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Abstract

We consider closed immersed surfaces in R^3 evolving by the geometric triharmonic heat flow. Using local energy estimates, we prove interior estimates and a positive absolute lower bound on the lifespan of solutions depending solely on the local concentration of curvature of the initial immersion in L^2. We further use an {\epsilon}-regularity type result to prove a gap lemma for stationary solutions. Using a monotonicity argument, we then prove that a blowup of the flow approaching a singular time is asymptotic to a non-umbilic embedded stationary surface. This allows us to conclude that any solution with initial L^2-norm of the tracefree curvature tensor smaller than an absolute positive constant converges exponentially fast to a round sphere with radius equal to the cube root of 3V_0/4{\pi}, where V_0 denotes the signed enclosed volume of the initial data.

Disciplines

Engineering | Science and Technology Studies

Publication Details

McCoy, J., Parkins, S. & Wheeler, G. (2017). The geometric triharmonic heat flow of immersed surfaces near spheres. Nonlinear Analysis, 161 44-86.

This journal article is available at Research Online: https://ro.uow.edu.au/eispapers1/369 arXiv:1501.07651v1 [math.AP] 30 Jan 2015

THE GEOMETRIC TRIHARMONIC HEAT FLOW OF IMMERSED SURFACES NEAR SPHERES

JAMES MCCOY, SCOTT PARKINS, AND GLEN WHEELER

Abstract. We consider closed immersed surfaces in $\mathbb{R}^3$ evolving by the geometric triharmonic heat flow. Using local energy estimates, we prove interior estimates and a positive absolute lower bound on the lifespan of solutions depending solely on the local concentration of curvature of the initial immersion in L^2. We further use an ε-regularity type result to prove a gap lemma for stationary solutions. Using a monotonicity argument, we then prove that a blowup of the flow approaching a singular time is asymptotic to a non-umbilic embedded stationary surface. This allows us to conclude that any solution with initial L^2-norm of the tracefree curvature tensor smaller than an absolute positive constant converges exponentially fast to a round sphere with radius equal to $\sqrt[3]{3V_0/4\pi}$, where V_0 denotes the signed enclosed volume of the initial data.

1. Introduction

Let $f : \Sigma \times [0,T) \to \mathbb{R}^3$, $T > 0$, be a one-parameter family of compact immersed surfaces $f(\cdot,t) = f_t : \Sigma \to f_t(\Sigma) = \Sigma_t$. We say f is evolving under the geometric triharmonic heat flow if it satisfies the following equation

$$\text{(GTHF)} \qquad \frac{\partial}{\partial t} f = -(\Delta^2 H)\nu\,.$$

with given smooth initial surface $f(\cdot,0) = f_0$. Above Δ and H are respectively the Laplace-Beltrami operator and mean curvature of the surface Σ_t, and ν is the outer unit normal. Further notation and conventions are set out in Section 2.

The mean curvature vector $\vec{H}$ satisfies $\Delta f = \vec{H} = -H\nu$. As $\vec{H}$ is a section of the normal bundle, it is natural to apply to this the induced Laplacian in the normal bundle $\Delta^\perp$. In this case one recovers $(\Delta^2 H)\nu = \Delta^\perp\Delta^\perp\Delta f$. This is natural from a geometric perspective as well as from the perspective of curvature flow: only normal terms in the speed of the flow affect geometric quantities which are invariant under the diffeomorphism group, and so any natural geometric operator should have image in the normal bundle. One may however wish to interpret $\vec{H}$ as a vector in $\mathbb{R}^3$ and instead apply the rough Laplacian Δ_R to it. In this case one recovers a lengthy expression for $\Delta_R\Delta_R\Delta f$ which has leading order term $(\Delta^2 H)\nu$ with many other terms contributing in tangential and normal directions. We have termed (GTHF) the geometric triharmonic heat flow in order to distinguish it from the possible second interpretation.

As (GTHF) is a sixth-order quasilinear degenerate parabolic system of partial differential equations, the local existence of a solution is standard (see, eg. [14, 15]) for regular enough initial data.

Theorem 1. *Suppose Σ is a complete, compact 2-manifold without boundary. Let $f_0 : \Sigma \to \mathbb{R}^3$ be a $C^{6,\alpha}$ isometric immersion. There exists a maximal $T > 0$ and a corresponding unique one-parameter family of smooth isometric immersions $f : \Sigma \times (0,T) \to \mathbb{R}^3$ satisfying* (GTHF) *and*

$$f(\cdot,t) \to f_0(\cdot) \quad \text{as} \quad t \searrow 0\,,$$

locally smoothly in the $C^{6,\alpha}$ topology.

In this paper we will always assume that the initial data f_0 is smooth, and so then the flow generated by f_0 is $f : \Sigma \times [0,T) \to \mathbb{R}^3$, which is a smooth family on the half-open interval $[0,T)$.

Date: ****Received 6 December 2013; received in final form *******.
1991 *Mathematics Subject Classification.* 53C44.
The research of the second author was supported by an Australian Postgraduate Award. The research of the first and third authors was supported by Discovery Project DP120100097 of the Australian Research Council. Results from this paper will appear in the PhD thesis of the second author.

Applications of equations involving a thrice-iterated Laplacian are growing, and include for example the modelling of ulcers [25], computer graphics [1, 24], and interactive design [8]. In each of these applications it is important that the Laplacian considered possesses some inherent geometric invariances, with the natural candidate being the Laplace-Beltrami operator. We hope in future work to generalise equation (GTHF) to include lower order terms; such sixth order equations appear as the phase field crystal equation proposed as a model for microstructure evolution of two-phase systems on atomic length and diffusive time scales (see [4] and the references contained therein) and in surface modelling [12, 23].

In Section 6 we study properly immersed surfaces which are not necessarily closed. Our main result is the following tracefree curvature estimate, which holds quite generally. In particular, the immersion does not need to satisfy $\Delta^2 H = 0$.

Theorem 2. *Let $x \in \mathbb{R}^3$, $\rho > 0$, and $f : \Sigma \to \mathbb{R}^3$ be a locally $W^{5,2}$ immersion satisfying*

$$\int_{f^{-1}(B_{2\rho}(x))} |A^o|^2 \, d\mu \le \varepsilon_0 \tag{1}$$

for $\varepsilon_0 > 0$ sufficiently small. Then there is a universal constant $c > 0$ such that

$$\|A^o\|^6_{\infty, f^{-1}(B_\rho(x))} \le c \|A^o\|^4_{2, f^{-1}(B_{2\rho}(x))} \left(\|\nabla \Delta H\|^2_{2, f^{-1}(B_{2\rho}(x))} + \rho^{-6} \right).$$

This has as an immediate corollary the following higher-order analogue of ε-regularity (see [9, 20, 26] for the corresponding result for the Willmore operator, minimal surfaces, and the surface diffusion operator respectively).

Corollary 3. *Let $f : \Sigma \to \mathbb{R}^3$ be a locally $W^{5,2}$ closed immersion satisfying $\Delta^2 H = 0$ weakly. Suppose that for $x \in \mathbb{R}^3$, $\rho > 0$, the local smallness condition* (1) *is satisfied for $\varepsilon_0 > 0$ sufficiently small. Then there is an absolute constant $c > 0$ such that*

$$\|A^o\|^6_{\infty, f^{-1}(B_\rho(x))} \le c \, \varepsilon^2 \rho^{-6}.$$

This estimate is a key first step in a larger regularity program for weak solutions, improving the a-priori regularity assumption dramatically under the natural assumption that the L^2-norm of the tracefree curvature tensor A^o is globally bounded.

If the L^2-norm of A^o is in fact globally small, then we may apply again Theorem 2 to obtain the following much stronger result than Corollary 3 above. It is proven in Section 8.

Theorem 4 (Gap Lemma)**.** *Suppose $f : \Sigma \to \mathbb{R}^3$ is a proper locally C^6 immersion with $\Delta^2 H \equiv 0$. Then, if f is closed and satisfies*

$$\int_\Sigma |A^o|^2 d\mu < \varepsilon_0 < 8\pi \tag{2}$$

for $\varepsilon_0 > 0$ sufficiently small, there exist $x \in \mathbb{R}^3$ and $\rho > 0$ such that

$$f(\Sigma) = \mathbb{S}^2_\rho(x),$$

where $\mathbb{S}^2_\rho(x)$ denotes a standard round sphere in $\mathbb{R}^3$ with radius ρ and centre $x \in \mathbb{R}^3$. If f is not closed, then under assumption (2) *we instead conclude that*

$$f(\Sigma) = \mathbb{P}^2,$$

where $\mathbb{P}^2$ denotes a standard flat plane in $\mathbb{R}^3$.

Gap Lemmata are known in a variety of contexts: for the Willmore operator [9], the surface diffusion operator [26], and a family of fourth-order geometric operators [27]. We expect Theorem 2 to enjoy further applications.

Using $\mathrm{Vol}(\Sigma_t)$ and $|\Sigma_t|$ to denote the signed enclosed volume and surface area respectively of Σ_t in $\mathbb{R}^3$, we compute that under the flow (GTHF)

$$\frac{d}{dt}\mathrm{Vol}(\Sigma_t) = -\int_\Sigma \Delta^2 H d\mu = 0 \tag{3}$$

and

$$\frac{d}{dt}|\Sigma_t| = -\int_\Sigma H \Delta^2 H d\mu = -\int_\Sigma |\Delta H|^2 d\mu \le 0. \tag{4}$$

From (3), (4), the flow (GTHF) is isoperimetrically natural. The flow may develop singularities while improving the isoperimetric ratio, manifested as curvature singularities. Our first main parabolic result is the following characterisation of the singular time.

Theorem 5 (Lifespan Theorem)**.** *Suppose $f : \Sigma \times [0,T) \longrightarrow \mathbb{R}^n$ satisfies (GTHF) with smooth initial data f_0. Then there are constants $\rho > 0, \varepsilon_0 > 0$ and $C < \infty$ such that if ρ is chosen with*

$$\int_{f^{-1}(B_\rho(x))} |A|^2 \, d\mu \Big|_{t=0} = \varepsilon(x) \le \varepsilon_0 \text{ for any } x \in \mathbb{R}^3, \tag{5}$$

then the maximal time T of smooth existence for the flow satisfies

$$T \ge \frac{1}{C}\rho^6, \tag{6}$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^2 \, d\mu \le C\varepsilon_0 \text{ for } 0 \le t \le \frac{1}{C}\rho^6. \tag{7}$$

By linearising (GTHF) around a round sphere, one finds that the spectrum is entirely real and non-positive. Using standard centre-manifold methods, one can conclude that any solution with initial data close to a sphere S_1 in $C^{6,\alpha}$ exists for all time and converges exponentially fast to a sphere S_2 in the C^∞ topology. Note that S_1 need not equal S_2 in any sense: the perturbation could be a small translation, in which case the centre changes, or it may change the enclosed volume of the initial data from that of S_1, in which case the radius changes. Using completely different methods, we are able to improve this statement, weakening the initial condition to geometric averaged closeness in L^2.

Let us briefly detail the argument. Note that if the initial immersion is locally smooth with finite total curvature then for any $\varepsilon_0 > 0$ it is always possible to find a positive $\rho = \rho(\varepsilon_0, \Sigma_0)$ such that assumption (5) is satisfied. Theorem 5 is a characterisation of singularities in the following sense. It tells us that the only way the flow can cease to exist and lose regularity in finite time $T < \infty$ is if we encounter a specific type of curvature singularity: if $\rho(t)$ denotes the largest radius such that (5) holds at time t, then $\rho(t) \le \sqrt[6]{C(T-t)}$ so at least ε_0 of the curvature concentrates in a ball $f^{-1}\left(B_{\rho(T)}(x)\right)$. That is,

$$\lim_{t \to T} \int_{f^{-1}\left(B_{\rho(t)}(x)\right)} |A|^2 \, d\mu \ge \varepsilon_0$$

where $x = x(t)$ is the centre of a ball where the integral is maximised at time t.

By considering a sequence of rescalings, we are able to extract a convergent subsequence asymptotic to a stationary solution of (GTHF). This relies on the monotonicity of the L^2-norm of the tracefree curvature tensor. The stationary solution extracted is entire, with small total tracefree curvature in L^2, and so by applying the Gap Lemma we are able to show that this is a contradiction. Subconvergence of the flow to a specific sphere results, which we then strengthen by a standard linearised stability argument. Our final result is the following.

Theorem 6 (Long time existence and exponential convergence to round spheres)**.** *There exists an absolute constant $\varepsilon_0 > 0$ such that a geometric triharmonic heat flow $f : \Sigma \times [0,T) \to \mathbb{R}^3$ with smooth initial data satisfying*

$$\int_\Sigma |A^o|^2 d\mu \Big|_{t=0} \le \varepsilon_0 < 8\pi \tag{8}$$

exists for all time, is a family of embeddings, and for some $x \in \mathbb{R}^3$ the family Σ_t converges to $\mathbb{S}^2_{\sqrt[3]{3V_0/4\pi}}(x)$ exponentially fast in the C^∞ topology, where $V_0 > 0$ is the signed enclosed volume of the initial immersion.

The paper is organised as follows. After fixing notation in Section 2 and stating evolution equations and interpolation inequalities in Section 3, we establish local control on the L^2-norm of iterated covariant derivatives of the curvature in Section 4. These provide us with all the requisite tools to prove the Theorem 5, which we do in Section 5. Section 6 is devoted to elliptic analysis, and culminates in the proof of Theorem 2. Section 7 establishes that $\|A^o\|_2^2$ is a Lyapunov functional for the flow if initially small enough. Section 8 contains the proof of the Gap Lemma. In Section 9 we use a compactness theorem and most of our earlier work to establish the existence and several key properties of the blowup of a singular time for the flow. This is applied in Section 10 to conclude global existence and subconvergence to a sphere, which is then strengthened by a standard linearised stability argument.

2. Preliminaries

We consider a surface Σ immersed into $\mathbb{R}^3$ via a smooth immersion $f : \Sigma \to \mathbb{R}^3$. The induced metric g on Σ is given by pulling back the standard Euclidean inner product on $\mathbb{R}^3$ along f. It is defined pointwise by

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle \,. \tag{9}$$

Here ∂ denotes the coordinate derivatives on Σ and $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product. This induces an inner product on all tensors along f of similar type, which are defined via traces over pairs of indices. For example, the inner product on the $(1, 2)$-tensors S, T is defined by

$$\langle S, T \rangle = g^{jm} g^{kn} g_{il} S^i_{jk} T^l_{mn}.$$

Here we adopt the Einstein convention: repeated indices are summed from 1 to 2. The norm squared of a tensor T is the inner product of T with itself and is denoted $|T|^2$. For tensors S and T we will also frequently utilise the notation of Hamilton [6], using $S \star T$ to denote an arbitrary linear combination of new tensors that is formed by contracting pairs of indices of S and T by g. For example, $\langle S, T \rangle = S \star T$. A useful property of $\star-$notation is that

$$S \star T \le c\, |S|\, |T|$$

for some constant c that depends only on the number of indices in S and T combined (recall the dimension and codimension are fixed). For a tensor T, we define

$$P^i_j (T) = \sum_{r_1 + \dots + r_j = i} \nabla_{(r_1)} T \star \dots \star \nabla_{(r_j)} T,$$

a sum of terms, each of which contains j factors of T with total order of covariant derivatives equal to i. This notation is particularly useful when only the number of factors of T and total number of derivatives is important. As an example

$$\left|\nabla_{(2)} T\right|^2 = \nabla_{(2)} T \star \nabla_{(2)} T = P^4_2 (T) \,.$$

The mean curvature is defined to be

$$H = g^{ij} A_{ij} = A^i_i$$

where A_{ij} are the components of the second fundamental form A:

$$A_{ij} = - \langle \partial_{ij} f, \nu \rangle \,.$$

The Gauss curvature K is the determinant of the Weingarten map $A^j_i := g^{ik} A_{jk}$,

$$K = \det\, A^j_i \,.$$

We also define the tracefree second fundamental form to be the symmetric tracefree part of A, denoted A^o, with components

$$A^o_{ij} = A_{ij} - \frac{1}{2} g_{ij} H.$$

A short calculation shows that

$$(\nabla^* A^o)_j := \nabla^i A^o_{ij} = g^{ip} \nabla_p \left(A_{ij} - \frac{1}{2} g_{ij} H \right) = \frac{1}{2} \nabla_j H; \tag{10}$$

note that $-\nabla^*$ is the geometric divergence operator or formal adjoint of ∇ in L^2. Here we have used the total symmetry of the $(0, 3)$-tensor ∇A, known as the Codazzi equations:

$$\nabla_i A_{jk} = \nabla_j A_{ki} = \nabla_k A_{ij}.$$

It follows that

$$|\nabla H|^2 = 4\, |\nabla^* A^o|^2 \le 4\, |\nabla A^o|^2 \,.$$

Moreover, for any $k \in \mathbb{N}$ we have

$$\left|\nabla_{(k)} H\right|^2 \le 4 \left|\nabla_{(k)} A^o\right|^2 \tag{11}$$

and hence

$$\left|\nabla_{(k)} A\right|^2 = \left|\nabla_{(k)} A^o\right|^2 + \frac{1}{2} \left|\nabla_{(k)} H\right|^2 \le 3 \left|\nabla_{(k)} A^o\right|^2 .$$

The Laplace-Beltrami operator acts on the components of an (m, n)-tensor S via

$$\Delta S^{i_1\dots i_m}_{j_1\dots j_n} = g^{pq}\nabla_{pq} S^{i_1\dots i_m}_{j_1\dots j_n}.$$

We define the integral of a compactly supported function $h : \Sigma \to \mathbb{R}$ as

$$\int_\Sigma h\, d\mu,$$

where $d\mu$ is the induced measure on Σ:

$$d\mu := \sqrt{\det(g_{ij})}\, d\mathcal{L}^2.$$

We denote the area by

$$\mu(\Sigma) = |\Sigma| = \int_\Sigma d\mu\,.$$

We will frequently employ the Divergence Theorem to 'integrate by parts' over a non-compact immersion. To do so we include a cut-off function $\gamma : \Sigma \to \mathbb{R}^3$ in the integrand. We keep this function arbitrary but sufficiently smooth with compact support so that we may eventually conclude global results from results that hold on the support of γ. In particular, we take $\gamma = \tilde{\gamma} \circ f : \Sigma \to [0, 1]$ for some C^3 function $\tilde{\gamma}$ satisfying

$$0 \le \tilde{\gamma} \le 1 \text{ and } \|\tilde{\gamma}\|_{C^3(\mathbb{R}^3)} \le c_{\tilde{\gamma}} < \infty.$$

Using the chain rule and the fact that $\{\partial_i f\}$ provides a basis for the tangent bundle $T\Sigma$, there is a universal, bounded constant $c_\gamma > 0$ such that γ satisfies

$$\|\nabla\gamma\|_\infty \le c_\gamma, \left\|\nabla_{(2)}\gamma\right\|_\infty \le c_\gamma\left(c_\gamma + |A|\right), \text{ and } \left\|\nabla_{(3)}\gamma\right\|_\infty \le c_\gamma\left(c_\gamma^2 + c_\gamma|A| + |\nabla A|\right). \tag{12}$$

3. Evolution equations for elementary geometric quantities

Under (GTHF), we have the following evolution equations for geometric quantities associated to Σ_t. The proof of Lemma 7 is standard and straightforward.

Lemma 7. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF). *Then*

$$\frac{\partial}{\partial t} g = -2\left(\Delta^2 H\right) A, \quad \frac{\partial}{\partial t} d\mu = -H\left(\Delta^2 H\right) d\mu, \quad \frac{\partial}{\partial t}\nu = \nabla\Delta^2 H,$$
$$\frac{\partial}{\partial t} A = \Delta^3 A + P_3^4(A)\,, \text{ and } \frac{\partial}{\partial t} H = \Delta^3 H + (\Delta^2 H)|A|^2\,.$$

By combining the evolution equations in Lemma 7 with the interchange of covariant derivatives formula, we obtain the following evolution equations.

Lemma 8. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF). *Then for any $k \in \mathbb{N}_0$:*

$$\frac{\partial}{\partial t}\nabla_{(k)} A = \Delta^3\nabla_{(k)} A + P_3^{k+4}(A)\,.$$

Corollary 9. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF). *Then for any $k \in \mathbb{N}_0$:*

$$\frac{\partial}{\partial t}\left|\nabla_{(k)} A\right|^2 = 2\left\langle\nabla_{(k)} A, \nabla^{i_3 i_2 i_1}\nabla_{i_1 i_2 i_3}\nabla_{(k)} A\right\rangle + P_3^{k+4}(A) \star \nabla_{(k)} A.$$

Using Corollary 9, Lemma 7, the product rule, and integrating by parts thrice we obtain the following.

Corollary 10. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF). *Then for any $k, s \in \mathbb{N}_0$:*

$$\begin{aligned}\frac{d}{dt}\int_\Sigma \left|\nabla_{(k)} A\right|^2\gamma^s d\mu + 2\int_\Sigma\left|\nabla_{(k+3)} A\right|^2 d\mu = \int_\Sigma \left|\nabla_{(k)} A\right|^2 \partial_t\gamma^s d\mu \\ - 2\sum_{j=1}^{3}\binom{3}{j}\int_\Sigma\left\langle\left(\nabla_{(j)}\gamma^s\right)\left(\nabla_{(k+3-j)} A\right), \nabla_{(k+3)} A\right\rangle d\mu + \int_\Sigma\left(P_3^{k+4}(A) \star \nabla_{(k)} A\right)\gamma^s d\mu.\end{aligned}$$

To deal with the extraneous intermediate terms above we will make use of (12) and an interpolation inequality of Kuwert and Schätzle [9, Corollary 5.3]. This gives us:

Lemma 11. *Let $2 \le p < \infty, k, m \in \mathbb{N}$ and $s \ge kp$. Then for $\delta > 0$ there exists a constant $c > 0$ depending only on δ, s and p such that*

$$\left(\int_\Sigma \left|\nabla_{(k)} A\right|^p \gamma^s d\mu\right)^{\frac{1}{p}} \le \delta\, c_\gamma^{-1} \left(\int_\Sigma \left|\nabla_{(k+1)} A\right|^p \gamma^{s+p} d\mu\right)^{\frac{1}{p}} + c_\delta\, c_\gamma^k \left(\int_{[\gamma>0]} |A|^2 \gamma^{s-kp}\, d\mu\right)^{\frac{1}{p}}.$$

The proof of Lemma 11 is essentially the same as in [9]. However, we have retained the derivative bounds c_γ to make more explicit the scale-invariance of each quantity. Using Lemma 11, the properties (12) and Corollary 10 we obtain the following estimate.

Proposition 12. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF)*. For any $\delta > 0$, $k \in \mathbb{N}_0$, and $s \ge 2(k+3)$ there exists a constant $c > 0$ depending only on δ, s and k such that*

$$\frac{d}{dt}\int_\Sigma \left|\nabla_{(k)} A\right|^2 \gamma^s d\mu + (2-\delta)\int_\Sigma \left|\nabla_{(k+3)} A\right|^2 \gamma^s d\mu \le \int_\Sigma \left(P_3^{k+4}(A) \star \nabla_{(k)} A\right) \gamma^s d\mu + c\, c_\gamma^{2(k+3)} \left\|A\right\|_{2,[\gamma>0]}^2.$$

In order to control the localised norms $\|\nabla_{(k)} A\|_{2,\gamma^s}^2 := \int_\Sigma |\nabla_{(k)} A|^2 \gamma^s d\mu$ we will employ standard interpolation and Sobolev inequalities, which we now state.

Theorem 13 (Michael-Simon Sobolev Inequality [19])**.** *Let Σ be an immersed surface and $u \in C_c^\infty(\Sigma)$. Then, for a universal, bounded constant $c_{MSS} > 0$,*

$$\int_\Sigma u^2 d\mu \le c_{MSS} \left(\int_\Sigma |\nabla u| d\mu + \int_\Sigma |u|\,|H| d\mu\right)^2$$

Theorem 14 ([10, Theorem 5.6])**.** *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion. For $n < p \le \infty$, $0 \le \beta \le \infty$ and $0 < \alpha \le 1$, where $\frac{1}{\alpha} = \left(\frac{1}{2} - \frac{1}{p}\right)\beta + 1$, there is a constant c depending on p and β such that for all $u \in C_c^1(\Sigma)$,*

$$\|u\|_\infty \le c\, \|u\|_p^{1-\alpha} \left(\|\nabla u\|_p + \|Hu\|_p\right)^\alpha. \tag{13}$$

Proposition 15 ([10, Corollary 5.5])**.** *Let $0 \le i_1, \dots, i_r \le k$, $\sum_{j=1}^r i_j = 2k$ and $s \ge 2k$. Then for any tensor T defined over an immersed surface $f : \Sigma \to \mathbb{R}^3$ we have*

$$\left|\int_\Sigma \nabla_{i_1} T \star \cdots \star \nabla_{i_r} T \gamma^s d\mu\right| \le c\, \|T\|_{\infty,[\gamma>0]}^{r-2} \left(\int_\Sigma \left|\nabla_{(k)} T\right|^2 \gamma^s d\mu + c_\gamma^{2k} \|T\|_{2,[\gamma>0]}^2\right).$$

Note that we have included c_γ explicitly on the right hand side above. This will be important later when we select a particular cutoff function. The next estimate below is an adaptation of [10, Lemma 4.3] to our situation. The proof is similar.

Proposition 16. *Suppose T is a tensor field, $s \ge 6$ and γ is as in* (12)*. There is an $\varepsilon > 0$ such that $\|A\|_{2,[\gamma>0]}^2 \le \varepsilon_0$ implies that there exists a universal, bounded constant $c > 0$ depending only on the order of T such that*

$$\|T\|_{\infty,[\gamma=1]}^6 \le c\, \|T\|_{2,[\gamma>0]}^4 \left(\left\|\nabla_{(3)} T \cdot \gamma^{\frac{s}{2}}\right\|_2^2 + \left\|\left|\nabla T\right|^2 |A|^4 \cdot \gamma^s\right\|_1 \right.$$
$$\left. + \left\|\left|\nabla_{(2)} T\right| |A| \cdot \gamma^{\frac{s}{2}}\right\|_2^2 + \left\||T|\,|\nabla A|\,|A| \cdot \gamma^{\frac{s}{2}}\right\|_{2,}^2 + c_\gamma^6 \left\|T \cdot \gamma^{\frac{s-6}{2}}\right\|_{2,[\gamma>0]}^2\right).$$

The lemma below is similar to [10, Lemma 4.2] and [9, Proposition 2.6]. We have incorporated the smallness assumption and since the proof involves some new estimates, we present it.

Lemma 17. *Let γ satisfy* (12)*. Then for an immersion $f : \Sigma \to \mathbb{R}^3$ satisfying*

$$\int_{[\gamma>0]} |A|^2\, d\mu \le \varepsilon_0 \tag{14}$$

for $\varepsilon_0 > 0$ sufficiently small, we have

$$\int_\Sigma \left|\nabla_{(2)} A\right|^2 |A|^2 \gamma^s d\mu + \int_\Sigma |\nabla A|^2 |A|^4 \gamma^s d\mu \;\le\; c \int_{[\gamma>0]} |A|^2\, d\mu \Big(\int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^s d\mu + c_\gamma^6 \int_{[\gamma>0]} |A|^2\, d\mu\Big).$$

Proof. The second term can be estimated via Theorem 13 with $u = |\nabla A|\,|A|^2\,\gamma^{\frac{s}{2}}$:

$$\begin{aligned}
&\int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^s d\mu \\
&\leq c\Big(\int_\Sigma \left|\nabla_{(2)}A\right|\,|A|^2\,\gamma^{\frac{s}{2}}d\mu + \int_\Sigma |\nabla A|^2\,|A|\,\gamma^{\frac{s}{2}}d\mu + c_\gamma\int_\Sigma |\nabla A|\,|A|^2\,\gamma^{\frac{s}{2}-1}d\mu + \int_\Sigma |\nabla A|\,|A|^3\,\gamma^{\frac{s}{2}}d\mu\Big)^2 \\
&\leq c\,\|A\|^2_{2,[\gamma>0]}\Big(\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^s d\mu + \int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^s d\mu + \int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^s d\mu \\
&\qquad + \int_\Sigma |\nabla A|^4\,\gamma^6 d\mu + c_\gamma^6\,\|A\|^2_{2,[\gamma>0]}\Big).
\end{aligned} \tag{15}$$

Here we have used integration by parts, the Cauchy-Schwarz inequality and Lemma 11. For the second last term in (15) we use integration by parts and Lemma 11:

$$\begin{aligned}
&\int_\Sigma |\nabla A|^4\,\gamma^s d\mu \leq 3\int_\Sigma \left|\nabla_{(2)}A\right|\,|\nabla A|^2\,|A|\,\gamma^s d\mu + s\,c_\gamma\int_\Sigma |\nabla A|^3\,|A|\,\gamma^{s-1}d\mu \\
&\leq \frac{1}{2}\int_\Sigma |\nabla A|^4\,\gamma^s d\mu + c\Big(\int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^s d\mu + \int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^s d\mu\Big) + c\,c_\gamma^4\int_\Sigma |\nabla A|^2\,\gamma^{s-4}d\mu,
\end{aligned}$$

which implies that

$$\int_\Sigma |\nabla A|^4\,\gamma^s d\mu \;\leq\; c\Big(\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^s d\mu + \int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^s d\mu + \int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^s d\mu + c_\gamma^6\,\|A\|^2_{2,[\gamma>0]}\Big).$$

For the second term above, we use Theorem 13 with $u = \left|\nabla_{(2)}A\right|\,|A|\,\gamma^{\frac{s}{2}}$:

$$\begin{aligned}
\int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^s d\mu \leq c\int_{[\gamma>0]}|A|^2\,d\mu\Big(&\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^s d\mu + c_\gamma^2\int_\Sigma \left|\nabla_{(2)}A\right|^2\gamma^{s-2}d\mu \\
&+ \int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^s d\mu\Big) + c\int_\Sigma \left|\nabla_{(2)}A\right|^2\gamma^4 d\mu\int_\Sigma |\nabla A|^2\,\gamma^2 d\mu.
\end{aligned} \tag{16}$$

The last term in (16) can be dealt with via integration by parts, the Cauchy-Schwarz inequality and the interpolation inequality from Lemma 11:

$$\begin{aligned}
&\int_\Sigma \left|\nabla_{(2)}A\right|^2\gamma^4 d\mu\int_\Sigma |\nabla A|^2\,\gamma^2 d\mu \\
&\leq c\left(\int_\Sigma \left|\nabla_{(3)}A\right|\,|\nabla A|\,\gamma^4 d\mu + c_\gamma\int_\Sigma \left|\nabla_{(2)}A\right|\,|\nabla A|\,\gamma^3 d\mu\right)\left(\int_\Sigma \left|\nabla_{(2)}A\right|\,|A|\,\gamma^2 d\mu + c_\gamma\int_\Sigma |\nabla A|\,|A|\,\gamma d\mu\right) \\
&\leq \frac{1}{2}\int_\Sigma \left|\nabla_{(2)}A\right|^2\gamma^4 d\mu\int_\Sigma |\nabla A|^2\,\gamma^2 d\mu + c\,\|A\|^2_{2,[\gamma>0]}\left(\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^6 d\mu + c_\gamma^6\,\|A\|^2_{2,[\gamma>0]}\right).
\end{aligned}$$

Hence

$$\int_\Sigma \left|\nabla_{(2)}A\right|^2\gamma^4 d\mu\int_\Sigma |\nabla A|^2\,\gamma^2 d\mu \leq c\,\|A\|^2_{2,[\gamma>0]}\left(\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^6 d\mu + c_\gamma^6\,\|A\|^2_{2,[\gamma>0]}\right).$$

Substituting this result into (16) and combining with (15) then yields

$$\begin{aligned}
&\int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^6 d\mu + \int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^6 d\mu \\
&\leq c\,\|A\|^2_{2,[\gamma>0]}\Big(\int_\Sigma \left|\nabla_{(3)}A\right|^2\gamma^6 d\mu + \int_\Sigma \left|\nabla_{(2)}A\right|^2|A|^2\,\gamma^6 d\mu + \int_\Sigma |\nabla A|^2\,|A|^4\,\gamma^6 d\mu + c_\gamma^6\,\|A\|^2_{2,[\gamma>0]}\Big),
\end{aligned}$$

so that absorbing and multiplying out yields the statement of the lemma. ☐

4. A-priori estimates for $\|\nabla_{(k)}A\|^2_{2,\gamma^s}$ along the flow

The structure of this argument is similar to that used for [10, Proposition 4.6], although the details are necessarily different. We begin by applying the evolution equations and interpolation inequalities above to obtain local control on the concentration of curvature $\|A\|^2_{2,\gamma^s}$.

Proposition 18. *Let $f : \Sigma \times [0, T^*] \to \mathbb{R}^3$ satisfy* (GTHF)*, and let γ be a cutoff function satisfying* (12)*. Then there is an absolute constant $\varepsilon_0 > 0$ such that if*

$$\sup_{[0,T^*]} \int_{[\gamma>0]} |A|^2 \, d\mu \leq \varepsilon_0$$

then for any $t \in [0, T^*]$ we have

$$\int_{[\gamma=1]} |A|^2 \, d\mu + \int_0^t \int_{[\gamma=1]} \left(\left|\nabla_{(3)} A\right|^2 + \left|\nabla_{(2)} A\right|^2 |A|^2 + |\nabla A|^2 |A|^4 \right) d\mu \, d\tau \leq \int_{[\gamma>0]} |A|^2 \, d\mu\Big|_{t=0} + c\, c_\gamma^6 \, \varepsilon_0 \, t.$$

Proof. Utilising Proposition 12 with $s = 6, k = 0$, yields

$$\frac{d}{dt} \int_\Sigma |A|^2 \gamma^6 d\mu + (2 - \delta) \int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu \leq \int_\Sigma \left[P_3^4(A) \star A \right] \gamma^6 d\mu + c\, c_\gamma^6 \, \|A\|_{2,[\gamma>0]}^2 . \tag{17}$$

We estimate the P-style terms via integration by parts and the Cauchy-Schwarz inequality, whilst utilising our inequalities from Lemma 17:

$$\begin{aligned}
\int_\Sigma \left[P_3^4(A) \star A \right] \gamma^6 d\mu &= \int_\Sigma \left(\nabla_{(4)} A \star A \star A \star A \right) \gamma^6 d\mu \\
&\quad + \int_\Sigma \left[\left(\nabla_{(3)} A \star \nabla A \star A + \nabla_{(2)} A \star \nabla_{(2)} A \star A + \nabla_{(2)} A \star \nabla A \star \nabla A \right) \star A \right] \gamma^6 d\mu \\
&\leq \int_\Sigma \left(\nabla_{(3)} A \star \nabla A \star A \star A \right) \gamma^6 d\mu + \int_\Sigma \left(\nabla_{(3)} A \star A \star A \star A \star \nabla\gamma \right) \gamma^5 d\mu \\
&\quad + \delta \int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu + c_\delta \int_\Sigma \left(\left|\nabla_{(2)} A\right|^2 |A|^2 + |\nabla A|^2 |A|^4 + |\nabla A|^4 \right) \gamma^6 d\mu \\
&\leq \delta \int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu + c_\delta \int_\Sigma \left(\left|\nabla_{(2)} A\right|^2 |A|^2 + |\nabla A|^2 |A|^4 + |\nabla A|^4 \right) \gamma^6 d\mu \\
&\quad + c\, c_\gamma^2 \int_\Sigma |A|^6 \gamma^4 d\mu.
\end{aligned} \tag{18}$$

The last term in (18) can be estimated via Theorem 13, the Cauchy-Schwarz inequality and Lemma 17:

$$\begin{aligned}
c_\gamma^2 \int_\Sigma |A|^6 \gamma^4 d\mu &\leq c\, c_\gamma^2 \left(\int_\Sigma |\nabla A| \, |A| \, \gamma^2 d\mu + c_\gamma \int_\Sigma |A|^3 \gamma d\mu + \int_\Sigma |A|^4 \gamma^2 d\mu \right)^2 \\
&\leq c \, \|A\|_{2,[\gamma>0]}^2 \left(\int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu + \int_\Sigma |\nabla A|^2 |A|^4 \gamma^6 d\mu + c_\gamma^2 \int_\Sigma |A|^6 \gamma^4 d\mu + c_\gamma^6 \, \|A\|_{2,[\gamma>0]}^2 \right).
\end{aligned}$$

Hence for ε_0 sufficiently small, Lemma 17 tells us that

$$c_\gamma^2 \int_\Sigma |A|^6 \gamma^4 d\mu \leq c \int_{[\gamma>0]} |A|^2 \, d\mu \left(\int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu + c_\gamma^6 \int_{[\gamma>0]} |A|^2 \, d\mu \right).$$

Combining this with (18) and substituting into (17) then yields

$$\frac{d}{dt} \int_\Sigma |A|^2 \gamma^6 d\mu + \frac{3}{2} \int_\Sigma \left|\nabla_{(3)} A\right|^2 \gamma^6 d\mu \leq c \int_\Sigma \left(\left|\nabla_{(2)} A\right|^2 |A|^2 + |\nabla A|^2 |A|^4 \right) \gamma^6 d\mu + c\, c_\gamma^6 \int_{[\gamma>0]} |A|^2 \, d\mu.$$

Using Lemma 17 and allowing ε_0 to be sufficiently small we then obtain

$$\frac{d}{dt} \int_\Sigma |A|^2 \gamma^6 d\mu + \int_\Sigma \left(\left|\nabla_{(3)} A\right|^2 + \left|\nabla_{(2)} A\right|^2 |A|^2 + |\nabla A|^2 |A|^4 \right) \gamma^6 d\mu \leq c\, c_\gamma^6 \int_{[\gamma>0]} |A|^2 \, d\mu.$$

Integrating over $[0, t]$ and using $[\gamma = 1] \subset [\gamma > 0]$, $0 \leq \gamma \leq 1$ completes the proof. □

We now control the evolution of $\|\nabla_{(k)} A\|_{2,\gamma^s}^2$ for $k \geq 1$. Our goal is to eventually apply the interpolation inequalities to the resultant estimate in order to prove that $\|\nabla_{(k)} A\|_{2,\gamma^s}^2$ satisfies a differential inequality and is therefore bounded via Gronwall's inequality.

Proposition 19. *Let $f : \Sigma \times [0,T) \to \mathbb{R}^3$ satisfy* (GTHF) *and let γ be a cutoff function satisfying* (12). *Then for $s \geq 2k+6$ the following estimate holds:*

$$\frac{d}{dt}\int_\Sigma \left|\nabla_{(k)}A\right|^2 \gamma^s d\mu + \int_\Sigma \left|\nabla_{(k+3)}A\right|^2 \gamma^s d\mu$$
$$\leq c\left\|A\right\|^6_{\infty,[\gamma>0]}\int_\Sigma \left|\nabla_{(k)}A\right|^2 \gamma^s d\mu + c\, c_\gamma^{2k}\int_{[\gamma>0]}\left|A\right|^2\, d\mu\left(c_\gamma^6 + \left\|A\right\|^6_{\infty,[\gamma>0]}\right).$$

Proof. We use integration by parts, Proposition 15 and the Cauchy-Schwarz inequality to deal with the P-style terms that appear in Proposition 12:

$$\begin{aligned}
&\int_\Sigma \left(P_3^{k+4}(A)\star\nabla_{(k)}A\right)\gamma^s d\mu\\
&= \int_\Sigma \left[\left(\nabla_{(k+4)}A\star A\star A + \nabla_{(k+3)}A\star\nabla A\star A\right)\star\nabla_{(k)}A\right]\gamma^s d\mu + \int_\Sigma P_4^{2k+4,k+2}(A)\cdot\gamma^s d\mu\\
&\leq \int_\Sigma \left[\nabla_{(k+3)}A\star\left(\nabla_{(k)}A\star\nabla A\star A + \nabla_{(k+1)}A\star A\star A\right)\gamma^s\right]d\mu\\
&\quad + c\, c_\gamma\int_\Sigma \left|\nabla_{(k+3)}A\right|\left|\nabla_{(k)}A\right|\left|A\right|^2\gamma^{s-1}d\mu + \int_\Sigma P_4^{2k+4,k+2}(A)\,\gamma^s d\mu\\
&\leq \delta\int_\Sigma\left|\nabla_{(3)}A\right|^2\gamma^s d\mu + \int_\Sigma\left(P_4^{2k+4,k+2}(A) + P_6^{2k+2,k+1}(A)\right)\gamma^s d\mu + c_\gamma^2\int_\Sigma P_6^{2k,k}(A)\,\gamma^{s-2}d\mu\\
&\leq \delta\int_\Sigma\left|\nabla_{(3)}A\right|^2\gamma^s d\mu + c\left\|A\right\|^2_{\infty,[\gamma>0]}\left(\int_\Sigma\left|\nabla_{(k+2)}A\right|^2\gamma^s d\mu + c_\gamma^{2(k+2)}\left\|A\right\|^2_{2,[\gamma>0]}\right)\\
&\quad + c\left\|A\right\|^4_{\infty,[\gamma>0]}\left(\int_\Sigma\left|\nabla_{(k+1)}A\right|^2\gamma^s d\mu + c_\gamma^{2(k+1)}\left\|A\right\|^2_{2,[\gamma>0]}\right)\\
&\quad + c\, c_\gamma^2\left\|A\right\|^4_{\infty,[\gamma>0]}\left(\int_\Sigma\left|\nabla_{(k)}A\right|^2\gamma^{s-2}d\mu + c_\gamma^{2k}\left\|A\right\|^2_{2,[\gamma>0]}\right).
\end{aligned}\tag{19}$$

Integrating by parts, applying the Cauchy-Schwarz inequality and absorbing and multiplying out similarly as in the proof of Lemma 11 we obtain the following interpolation inequalities:

$$\begin{aligned}
\left\|A\right\|^2_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k+2)}A\right|^2\gamma^s d\mu &\leq \delta\int_\Sigma\left|\nabla_{(k+3)}A\right|^2\gamma^s d\mu\\
&\quad + c\left\|A\right\|^4_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k+1)}A\right|^2\gamma^s d\mu + c\, c_\gamma^{2k}\left\|A\right\|^2_{2,[\gamma>0]}\left(c_\gamma^6 + \left\|A\right\|^6_{\infty,[\gamma>0]}\right),
\end{aligned}\tag{20}$$

$$\begin{aligned}
\left\|A\right\|^4_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k+1)}A\right|^2\gamma^s d\mu &\leq \delta\left\|A\right\|^2_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k+2)}A\right|^2\gamma^s d\mu\\
&\quad + c\left\|A\right\|^6_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k)}A\right|^2\gamma^s d\mu + c\, c_\gamma^{2k}\left\|A\right\|^2_{2,[\gamma>0]}\left(c_\gamma^6 + \left\|A\right\|^6_{\infty,[\gamma>0]}\right),
\end{aligned}\tag{21}$$

and

$$c_\gamma^2\left\|A\right\|^4_{\infty,[\gamma>0]} \leq \delta\int_\Sigma\left|\nabla_{(k+3)}A\right|^2\gamma^s d\mu + c\left\|A\right\|^6_{\infty,[\gamma>0]}\int_\Sigma\left|\nabla_{(k)}A\right|^2\gamma^s d\mu + c\, c_\gamma^{2(k+3)}\left\|A\right\|^2_{2,[\gamma>0]}.\tag{22}$$

Combining (20) , (21) and (22) and substituting into (19) estimates of all the P-style terms. Substituting this result into Proposition 19 then yields the desired result. □

We are now in a position to prove an analogue of [10, Proposition 4.6].

Proposition 20. *Let $f : \Sigma \times [0,T^*] \to \mathbb{R}^3$ satisfy* (GTHF) *and let γ be a cutoff function satisfying* (12). *Then there is a $\varepsilon_0 > 0$ such that if*

$$\sup_{[0,T^*]}\int_{[\gamma>0]}\left|A\right|^2\, d\mu \leq \varepsilon_0,\tag{23}$$

we can conclude that for all $k \in \mathbb{N}$ there are constants $\tilde{c}_k$, depending only on T^*, c_γ, ε_0 and

$$\alpha_0\left(k+3\right) := \sum_{j=0}^{k+3} \left\|\nabla_{(j)}A\right\|^2_{2,[\gamma>0]}\Big|_{t=0} \tag{24}$$

such that

$$\left\|\nabla_{(k)}A\right\|^2_{\infty,[\gamma=1]} \leq \tilde{c}_k. \tag{25}$$

Proof. We introduce a set of cutoff functions $\gamma_{\sigma,\tau}$ for $0 \leq \sigma < \tau \leq 1$ satisfying $\gamma_{\sigma,\tau} = 1$ for $\gamma \geq \tau$ and $\gamma_{\sigma,\tau} = 0$ for $\gamma \leq \sigma$. Applying Proposition 18 with $\gamma = \gamma_{0,\frac{1}{2}}$ yields

$$\int_0^{T^*}\int_{\left[\gamma\geq\frac{1}{2}\right]} \left|\nabla_{(3)}A\right|^2 + \left|\nabla_{(2)}A\right|^2|A|^2 + |\nabla A|^2|A|^4\, d\mu\, d\tau \leq c\varepsilon_0\left(1 + c_\gamma^6\, T^*\right).$$

Next, we apply Proposition 16 with $T = A$ and $\gamma_{\frac{1}{2},\frac{3}{4}}$. This gives

$$\int_0^{T^*} \|A\|^6_{\infty,\left[\gamma\geq\frac{3}{4}\right]}\, d\tau \leq c\varepsilon_0^3\left(1 + c_\gamma^6\, T^*\right).$$

Here we have used $\left[\gamma \geq \frac{3}{4}\right] \subseteq \left[\gamma_{\frac{1}{2},\frac{3}{4}} = 1\right]$, $\left[\gamma_{\frac{1}{2},\frac{3}{4}}\right] \subseteq \left[\gamma \geq \frac{1}{2}\right]$ and (26). Next we look at Proposition 19 with $\gamma_{\frac{3}{4},\frac{7}{8}}$ and integrate over $[0,T^*]$. This gives us for any $t \in [0,T^*]$:

$$\int_\Sigma \left|\nabla_{(k)}A\right|^2 \gamma^s_{\frac{3}{4},\frac{7}{8}} d\mu \leq \left\|\nabla_{(k)}A\right\|^2_{2,[\gamma>0]}\Big|_{t=0} + c\, c_\gamma^{2k}\varepsilon_0\left(1 + c_\gamma^6\, T^*\right) + c\int_0^t \|A\|^6_{\infty,\left[\gamma\geq\frac{3}{4}\right]}\int_\Sigma \left|\nabla_{(k)}A\right|^2 \gamma^s_{\frac{3}{4},\frac{7}{8}} d\mu\, d\tau.$$

Applying Gromwall's Inequality whilst taking into account identity (26) gives

$$\begin{aligned}\int_{\left[\gamma\geq\frac{7}{8}\right]} \left|\nabla_{(k)}A\right|^2\, d\mu &\leq c\left(\left\|\nabla_{(k)}A\right\|^2_{2,[\gamma>0]}\Big|_{t=0} + c\, c_\gamma^{2k}\varepsilon_0\left(1 + c_\gamma^6\, T^*\right)\right)e^{c\varepsilon_0^3\left(1+c_\gamma^6\, T^*\right)}\\ &\leq c_k\left(\alpha_0\left(k\right), c_\gamma, \varepsilon_0, T^*\right).\end{aligned}$$

Combining this with Proposition 16 with $\gamma_{\frac{7}{8},\frac{15}{16}}$,

$$\begin{aligned}\left\|\nabla_{(k)}A\right\|^6_{\infty,\left[\gamma\geq\frac{15}{16}\right]} &\leq c\, c_k^2\Big(c_{k+3} + \int_\Sigma \left[P_4^{2(k+2),k+2}\left(A\right) + P_6^{2(k+1),k+1}\left(A\right)\right]\gamma^s_{\frac{7}{8},\frac{15}{16}} d\mu + c_\gamma^{2(k+3)}\varepsilon_0\Big)\\ &\leq c\left(c_3, c_k, c_{k+2}, c_{k+3}, \varepsilon_0, T^*, c_\gamma\right),\end{aligned}$$

which proves (25). Here we have used the inequality

$$\|A\|^6_{\infty,[\gamma=1]} \leq c\,\varepsilon_0^2\left(c_3 + c_\gamma^6\varepsilon_0\right)$$

which can be obtained by combining Proposition 16 and Lemma 17. □

We now have sufficient tools to prove Theorem 5. However, we will first sharpen our derivative bounds from Proposition 20 for later application in combination with Theorem 32 where we construct a blowup. Theorem 20 will also be used to obtain our preliminary compactness result (Lemma 38) for the flow once global existence has been obtained.

Theorem 21 (Interior Estimates)**.** *Suppose $f : \Sigma \times (0,T^*] \to \mathbb{R}^3$ satisfies* (GTHF) *and*

$$\sup_{0<t\leq T^*}\int_{f^{-1}(B_\rho(x))} |A|^2\, d\mu \leq \varepsilon_0 \textit{ for } T^* \leq c\rho^6. \tag{26}$$

Then for any $m \in \mathbb{N}_0$ we have at time $t \in (0,T^*]$ we have the estimates

$$\left\|\nabla_{(m)}A\right\|_{2,f^{-1}\left(B_{\frac{\rho}{2}}(x)\right)} \leq c_m\sqrt{\varepsilon_0}\,t^{-\frac{m}{6}} \textit{ and } \left\|\nabla_{(m)}A\right\|_{\infty,f^{-1}\left(B_{\frac{\rho}{2}}(x)\right)} \leq c_k\sqrt{\varepsilon_0}\,t^{-\frac{m+1}{6}},$$

where the constants c_m depend only upon m, ρ, T^* and $\left|\nabla_{(m)}A\right|_{2,f^{-1}(B_\rho(x_0))}\Big|_{t=0}$.

The idea of the proof is to consider a family of cutoff functions in time and integrate over $(0,T^*]$. An inductive argument then gives the desired the result. Since this is similar to [9, Theorem 3.5], we have omitted it.

5. Proof of the lifespan theorem

Now that we have pointwise bounds on $\|\nabla_{(k)}A\|^2_{2,\gamma^s}$ when the concentration of curvature is small, we are able to proceed with our proof of Theorem 5. The proof will rest upon contradicting the maximal time T of smooth existence. We may assume $\rho = 1$ in assumption (5) due to scale invariance of the $\|A\|_2^2$. That is, under the rescaling $\tilde{f}(p,t) = f\left(\frac{p}{\rho}, \frac{t}{\rho^6}\right)$ we have

$$\int_{f^{-1}(B_\rho)} |A|^2 \, d\mu = \int_{\tilde{f}^{-1}(B_1)} |\tilde{A}|^2 \, d\tilde{\mu}.$$

Under such a rescaling, we aim to show that

$$T \geq \frac{1}{C}$$

and that for $0 \leq t \leq \frac{1}{C}$ the estimate

$$\int_{f^{-1}(B_1)} |A|^2 \, d\mu \leq C\,\varepsilon_0$$

holds. We make the definition

$$\alpha(t) = \sup_{x \in \mathbb{R}^3} \|A\|^2_{2, f^{-1}(B_1(x))}. \tag{27}$$

We can cover the ball B_1 with a number of translated copies of $B_{\frac{1}{2}}$. It follows that there is an absolute constant C_α such that

$$\alpha(t) \leq C_\alpha \sup_{x \in \mathbb{R}^3} \|A\|^2_{2, f^{-1}(B_{\frac{1}{2}}(x))}. \tag{28}$$

Now because f is smooth by the definition of our family of compact immersions Σ_t we have $f(\Sigma \times [0,t])$ compact for $t < T$, and so $\alpha \in C([0,T))$. Firstly we define C_0 to be the constant on the right hand side of Proposition 18. From this we make the definition $\lambda = \frac{1}{C_0 c_\gamma^6}$, and

$$t_0 = \sup\{0 \leq t \leq \min\{T, \lambda\} : \alpha(\tau) \leq 3\,C_\alpha\,\varepsilon_0 \text{ for } 0 \leq \tau \leq t\}. \tag{29}$$

The reason for this parameter λ will become apparent later when we establish a contradiction.

We will go through this proof in 3 steps, labelled (30) – (32). These are as follows:

$$\text{Show that } t_0 = \min\{T, \lambda\}, \tag{30}$$

$$\text{Show that if } t_0 = \lambda \text{ then we have the Lifespan Theorem,} \tag{31}$$

and

$$\text{Show that if } T \neq \infty \text{ then } t_0 \neq T. \tag{32}$$

Note that (32) is equivalent to showing that $t_0 = T \implies T = \infty$. We claim that the three statements $(30), (31)$ and (32) together prove the Lifespan Theorem. To see this, note that if (32) holds then we must have either $t_0 = T$ or $t_0 = \lambda$. If $t_0 = T$, then (32) implies that $T = \infty$, meaning that the flow exists for all time, which proves the Lifespan Theorem. If instead $t_0 = \lambda$, then (31) will directly give us the Lifespan Theorem.

To begin, we note that by one of the assumptions of the theorem we have

$$\alpha(0) = \sup_{x \in \mathbb{R}^3} \|A\|^2_{2, f^{-1}(B_1(x))}\Big|_{t=0} \leq \varepsilon_0 < 3\,C_\alpha\,\varepsilon_0,$$

and so we must have t_0 strictly positive. To see this, we note that the continuity of α and the fact that $\alpha(0) < 3\,C_\alpha\,\varepsilon_0$ forces $\alpha(t) < 3\,C_\alpha\,\varepsilon_0$ for some strictly positive time period. The definition of t_0 then guarantees $t_0 > 0$. Also, by the definition of t_0 and by continuity of α, we must have

$$\alpha(t_0) = 3\,C_\alpha\,\varepsilon_0. \tag{33}$$

Let us assume that $t_0 < \min\{\lambda, T\}$ and aim for a contradiction. We choose a cutoff function γ such that

$$\chi_{B_{\frac{1}{2}}(x)} \leq \tilde{\gamma} \leq \chi_{B_1(x)} \text{ for any } x \in \Sigma_t.$$

By the initial smoothness of $|A|^2$, we can always find a $\rho^\star > 0$ in assumption (5) that is small enough to guarantee the smallness condition (23) holds for every $T^* < t_0$. Note that this is equivalent to assuming the hypothesis of Proposition 18 on the interval $[0, t_0)$. The results of Proposition 18 then imply that

$$\|A\|^2_{2,[\gamma=1]} \leq \|A\|^2_{2,[\gamma>0]}\Big|_{t=0} + C_0\, c_\gamma^6 \, \|A\|^2_{2,[\gamma>0]}\, t,$$

from which we conclude

$$\int_{f^{-1}(B_{\frac{1}{2}}(x))} |A|^2 \, d\mu \leq \int_{f^{-1}(B_1(x))} |A|^2 \, d\mu\bigg|_{t=0} + C_0\, c_\gamma^6 \int_{f^{-1}(B_1(x))} |A|^2 \, d\mu \leq \varepsilon_0 + C_0\, c_\gamma^6\, \varepsilon_0\, t_0$$

for $t \in [0, t_0)$ and $x \in \mathbb{R}^3$. It follows from the assumption that $t_0 < \lambda$ and the definition of λ that for $t \in [0, t_0)$ and $x \in \mathbb{R}^3$ that we have

$$\int_{f^{-1}(B_{\frac{1}{2}}(x))} |A|^2 \, d\mu \leq \varepsilon_0 + C_0\, c_\gamma^6\, \varepsilon_0\, t_0 < 2\, \varepsilon_0.$$

We deduce from (28) that

$$\alpha(t) \leq C_\alpha \sup_{x\in\mathbb{R}^3} |A|^2_{2,f^{-1}(B_{\frac{1}{2}}(x))} \leq 2\, C_\alpha\, \varepsilon_0 \text{ for } t \in [0, t_0)\,.$$

By the continuity of α this means that

$$\lim_{t\nearrow t_0} \alpha(t) \leq 2\, C_\alpha\, \varepsilon_0\,.$$

This contradicts statement (33) where we said that $\alpha(t_0) = 3\, C_\alpha\, t_0$. Hence (30) holds.

We now note that under assumption (30) (which was just proven) we must have either $t_0 = T$ or $t_0 = \lambda$. If $t_0 = \lambda$ then because $t_0 = \min\{\lambda, T\}$ it follows that

$$T \geq t_0 = \lambda = \frac{1}{C_0\, c_\gamma^6},$$

and the definition of t_0 tells us that

$$\int_{f^{-1}(B_1(x))} |A|^2 \, d\mu \leq 3\, C_\alpha\, \varepsilon_0$$

for $x \in \Sigma_t$ and $t \in [0, \lambda]$. Together these two statements give us both statements (6) and (7) of the theorem with $C = \max\left\{C_0\, c_\gamma^6, 3\, C_\alpha\right\}$. That is, assuming that $t_0 = \lambda$, we have the Lifespan Theorem. This is (31).

Finally, we turn our attention to (32). We will assume

$$t_0 = T \neq \infty$$

and then aim to contradict the maximality of T. We note that we have not included the case $T = \infty$ because in that case the second part of the Lifespan Theorem (6) would hold automatically. Additionally, we would have $\lambda \leq T$ so that $t_0 = \lambda = T$ and then statement (7) with $C = 2\, C_\alpha$ would directly follow from our earlier estimate:

$$\alpha(t) \leq 2\, C_\alpha\, \varepsilon_0, t \in [0, t_0)\,.$$

We can also exclude the case $T < \lambda$ for the sake of our argument, because by (30) it would follow that $t_0 = T$, from which we could conclude (using (31)) the Lifespan Theorem. Next, Proposition 20 allows us to establish a uniform bound on all derivatives of A up to and including the final time T. A standard argument similar to that of Hamilton [6] allows us to conclude that $\Sigma_t \to \Sigma_T$ in the C^∞ topology and that each $f(\cdot, t)$ induces an equivalent family of metrics, implying the uniqueness (up to reparametrisation) of our limiting object Σ_T up to rigid motions. Hence we may define a new family of immersions $h : \Sigma \times [0, \delta) \to \mathbb{R}^3$ satisfying (GTHF), given by $h(\cdot, t) = f(\cdot, t + T)$, so that h has smooth initial data $f(\cdot, T)$. Theorem 1 implies existence of the flow h for some positive time $\delta > 0$, extending the smooth existence of our original flow to the interval $[0, T + \delta] \supset [0, T]$, contradicting the maximality of T. Hence we have established (32). This means we have proven steps (30), (31) and (32), which as noted earlier completes the proof of Theorem 5.

6. A pointwise estimate for the tracefree curvature

In this section we present some new uses of geometric quantities involving iterated covariant derivatives of A, A^o and H. These will include local integral estimates and a multiplicative Sobolev-type inequality, aimed at the establishemnt of a pointwise bound for the tracefree curvature A^o that only depends on $\|\nabla\Delta H\|^2_{2,[\gamma>0]}$, $\|A^o\|^2_{2,[\gamma>0]}$, and the gradient bound for γ.

This holds for any immersed surface with sufficient weak regularity, and so is of independent interest. For our purposes in this paper the estimate will also be useful in the proof of Theorem 4 in the next section.

We begin with Simons' identity-type relations.

Lemma 22. *The following identities hold:*

$$\Delta A^o = S^o\left(\nabla_{(2)}H\right) + \frac{1}{2}H^2A^o - |A^o|^2 A^o \tag{34}$$

and

$$\Delta\nabla H = \nabla\Delta H + \frac{1}{4}\nabla H\,|H|^2 + \nabla H \star (HA^o + A^o \star A^o)\,. \tag{35}$$

Consequently, there is a universal constant c such that

$$-\langle\Delta\nabla H, \nabla H\rangle \le -\langle\nabla\Delta H, \nabla H\rangle - \frac{1}{8}|\nabla H|^2 H^2 + c\,|\nabla H|^2 |A^o|^2\,. \tag{36}$$

Proof. The first identity is identical to one in the Appendix of [18]. For the second identity, we employ the Riemann curvature tensor:

$$\nabla_{ijk}H = \nabla_{ikj}H = \nabla_{kij}H + R_{ikjs}g^{st}\nabla_t H = \nabla_{kij}H + A^t_kA_{ij}\nabla_tH - A^t_iA_{kj}\nabla_tH.$$

This immediately gives

$$\Delta\nabla_kH = \nabla_k\Delta H + HA^t_k\nabla_tH - A^t_iA^i_k\nabla_tH. \tag{37}$$

Evaluating the second term on the right, we have

$$HA^t_k\nabla_tH = H\left((A^o)^t_k + \frac{1}{2}\delta^t_kH\right)\nabla_tH = \frac{1}{2}\nabla_kH\,|H|^2 + H\nabla H\star A^o.$$

The last term in (37) is dealt with similarly:

$$A^t_iA^i_k\nabla_tH = \left((A^o)^t_i + \frac{1}{2}\delta^t_iH\right)\left((A^o)^i_k + \frac{1}{2}\delta^i_kH\right)\nabla_tH = \frac{1}{4}\nabla_kH\,|H|^2 + \nabla H\star(HA^o + A^o\star A^o)\,.$$

Substituting the two preceding results into (37) then yields the second statement of the lemma. For the last statement, we take the results of (37) and apply the Peter-Paul inequality with $p=8, q=\frac{8}{7}$:

$$\begin{aligned}-\langle\Delta\nabla H, \nabla H\rangle &\le -\langle\nabla\Delta H, \nabla H\rangle - \frac{1}{4}|\nabla H|^2H^2 + c\,|\nabla H|^2\left(|A^o|\,|H| + |A^o|^2\right)\\ &\le -\langle\nabla\Delta H, \nabla H\rangle - \frac{1}{8}|\nabla H|^2H^2 + c\,|\nabla H|^2|A^o|^2\,.\end{aligned}$$

□

We now prove a multiplicative Sobolev inequality.

Lemma 23. *If $f:\Sigma\to\mathbb{R}^3$ is an immersion satisfying*

$$\int_{[\gamma>0]}|A^o|^2\,d\mu \le \varepsilon_0 \tag{38}$$

for $\varepsilon_0>0$ sufficiently small, then for a universal constant $c>0$,

$$\int_\Sigma|\nabla A^o|^2\gamma^2d\mu + \int_\Sigma|A^o|^2H^2\gamma^2d\mu \le c\,\|A^o\|^{\frac{4}{3}}_{2,[\gamma>0]}\left(\int_\Sigma|\nabla\Delta H|^2\gamma^6d\mu\right)^{\frac{1}{3}} + c\,c_\gamma^2\,\|A^o\|^2_{2,[\gamma>0]}\,, \tag{39}$$

$$\int_\Sigma\left|\nabla_{(2)}H\right|^2\gamma^4d\mu + \int_\Sigma|\nabla H|^2H^2\gamma^4d\mu \le c\,\|A^o\|^{\frac{2}{3}}_{2,[\gamma>0]}\left(\int_\Sigma|\nabla\Delta H|^2\gamma^6d\mu\right)^{\frac{2}{3}} + c\,c_\gamma^4\,\|A^o\|^2_{2,[\gamma>0]}\,, \tag{40}$$

and

$$\int_\Sigma |A^o|^4\gamma^2 d\mu \le c\,\|A^o\|_{2,[\gamma>0]}^{\frac{10}{3}}\left(\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac{1}{3}} + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^4\,. \tag{41}$$

Proof. For the first inequality we combine integration by parts, identity (34) and Theorem 13 with $u = |A^o|^2\gamma$:

$$\begin{aligned}
\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu &\le -\int_\Sigma \langle A^o,\Delta A^o\rangle\gamma^2 d\mu + 2\,c_\gamma\int_\Sigma |\nabla A^o|\,|A^o|\,\gamma d\mu\\
&\le \|A^o\|_{2,[\gamma>0]}\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu\right)^{\frac12} - \frac12\int_\Sigma |A^o|^2 H^2\gamma^2 d\mu\\
&\quad + c_{\mathrm{MSS}}\left(2\int_\Sigma |\nabla A^o|\,|A^o|\,\gamma d\mu + c_\gamma\,\|A^o\|_{2,[\gamma>0]}^2 + \int_\Sigma |A^o|^2\,|H|\,\gamma d\mu\right)^2\\
&\quad + \frac12\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + 2\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^2\\
&\le \|A^o\|_{2,[\gamma>0]}\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu\right)^{\frac12} + \left(c\,\|A^o\|_{2,[\gamma>0]}^2 - \frac12\right)\int_\Sigma |A^o|^2 H^2\gamma^2 d\mu\\
&\quad + \left(c\,\|A^o\|_{2,[\gamma>0]}^2 + \frac12\right)\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^2\,.
\end{aligned}$$

Then for $\varepsilon_0 > 0$ sufficiently small we can absorb and multiply out, yielding

$$\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2\gamma^2 d\mu \le 2\,\|A^o\|_{2,[\gamma>0]}\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu\right)^{\frac12} + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^2\,. \tag{42}$$

We will leave this identity for the moment. For the second inequality we combine integration by parts, identity (39), Lemma 22 and Theorem 13 with $u = |\nabla H|\,|A^o|\,\gamma^2$ to obtain:

$$\begin{aligned}
\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu &\le -\int_\Sigma \langle\nabla H,\Delta\nabla H\rangle\gamma^4 d\mu + 4\,c_\gamma\int_\Sigma \left|\nabla_{(2)}H\right|\,|\nabla H|\,\gamma^3 d\mu\\
&\le 2\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\cdot\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac12} - \frac18\int_\Sigma |\nabla H|^2 H^2\gamma^4 d\mu\\
&\quad + c\int_\Sigma |\nabla H|^2\,|A^o|^2\gamma^4 d\mu + \frac12\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + 32\,c_\gamma^2\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\\
&\le 2\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\cdot\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac12} - \frac18\int_\Sigma |\nabla H|^2 H^2\gamma^4 d\mu\\
&\quad + c\Bigg(\int_\Sigma \left|\nabla_{(2)}H\right|\,|A^o|\,\gamma^2 d\mu + \int_\Sigma |\nabla H|\,|\nabla A^o|\,\gamma^2 d\mu + c_\gamma\int_\Sigma |\nabla H|\,|A^o|\,\gamma d\mu\\
&\qquad + \int_\Sigma |\nabla H|\,|A^o|\,|H|\,\gamma^2 d\mu\Bigg)^2 + \frac12\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + 32\,c_\gamma^2\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\\
&\le 2\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\cdot\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac12} + c\,c_\gamma^2\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\\
&\quad + \left(c\,\|A^o\|_{2,[\gamma>0]}^2 + \frac12\right)\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + c\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\right)^2\\
&\quad + \left(c\,\|A^o\|_{2,[\gamma>0]}^2 - \frac18\right)\int_\Sigma |\nabla H|^2 H^2\gamma^4 d\mu.
\end{aligned}$$

Absorbing and multiplying out, whilst combining with (42) gives for ε_0 sufficiently small:

$$\begin{aligned}
\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + \int_\Sigma |\nabla H|^2 H^2\gamma^4 d\mu &\le c\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\cdot\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac12}\\
&\quad + c\,c_\gamma^2\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + c\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu\right)^2.
\end{aligned} \tag{43}$$

Combining $(42), (43)$ and using the Cauchy-Schwarz inequality then yields

$$\int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2\gamma^2 d\mu \le \left(c\,|A^o|_{2,[\gamma>0]} + \frac{1}{2}\right)\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu$$
$$+ c\,\|A^o\|_{2,[\gamma>0]}^{\frac{4}{3}}\left(\int_\Sigma |\nabla \Delta H|^2\gamma^6 d\mu\right)^{\frac{1}{3}} + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^2.$$

For ε_0 sufficiently small we may then absorb and multiply out to give (39). Using $(39), (43)$ and the Cauchy-Schwarz inequality yields (40). Finally for the third statement of the lemma we combine (39) and Theorem 13 with $u = |A^o|^2\gamma$ to obtain:

$$\begin{aligned}\int_\Sigma |A^o|^4\gamma^2 d\mu &\le c_{\mathrm{MSS}}\left(2\int_\Sigma |\nabla A^o|\,|A^o|\,\gamma d\mu + c_\gamma\,\|A^o\|_{2,[\gamma>0]}^2 + \int_\Sigma |A^o|^2\,|H|\,\gamma d\mu\right)^2\\ &\le c\,\|A^o\|_{2,[\gamma>0]}^2\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2\gamma^2 d\mu\right) + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^4\\ &\le c\,\|A^o\|_{2,[\gamma>0]}^{\frac{10}{3}}\left(\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac{1}{3}} + c\,c_\gamma^2\,\|A^o\|_{2,[\gamma>0]}^4.\end{aligned}$$

This completes the proof. □

The following four lemmata are an adaptation of methods from standard energy estimates for elliptic PDE to the manifold setting. Difficulties arise due to the geometry. Since we are working in terms of curvature norms, such problems can be absorbed by good terms in the estimates.

Lemma 24. *If $f : \Sigma \to \mathbb{R}^3$ is an immersion satisfying* (38) *for $\varepsilon_0 > 0$ sufficiently small, then there is a universal constant $c > 0$ such that*

$$\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu + \int_\Sigma |\nabla A^o|^2 H^2\gamma^4 d\mu$$
$$\le c\int_\Sigma |\Delta A^o|^2\gamma^4 d\mu + c\,\|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}}\left(\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu\right)^{\frac{2}{3}} + c\,c_\gamma^4\,\|A^o\|_{2,[\gamma>0]}^2. \tag{44}$$

Proof. Using interchange of covariant derivatives:

$$\begin{aligned}\Delta\nabla_k A^o_{lm} &= g^{ij}\nabla_{ikj}A^o_{lm} + g^{ij}g^{st}\nabla_i\left(R_{jkls}A^o_{tm} + R_{jkms}A^o_{lt}\right)\\ &= g^{ij}\nabla_{ikj}A^o_{lm} + \frac{1}{4}\left(\nabla_l A^o_{km} + \nabla_m A^o_{kl}\right)H^2\\ &\quad - \frac{1}{4}\left(g_{kl}\nabla_i\left(A^o\right)^i_m + g_{km}\nabla_i\left(A^o\right)^i_l\right)H^2 + \nabla\left(HA^o \star A^o + A^o\star A^o\star A^o\right).\end{aligned}\tag{45}$$

Here R_{ijkl} is the Riemann curvature tensor. We also used (10) and

$$H\nabla H \star A^o = H\nabla A^o \star A^o = \nabla\left(HA^o\star A^o\right).$$

Performing a similar operation, we have

$$\begin{aligned}g^{ij}\nabla_{ikj}A^o_{lm} &= \nabla_k\Delta A^o_{lm} + \frac{1}{4}\left(\nabla_k A^o_{lm} + \nabla_l A^o_{km} + \nabla_m A^o_{kl}\right)H^2\\ &\quad -\frac{1}{4}\left(g_{kl}\nabla_i\left(A^o\right)^i_m + g_{km}\nabla_i\left(A^o\right)^i_l\right)H^2 + \nabla\left(HA^o\star A^o + A^o\star A^o\star A^o\right).\end{aligned}\tag{46}$$

Combining (45) and (46) then gives us

$$\begin{aligned}\Delta\nabla_k A^o_{lm} &= \nabla_k\Delta A^o_{lm} + \frac{1}{4}\left(\nabla_k A^o_{lm} + 2\left(\nabla_l A^o_{km} + \nabla_m A^o_{kl}\right)\right)H^2\\ &\quad - \frac{1}{2}\left(g_{kl}\nabla_i\left(A^o\right)^i_m + g_{km}\nabla_i\left(A^o\right)^i_l\right)H^2 + \nabla\left(HA^o\star A^o + A^o\star A^o\star A^o\right).\end{aligned}\tag{47}$$

Next tracing the term $\nabla_l A^o_{km}$ with $\nabla^k\left(A^o\right)^{lm}$ gives

$$\nabla^k\left(A^o\right)^{lm}\nabla_l A^o_{km} = g^{ks}g^{lt}g^{mu}\left(\nabla_s A_{tu} - \frac{1}{2}g_{tu}\nabla_s H\right)\left(\nabla_l A_{km} - \frac{1}{2}g_{km}\nabla_l H\right) = |\nabla A|^2 - \frac{3}{4}|\nabla H|^2.$$

Substituting these results back into (47) and taking inner products with ∇A^o yields

$$\begin{aligned}\langle \nabla A^o, \Delta\nabla A^o\rangle &= \langle \nabla A^o, \nabla\Delta A^o\rangle + \frac{1}{4}\left|\nabla A^o\right|^2 H^2 + \left(\left|\nabla A\right|^2 - \frac{3}{4}\left|\nabla H\right|^2\right) H^2 \\ &\quad - \frac{1}{4}\nabla^k \left(A^o\right)^{lm}\left(g_{kl}\nabla_m H + g_{km}\nabla_l H\right) H^2 + \nabla A^o \star \nabla\left(H A^o \star A^o + A^o\star A^o\star A^o\right) \\ &= \langle \nabla A^o, \nabla\Delta A^o\rangle + \frac{5}{4}\left|\nabla A^o\right|^2 H^2 - \frac{1}{2}\left|\nabla H\right|^2 H^2 + \nabla A^o \star \nabla\left(H A^o \star A^o + A^o\star A^o\star A^o\right).\end{aligned}$$

Taking the negative of the previous inequality, integrating, and applying the Cauchy-Schwarz inquality gives

$$\begin{aligned}\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu &\le -\int_\Sigma \langle \nabla A^o, \Delta\nabla A^o\rangle\gamma^4 d\mu + 4\,c_\gamma\int_\Sigma \left|\nabla A^o\right|\left|\nabla_{(2)}A^o\right|\gamma^3 d\mu \\ &\le -\int_\Sigma \langle \nabla A^o, \nabla\Delta A^o\rangle\gamma^4 d\mu - \frac{5}{4}\int_\Sigma \left|\nabla A^o\right|^2 H^2\gamma^4 d\mu + \frac{1}{2}\int_\Sigma\left|\nabla H\right|^2 H^2\gamma^4 d\mu \\ &\quad + c\int_\Sigma \left|\nabla A^o\right|\Big(\left|\nabla A^o\right|\left|A^o\right|^2 + \left|\nabla A^o\right|\left|A^o\right|\left|H\right|\Big)\gamma^4 d\mu \\ &\quad + \frac{1}{2}\int_\Sigma\left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu + 8\,c_\gamma^2\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu \\ &\le c\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + c\,c_\gamma^2\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu - \int_\Sigma\left|\nabla A^o\right|^2 H^2\gamma^4 d\mu \\ &\quad + \frac{1}{2}\int_\Sigma\left|\nabla H\right|^2 H^2\gamma^4 d\mu + c\int_\Sigma\left|\nabla A^o\right|^2\left|A^o\right|^2\gamma^4 d\mu + \frac{1}{2}\int_\Sigma\left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu.\end{aligned}$$

We then absorb, multiply by 2, and apply Theorem 13 with $u = \left|\nabla A^o\right|\left|A^o\right|\gamma^2$ to obtain:

$$\begin{aligned}\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu &\le c\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + c\,c_\gamma^2\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu - 2\int_\Sigma\left|\nabla A^o\right|^2 H^2\gamma^4 d\mu + \int_\Sigma\left|\nabla H\right|^2 H^2\gamma^4 d\mu \\ &\quad + c\left(\int_\Sigma\left|\nabla_{(2)}A^o\right|\left|A^o\right|\gamma^2 d\mu + \int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu + c_\gamma\int_\Sigma\left|\nabla A^o\right|\left|A^o\right|\gamma d\mu\right. \\ &\quad \left. + \int_\Sigma\left|\nabla A^o\right|\left|A^o\right|\left|H\right|\gamma^2 d\mu\right)^2 \\ &\le c\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + c\,c_\gamma^2\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu + \frac{1}{2}\int_\Sigma\left|\nabla H\right|^2 H^2\gamma^4 d\mu + c\left(\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu\right)^2 \\ &\quad + \left(c\left\|A^o\right\|^2_{2,[\gamma>0]} - 1\right)\int_\Sigma\left|\nabla A^o\right|^2 H^2\gamma^4 d\mu + c\left\|A^o\right\|^2_{2,[\gamma>0]}\int_\Sigma\left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu.\end{aligned}$$

Absorbing and multiplying out, using Lemma 23 and the Cauchy-Schwarz inequality then gives

$$\begin{aligned}&\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu + \int_\Sigma\left|\nabla A^o\right|^2 H^2\gamma^4 d\mu \\ &\le c\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + c\,c_\gamma^2\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu + c\int_\Sigma\left|\nabla H\right|^2 H^2\gamma^4 d\mu + c\left(\int_\Sigma\left|\nabla A^o\right|^2\gamma^2 d\mu\right)^2 \\ &\le c\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + c\left\|A^o\right\|^{\frac{2}{3}}_{2,[\gamma>0]}\left(\int_\Sigma\left|\nabla\Delta H\right|^2\gamma^6 d\mu\right)^{\frac{2}{3}} + c\,c_\gamma^4\left\|A^o\right\|^2_{2,[\gamma>0]},\end{aligned}$$

which is the statement of the lemma. □

Lemma 25. *If $f:\Sigma\to\mathbb{R}^3$ is an immersion satisfying (38) for $\varepsilon_0>0$ sufficiently small then there is a universal constant $c>0$ such that*

$$\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu + \int_\Sigma\left|\nabla A^o\right|^2 H^2\gamma^4 d\mu \le c\left\|A^o\right\|^{\frac{2}{3}}_{2,[\gamma>0]}\left(\int_\Sigma\left|\nabla\Delta H\right|^2\gamma^6 d\mu\right)^{\frac{2}{3}} + c\,c_\gamma^4\left\|A^o\right\|^2_{2,[\gamma>0]}. \tag{48}$$

Proof. Integration by parts, estimate (34), Theorem 13, and Simons' identity for A^o together yield

$$\int_\Sigma\left|\Delta A^o\right|^2\gamma^4 d\mu = \int_\Sigma\left\langle \Delta A^o, \nabla_{(2)}H + \frac{1}{2}H^2 A^o - \left|A^o\right|^2 A^o\right\rangle\gamma^4 d\mu$$

$$
\begin{aligned}
&\leq \left(\int_\Sigma |\Delta A^o|^2 \gamma^4 d\mu \cdot \int_\Sigma \left|\nabla_{(2)} H\right|^2 \gamma^4 d\mu\right)^{\frac{1}{2}} - \frac{1}{4}\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu \\
&\quad + c\int_\Sigma |\nabla A^o|^2 |A^o|^2 \gamma^4 d\mu + c\, c_\gamma^2 \left(\int_\Sigma |A^o|^2 H^2 \gamma^2 d\mu + \int_\Sigma |A^o|^4 \gamma^2 d\mu\right) \\
&\leq \left(\int_\Sigma |\Delta A^o|^2 \gamma^4 d\mu \cdot \int_\Sigma \left|\nabla_{(2)} H\right|^2 \gamma^4 d\mu\right)^{\frac{1}{2}} - \frac{1}{4}\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu \\
&\quad + c\Big(\int_\Sigma \left|\nabla_{(2)} A^o\right| |A^o| \gamma^2 d\mu + \int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu + c_\gamma \int_\Sigma |\nabla A^o| |A^o| \gamma d\mu \\
&\qquad + \int_\Sigma |\nabla A^o| |A^o| |H| \gamma^2 d\mu\Big)^2 + c\, c_\gamma^2 \left(\int_\Sigma |A^o|^2 H^2 \gamma^2 d\mu + \int_\Sigma |A^o|^4 \gamma^2 d\mu\right) \\
&\leq \frac{1}{2}\int_\Sigma |\Delta A^o|^2 \gamma^4 d\mu + \frac{1}{2}\int_\Sigma \left|\nabla_{(2)} H\right|^2 \gamma^4 d\mu + c\left(\int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu\right)^2 \\
&\quad + \left(c\, \|A^o\|_{2,[\gamma>0]}^2 + \frac{1}{2}\right)\int_\Sigma \left|\nabla_{(2)} A^o\right|^2 \gamma^4 d\mu + \left(c\, \|A^o\|_{2,[\gamma>0]}^2 - \frac{1}{4}\right)\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu \\
&\quad + c\, c_\gamma^2 \left(\int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2 \gamma^2 d\mu + \int_\Sigma |A^o|^4 \gamma^2 d\mu\right).
\end{aligned}
$$

Once again, if $\varepsilon_0 > 0$ is sufficiently small, we can absorb and multiply out, utilising the results of Lemmata 23 and 24 as well as the Cauchy-Schwarz inequality to derive:

$$
\begin{aligned}
&\int_\Sigma |\Delta A^o|^2 \gamma^4 d\mu + \int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu \\
&\leq c\, c_\gamma^2 \left(\int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2 \gamma^2 d\mu + \int_\Sigma |A^o|^4 \gamma^2 d\mu\right) + c\left(\int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu\right)^2 + c\int_\Sigma \left|\nabla_{(2)} H\right|^2 \gamma^4 d\mu \\
&\leq c\, \|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}} \left(\int_\Sigma |\nabla \Delta H|^2 \gamma^6 d\mu\right)^{\frac{2}{3}} + c\, c_\gamma^4 \, \|A^o\|_{2,[\gamma>0]}^2 .
\end{aligned}
$$

This is the statement of the lemma. $\square$

Lemma 26. *If $f : \Sigma \to \mathbb{R}^3$ is an immersion satisfying* (38) *for $\varepsilon_0 > 0$ sufficiently small then there is a universal constant $c > 0$ such that*

$$
\int_\Sigma \left|\nabla_{(2)} A^o\right|^2 \gamma^4 d\mu + \int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu \leq c\, \|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}} \left(\int_\Sigma |\nabla \Delta H|^2 \gamma^6 d\mu\right)^{\frac{2}{3}} + c\, c_\gamma^4 \, \|A^o\|_{2,[\gamma>0]}^2 .
$$

Proof. Combining the results of Lemmata 24 and 25 instantly gives us the result. $\square$

Lemma 27. *If $f : \Sigma \to \mathbb{R}^3$ is an immersion satisfying* (38) *for $\varepsilon_0 > 0$ sufficiently small then there is a universal constant $c > 0$ such that*

$$
\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu + \int_\Sigma |A^o|^2 H^4 \gamma^4 d\mu \leq c\, \|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}} \left(\int_\Sigma |\nabla \Delta H|^2 \gamma^6 d\mu\right)^{\frac{2}{3}} + c\, c_\gamma^4 \, \|A^o\|_{2,[\gamma>0]}^2 . \tag{49}
$$

Proof. Using integration by parts, the Cauchy-Schwarz inequality, (34) and (11) yields

$$
\begin{aligned}
\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu &\leq -\int_\Sigma \left\langle A^o, \nabla_{(2)} H + \frac{1}{2} H^2 A^o - |A^o|^2 A^o \right\rangle H^2 \gamma^4 d\mu + 4\int_\Sigma |\nabla A^o|^2 |A^o| |H| \gamma^4 d\mu \\
&\quad + 4\, c_\gamma \int_\Sigma |\nabla A^o| |A^o| |H| \gamma^3 d\mu \\
&\leq \left(32\int_\Sigma \left|\nabla_{(2)} H\right|^2 \gamma^4 d\mu + \frac{1}{8}\int_\Sigma |A^o|^2 H^4 \gamma^4 d\mu\right) - \frac{1}{2}\int_\Sigma |A^o|^2 H^4 \gamma^4 d\mu \\
&\quad + \int_\Sigma |A^o|^4 H^2 \gamma^4 d\mu + \left(\frac{1}{2}\int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 d\mu + 8\int_\Sigma |\nabla A^o|^2 |A^o|^2 \gamma^4 d\mu\right) \\
&\quad + \left(\frac{1}{8}\int_\Sigma |A^o|^2 H^4 \gamma^4 d\mu + 32\, c_\gamma^2 \int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu\right).
\end{aligned}
$$

Absorbing and applying Theorem 13 twice with $u = |\nabla A^o|\,|A^o|\,\gamma^2, u = |A^o|^2\,|H|\,\gamma^2$ and the Cauchy-Schwarz inequality, and using Lemmata 23 and 26 gives

$$
\begin{aligned}
&\frac{1}{2}\int_\Sigma |\nabla A^o|^2\,H^2\gamma^4 d\mu + \frac{1}{4}\int_\Sigma |A^o|^2\,H^4\gamma^4 d\mu \\
&\le 32\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + c_\gamma^2\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\right) + 8\int_\Sigma |\nabla A^o|^2\,|A^o|^2\,\gamma^4 d\mu + \int_\Sigma |A^o|^4\,H^2\gamma^4 d\mu \\
&\le 32\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + c_\gamma^2\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\right) \\
&\quad + 8\,c_{\mathrm{MSS}}\Big(\int_\Sigma \left|\nabla_{(2)}A^o\right|\,|A^o|\,\gamma^2 d\mu + \int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu + 2\,c_\gamma\int_\Sigma |\nabla A^o|\,|A^o|\,\gamma d\mu \\
&\quad + \int_\Sigma |\nabla A^o|\,|A^o|\,|H|\,\gamma^2 d\mu\Big)^2 + c_{\mathrm{MSS}}\Big(2\int_\Sigma |\nabla A^o|\,|A^o|\,|H|\,\gamma^2 d\mu \\
&\quad + \int_\Sigma |A^o|^2\,|\nabla H|\,\gamma^2 d\mu + 2\,c_\gamma\int_\Sigma |A^o|^2\,|H|\,\gamma d\mu + \int_\Sigma |A^o|^2\,H^2\gamma^2 d\mu\Big)^2 \\
&\le 32\left(\int_\Sigma \left|\nabla_{(2)}H\right|^2\gamma^4 d\mu + c_\gamma^2\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\right) + c\,\|A^o\|_{2,[\gamma>0]}^2\left(\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu\right. \\
&\quad \left. + c_\gamma^2\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu + c_\gamma^2\int_\Sigma |A^o|^2\,H^2\gamma^4 d\mu\right) + c\left(\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\right)^2 \\
&\quad + c\,\|A^o\|_{2,[\gamma>0]}^2\left(\int_\Sigma |\nabla A^o|^2\,H^2\gamma^4 d\mu + \int_\Sigma |A^o|^2\,H^4\gamma^4 d\mu\right) \\
&\le c\,\|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}}\left(\int_\Sigma |\nabla\Delta H|^2\,\gamma^6 d\mu\right)^{\frac{2}{3}} + c c_\gamma^4\,\|A^o\|_{2,[\gamma>0]}^2 \\
&\quad + c\,\|A^o\|_{2,[\gamma>0]}^2\left(\int_\Sigma |\nabla A^o|^2\,H^2\gamma^4 d\mu + \int_\Sigma |A^o|^2\,H^4\gamma^4 d\mu\right).
\end{aligned}
$$

Absorbing and multiplying out then gives the statement of the lemma. □

Now we are in a position to prove the pointwise estimate.

Theorem 28. *Let $f:\Sigma\to\mathbb{R}^3$ be an immersion satisfying (38) for $\varepsilon_0>0$ sufficiently small. Then there is a universal constant $c>0$ such that*

$$
\|A^o\|_{\infty,[\gamma=1]}^6 \le c\,\|A^o\|_{2,[\gamma>0]}^4\left(\int_\Sigma |\nabla\Delta H|^2\,\gamma^6 d\mu + c_\gamma^6\right). \tag{50}
$$

Proof. Using the Sobolev inequality from Theorem 14 we have

$$
\|\Psi\|_\infty^6 \le C\,\|\Psi\|_2^2\left(\|\nabla\Psi\|_4^4 + \|\Psi H\|_4^4\right)
$$

for any function Ψ defined on Σ. Letting $\Psi = |A^o|\,\gamma^{\frac{3}{2}}$ it follows that

$$
\|A^o\|_{\infty,[\gamma=1]}^6 \le c\,\|A^o\|_{2,[\gamma>0]}^2\left(\int_\Sigma |\nabla A^o|^4\,\gamma^6 d\mu + \int_\Sigma |A^o|^4\,H^4\gamma^6 d\mu + c_\gamma^4\int_\Sigma |A^o|^4\,\gamma^2 d\mu\right). \tag{51}
$$

Since the last term was taken care of in Lemma 23, we just need to look at the first two terms. For the first we apply Theorem 13 with $u = |\nabla A^o|^2\,\gamma^3$ and the results of Lemma 23 and Lemma 27:

$$
\begin{aligned}
\int_\Sigma |\nabla A^o|^4\,\gamma^6 d\mu &\le c\left(\int_\Sigma \left|\nabla_{(2)}A^o\right|\,|\nabla A^o|\,\gamma^3 d\mu + c_\gamma\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu + \int_\Sigma |\nabla A^o|^2\,|H|\,\gamma^3 d\mu\right)^2 \\
&\le c\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\cdot\left(\int_\Sigma \left|\nabla_{(2)}A^o\right|^2\gamma^4 d\mu + \int_\Sigma |\nabla A^o|^2\,H^2\gamma^4 d\mu + c_\gamma^2\int_\Sigma |\nabla A^o|^2\,\gamma^2 d\mu\right) \\
&\le c\,\|A^o\|_{2,[\gamma>0]}^2\int_\Sigma |\nabla\Delta H|^2\,\gamma^6 d\mu + c\,c_\gamma^6\,\|A^o\|_{2,[\gamma>0]}^2.
\end{aligned}
$$

We approach the second term in (51) similarly, this time utilising Theorem 13 with $u = |A^o|^2 H^2\gamma^3$:

$$\begin{aligned}\int_\Sigma |A^o|^4 H^4\gamma^6 d\mu &\le c\Big(\int_\Sigma |\nabla A^o|\,|A^o|\,H^2\gamma^3 d\mu + \int_\Sigma |A^o|^2\,|\nabla H|\,|H|\,\gamma^3 d\mu + c_\gamma\int_\Sigma |A^o|^2 H^2\gamma^2 d\mu \\ &\qquad + \int_\Sigma |A^o|^2\,|H|^3\,\gamma^3 d\mu\Big)^2 \\ &\le c\int_\Sigma |A^o|^2 H^4\gamma^4 d\mu\cdot\left(\int_\Sigma |\nabla A^o|^2\gamma^2 d\mu + \int_\Sigma |A^o|^2 H^2\gamma^2 d\mu + c_\gamma^2\,\|A^o\|^2_{2,[\gamma>0]}\right) \\ &\quad + c\int_\Sigma |\nabla H|^2 H^2\gamma^4 d\mu\cdot\int_\Sigma |A^o|^4\gamma^2 d\mu \\ &\le c\,\|A^o\|^2_{2,[\gamma>0]}\int_\Sigma |\nabla\Delta H|^2\gamma^6 d\mu + c\,c_\gamma^6\,\|A^o\|^2_{2,[\gamma>0]}\,.\end{aligned}$$

Substituting these into (51) and again utilising Lemma 23 and the Cauchy-Schwarz inequality gives the desired result. □

Choosing a particular cutoff function gives Theorem 2 from Theorem 28.

7. Preserved closeness to the sphere

In this section we show that any immersion with small initial energy (in the sense of (8)) evolving under (GTHF) drives the energy monotonically toward zero. As the energy is in a sense the average deviation of the initial immersion from an embedded round sphere, the result will essentially show that an immersion with initially small energy will become more spherical under the flow.

We will first state a result of Li and Yau [11]. This result will later allow us, under a similar small energy condition, to establish that $f(\Sigma, t), t\in[0,T)$ is a one-parameter family of embeddings.

Theorem 29 ([11, Theorem 6])**.** *If an immersion $f:\Sigma\to\mathbb{R}^3$ has the property that*

$$\frac{1}{4}\int_\Sigma H^2 d\mu < 8\pi, \tag{52}$$

then f is an embedding.

Theorem 30 (Preserved Closeness to Spheres)**.** *Let $f:\Sigma\times[0,T)\to\mathbb{R}^3$ satisfy* (GTHF)*. Then there exists an $\varepsilon_0>0$ such that if*

$$\int_\Sigma |A^o|^2 d\mu\Big|_{t=0} \le \varepsilon_0 < 8\pi \tag{53}$$

then for $t<T$ we have the estimate

$$\frac{d}{dt}\int_\Sigma |A^o|^2 d\mu \le -\frac{1}{2}\int_\Sigma |\nabla\Delta H|^2 d\mu.$$

In particular, $\Sigma_t = f(\Sigma,t), t\in[0,T)$ is a one-parameter family of embeddings.

Proof. By continuity, given a flow satisfying the condition (53), there exists a time interval $I_{0,\delta_1} = [0,\delta_1), \delta_1 > 0$ such that for $\tau\in I_{0,\delta_1}$,

$$\int_\Sigma |A^o|^2 d\mu\Big|_{t=\tau} \le 2\,\varepsilon_0. \tag{54}$$

Let us now look at the evolution equation associated to this integral. We first choose a normalised frame as in the proof of Lemma 22. In this frame, we compute

$$|A^o|^2 = \frac{1}{2}(\kappa_1-\kappa_2)^2 = \frac{1}{2}\left(\kappa_1^2+\kappa_2^2+2\kappa_1\kappa_2-4\kappa_1\kappa_2\right) = \frac{1}{2}H^2 - 2K.$$

Applying the Gauss-Bonnet formula gives

$$\frac{d}{dt}\int_\Sigma |A^o|^2 d\mu = \frac{1}{2}\frac{d}{dt}\int_\Sigma H^2 d\mu. \tag{55}$$

The evolution equation for mean curvature from Lemma 7 now implies

$$\frac{d}{dt}\int_\Sigma |A^o|^2 d\mu = \int_\Sigma H\Delta^3 H d\mu + \int_\Sigma H\Delta^2 H\, |A^o|^2 d\mu.$$

Performing integration by parts yields

$$\begin{aligned}\frac{d}{dt}\int_\Sigma |A^o|^2 d\mu &\le -\int_\Sigma |\nabla\Delta H|^2 d\mu + \|A^o\|_\infty^2 \left|\int_\Sigma H\Delta^2 H d\mu\right| \\ &= -\int_\Sigma |\nabla\Delta H|^2 d\mu + \|A^o\|_\infty^2 \int_\Sigma |\Delta H|^2 d\mu.\end{aligned}$$

Combining this with Lemma 23 and Theorem 2 with $\gamma \equiv 1$ then gives

$$\frac{d}{dt}\int_\Sigma |A^o|^2 d\mu \le \left(c\,\|A^o\|_2^2 - 1\right)\int_\Sigma |\nabla\Delta H|^2 d\mu \le -\frac{1}{2}\int_\Sigma |\nabla\Delta H|^2 d\mu$$

for ε_0 sufficiently small. So, we know that if (53) holds for ε_0 sufficiently small, then it will decrease monotonically on the interval I_{0,δ_1}. We may repeat this process repeatedly right up to the maximal time of existence of the flow, allowing us deduce that the integral $\int_\Sigma |A^o|^2 d\mu$ decreases monotonically on the interval $[0, T)$. Finally, a quick check shows that under our initial smallness condition (52) is satisfied, so Theorem 29 applies on $[0, T)$, completing the proof. □

Remark 1. Theorem 30 in particular implies, in view of (55), that the Willmore energy is non-incresing under any flow (GTHF) satisfying (53).

8. The gap lemma

In this section we do not assume that the immersion $f : \Sigma \to \mathbb{R}^3$ is closed and compact; instead, we only assume that it is proper. Our goal is to prove Theorem 4. We begin with some remarks on the theorem.

Remark 2. If we do not know a-priori whether or not f is compact, but we do know that (2) is satisfied, then we may still conclude that f is either a plane or a sphere, i.e. an embedded umbilic.

Remark 3. Clearly the regularity assumption that f is locally C^6 is much more than what is required; indeed, Corollary 3 indicates that weak solutions in $W^{5,2}_{\text{loc}}$ satisfying (2) are classical and smooth.

Proof of the Gap Lemma. We shall consider the compact and noncompact cases separately. In each case we will prove that $|A^o|^2 \equiv 0$. The compact case is relatively straightforward: under the assumption $\Delta^2 H \equiv 0$ integration by parts gives us

$$\int_\Sigma |\nabla\Delta H|^2 d\mu = -\int_\Sigma \Delta H \cdot \Delta^2 H d\mu = 0,$$

forcing $\nabla\Delta H \equiv 0$. Our pointwise bound from Theorem 2 with $\gamma \equiv 1$ then yields

$$\|A^o\|_\infty^6 \le c\,\|A^o\|_2^4 \int_\Sigma |\nabla\Delta H|^2 d\mu = 0,$$

and so

$$|A^o| \equiv 0. \tag{56}$$

For the noncompact case we will have to work a little harder. Performing integration by parts followed by using the Cauchy-Schwarz inequality yields

$$\int_\Sigma |\nabla\Delta H|^2 \gamma^6 d\mu \le \frac{1}{2}\int_\Sigma |\nabla\Delta H|^2 \gamma^6 d\mu + 18\, c_\gamma^2 \int_\Sigma |\Delta H|^2 \gamma^4 d\mu,$$

so that absorbing and multiplying out yields

$$\int_\Sigma |\nabla\Delta H|^2 \gamma^6 d\mu \le 36\, c_\gamma^2 \int_\Sigma |\Delta H|^2 \gamma^4 d\mu. \tag{57}$$

Combining this with Lemma 26 and using the Cauchy-Schwarz inequality,

$$\int_\Sigma |\nabla\Delta H|^2 \gamma^6 d\mu \le c\, c_\gamma^2 \left[\|A^o\|_{2,[\gamma>0]}^{\frac{2}{3}} \left(\int_\Sigma |\nabla\Delta H|^2 \gamma^6 d\mu\right)^{\frac{2}{3}} + c_\gamma^4 \|A^o\|_{2,[\gamma>0]}^2\right]$$

$$\leq \frac{1}{2}\int_{\Sigma} |\nabla \Delta H|^2 \gamma^6 d\mu + c\, c_\gamma^6 \|A^o\|^2_{2,[\gamma>0]}\,,$$

implying that

$$\int_{\Sigma} |\nabla \Delta H|^2 \gamma^6 d\mu \leq c\, c_\gamma^6 \|A^o\|^2_{2,[\gamma>0]}\,. \tag{58}$$

Now choose γ such that

$$\gamma(p) = \Phi\left(\frac{1}{\rho}|f(p)|\right),$$

where $\Phi \in C^1(\mathbb{R})$ and

$$\chi_{B_{\frac{1}{2}}(0)} \leq \Phi \leq \chi_{B_1(0)}.$$

With this choice for γ we have $c_\gamma = \frac{c}{\rho}$ for some universal constant $c > 0$. Taking $\rho \to \infty$ in (58) yields

$$\int_{\Sigma} |\nabla \Delta H|^2 d\mu \leq c \liminf_{\rho\to\infty} \frac{1}{\rho^6}\int_{f^{-1}(B_\rho(0))} |A^o|^2\, d\mu = 0.$$

Here we have used the smallness assumption (2) in the last step. This tells us that $\nabla\Delta H \equiv 0$, from which we can utilise our pointwise bound from Theorem 2 with $\rho \to \infty$ to obtain

$$\|A^o\|^6_\infty \leq c \liminf_{\rho\to\infty} \frac{1}{\rho^6}\int_{f^{-1}(B_\rho(0))} |A^o|^2\, d\mu = 0,$$

which again implies

$$|A^o| \equiv 0. \tag{59}$$

By observing (56) and (59) we conclude that, compact or not, we must have $A^o \equiv 0$ on Σ. In terms of principal curvatures at any particular point, this means that

$$|A^o|^2 = \frac{1}{2}(\kappa_1 - \kappa_2)^2 \equiv 0 \text{ on } \Sigma.$$

Let us note that since f is of class C^6_{loc}, non-smooth unions of planes and spheres are not possible. A classical result from Codazzi then tells us that $f(\Sigma)$ must be a standard flat plane or standard round sphere. We conclude that if $f(\Sigma)$ is a round sphere if compact, and a flat plane if noncompact. □

9. Construction of the blowup

Let $f : \Sigma \times [0,T) \to \mathbb{R}^3$ satisfy (GTHF). We make the definition

$$\alpha(p,\tau) = \sup_{x\in\mathbb{R}^3}\int_{f^{-1}(B_r(x))} |A|^2\, d\mu\Big|_{t=\tau}, \tag{60}$$

and pick an arbitary decreasing sequence of radii $\{r_j\} \searrow 0$. We assume that the curvature concentrates in then sense that for each j

$$t_j := \inf\{t \geq 0 : \alpha(r_j, t) > \varepsilon_1\} < T.$$

Here $\varepsilon_1 := \frac{\varepsilon_0}{C}$, where ε_0, C are the same constants as in the Lifespan Theorem. Note that by construction, $\{t_j\}$ is a monotonically increasing sequence. By the definition of t_j, this means that

$$\int_{f^{-1}\left(B_{r_j}(x)\right)} |A|^2\, d\mu\Big|_{t=t_j} \leq \varepsilon_1 \text{ for any } x \in \mathbb{R}^3. \tag{61}$$

However, for each j it is possible to find a point $x_j \in \mathbb{R}^3$ such that

$$\int_{f^{-1}\overline{\left(B_{r_j}(x_j)\right)}} |A|^2\, d\mu\Big|_{t=t_j} \geq \varepsilon_1. \tag{62}$$

To see this, we can take a sequence $\{v_j\} \nearrow \infty, v_j > 1$ and consider the times $\tau_j = t_j + v_j^{-1}$ and radii $\lambda_j = r_j + v^{-2}$. By construction we have

$$\tau_j \searrow t_j \text{ and } \lambda_j \nearrow r_j \text{ as } j \to \infty.$$

Then by continuity, the definition of t_j, and the fact that $\tau_j > t_j$, for each v we can choose a point $x_{j+v^{-1}} \in \mathbb{R}^3$ such that

$$\int_{f^{-1}\left(B_{\lambda_j}\left(x_{j+v^{-1}}\right)\right)} |A|^2 \, d\mu \Big|_{t=\tau_j} \geq \varepsilon_1,$$

so that taking $v \to \infty$ yields (62).

Now consider the sequence of rescaled immersions

$$f_j : \Sigma \times \left[-r_j^{-6} t_j, r_j^{-6}\left(T - t_j\right)\right) \to \mathbb{R}^3, \text{ where } f_j(p,t) = \frac{1}{r_j}\left(f\left(p, t_j + r_j^6 t\right) - x_j\right) .$$

Let us define α_j to be the value of α, computed as in (60), with the immersion f_j in place of f. That is

$$\alpha_j\left(p,\tau\right) = \sup_{x\in\mathbb{R}^3} \int_{f_j^{-1}(B_r(x))} \left|A_j\right|^2 \, d\mu_j \Big|_{t=\tau} .$$

Quantities decorated with the subscript j correspond to the immersion f_j. One may check that

$$f_j^{-1}\overline{\left(B_1\left(0\right)\right)}\Big|_{t=0} = f^{-1}\overline{\left(B_{r_j}\left(x_j\right)\right)}\Big|_{t=t_j},$$

so that by (62) we conclude

$$\int_{f_j^{-1}\overline{(B_1(0))}} \left|A_j\right|^2 \, d\mu_j \Big|_{t=0} \geq \varepsilon_1.$$

Similarly, one may check that for $\tau \leq 0$:

$$\alpha_j\left(1,\tau\right) \leq \sup_{x\in\mathbb{R}^3} \int_{f^{-1}\left(B_{r_j}(x)\right)} |A|^2 \, d\mu \Big|_{t=t_j+r_j^6\tau \leq t_j} \leq \varepsilon_1. \tag{63}$$

Recall that the Lifespan Theorem tells us that the original flow f will exist up until such time as the curvature concentrates, with T satisfying $T \geq \frac{1}{C}$. For the rescaled immersions, this implies

$$r_j^{-6}\left(T - t_j\right) \geq \frac{1}{C} \text{ for each } j.$$

Additionally, statement (7) of the Lifespan Theorem tells us that for $0 \leq \tau \leq \frac{1}{C}$ we have

$$\alpha_j\left(1,\tau\right) \leq C\varepsilon_1 = \varepsilon_0 \text{ for } 0 < \tau \leq \frac{1}{C}.$$

We then apply the interior estimates from Theorem 21 on the cylinder $B_1\left(x\right) \times \left(t-1, t\right]$ to conclude that

$$\left\|\nabla_{(k)} A_j\right\|_{\infty, f_j} \leq c\left(k\right) \text{ for } -r_j^{-6} t_j + 1 \leq t \leq \frac{1}{C} . \tag{64}$$

Note that the covariant derivative above is that associated with f_j. Consider the sets

$$\Sigma_j\left(R\right) := \left\{p \in \Sigma : \left|f_j\left(p\right)\right| < R\right\} = \Sigma \cap f_j^{-1}\left(B_R\right) .$$

Let us now show that $\mu_j\left(\Sigma\left(R\right)\right) \leq c\left(R\right)$ for any $R > 0$. Recall the following monotonicity formula due to Simon.

Lemma 31 ([21, Equation (1.3)]). *Let $f : \Sigma \to \mathbb{R}^3$ be an immersed surface. Then for $0 < \sigma \leq \rho < \infty$ we have*

$$\frac{|\Sigma_\sigma|}{\sigma^2} \leq c\left(\frac{|\Sigma_\rho|}{\rho^2} + \int_{\Sigma_\rho} H^2 \, d\mu\right).$$

By Theorem 30, the Willmore energy is monotonically decreasing along the flow, and so Lemma 31 with $\sigma = R$, $\rho = R\sqrt{2c}$ gives

$$\frac{\mu_j(\Sigma_j(R))}{R^2} \leq \frac{c\mu_j(\Sigma_j(R))}{2cR^2} + c\int_{\Sigma_\rho} H^2 \, d\mu .$$

Absorbing yields

$$\frac{\mu_j(\Sigma_j(R))}{R^2} \leq c\left(\varepsilon_0 + 4\pi\chi\left(\Sigma\right)\right). \tag{65}$$

We now state the following compactness theorem.

Theorem 32 ([9, Theorem 4.2])**.** *Let $f_j : \Sigma_j \to \mathbb{R}^3$ be a sequence of proper immersions, where each Σ_j is a surface without boundary. For $R > 0$ define*

$$\Sigma_j\left(R\right) := \left\{p \in \Sigma_j : \left|f_j\left(p\right)\right| < R\right\} = \Sigma_j \cap f_j^{-1}\left(B_R\right).$$

Assume the bounds

$$\mu_j\left(\Sigma_j\left(R\right)\right) \leq c\left(R\right) \text{ for any } R > 0, \text{ and } \left\|\nabla_{(k)}A\right\|_\infty \leq c\left(k\right) \text{ for any } k \in \mathbb{N}_0 \tag{66}$$

hold. Then there is a proper immersion $\tilde{f} : \tilde{\Sigma} \to \mathbb{R}^3$ (where $\tilde{\Sigma}$ is also a $2-$manifold without boundary) such that after passing to a subsequence one has

$$f_j \circ \phi_j = \tilde{f} + u_j \text{ on } \tilde{\Sigma}\left(j\right) := \tilde{\Sigma} \cap \tilde{f}^{-1}\left(B_j\right), \tag{67}$$

satisfying the following properties:

$$\begin{cases} \phi_j : \tilde{\Sigma}\left(j\right) \to U_j \subset \Sigma_j \text{ is a diffeomorphism,} \\ \Sigma_j\left(R\right) \subset U_j \text{ if } j \geq j\left(R\right), \\ u_j \in C^\infty\left(\tilde{\Sigma}\left(j\right), \mathbb{R}^3\right) \text{ is normal along } \tilde{f}\left(\tilde{\Sigma}\right), \text{ and} \\ \left\|\tilde{\nabla}_{(k)}u_j\right\|_{\infty, \Sigma_j} \to 0 \text{ as } j \to \infty \text{ for any } k \in \mathbb{N}_0. \end{cases}$$

The theorem says that on any ball $B_R, R > 0$, for sufficiently large j our sequence of immersions can be written as a normal graph $\tilde{f} + u_j$ over our limit immersion $\tilde{f}$ with small norm (after reparametrising with the diffeomorphisms ϕ_j).

Our work above, in particular (64) and (65) imply that we may apply the compactness theorem above to the sequence $\{f_j\}$ in order to extract a convergent subsequence asymptotic to a proper immersion $\tilde{f}_0 : \tilde{\Sigma} \to \mathbb{R}^3$. We let $\phi_j : \tilde{\Sigma}\left(j\right) \to U_j \subset \Sigma$ be a sequence of diffeomorphisms as in (67). Then the reparametrisation

$$f_j\left(\phi_j, \cdot\right) : \tilde{\Sigma}\left(j\right) \times \left[0, C^{-1}\right] \to \mathbb{R}^3 \tag{68}$$

satisfies (GTHF). Also, by (67), the reparametrised flows $\{f_j\left(\phi_j, \cdot\right)\}$ have initial data

$$f_j\left(\phi_j, 0\right) = \tilde{f}_0 + u_j : \tilde{\Sigma}\left(j\right) \to \mathbb{R}^3, \tag{69}$$

converging locally in C^∞ to the immersion $\tilde{f}_0 : \Sigma \to \mathbb{R}^3$. By converting our covariant derivatives of curvature into partial derivatives of our immersion functions f_j, we conclude from (64) that

$$f_j\left(\phi_j, \cdot\right) \to \tilde{f}$$

locally in C^∞ where $\tilde{f} : \tilde{\Sigma} \times \left[0, \frac{1}{c}\right] \to \mathbb{R}^3$ satisfies (GTHF) with initial data f_0. Hence the convergence is locally smooth. We now prove four key properties of the blowup.

Theorem 33. *Let $f : \Sigma \times [0, T) \to \mathbb{R}^3$ satisfy* (GTHF) *and* (2)*. Then the blowup $\tilde{f}$ constructed above is stationary.*

Proof. By Theorem 30 and the fact that each of the maps ϕ_j is a diffeomorphism, we have for $t \in \left[0, \frac{1}{C}\right]$:

$$\int_{U_j} \left|\nabla \Delta H_j\right|^2 d\mu_j\Big|_{t=\tau} \leq -\frac{d}{dt}\int_\Sigma \left|A_j\right|^2 d\mu_j\Big|_{t=\tau}. \tag{70}$$

By scale invariance we have

$$\int_\Sigma \left|A_j\right|^2 d\mu_j\Big|_{t=0} = \int_\Sigma |A|^2 d\mu\Big|_{t=t_j}, \text{ and } \int_\Sigma \left|A_j\right|^2 d\mu_j\Big|_{t=C^{-1}} = \int_\Sigma |A|^2 d\mu\Big|_{t=t_j+r_j^6C^{-1}}.$$

Thus integrating identity (70) over the interval $\left[0, \frac{1}{C}\right]$ then yields

$$\int_0^{\frac{1}{C}} \int_{\tilde{\Sigma}(j)} \left|\nabla \Delta H_j\right|^2 d\mu_j\, d\tau \leq \int_\Sigma |A|^2 d\mu\Big|_{t=t_j} - \int_\Sigma |A|^2 d\mu\Big|_{t=t_j+r_j^6C^{-1}},$$

so that taking $j \to \infty$ yields

$$\lim_{j\to\infty} \int_0^{\frac{1}{C}} \int_{\tilde{\Sigma}(j)} \left|\nabla \Delta H_j\right|^2 d\mu_j\, d\tau = 0.$$

Hence as $f_j(\phi_j,\cdot) \to \tilde{f}$ locally smoothly, we find that $\nabla\Delta H \equiv 0$ with respect to the immersion $\tilde{f}$. This tells us immediately that $\tilde{f}$ satisfies

$$\Delta^2 H \equiv 0\,.$$

□

Lemma 34. *The blowup $\tilde{f}$ constructed above is not a union of planes.*

Proof. By identity (62) and construction of $\tilde{f}$, we have

$$\int_{\tilde{f}^{-1}\overline{(B_1(0))}} |A|^2 \, d\mu \geq \varepsilon_1 > 0,$$

This tells us that $\tilde{f}$ has a nonplanar component. □

Lemma 35. *Let $\tilde{f} : \tilde{\Sigma} \to \mathbb{R}^3$ be the blowup constructed above. If $\tilde{\Sigma}$ contains a compact component $\mathcal{C}$, then $\tilde{\Sigma} = \mathcal{C}$ and Σ is diffeomorphic to $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be the aforementioned compact component of $\tilde{\Sigma}$. Then for sufficiently large j, $\phi_j(\mathcal{C})$ is both closed and open (clopen) in Σ. Since Σ is connected and $\mathcal{C} \neq \emptyset$, we conclude that $\Sigma = \phi_j(\mathcal{C})$. Hence Σ is diffeomorphic to $\mathcal{C}$. Taking $j \nearrow \infty$ then allows us to conclude that

$$\mathcal{C} = \lim_{j\to\infty} \phi_j^{-1}(\Sigma) = \lim_{j\to\infty} \tilde{\Sigma}(j) = \tilde{\Sigma}\,.$$

□

Theorem 36. *Let $f : \Sigma \to \mathbb{R}^3$ satisfy* (GTHF) *and* (53). *Let $\tilde{f}$ be the blowup constructed above. Then none of the components of $\tilde{f}(\tilde{\Sigma})$ are compact, and the blowup has a component which is nonumbilic and satisfies $\Delta^2 H \equiv 0$.*

Proof. We claim that the surface area $\mu(\Sigma_t)$ is uniformly bounded away from zero. To see this, note that by Theorem 30 each f_t is an embedding for $t \in [0,T)$, and (3) implies that the enclosed volume of the flow is constant. Combining this with the isoperimetric inequality we conclude that for t in this time interval,

$$\mu(\Sigma_t) \geq \sqrt[3]{36\pi \mathrm{Vol}(\Sigma_0)} > 0.$$

Next assume (for the sake of contradiction) that $\tilde{f}(\tilde{\Sigma})$ has a compact component, say $\mathcal{D}$. The properness of $\tilde{f}$ implies that $\tilde{f}^{-1}(\mathcal{D}) \subseteq \tilde{\Sigma}$ is also compact. We infer from Lemma 34 that $\mathcal{D} = \tilde{f}(\tilde{\Sigma})$, so that $\tilde{f}(\tilde{\Sigma})$ must consist of a single compact component. Hence

$$|\tilde{f}(\tilde{\Sigma})| = \lim_{j\to\infty} \mu_j(\Sigma_t) < \infty. \tag{71}$$

We next use the definition of the sequence of immersions $\{f_j\}$ to compute the measure of the blowup. Firstly, a quick computation gives

$$g\Big|_{t=t_j} = r_j^2 g_j\Big|_{t=0}$$

where g_j denotes the metric induced by the immersion f_j. The measure of the blowup can then be calculated from the formula

$$\mu(\Sigma_t)\Big|_{t=t_j} = \int_\Sigma d\mu\Big|_{t=t_j} = r_j^2 \int_\Sigma d\mu_j\Big|_{t=0} = r_j^2 \mu_j(\Sigma_t)\Big|_{t=0}.$$

Thus by our choice of r_j, we know that

$$\mu(\Sigma_t)\Big|_{t=T} = \lim_{j\to\infty} r_j^2 \mu_j(\Sigma_t)\Big|_{t=0} = 0. \tag{72}$$

This of course contradicts our statement above regarding the enclosed area being strictly positive. Thus we conclude that $\tilde{f}(\tilde{\Sigma})$ has no compact components. Recall that the locally smooth convergence rules out non-smooth combinations of pieces of spheres and planes. Now Lemma 34 tells us that there must be a component of $\tilde{f}(\tilde{\Sigma})$ with nonzero curvature, ruling out flat planes; the only umbilics with unbounded area. This tells us that the component above is noncompact and nonumbillic, which is what we wished to show. □

10. Global analysis of the flow

We begin by proving that a flow (GTHF) with initial data satisfying (53) exists for all time.

Proposition 37. *Suppose* $f : \Sigma \times [0,T) \to \mathbb{R}^3$ *satisfies* (GTHF) *and* (53). *Then* $T = \infty$.

Proof. Suppose that $T < \infty$. Then Theorem 5 implies that curvature must at time T. Next, by Theorem 30 we know that A^o remains small in L^2 on the time interval $[0,T)$. We construct a blowup $\tilde{f}$ as in Section 9. We proved that the blowup $\tilde{f}$ satisfies the hypothesis of Theorem 4, and so we conclude that $\tilde{f}(\tilde{\Sigma})$ is either a plane or a sphere. But Theorem 36 tells us that $\tilde{f}(\tilde{\Sigma})$ is noncompact with a nonumbilic component, and so we have reached a contradiction. We conclude that $T = \infty$. □

Let us now identify a subsequence of times along which the flow is asymptotic to a round sphere.

Lemma 38. *Let* $f : \Sigma \times [0,T) \to \mathbb{R}^3$ *satisfy* (GTHF) *and* (53). *Then for any sequence* $t_j \nearrow \infty$ *we may choose* $x_j \in \mathbb{R}^3, \phi_j \in$ *Diff*$\left(\Sigma, \mathbb{R}^3\right)$ *such that after passing to a subsequence, the immersions* $f(\phi_j, t) - x_j$ *converge in* C^∞ *to an embedded round sphere.*

Proof. Proposition 37 implies that $T = \infty$. Let $p \in \Sigma$ and set $x_j = f(p, t_j)$. By Theorem 21, we conclude that for every t_j we have

$$\left\|\nabla_{(k)} A\right\|_\infty \Big|_{t=t_j} \leq c(k).$$

By (4), we know that if $\mu(\Sigma_0)$ it remains so, that is:

$$\mu(\Sigma)\Big|_{t=t_j} \leq \mu(\Sigma_0) < \infty. \tag{73}$$

In particular, if we consider the sequence of immersions $f_j : \Sigma \to \mathbb{R}^3$ given by $f_j(p,t) = f(p,t_j) - x_j$ then Theorem 32 guarantees the existence of a proper immersion $\tilde{f} : \tilde{\Sigma} \to \mathbb{R}^3$ (where, of course, $\tilde{\Sigma}$ is a surface without boundary) and a sequence $\phi_j \in \mathrm{Diff}\left(\Sigma, \mathbb{R}^3\right)$ such that

$$f_j(\phi_j, t) = f(\phi_j, t_j) - x_j \to \tilde{f} \text{ as } j \nearrow \infty. \tag{74}$$

Here the convergence is in the smooth topology. We define a new sequence of flows $h_j : \tilde{\Sigma}(j) \times [-t_j, \infty) \to \mathbb{R}^3$ defined by

$$h_j(p,t) = f(\phi_j(p), t_j + t) - x_j.$$

Then each h_j also satisfies the interior estimates and bounded area hypothesis of Theorem 21, and from (74) we conclude that

$$h_j(p,0) = f(\phi_j(p), t_j) - x_j \to \tilde{f} \text{ as } j \nearrow \infty.$$

That is to say, at initial time our sequence h_j converges locally in C^∞ to $\tilde{f}$. Following the same line of argument as in the proof of Theorem 33 we conclude that

$$\int_{t_j}^{t_{j+1}} \int_\Sigma |\nabla \Delta H|^2 d\mu\, d\tau \leq \int_\Sigma |A|^2 d\mu\Big|_{t=t_j} - \int_\Sigma |A|^2 d\mu\Big|_{t=t_{j+1}} \searrow 0 \text{ as } j \nearrow \infty,$$

which tells us that $\tilde{f}$ satisfies $\Delta^2 H \equiv 0$. Noting that $\tilde{f}$ here is compact, Theorem 4 then implies that $\tilde{f}$ must be a sphere. □

We now use a standard argument to obtain exponentially fast convergence of the family of immersions f in the C^∞ topology.

Proposition 39. *Suppose* $f : \Sigma \times [0,T) \to \mathbb{R}^3$ *satisfies* (GTHF) *and* (53). *Then for all* t *sufficiently large*

$$\left\|\nabla_{(k)} A\right\|_\infty \leq c_k e^{-\xi t} \text{ and } \|A^o\|_\infty \leq c_0 e^{-\xi t},$$

where $k \in \mathbb{N}$.

Proof. Because the flow (GTHF) preserves volume, the radius of the limiting sphere is

$$\rho_\infty = \sqrt[3]{\frac{3\mathrm{Vol}(\Sigma_0)}{4\pi}}.$$

By continuity, there is a time after which $f(\cdot, t)$ remains a radial graph; we linearise the radial graph evolution corresponding to (GTHF) over the limiting stationary immersion and apply the principle of linearised stability, along with a result of Lunardi [13] to obtain exponential convergence to the sphere. Specifically, for t sufficiently large, we write our surface as a radial graph over the unit 2−sphere:

$$f(z,t) = \rho(z,t)\, z,$$

for $z \in \mathbb{S}^2$, adding a tangential diffeomorphism to (GTHF) such that the parametrisation is preserved. The components of the induced metric g are

$$g_{ij} = \rho^2 \sigma_{ij} + \overline{\nabla}_i \rho \overline{\nabla}_j \rho.$$

Here σ is the metric on $\mathbb{S}^2$. We have used the identities $z \perp \overline{\nabla}_i z$ and $|z|^2 = 1$. We compute (see [16] for details)

$$A_{ij} = -\Phi(\rho)^{-\frac{1}{2}} \left(\rho \overline{\nabla}_{ij}\rho - 2\overline{\nabla}_i \rho \overline{\nabla}_j \rho - \sigma_{ij}\rho^2\right) \text{ and } g^{ij} = \rho^{-2}\left(\sigma^{ij} - \Phi(\rho)^{-1} \overline{\nabla}^i \rho \overline{\nabla}^j \rho\right),$$

where

$$\Phi(\rho) := \rho^2 + \left|\overline{\nabla}\rho\right|^2$$

and $\overline{\Delta}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^2$. Hence

$$H(\rho) = -\rho^{-1}\Phi(\rho)^{-\frac{1}{2}}\, \overline{\Delta}\rho + \rho^{-1}\Phi(\rho)^{-\frac{3}{2}}\, \overline{\nabla}^i \rho \overline{\nabla}^j \rho \overline{\nabla}_{ij}\rho + 2\Phi(\rho)^{-\frac{1}{2}} + \Phi(\rho)^{-\frac{3}{2}} \left|\overline{\nabla}\rho\right|^2, \tag{75}$$

Now consider a variation of ρ centred around the stationary spherical solution. That is,

$$\rho \mapsto \rho_\epsilon = \rho_\infty + \epsilon\eta. \tag{76}$$

Taking repeated covariant derivatives of our expression for the mean curvature H in (75) yields

$$\Delta^2 H(\rho_\epsilon) = -\rho_\epsilon^{-4}\left(\rho_\epsilon^{-1}\Phi(\rho_\epsilon)^{-\frac{1}{2}}\, \overline{\Delta}^3 \rho_\epsilon + \Phi(\rho_\epsilon)^{-\frac{3}{2}}\, \overline{\Delta}^2 \Phi(\rho_\epsilon)\right) + Q(\rho_\epsilon, \eta, \epsilon), \tag{77}$$

where Q satisfies $\frac{d}{d\epsilon}Q\Big|_{\epsilon=0} = 0$. Hence

$$\begin{aligned}
\frac{d}{d\epsilon}\Delta^2 H(\rho_\epsilon)\Big|_{\epsilon=0} &= -\frac{d}{d\epsilon}\rho_\epsilon^{-4}\left(\rho_\epsilon^{-1}\Phi(\rho_\epsilon)^{-\frac{1}{2}}\, \overline{\Delta}^3\rho_\epsilon + \Phi(\rho_\epsilon)^{-\frac{3}{2}}\, \overline{\Delta}^2\Phi(\rho_\epsilon)\right)\Big|_{\epsilon=0} \\
&= -\rho_\epsilon^{-5}\Phi(\rho_\epsilon)^{-\frac{1}{2}}\, \overline{\Delta}^3\eta\Big|_{\epsilon=0} - 2\rho_\epsilon^{-3}\Phi(\rho_\epsilon)^{-\frac{3}{2}}\, \overline{\Delta}^2\eta\Big|_{\epsilon=0} \\
&= -\rho_\infty^{-6}\left(\overline{\Delta}^3\eta + 2\overline{\Delta}^2\eta\right).
\end{aligned}$$

Hence the linearisation of (GTHF) about the stationary sphere solution with radius ρ_∞ is

$$\frac{\partial\eta}{\partial t} = \rho_\infty^{-6}\left(\overline{\Delta}^3\eta + 2\overline{\Delta}^2\eta\right) =: \mathcal{L}\eta. \tag{78}$$

It is well-known (see [2] for example) that the set of eigenvalues of the Laplacian $\overline{\Delta}$ on $\mathbb{S}^2$ is

$$\sigma\left(\overline{\Delta}\right) = \{\lambda_l : l \in \mathbb{N}_0\} = \{-l(l+1) : l \in \mathbb{N}_0\} \subset (-\infty, 0],$$

with the algebraic multiplicity of each eigenvalue λ_l being equal to the dimension of the space of homogenous, harmonic polynomials of degree l on the sphere. In particular, the multiplicities of λ_0 and λ_1 are 1 and 3 respectively. It is also well-known that the eigenvalue of the p-times iterated Laplacian $\overline{\Delta}^p$ ($p \in \mathbb{N}$), is given by

$$\sigma\left(\overline{\Delta}^p\right) = \left\{\lambda_l^p : \lambda_l \in \sigma\left(\overline{\Delta}\right)\right\}.$$

Hence it follows that the set of eigenvalues of the operator $\mathcal{L}$ in (78) is

$$\sigma(\mathcal{L}) = \left\{-\rho_\infty^{-6}(l+2)(l+1)^2 l^2 (l-1) : l \in \mathbb{N}_0\right\}.$$

The zero eigenvalue (corresponding to $l = 0, 1$) of $\mathcal{L}$ has algebraic multiplicity 4. One may follow a set of steps completely analogous to [3, pp. 1428–1430], quotienting out the zero eigenvalues and proving that [22] can be applied to conclude exponentially fast convergence to ρ_∞ in the C^∞ topology. Converting the norms on the derivatives of our immersion function into covariant derivatives of curvature allows us to establish the desired exponential convergence result of the theorem. □

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