in L^p(\widehat{G}, \Psi d\mu_{\widehat{G}})$. Then, from (4.6) we see that

$$\widehat{f}(\gamma) = 0 \quad \text{for } \mu_{\widehat{G}}\text{-almost all } \gamma \in \bigcup_{j=1}^{q} \psi_j^{-1}(\{0\}).$$

Since $|\psi_j| = |\phi_j|$ and since $\eta_k = \phi_j$ for all $k \in J_j$, it follows that

$$\widehat{f}(\gamma) = 0 \quad \text{for } \mu_{\widehat{G}}\text{-almost all } \gamma \in \bigcup_{k=1}^{m} \eta_k^{-1}(\{0\}). \tag{4.12}$$

Also, (4.11) implies that

$$\int_{\widehat{G}} |\widehat{f}(\gamma)|^p \, \mathcal{G}\big(c_1|\eta_1(\gamma)|, \ldots, c_m|\eta_m(\gamma)|\big) \left(\prod_{k=1}^{m} |\eta_k(\gamma)|^{-1} \right) d\mu_{\widehat{G}}(\gamma) \;\leq\; L(d_1, \ldots, d_q) \int_{\widehat{G}} |\widehat{f}|^p \Psi \, d\mu_{\widehat{G}} \;<\; \infty. \tag{4.13}$$

By using (4.12) and (4.13), Theorem 4.1 may be applied, with $|\widehat{f}|^p$ in place of f and with η_k in place of ϕ_k. We then deduce that

$$\int_{\widehat{G}} |\widehat{f}(\gamma)|^p \; g\big(\langle \eta_1(\gamma), y_1 \rangle, \langle \eta_2(\gamma), y_2 \rangle, \ldots, \langle \eta_m(\gamma), y_m \rangle\big) \, d\mu_{\widehat{G}}(\gamma) < \infty,$$

for $\big(\prod_{k=1}^{m} \mu_{W_k}\big)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{k=1}^{m} B_{W_k}(c_k)$. In other words,

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\displaystyle\prod_{j=1}^{q} \left(\sum_{k \in J_j} |\alpha_k(\langle y_k, \phi_j(\gamma) \rangle)|^p \right)} < \infty, \tag{4.14}$$

for $\big(\prod_{k=1}^{m} \mu_{W_k}\big)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{k=1}^{m} B_{W_k}(c_k)$. From the definitions of m_j, W_k and c_k, it now follows immediately that (4.14) holds for $\big(\prod_{j=1}^{q} \mu_{V_j}^{m_j}\big)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} B_{V_j}(d_j)^{m_j}$.

Let $k \in \{1, 2, \ldots, m\}$ and take a unique $j \in \{1, 2. \ldots, q\}$ with $k \in J_j$. Since α_k is bounded by assumption, given $y_k \in V_j$ there is a pseudomeasure $\mu_{\alpha_k, \phi_j, y_k} \in \mathcal{F}^\infty(G)$ such that

$$\widehat{\mu}_{\alpha_k, \phi_j, y_k}(\gamma) = 1 - \alpha_k(\langle \phi_j(\gamma), y_k \rangle), \quad \text{for all } \gamma \in \widehat{G}.$$

Hence, we can rewrite (4.14) as

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\displaystyle\prod_{j=1}^{q} \left(\sum_{k \in J_j} |1 - \widehat{\mu}_{\alpha_k, \phi_j, y_k}(\gamma)|^p \right)} < \infty,$$

for $\left(\prod_{j=1}^{q}\mu_{V_j}^{m_j}\right)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} B_{V_j}(d_j)^{m_j}$. Then, by Theorem 4.2, for $\left(\prod_{j=1}^{q}\mu_{V_j}^{m_j}\right)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} B_{V_j}(d_j)^{m_j}$, it holds that for each $(k_1, k_2, \ldots, k_q) \in \prod_{j=1}^{q} J_j$ there is a vector $f_{k_1 k_2 \ldots k_q} \in \mathcal{F}^p(G)$ so that (4.7) is satisfied. Because this statement is true for $\left(\prod_{j=1}^{q}\mu_{V_j}^{m_j}\right)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} B_{V_j}(d_j)^{m_j}$ and because the $d_1, d_2, \ldots, d_m$ are *any* numbers strictly greater than 0, we conclude that, in fact, for $\left(\prod_{j=1}^{q}\mu_{V_j}^{m_j}\right)$-almost all $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} V_j^{m_j}$, it holds that for each $(k_1, k_2, \ldots, k_q) \in \prod_{j=1}^{q} J_j$ there is $f_{k_1 k_2 \ldots k_q} \in \mathcal{F}^p(G)$ such that (4.7) is valid; compare this with [9, p.55]. Therefore, (1) implies (2).

It is obvious that (3) follows from (2).

Finally, we prove that (3) implies (1). To this end, let $f \in \mathcal{F}^p(G)$ and assume that for some $(y_1, y_2, \ldots, y_m) \in \prod_{j=1}^{q} V_j^{m_j}$ and for each $(k_1, k_2, \ldots, k_q) \in \prod_{j=1}^{q} J_j$ there is a vector $f_{k_1 k_2 \ldots k_q} \in \mathcal{F}^p(G)$ such that (4.7) holds. Let $A \subseteq \{1, 2, \ldots, q\}$. Then we need to show that

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod\limits_{j\in A} |\psi_j(\gamma)|^{ps_j}} < \infty.$$

By renumbering the parameters, if necessary, there is no loss of generality in assuming that $A = \{1, 2, \ldots, r\}$ for some $r \in \mathbb{N}$. Now in (4.7), put

$$h_{k_1 k_2 \ldots k_r} = \sum_{(k_{r+1}, k_{r+2}, \ldots, k_q) \in \prod\limits_{j=r+1}^{q} J_j} \left(\prod_{j=r+1}^{q} * \, (\delta_e - \mu_{\alpha_{k_j}, \phi_j, y_{k_j}}) \right) * f_{k_1 k_2 \ldots k_q},$$

so that

$$f = \sum_{(k_1, k_2, \ldots, k_r) \in \prod\limits_{j=1}^{r} J_j} \left(\prod_{j=1}^{r} * \, (\delta_e - \mu_{\alpha_{k_j}, \phi_j, y_{k_j}}) \right) * h_{k_1 k_2 \ldots k_r}.$$

In this equation, each vector $h_{k_1 k_2 \ldots k_r}$ belongs to $\mathcal{F}^p(G)$. Then an application of Theorem 4.2 and the fact that $1 - \widehat{\mu}_{\alpha_{k_j}, \phi_j, y_{k_j}}(\gamma) = \alpha_{k_j}(\langle \phi_j(\gamma), y_{k_j} \rangle)$ for every $j \in \{1, 2, \ldots, r\}$ and every $g \in \widehat{G}$ give

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod\limits_{j=1}^{r} \left(\sum\limits_{k\in J_j} |\alpha_{k_j}(\langle \phi_j(\gamma), y_{k_j} \rangle)|^p \right)} < \infty. \tag{4.15}$$

Now let $j \in \{1, 2, \ldots, r\}$ and $k \in J_j$. By Lemma I.3.2 in [9] (for example), for the vector $y_k \in V_j$ there is a constant $a > 0$ such that

$$a \left(\sum_{k\in J_j} |\langle u, y_k \rangle|^{s_j p} \right) \le |u|^{s_j p}, \quad \text{for all } u \in V_j. \tag{4.16}$$

Then, using inequalities (4.4), (4.15) and (4.16), and the inequality that $|\langle u, v \rangle| \le |u| \cdot |v|$ for all $u, v \in V_j$, enables us to deduce that

$$\int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod\limits_{j\in A} |\psi_j(\gamma)|^{s_j p}} = \int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod\limits_{j\in A} |\phi_j(\gamma)|^{s_j p}},$$

$$
\begin{aligned}
&= \int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod_{j=1}^{r} |\phi_j(\gamma)|^{s_j p}}, \\
&\le \frac{1}{a} \int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod_{j=1}^{r} \left(\sum_{k \in J_j} |\langle \phi_j(\gamma), y_k \rangle|^{s_j p} \right)}, \\
&\le \frac{c}{a} \int_{\widehat{G}} \frac{|\widehat{f}(\gamma)|^p \, d\mu_{\widehat{G}}(\gamma)}{\prod_{j=1}^{r} \left(\sum_{k \in J_j} |\alpha_k(\langle \phi_j(\gamma), y_k \rangle)|^{p} \right)}, \\
&< \infty.
\end{aligned}
$$

This proves that (2) implies (1) and completes the proof of Theorem 4.3. □

It should be noted that the case described in the first part of this section corresponds to the special case of Theorem 4.3 which arises when $q = 1$.

Note that for a given p the function $\Psi = \sum_{A \subseteq \{1,2,\ldots,q\}} \left(\prod_{j \in A} |\psi_j|^{-ps_j} \right)$ depends only upon the functions $\psi_1, \psi_2, \ldots, \psi_q$ and the numbers $s_1, s_2, \ldots, s_q$. Thus, superficially it would appear that an abstract distribution $f \in \mathcal{F}^p(G)$ which can be expressed in the form (4.7) for some particular choice of functions α_k would not necessarily be expressible in the form (4.7) for another, *different*, choice of functions α_k. However, in fact, Theorem 4.3 shows that if f can be expressed in the form (4.7) using *one* choice of functions α_k, then it can be expressed in the form (4.7) for *any* choice of functions α_k but subject, of course, to the stated conditions of Theorem 4.3. Theorem 4.3 shows that the functions or abstract distributions which are expressible in the form (4.7) for some choice of α_k form a vector space which is isomorphic, under the Fourier transform, to the space $L^p(\widehat{G}, \Psi d\mu_{\widehat{G}})$, a space which is independent of the choice of the α_k. Accordingly, the vector space of abstract distributions in $\mathcal{F}^p(G)$ which can be written in the form (4.7) for some choice of α_k may be denoted by $\mathcal{D}_{s_1,\ldots,s_q,\psi_1,\ldots,\psi_q}(\mathcal{F}^p(G))$. Also from Theorem 4.3, and using the same notations and assumptions therein, we have the following result.

4.4 Theorem. *Let* $1 \le p < \infty$. *A vector* $f \in \mathcal{F}^p(G)$ *belongs to* $\mathcal{D}_{s_1,\ldots,s_q,\psi_1,\ldots,\psi_q}(\mathcal{F}^p(G))$ *if and only if* $\widehat{f} \in L^p(\widehat{G}, \Psi d\mu_{\widehat{G}})$. *The space* $\mathcal{D}_{s_1,\ldots,s_q,\psi_1,\ldots,\psi_q}(\mathcal{F}^p(G))$ *is a Banach space in the norm* $|||\cdot|||_\Psi$ *given by*

$$
|||f|||_\Psi = \left(\int_{\widehat{G}} |\widehat{f}|^p \Psi \, d\mu_{\widehat{G}} \right)^{1/p} = \left(\int_{\widehat{G}} |\widehat{f}|^p \left(\sum_{A \subseteq \{1,2,\ldots,q\}} \left(\prod_{j \in A} |\psi_j|^{-ps_j} \right) \right) d\mu_{\widehat{G}} \right)^{1/p},
$$

for all $f \in \mathcal{D}_{s_1,\ldots,s_q,\psi_1,\ldots,\psi_q}(\mathcal{F}^p(G))$.

Note that for a given $f \in \mathcal{F}^p(G)$, the choice of the abstract distributions $f_{k_1 k_2 \ldots k_q}$ so that (4.7) holds depends upon the particular choice of $(y_1, y_2, \ldots, y_m)$ in $\prod_{j=1}^{q} V_j^{m_j}$.

The question arises as to when the functions α_k will satisfy the assumed condition (4.3).

4.5 Example. Let $j \in \{1, 2, \ldots, q\}$. Suppose that for each $k \in J_j$, $\alpha_k : \mathbb{R} \longrightarrow \mathbb{C}$ is a continuous 2π-periodic function such that

$$
\int_{(-\pi,\pi)^{m_j}} \frac{dx_1 \, dx_2 \ldots dx_{m_j}}{\sum_{k \in J_j} |\alpha_k(x_k)|^p} < \infty, \tag{4.17}
$$

and such that for some c', $\varepsilon > 0$,

$$|x|^{s_j} \leq c'|\alpha_k(x)|, \quad \text{whenever } |x| < \varepsilon. \tag{4.18}$$

Then under the assumptions (4.17) and (4.18), Lemma II.3.3 in [9] (see also [10]) shows that (4.3) holds. Thus, if the functions α_k satisfy conditions (4.4), (4.17) and (4.18), they may be taken as appropriate functions for forming the pseudomeasures $\mu_{\alpha_{k_j},\phi_j,y_{k_j}}$ in Theorem 4.3. An example of functions α_k satisfying (4.4), (4.17) and (4.18) is given by $\alpha(x) = (1 - e^{-ix})^{s_j}$ for every $x \in \mathbb{R}$. □

It should be noted that the conditions of (4.3) and (4.4) may be satisfied by the functions α_k even when (4.17) is not satisfied (see [9, Theorem II.3.4]).

5 Ranges of multiplier operators

Let G be a locally compact Hausdorff abelian group, with dual group $\widehat{G}$. Let n, $q \in \mathbb{N}$ and let $s_1, s_2, \ldots, s_q$ be q strictly positive real numbers. Let Υ be a Borel measurable function from $\widehat{G}$ into $\mathbb{C}$. For each $j \in \{1, 2, \ldots, q\}$ let $\psi_j : \widehat{G} \longrightarrow \mathbb{C}$ be a Borel measurable function and assume that

$$\Upsilon = \prod_{j=1}^{q} \psi_j^{s_j}. \tag{5.1}$$

This will be called a *factorization* of Υ and the functions $\psi_1, \psi_2, \ldots, \psi_q$ will be called *factors* of Υ. It will be seen that each factorization of Υ leads to a corresponding multiplier operator whose multiplier is Υ. In general, the multiplier operators which arise from factorizations of Υ have different domains. The main aims in this section are to identify these domains and identify the ranges of the different multiplier operators.

To this end, adopt the convention that when A is the empty set any product $\prod_{j\in A}$ is taken to be 1. Define functions $\Theta : \widehat{G} \longrightarrow \mathbb{R}_+$ and $\Psi : \widehat{G} \longrightarrow [0, \infty]$ by putting

$$\Theta = \sum_{A\subseteq\{1,2,\ldots,q\}} \left(\prod_{j\in A} |\psi_j|^{ps_j} \right) \quad \text{and} \quad \Psi = \sum_{A\subseteq\{1,2,\ldots,q\}} \left(\prod_{j\in A} |\psi_j|^{-ps_j} \right). \tag{5.2}$$

Strictly speaking, the functions Ψ and Θ depend upon p as well as upon the factorization of Υ, but as p is regarded as being given, Θ and Ψ may be regarded for our purposes as depending only upon the factorization of Υ. Then, from the definitions, it is clear that both Θ and Ψ are essentially bounded away from 0, so $\mathcal{F}^{p,\Theta}(G)$ and $\mathcal{F}^{p,\Psi}(G)$ are Banach spaces in their respective norms (see Lemma 3.1).

The following result now describes an appropriate domain for the multiplier operator corresponding to a given factorization of Υ, and gives conditions which ensure that the operator is an isometry onto a difference space.

5.1 Theorem. *Let $1 \leq p < \infty$ and let the functions Υ, Θ and Ψ on $\widehat{G}$ be given as in* (5.1) *and* (5.2) *above. Then the following statements hold.*

(1) $\left(\mathcal{D}_{s_1,\ldots,s_q,\psi_1,\ldots\psi_q}(\mathcal{F}^p(G))\right)^\wedge = L^p(\widehat{G}, \Psi d\mu_{\widehat{G}})$.

(2) *There is a uniquely defined multiplier operator* T *which maps* $\mathcal{F}^{p,\Theta}(G)$ *onto* $\mathcal{D}_{s_1,\dots,s_q,\psi_1,\dots,\psi_q}(\mathcal{F}^p(G))$ *and which has multiplier* Υ.

(3) *If* Υ *is further assumed to have the property that* $\Upsilon(g) \neq 0$ *for* $\mu_{\widehat{G}}$*-almost all* $g \in \widehat{G}$, *then the operator* T *in* (2) *is an isometry from* $\mathcal{F}^{p,\Theta}(G)$ *onto* $\mathcal{D}_{s_1,\dots,s_q,\psi_1,\dots,\psi_q}(\mathcal{F}^p(G))$.

Proof. The first conclusion is immediate on realizing that in view of Theorem 4.4, the spaces $\mathcal{F}^{p,\Psi}(G)$ and $\mathcal{D}_{s_1,\dots,s_q,\psi_1,\dots,\psi_q}(\mathcal{F}^p(G))$ coincide and have identical norms.

To prove (2), note that it is clear from the definitions that if $\gamma \in \widehat{G}$ and $\Upsilon(\gamma) \neq 0$, then $|\Upsilon(\gamma)|^p\Psi(\gamma) = \Theta(\gamma)$. Thus, with the convention that $0 \cdot \infty = 0$, it follows that $|\Upsilon|^p\Psi \leq \Theta$ on $\widehat{G}$. So, if $f \in \mathcal{F}^{p,\Theta}(G)$, then

$$\int_{\widehat{G}} |\widehat{f}|^p|\Upsilon|^p\Psi\, d\mu_{\widehat{G}} \leq \int_{\widehat{G}} |\widehat{f}|^p\Theta\, d\mu_{\widehat{G}} = |||f|||_{p,\Theta}^p < \infty, \tag{5.3}$$

which shows that $\left(\Upsilon\widehat{f}\right)^\vee \in \mathcal{F}^{p,\Psi}(G)$. Thus, we can make the definition that

$$T(f) = \left(\Upsilon\widehat{f}\right)^\vee, \quad \text{for all } f \in \mathcal{F}^{p,\Theta}(G).$$

It then follows from (5.3) that the operator T is a linear map from $\mathcal{F}^{p,\Theta}(G)$ into $\mathcal{F}^{p,\Psi}(G)$ and it is clear from the definition of T that T is a multiplier operator whose multiplier is Υ.

To prove the surjectivity of T, let $g \in \mathcal{F}^{p,\Psi}(G)$. Let $B = \Upsilon^{-1}(\{0\})$. Since $\Psi(\gamma) = \infty$ for every $\gamma \in B$ and since $\widehat{g}|\Psi|^{1/p} \in L^p(\widehat{G})$ it follows that $\widehat{g}\chi_B$ is $\mu_{\widehat{G}}$-null, that is, $\widehat{g} = \widehat{g}\chi_{B^c}$ in $L^p(\widehat{G})$. Moreover, $(\widehat{g}\chi_{B^c})/\Upsilon$ belongs to $L^p(\widehat{G})$. Then the function $f = \left((\widehat{g}\chi_{B^c})/\Upsilon\right)^\vee$ belonging to $\mathcal{F}^p(G)$ satisfies

$$\int_{\widehat{G}} |\widehat{f}|^p\Theta\, d\mu_{\widehat{G}} = \int_{B^c} |\widehat{g}|^p\Psi\, d\mu_{\widehat{G}} = \int_{\widehat{G}} |\widehat{g}|^p\Psi\, d\mu_{\widehat{G}} < \infty.$$

So $f \in \mathcal{F}^{p,\Theta}(G)$ and $T(f)^\wedge = \widehat{g}\chi_{B^c} = \widehat{g}$, which implies that T is surjective. This proves (2).

Now when Υ is assumed to vanish only on a set of measure zero, it follows from above that $|\Upsilon|^p\Psi = \Theta$ holds as an equation in $L^p(\widehat{G})$. In this case the first inequality (5.3) becomes an equality and then T will be an isometry from $\mathcal{F}^{p,\Theta}(G)$ onto $\mathcal{F}^{p,\Psi}(G)$, which again is the space $\mathcal{D}_{s_1,\dots,s_q,\psi_1,\dots,\psi_q}(\mathcal{F}^p(G))$. □

Now if Υ is given as in (5.1), the simplest factorization for Υ is given by taking $q = 1$, $\psi_1 = \Upsilon$ and $s = 1$. Then from (5.2) we have

$$\Theta = 1 + |\Upsilon|^p \quad \text{and} \quad \Psi = 1 + |\Upsilon|^{-p}.$$

With this factorization we associate the operator T whose domain is

$$\left\{f : f \in \mathcal{F}^p(G) \text{ and } \int_{\widehat{G}} |\widehat{f}|^p \left(1 + |\Upsilon|^p\right) d\mu_{\widehat{G}} < \infty\right\},$$

whose range $\mathcal{R}(T)$ is

$$\left\{f : f \in \mathcal{F}^p(G) \text{ and } \int_{\widehat{G}} |\widehat{f}|^p \left(1 + |\Upsilon|^{-p}\right) d\mu_{\widehat{G}} < \infty\right\},$$

and which has the property that for all f in the domain of T,

$$T(f)^\wedge = \Upsilon\widehat{f}. \tag{5.4}$$

Let us consider the special case when $p = 2$, in which case $\mathcal{F}^2(G) = L^2(G)$.

5.2 Theorem. *Let $p = 2$. Let T be the operator as described by (5.4), and if $a \in \mathbb{R}$ let ν_a be the pseudomeasure such that*

$$\widehat{\nu}_a = \exp(-ia|\Upsilon|).$$

Then

(1) $\nu_a * \nu_b = \nu_{a+b}$, *for all* $a, b \in \mathbb{R}$, *and*

(2) *a function f in $L^2(G)$ is in the range $\mathcal{R}(T)$ of T if and only if there are numbers $a_1, a_2, a_3 \in \mathbb{R}$ and functions $f_1, f_2, f_3 \in L^2(G)$ such that*

$$f = \sum_{j=1}^{3} \left(f_j - \nu_{a_j} * f_j\right). \tag{5.5}$$

Proof. Let $\phi = |\Upsilon|$. Let $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ be the function defined by $\alpha(x) = 1 - e^{-ix}$ for every $x \in \mathbb{R}$; see Example 4.5. Then

$$\widehat{\mu}_{\alpha,\phi,a}(\gamma) = \exp\left(-ia\phi(\gamma)\right) = \exp(-ia|\Upsilon(\gamma)|), \quad \text{for all } \gamma \in \widehat{G},$$

so that $\mu_{\alpha,\phi,a} = \nu_a$ for every $a \in \mathbb{R}$. Clearly (1) holds.

Let $m = 3$; that is, m is the smallest integer satisfying $m > ps = 2$. Then (2) follows from Theorems 4.3 and 5.1(2). □

Note that if $p \neq 2$ in Theorem 5.2 then each $f \in \mathcal{R}(T)$ would then be expressed as the sum of m terms, provided m is the smallest integer such that $m > p$.

In Theorem 5.2 we have considered the case when $q = 1$ and $s = 1$, and each function in the range of T is then expressed as the sum of *three* terms with pseudomeasures as in (5.5). If $q > 1$, then we may have more terms, as illustrated in the following example.

5.3 Example. Let $p = 2$. Assume that $\Upsilon = \psi_1\psi_2$ for some $\mathbb{C}$-valued, Borel measurable functions ψ_1 and ψ_2 on $\widehat{G}$ satisfying $|\psi_1| = |\psi_2|$; that is, $q = 2$ and $s_1 = s_2 = 1$. Let $m_1 = m_2 = 3$ so that $J_1 = \{1, 2, 3\}$ and $J_2 = \{4, 5, 6\}$. Define Θ and Ψ as in (5.2), and let $T : \mathcal{F}^{2,\Theta}(G) \longrightarrow \mathcal{F}^{2,\Psi}(G)$ be the multiplier operator with multiplier Υ. By Theorems 5.1 the operator T is surjective and $\mathcal{F}^{2,\Psi}(G) = \mathcal{D}_{s_1,s_2,\psi_1,\psi_2}(G)$. Take a Borel measurable function $\phi : \widehat{G} \longrightarrow \mathbb{R}$ such that $|\phi| = |\psi_1|$. Let $\alpha(x) = 1 - e^{-ix}$ for every $x \in \mathbb{R}$, so that $\widehat{\mu}_{\alpha,\phi,a} = \exp(-ia\phi)$ for every $a \in \mathbb{R}$. Define $\nu_a = \mu_{\alpha,\phi,a}$ for every $a \in \mathbb{R}$, which gives $\nu_a * \nu_b = \nu_{a+b}$ for all $a, b \in \mathbb{R}$. By Theorems 4.3 and 5.1, a function $f \in \mathcal{F}^2(G) = L^2(G)$ belongs to the range of T if and only if there exist a vector $(a_1, a_2, \ldots, a_6) \in \mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ and functions $f_{j,k} \in L^2(G)$ such that

$$f = \sum_{j=1}^{3}\sum_{k=4}^{6}(7_e - \nu_{a_j}) * (7_e - \nu_{a_k}) * f_{j,k} = \sum_{j=1}^{3}\sum_{k=4}^{6}(f_{j,k} - \nu_{a_j} * f_{j,k} - \nu_{a_k} * f_{j,k} + \nu_{a_j+a_k} * f_{j,k}). \qquad \square$$

5.4 Example. Let $p = 2$. Let $G = \mathbb{R}$, so that $\widehat{G} = \mathbb{R}$. Let $\Upsilon(\gamma) = \gamma^2$ for every $\gamma \in \mathbb{R}$. Theorem 5.2 and Example 5.3 give different descriptions of elements of the range of the multiplier operator T with multiplier Υ.

(i) Let us consider the case when $q = 1$ as in Theorem 5.2. Define $\Theta_1 = 1 + |\Upsilon|^2$ and $\Psi_1 = 1 + |\Upsilon|^{-2}$. Let $T : \mathcal{F}^{2,\Theta_1}(\mathbb{R}) \longrightarrow \mathcal{F}^{2,\Psi_1}(\mathbb{R})$ be the multiplier operator with multiplier Υ. Given $a \in \mathbb{R}$, let ν_a be the pseudomeasure such that $\widehat{\nu}_a(\gamma) = \exp(-ia\gamma^2)$ for every $\gamma \in \mathbb{R}$. When $a \neq 0$, the function $\widehat{\nu}_a$ is not uniformly continuous and hence it is not the Fourier transform of any measure on $\mathbb{R}$. So, the expression (5.5) of each $f \in \mathcal{R}(T)$ involves genuine pseudomeasures only, in a sum of three terms.

(ii) Let us consider the factorization $\Upsilon(\gamma) = \gamma \cdot \gamma$ for every $\gamma \in \mathbb{R}$, in which case $q = 2$ and $s_1 = s_2 = 1$. This is a special case of Example 5.3. Let

$$\Theta_2(\gamma) = 1 + 2|\gamma|^2 + |\gamma|^4 \quad \text{and} \quad \Psi_2(\gamma) = 1 + 2|\gamma|^{-2} + |\gamma|^{-4}, \quad \text{for every } \gamma \in \mathbb{R}.$$

Then $\mathcal{F}^{2,\Theta_2}(\mathbb{R}) = \mathcal{F}^{2,\Theta_1}(\mathbb{R})$ and $\mathcal{F}^{2,\Psi_2}(\mathbb{R}) = \mathcal{F}^{2,\Psi_1}(\mathbb{R})$. In other words, the multiplier operator with multiplier Υ in this setting is the same as T given in (i).

Our aim is to give a description of each function in $\mathcal{R}(T)$ by using measures. Let $\alpha(x) = 1 - e^{-ix}$ for every $x \in \mathbb{R}$ and $\phi(\gamma) = \gamma$ for every $\gamma \in \mathbb{R}$. Then

$$\widehat{\mu}_{\alpha,\phi,a}(\gamma) = \exp(-ia\phi(\gamma)) = \widehat{\delta}_a(\gamma), \quad \text{for all } a \in \mathbb{R} \text{ and } \gamma \in \mathbb{R}.$$

By Example 5.3, a function $f \in L^2(\mathbb{R})$ belongs to $\mathcal{R}(T)$ if and only if there exist numbers $a_1, a_2, \ldots, a_6 \in \mathbb{R}$ and functions $f_{j,k} \in L^2(\mathbb{R})$ $(j = 1, 2, 3,\ k = 4, 5, 6)$ such that

$$f = \sum_{j=1}^{3} \sum_{k=4}^{6} (f_{j,k} - \delta_{a_j} * f_{j,k} - \delta_{a_k} * f_{j,k} + \delta_{a_j + a_k} * f_{j,k}).$$

So we have obtained a description of every $f \in \mathcal{R}(T)$ by using (Dirac) measures, but where f is the sum of *nine* terms. □

Finally observe that a vector f is in the range of the operator T considered in Example 5.4 if and only if f is the second derivative of a function in $L^2(\mathbb{R})$. This observation has been adapted in [10, p.74] to deduce that such a function f belongs to $\mathcal{R}(T)$ if and only if

$$f = \sum_{k=1}^{5} \left(f_k - 2^{-1}(\delta_{a_k} + \delta_{-a_k}) * f_k \right)$$

for some numbers $a_1, a_2, \ldots, a_5 \in \mathbb{R}$ and some functions $f_1, f_2, \ldots, f_5 \in L^2(\mathbb{R})$; that is, f is the sum of *five* second order differences. We can obtain this result also by applying Theorems 4.3 and 5.1. In fact, consider the factorization $\Upsilon(g) = (g)^2$ for every $x \in \mathbb{R}$, so that $q = 1$ and $s = 2$. The use of the function α given by $\alpha(x) = (1 - e^{-ix})^2$ for every $x \in \mathbb{R}$ (see Example 4.5) and the identity function ϕ on $\mathbb{R}$ yields the result.

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