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New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function - a review

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The book, Orthogonal Designs: Quadratic Forms and Hadamard Matrices, Marcel Dekker, New York-Basel, 1979, by A. V. Geramita and Jennifer Seberry, has now been out of print for almost two decades. Many of the results on weighing matrices presented therein have been greatly improved. Here we review the theory, restate some results which are no longer available and expand on the existence of many new weighing matrices and orthogonal designs of order 2n where n is odd. We give a number of new constructions for orthogonal designs. Then using number theory, linear algebra and computer searches we find new weighing matrices and orthogonal designs. We have reviewed completely the weighing matrix conjecture for orders 2n, n ≤ 35, n odd. The previously known results for weighing matrices for n ≤ 21 are summarized here, and new result given, leaving 3 unresolved cases. The results for weighing matrices for n ≥ 23 are presented here for the first time. For orders n, 23 ≤ n ≤ 25, 3 remain unsolved as do a further 106 cases for orders 27 ≤ n ≤ 49. We also review completely the orthogonal design conjecture for two variables in orders ≡ 2(mod 4). The results for orders 2n, n odd, 15 ≤ n ≤ 33 being given here for the first time.

Keywords

Weighing matrices, orthogonal design, sequences, autocorrelation, construction, algorithm, AMS Subject Classification: Primary 05B20, Secondary 62K05, 62K10

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New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function - a review

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Abstract

The book, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979, by A. V. Geramita and Jennifer Seberry, has now been out of print for almost two decades. Many of the results on weighing matrices presented therein have been greatly improved. Here we review the theory, restate some results which are no longer available and expand on the existence of many new weighing matrices and orthogonal designs of order $2n$ where n is odd.

We give a number of new constructions for orthogonal designs. Then using number theory, linear algebra and computer searches we find new weighing matrices and orthogonal designs.

We have reviewed completely the weighing matrix conjecture for orders $2n$, $n \leq 35$, n odd. The previously known results for weighing matrices for $n \leq 21$ are summarized here, and new result given, leaving 3 unresolved cases.

The results for weighing matrices for $n \geq 23$ are presented here for the first time. For orders n, $23 \leq n \leq 25$, 3 remain unsolved as do a further 106 cases for orders $27 \leq n \leq 49$. We also review completely the orthogonal design conjecture for two variables in orders $\equiv 2 (mod\ 4)$. The results for orders $2n$, n odd, $15 \leq n \leq 33$ being given here for the first time.

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1 Introduction

An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$, is a square matrix of order n with entries $\pm x_k$ where for each k x_k occurs s_k times in each row and column and such that the distinct rows are pairwise orthogonal.

In other words

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, defined by $\rho(n) = 8c + 2^d$, where $n = 2^a b$, b odd, and $a = 4c + d, 0 \leq d < 4$.

A weighing matrix $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n;k)$. The number k is called the *weight* of W.

Weighing matrices have long been studied because of their use in weighing experiments as first studied by Hotelling [14] and later by Raghavarao [17] and others [3, 24].

Given a set of ℓ sequences, the sequences $A_j = \{a_{j1}, a_{j2}, ..., a_{jn}\}$, $j = 1, \ldots, \ell$, of length n the *non-periodic autocorrelation function* $N_A(s)$ is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{1}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \tag{2}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{3}$$

For the results of this paper generally PAF is sufficient. However NPAF sequences imply PAF sequences exist, the NPAF sequence being padded at the end with sufficient zeros to make longer lengths. Hence NPAF can give more general results.

Notation. We use the following notation throughout this paper

1. We use $\bar{a}$ to denote $-a$.
2. Z* means the reverse of the sequence Z, for example

$$Z = \{z_1, z_2, ..., z_n\} \qquad \text{and} \qquad Z^* = \{z_n, z_{n-1}, ..., z_2, z_1\}.$$

3. $[X/Y]$ means the interleaved sequence

$$x_1,\ y_1,\ x_2,\ y_2, \cdots,\ x_n,\ y_n$$

and $[0/Y/0_m]$ means the interleaved sequence

$$0,\ y_1,\ 0_m,\ 0,\ y_2,\ 0_m, \cdots,\ 0,\ y_n,\ 0_m.$$

4. We will say that two sequences of variables are *directed* if the sequences have zero autocorrelation function independently from the properties of the variables, i.e. they do not rely on commutativity to ensure zero autocorrelation, For example $\{a, b\}$ and $\{a, -b\}$ are directed sequences while $\{a, b\}$ and $\{b, -a\}$ are not directed.

5. $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$ are two disjoint sequences if at least one of each pair x_i, y_i is non-zero.

6. Two sequences, of length n, will be said to be of type (s, t) if the sequences are composed of two variables, say a and b, and a and $-a$ occur a total of s times and b and $-b$ occur a total of t times. Such sequences will be used as the first rows of two circulant matrices in Theorem 5 to obtain an $OD(2n; s, t)$. The sequences are said to be of type $(0, \pm 1)$ and weight w if they have a total of w non-zero elements and will be used as the first rows of two circulant matrices in Theorem 5 to obtain a $W(2n, w)$.

2 Necessary Conditions and Conjectures

In this paper we concentrate on weighing matrices in orders $2n \equiv 2(mod\ 4)$. The first work towards this conjecture appears in [25].

We have two theoretical necessary conditions

Theorem 1 (Geramita and Seberry [10]) *If there is an $OD(2n; s_1, s_2)$, n odd, then $s_1 + s_2 < n$. Also s_1, s_2 and $s_1 + s_2$ are each the sum of two squares.*

Corollary 1 *If there is a $W(2n; k)$, n odd, then $k < n$ and k is the sum of two squares.*

Theorem 2 (Eades sum-fill theorem) *An $OD(2n; a, b)$ constructed from two circulants exists only if there is a 2×2 integer matrix P satisfying $PP^T =$ diag (a, b).*

The first asymptotic result of this kind is

Theorem 3 (Geramita and Seberry [9]) *Given any positive integer k there exists an integer n so that a $W(N, k^2)$ exists for all $N > n$.*

Conjecture 1 (Geramita and Seberry [10]) *For all odd n, there exists a weighing matrix $W(2n, k)$ for $k \leq 2n - 1$, $k = a^2 + b^2$, a, b integers.*

This conjecture was proved true for orders $2n = 6, 10, 14, \ldots, 30$ in [10, p331], where for all the cases except $W(18, 9)$, $W(30, 9)$, $W(30, 18)$, and $W(30, 25)$, the matrices can be constructed using two circulant matrices. We give here a two circulant construction for $W(30, 9)$, $W(30, 18)$ and $W(30, 25)$. A complete search [10] did not give a $W(18, 9)$ constructed using two circulants. We venture to conjecture

Conjecture 2 *For all odd n, there exists a weighing matrix $W(2n, k)$ for $k \leq 2n - 1$, $k = a^2 + b^2$, a, b integers constructed from two circulants for almost all k.*

In fact we believe that if $2n - k$ is very small compared with n, but not one, then there may not be a two circulants solution. If $2n - k$ is small, compared with n, the two circulants will be able to be constructed from sequences with zero PAF. Otherwise the two circulants will be able to be constructed from sequences with zero $NPAF$. Table 9 illustrates this observation when the variables are replaced by ± 1.

Eades [4] has proved asymptotic existence of many $OD(2n; a, b)$. He has in fact proved that

Theorem 4 (Eades [4]) *There exists and integer N so that for $a + b \leq 2n - 1$ and $a = (x^2 + y^2)z^2$, $b = (x^2 + y^2)w^2$ for x, y, z, w integers, an $OD(2n; a, b)$ exists for all $n \geq N$.*

One of us has proved many of these results can be constructed using two circulant matrices alone. Hence we conjecture

Conjecture 3 *Except for a small number of cases an $OD(2n; a, b)$ can be constructed from two circulants exists whenever $a + b \leq 2n - 1$ and $a = (x^2 + y^2)z^2$, $b = (x^2 + y^2)w^2$ for x, y, z, w integers.*

3 Preliminary Results

As the book of Geramita and Seberry [10] is out of print, we quote the following theorems and lemmas and totally review the existence of weighing matrices, especially though constructed using two circulant matrices.

Lemma 1 [10, Lemmas 4.21 and 4.22] *Let A and B be circulant matrices of order n and $R = (r_{ij})$ where $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise. Then $A(BR)^T = (BR)A^T$.*

The following theorem is one of the most important for construction of weighing matrices and orthogonal designs.

Theorem 5 (The two circulant construction)[10, Theorem 4.46] *If there exist two circulant matrices A_1, A_2 of order n satisfying*

$$\sum_{i=1}^{2} A_i A_i^T = fI.$$

If f is the quadratic form $\sum_{j=1}^{2} s_j x_j^2$, then there is an orthogonal design $OD(2n; s_1, s_2)$. If f is an integer there exists a $W(2n, f)$.

Proof. We use the matrices as follows

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2R \\ -A_2R & A_1 \end{pmatrix}.$$

□

We use the following known results, proved in [10], to produce asymptotic results.

Theorem 6 (Kronecker Product) *If there exists a $W(n, k)$ and a $W(m, \ell)$ then there exists a $W(mn, k\ell)$ by taking the Kronecker product of the two matrices.*

Theorem 7 (Circulant Kronecker Product) *If there exists a circulant $W(n,k)$ and a circulant $W(m,\ell)$ where $\gcd(n,m)=1$ then there exists a circulant $W(mn,k\ell)$ by taking the Circulant Kronecker product of the two matrices.*

Theorem 8 [10, Whiteman, Theorems 4.124 and 4.41] *Let q be a prime power. Then there is a circulant $W = W(q^2+q+1,\ q^2)$. Let $p \equiv 1(mod\ 4)$, p a prime power. Then there are two circulant symmetric matrices R, S of order $(p+1)/2$, with R having zero diagonal and all other entries of R and S ± 1, satisfying*

$$RR^T + SS^T = pI.$$

We note that using the R and S of Theorem 8 in Theorem 5 gives a $W(p+1,p)$ for all prime powers p.

Theorem 9 [10, Paley, Proof of Lemma 4.34] *Let q be a prime. Then there is a circulant matrix Q, of order q, with zero diagonal and other entries ± 1. Q satisfies $QQ^T = qI - J$, $QJ = JQ = 0$, $Q^T = (-1)^{(q-1)/2}Q$.*

Theorem 10 [10, (Seberry) Wallis-Whiteman, Theorem 4.124] *Let q be a prime power. Then there exists a circulant $W(q^2+q+1, q^2)$.*

The next useful lemma is well known.

Lemma 2 *If there exists a circulant $W(n,k)$ there exists a circulant $W(pn,k)$ for all $p \geq 1$.*

Proof. We consider the sequence $\{w_1, w_2, \ldots, w_n\}$. Then, writing 0_{p-1} for the sequence of $p-1$ zeros, the sequence $\{w_1, 0_{p-1}, w_2, 0_{p-1}, \ldots, w_n, 0_{p-1}\}$ gives the answer. □

Lemma 3 *If there exist two circulant matrices which give an $OD(2n;a,b)$ there exist two circulant matrices which give an $OD(2pn;a,b)$ and an $OD(2pn;2a,2b)$ for all integers $p \geq 0$.*

Proof. Write 0_{p-1} for the sequence of $p-1$ zeros. Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are the two sequences of type (a,b) and length n with zero PAF. The sequences of type (a,b) and length pn can be found by considering the sequences $X' = \{x_1, 0_{p-1}, x_2, 0_{p-1}, \ldots, x_n, 0_{p-1}\}$ and $Y' = \{y_1, 0_{p-1}, y_2, 0_{p-1}, \ldots, y_n, 0_{p-1}\}$. If we now form another sequence Y'' by permuting the first row of Y' by one position (i.e. $y'_i := y''_{i+1}$). Then X' and Y'' are disjoint. Hence $X'+Y''$ and $X'-Y''$ are two PAF sequences of length pn and type $(2a,2b)$. □

Lemma 4 *Let a and b be prime powers. Further let $p = \mathrm{lcm}(a^2+a+1, b^2+b+1)$, where $\gcd(a^2+a+1, b^2+b+1) = 1$, and $4m \geq a^2+b^2$. Then there exists and $OD(n;a^2,b^2)$ for $n \geq 4m+p-1$.*

Proof. Use circulants in the appropriate order from Theorem 10. This gives circulant $W(a^2+a+1, a^2)$ and $W(b^2+b+1, b^2)$. Now use Lemma 2 to construct circulant $W(p,a^2)$ and $W(p,b^2)$, where $p = \mathrm{lcm}(a^2+a+1, b^2+b+1)$. Hence there exists an $OD(2p;a^2,b^2)$ constructed from two circulants.

We now assume that an $OD(4m;a^2,b^2)$ exists for all $4m \geq a^2+b^2$. Hence, by direct sum there exists an $OD(n;a^2,b^2)$ for $n \geq 4m+p-1$. □

Example 1 This gives the result in Table 11 for $OD(2n;4,25)$, $n \geq 248$, $OD(2n;4,49)$, $n \geq 426$, $OD(2n;9,16)$, $n \geq 286$ and $OD(2n;9,25)$, $n \geq 420$.

Remark 1 The $W(31,16)$ found by Eades [7] is described by the sets

$$P = \{8,12,14,15,20,22,23,26,27,29\}$$

which signify the + elements and

$$N = \{1,6,10,11,19,21\}$$

which signify the $-$ elements in the first row.

Dina Torban [22] is described by the sets

$$P = \{1,4,5,6,14,16,18,20,21,23,24,25,26,30,31\}$$

which signify the + elements and

$$N = \{2,7,8,10,13,17,19,28,29,32\}$$

which signify the $-$ elements in the first row.

At about the same time as Torban found her matrix Arasu [1] also found a circulant $W(33,25)$.

The $W(87,49)$ found by Strassler [21] is described by the sets

$$P = \{5,10,12,15,18,20,23,27,35,37,38,39,53,55,59,62,65,66,70,71, 73,74,76,80,83,84,85,86\}$$

which signify the + elements and

$$N = \{1,2,7,11,14,16,17,24,25,30,32,36,45,49,50,52,54,77,78,81,82\}$$

which signify the $-$ elements in the first row.

Strassler [18, 20, 21, 22] also found circulant $W(71,25)$, $W(73,49)$, $W(87,49)$ and $W(127,64)$ exist and that circulant weighing matrices $W(57,25)$, $W(73,16)$, $W(91,16)$ and $W(91,25)$ do not exist.

The existence of circulant $W(n,k)$, which must have n odd if k is odd is of considerable independent interest. The interested reader is referred to [2, 3, 4, 5, 6, 8, 18, 19, 20].

Lemma 5 [10, Proof of Theorem 4.142] *Suppose there exists a circulant $W(n,k)$, $k \leq n$. Then there exist $OD(2n;1,k)$, $OD(2n;2,2k)$, $OD(2n;k,k)$ and $OD(2n;k+1,k+1)$ constructed from two circulants.*

Corollary 2 *Suppose q is a prime power. Then there exist $OD(2(q^2+q+1);a,b)$, constructed from two circulants, for $(a,b) = (1,q^2)$, $(2,2q^2)$, (q^2,q^2) and (q^2+1,q^2+1).*

Remark 2 Theorem 7, Theorem 10 and Remark 1 give us the following circulant matrices of small order to use in the Lemma 5 and Corollary 2: $W(7,4)$, $W(13,9)$, $W(21,16)$, $W(31,16)$, $W(31,25)$, $W(33,25)$, $W(57,49)$, $W(71,25)$, $W(73,49)$, $W(73,64)$, $W(87,49)$, $W(91,4)$, $W(91,9)$, $W(91,36)$, $W(91,81)$, $W(127,64)$, $W(133,121)$, and $W(183,169)$.

Using these results we obtain:

1. $OD(14;1,4)$, $OD(14;2,8)$, $OD(14;4,4)$, $OD(14;5,5)$,
2. $OD(26;1,9)$, $OD(26;2,18)$, $OD(26;9,9)$, $OD(26;10,10)$,
3. $OD(42;1,16)$, $OD(42;2,32)$, $OD(42;16,16)$, $OD(42;17,17)$,
4. $OD(62;1,16)$, $OD(62;1,25)$, $OD(62;2,32)$, $OD(62;2,50)$, $OD(62;16,16)$, $OD(62;17,17)$, $OD(62;25,25)$, $OD(62;26,26)$,
5. $OD(66;1,25)$, $OD(66;2,50)$, $OD(66;25,25)$, $OD(66;26,26)$,
6. $OD(114;1,49)$, $OD(114;2,98)$, $OD(114;49,49)$, $OD(114;50,50)$,
7. $OD(142;1,25)$, $OD(142;2,50)$, $OD(142;25,25)$, $OD(142;26,26)$,
8. $OD(146;1,49)$, $OD(146;2,98)$, $OD(146;49,49)$, $OD(146;50,50)$, $OD(146;1,64)$, $OD(146;2,128)$, $OD(146;64,64)$, $OD(146;65,65)$,
9. $OD(182;1,4)$, $OD(182;1,9)$, $OD(182;1,26)$, $OD(182;2,8)$, $OD(182;2,18)$, $OD(182;2,72)$, $OD(182;4,4)$, $OD(182;5,5)$, $OD(182;9,9)$, $OD(182;10,10)$, $OD(182;36,36)$, $OD(182;37,37)$,
10. $OD(254;1,64)$, $OD(254;2,128)$, $OD(254;64,64)$, $OD(254;65,65)$,
11. $OD(266;1,121)$, $OD(266;2,242)$, $OD(266;122,122)$, $OD(266;121,121)$,
12. $OD(366;1,169)$, $OD(366;2,338)$, $OD(366;169,169)$ and $OD(366;170,170)$.

Theorem 11 *Suppose q is a prime power and q^2+q+1 is a prime. Then there exists a $W(2(q^2+q+1),(q+1)^2)$ constructed from two circulants.*

If, in addition, $q^2+q+1 \equiv 3(mod\ 4)$ then an $OD(2(q^2+q+1);1,(q+1)^2)$ exists.

Proof. We form two matrices to use in Theorem 5. Use as one of the matrices A, the circulant incidence matrix of the $SBIBD(q^2+q+1,q+1,1)$ which has inner product 1 between each pair of rows. As the second matrix use the circulant matrix Q constructed in Theorem 9 which has weight q^2+q and inner product between rows of -1. These two matrices are used in Theorem 5 to give the $W(2(q^2+q+1),(q+1)^2)$.

If $q^2+q+1 \equiv 3(mod\ 4)$ then the matrix Q described in Theorem 9 is skew symmetric and so we can multiply A by a variable say x and use another variable, say y to form $yI+xQ$. These two matrices, A and $yI+xQ$ used in Theorem 5 give the $OD(2(q^2+q+1);1,(q+1)^2)$.

Corollary 3 *There exist $W(26,16)$, $W(62,36)$, $W(146,81)$, $W(266,144)$ and $OD(62;1,36)$. There also exist $OD(14;1,9)$, $OD(62;1,36)$ and $OD(146;1,81)$. All these are constructed using two circulant matrices.*

An interesting extra property to the next theorem appears in Geramita and Verner [11].

Theorem 12 [10, Theorems 2.19 and 2.20] *The existence of a skew-symmetric $W(n,k)$ is equivalent to the existence of an $OD(n;1,k)$.*

The same theorems and paper [11], which lead to the proof of non-existence of an $OD(18; 1, 16)$, leads us to conjecture:

Conjecture 4 *There is no $OD(k^2 + 2; 1, k^2)$ for k even.*

Definition 1 We define the sets $S = \{0, 1, 2, 4, 5, 8, 10, 13, 16, 20, 26\}$, $U = S \cup \{17, 25, 32, 34, 40\}$, $T = U \cup \{50, 52, 64, 65, 80\}$, $W = \{29, 36, 37, 41, 45, 49\}$, $X = W \cup \{53, 54, 58, 65, 68\}$ and $Y = X \cup \{72, 73, 74, 81, 82, 85\}$.

The results from Geramita and Seberry [10, pp162-163,pp329-336] for $OD(2n; 1, 4)$, and $OD(2n; 2, 8)$ are given in Table 14. We also note, using the constructions of Geramita and Seberry [10, pp162-163,pp329-336] and the results of Table 9, that if there exist two sequences with zero $NPAF$ of length n and type k, there exist $W(2n, k)$ or $OD(2n; k, k)$ for all orders $2m$, $m \geq n$, we obtain

Lemma 6 *There exist weighing matrices $W(2n, 2k)$ and directed orthogonal designs, $OD(2n; k, k)$ for $n = 2^a 6^b 10^c 9^d 14^e 18^f 26^g 24^h 34^i$ and $k = 2^a 5^b 10^c 13^d 17^e 25^f 26^g 34^h 50^i$, a, b, c, d, e, f, g, h, i non negative integers, constructed from two circulant matrices.*

Hence we have as a corollary the results from Geramita and Seberry [10, Lemma 4.99], that

Corollary 4 *There exist weighing matrices $W(2n, k)$ and orthogonal designs, $OD(2n; k, k)$ for $n \geq k$, $k \in S$, S given in Definition 1, $n \geq 6$ for $k = 5$, and $n \geq 9$ for $k = 13$, constructed from two circulant matrices.*

4 Sequences with Zero Autocorrelation

We now consider a few constructions which allow new sequences to be constructed from those that are already known. The last two theorems are extensions of theorems used in [10]. Other results, which are new, are inspired by [12].

Theorem 13 *Suppose X and Y are two disjoint sequences of commuting variables of length n and type (s, t) or of type $(0, \pm 1)$ and weight w with zero PAF or $NPAF$, then there are sequences of length n and type $(2s, 2t)$ or (w, w) respectively. These sequences can be used to construct $OD(2n; 2s, 2t)$ or $OD(2n; w, w)$ respectively. If the sequences have zero $NPAF$ the new sequences will have zero $NPAF$ and length $N \geq n$, however if the sequences have zero PAF the new sequences will have zero PAF and length n. Furthermore the new $OD(2n; w, w)$ sequences are directed.*

Proof. If the original sequences were of type (s, t) use $X + Y$, $X - Y$ to get the sequences of type $(2s, 2t)$. If the original sequences were of type $(0, \pm 1)$ and weight w use $aX + bY$ and $bX^* - aY^*$ to obtain the directed sequences of type (w, w). □

Corollary 5 *$OD(2n; 9, 9)$, $n = 15, 17, 19, 21, 23, 25$ exist constructed from two circulants.*

Proof. Using the sequences in Table 9 we find the result.

Example 2 The sequences $X =$`a 0 0 0 a 0` and $Y =$ `0 a 0 b 0-a` yield the following sequences with $NPAF = 0$

$X + Y =$`a a 0 b a-a`; and $X - Y =$ `a-a 0-b a a`

which give sequences with NPAF $= 0$ and type $(2, 8)$ and $OD(2n; 2, 8)$ for all $n \geq 6$. The sequences $X =$`1 0 1 0 - 0` and $Y =$ `0 1 0 0 0 1` yield the following sequences with $NPAF = 0$

$aX + bY =$`a b a 0-a b`; and $bX^* - aY^* =$`-a-b 0 b-a b`

which give directed sequences with NPAF $= 0$ and type $(5, 5)$ and $OD(2n; 5, 5)$ for all $n \geq 6$.

Lemma 7 *Suppose* $X = \{x_1, x_2, \ldots, x_n\}$ *and* $Y = \{y_1, y_2, \ldots, y_n\}$ *are two sequences of commuting variables of type* (s, t) *or of type* $(0, \pm 1)$ *and weight* $s + t$ *with zero PAF or NPAF. Then there are two directed sequences of length* $N = n(m + 2)$, $m \geq 0$ *which give* $OD(2N; s + t, s + t)$. *If the sequences have* $NPAF = 0$ *the last* 0_p *in the construction can be discarded and we get directed* $NPAF$ *sequences of length* $N \geq n(p + 2) - p$, $p \geq 0$. *If the sequences have* $PAF = 0$ *we get directed* PAF *sequences of length* $n(p + 2)$.

Proof. Let a and b be commuting variables. Then the sequences

$$ax_1,\ by_1,\ 0_p,\ ax_2,\ by_2,\ 0_p, \cdots,\ ax_n,\ by_n,\ 0_p$$

$$\bar{a}y_n,\ bx_n, 0_p,\ \bar{a}y_{n-1},\ bx_{n-1},\ 0_p, \cdots,\ \bar{a}y_1,\ bx_1,\ 0_p$$

are the sequences required if X and Y have type $(0, \pm 1,)$ and weight $s + t$. If X, Y have variables then setting $a = b = 1$ gives the required sequences.

We note that the term in the autocorrelation function which arises from the x_i of X with the y_j of Y will appear as ax_iby_j in $a[X/0_{p+1}] + b[0/Y/0_p]$ and $-ay_jbx_i$ in $\bar{a}[Y^*/0_{p+1}] + b[0/X/0_p]$, where $[0/Y/0_p]$ means the interleaved sequence.

The construction ensures these terms cancel. □

As the derived sequences are directed their variables can be replaced by sequences.

Example 3 1) The sequences `1 1 -` and `1 0 1` for X and Y, yield the following directed sequences of length 9 and type $(5, 5)$ with $NPAF = 0$

`a b 0 a 0 0-a b 0; -a-b 0 0 b 0-a b 0.`

2) The sequences `- 1 1 1 1` and `- 1 1 - 0` for X and Y, yield the following directed sequences of length 15 and type $(9, 9)$ with $PAF = 0$

`0 b 0 a b 0-a b 0-a b 0 a-b 0;-a-b 0 a b 0 a b 0 a-b 0 a 0 0.`

Corollary 6 *Suppose* $q \equiv 1 (\mathrm{mod}\ 4)$ *is a prime power. Then there exist* $OD(m(q + 1); q, q)$ *for all* $m \geq 1$.

Proof. From Theorem 10, for $q \equiv 1 (\mathrm{mod}\ 4)$ a prime power we have two $0, \pm 1$ sequences of length $\frac{1}{2}(q+1)$ and weight q. We now use these sequences in the previous lemma to obtain the result.

Example 4 Using the appropriate $0, \pm 1$ sequences in the previous lemma, and from Table 9, we multiply the length by 3 and 5 to obtain the new sequences of Table 1.

Type	Length	Weight	New Length	New Type and Property
$0, \pm 1$	3	5	$3m,\ m \geq 2$	(5,5) NPAF(D)
$0, \pm 1$	5	9	$5m,\ m \geq 2$	(9, 9) PAF(D)
$0, \pm 1$	7	13	21	(13,13) PAF(D)
$0, \pm 1$	9	13	$9m,\ m \geq 2$	(13,13) PAF(D)
$0, \pm 1$	11	13	$11m,\ m \geq 2$	(13,13) PAF(D)
$0, \pm 1$	9	17	27	(17,17) PAF(D)
$0, \pm 1$	9	17	$9m,\ m \geq 2$	(17,17) PAF(D)
$0, \pm 1$	11	17	$11m,\ m \geq 2$	(17,17) PAF(D)
$0, \pm 1$	13	25	39	(25,25) PAF(D)
$0, \pm 1$	15	29	45	(29,29) PAF(D)
$0, \pm 1$	15	29	75	(29,29) PAF(D)
$0, \pm 1$	19	37	57	(37,37) PAF(D)
$0, \pm 1$	19	37	95	(37,37) PAF(D)
$0, \pm 1$	21	43	63	(41,41) PAF(D)
$0, \pm 1$	25	49	75	(49,49) PAF(D)
$0, \pm 1$	27	53	81	(53,53) PAF(D)
$0, \pm 1$	31	61	93	(61,61) PAF(D)
$0, \pm 1$	37	73	111	(73,73) PAF(D)

Table 1: Sequences constructed using Theorems of this section.

Theorem 14 *Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two sequences of commuting variables of type (s,t) or of type $(0, \pm 1)$ and weight w with zero PAF or NPAF, where at least one of y_i and y_{n+2-i} is zero for each i and y_1 is zero. Then there are two sequences of length $3n$ which give $OD(6n; w, w)$, $OD(6n; 2w, 2w)$, $OD(6n; 2s, 2w-2s)$ and $OD(6n; 4s, 4w-4s)$. Furthermore if the original sequences were directed, and both begin with zero, then the new sequences are also directed. (If X and Y have $NPAF = 0$, and do not necessarily start with zero, then the sequences obtained may be used to construct an $OD(2N; s+t, s+t)$ for all $N \geq 3n$.)*

Proof. Let a and b be commuting variables. Then, with X and Y of type $(0, \pm 1)$ and weight w, the sequences

$$ax_1,\ by_1,\ 0_p,\ ax_2,\ by_2,\ 0_p, \cdots,\ ax_n,\ by_n,\ 0$$

$$0,\ \bar{a}y_1,\ bx_1,\ 0_p,\ \bar{a}y_n,\ bx_n,\ 0_p, \cdots,\ \bar{a}y_2,\ bx_2$$

are disjoint the sequences which can be used to construct the $OD(N; w, w)$, $N = (n-1_(p+2)+3,\ p \geq 0$. If X and Y of type (s,t) on the variables c and d, set

$a = b = 1$ and using the variables in the sequences we obtain disjoint sequences which give an $OD(6n; 2s, 2t)$. Furthermore, because of the properties of the y_i

$$ax_1,\ by_1 + \bar{a}y_1,\ bx_1,\ ax_2,\ by_2 + \bar{a}y_n,\ bx_n,\ ax_3, \cdots,\ ax_n,\ by_n + \bar{a}y_2,\ bx_2$$

$$ax_1,\ by_1 + ay_1,\ \bar{b}x_1,\ ax_2,\ by_2 + ay_n,\ \bar{b}x_n,\ ax_3, \cdots,\ ax_n,\ by_n + ay_2,\ \bar{b}x_2$$

are the sequences required to construct the $OD(3n; 2w, 2w)$ or setting $a = b = 1$ and using the variables in the sequences we obtain an $OD(2N; 4s, 4w - 4s)$. □

Example 5 Consider the sequences X and Y to be `0 1 1 - 0` and `0 0 1 0 1`. These are of length 5, type $(0, \pm 1)$ and weight 5 and yield the following sequences with $NPAF = 0$

```
0 0 0 a 0 0 a b 0-a 0 0 0 b 0; 0 0 0 0-a 0 0 0-b 0-a b 0 0 b
0 0 0 a-a 0 a b-b-a-a b 0 b b; 0 0 0 a a 0 a b b-a a-b 0 b-b
```

which may be used to give an $OD(4p + 11; 5, 5)$, $p \geq 0$, and an $OD(2n; 10, 10)$, $n \geq 12$. The sequences `a b-a 0 0 ; 0 0 a 0 a` which yield an $OD(10; 1, 4)$ give the following sequences which have zero $NPAF$

```
0 0 0 a 0 0 b a 0-a 0 0 0 a 0; 0 0 0 0-a 0 0 0-a 0-a b 0 0 a
0 0 0 a-a 0 b a-a-a-a b 0 a a; 0 0 0 a a 0 b a a-a a-b 0 a-a
```

and these may be used to construct an $OD(4p + 11; 2, 8)$, $p \geq 0$, and an $OD(2n; 4, 16)$, $n \geq 12$.

Use the $OD(62; 1, 36)$ to make $OD(186; s, t)$ for $(s, t) = (36, 36)$, $(37, 37)$, $(72, 72)$ and $(74, 74)$. None of these are new but illustrate the potential.

Theorem 15 *Suppose there exist two disjoint sequences P and Q of length m with zero autocorrelation function and type (a, b) or of type $(0, \pm 1)$ and weight $a + b$. Further suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two sequences of commuting variables of type (s, t) or of type $(0, \pm 1)$ and weight $s + t$ with zero PAF or NPAF. Then*

$$Px_1 + Qy_1,\ Px_2 + Qy_2, \cdots, Px_n + Qy_n$$

$$Qx_1 - Py_1,\ Qx_2 - Py_2, \cdots, Qx_n - Py_n$$

are two sequences of length nm. They are directed if P, Q, X, Y are directed. The new sequences are of type $(a(s + t), b(s + t))$ if X and Y are of type $0, \pm 1$ or $((a + b)s, (a + b)t)$ if P and Q are of type $0, \pm 1$. (If P and Q, and X and Y, have $NPAF = 0$ then the sequences obtained may be used to construct an $OD(2N; a(s + t), b(s + t))$ and an $OD((a + b)s, (a + b)t)$, respectively for all $N \geq mn$.)

Corollary 7 *Suppose there exist two disjoint sequences P and Q of length n with zero autocorrelation function and type (a, b). Then*

$$A = \{P + Q,\ P,\ -P + Q\} \quad \text{and} \quad B = \{P + Q,\ -Q,\ P - Q\}$$

are two sequences of length $3n$ and type $(5a, 5b)$ which may be used to form an $OD(6n; 5a, 5b)$. If P and Q are directed A and B are directed. If P and Q are *two disjoint sequences of length n with zero autocorrelation function and of type $0, \pm 1$ and weight a then*

$$A = \{xP + xQ,\ yP,\ -xP + xQ\} \quad \text{and} \quad B = \{xP^* + xQ^*,\ -yQ^*,\ xP^* - xQ^*\}$$

are two directed sequences of length $3n$ and type $(2N; a, 4a)$ which may be used to form an $OD(6n; a, 4a)$. (If P and Q have $NPAF = 0$ then the sequences obtained may be used to construct an $OD(2N; 5a, 5b)$ and an $OD(2N; a, 4a)$, respectively for all $N \geq 3n$.)

Example 6 1) $P = \{$`a 0 b 0-a 0`$\}$ and $Q = \{$`0 a 0 0 0 a`$\}$ are disjoint sequences with zero NPAF of type (1,4) and length 6. Then there exist sequences of length $n \geq 18$ and type $(5, 20)$. 2) $P = \{$`a a-a b 0 b 0 0 0 0`$\}$ and $Q = \{$`0 0 0 0 a 0 a b-b-b`$\}$ are disjoint directed sequences with zero NPAF of type (5,5) and length 10. Then there exist directed sequences of length 30 and type $(25, 25)$. Hence there exists an $OD(2n; 25, 25)$ and an $OD(2n; 10, 40)$ for every $n \geq 30$, and an $OD(2m; 25, 100)$ for every $m \geq 90$. □

Corollary 8 *Suppose there exist two disjoint sequences P and Q of length n with zero autocorrelation function and type (a, b). Then there are two sequences of length $14n$ and type $(13a, 13b)$, used to form an $OD(28n; 13a, 13b)$, if P and Q had $PAF = 0$ and an $OD(2n; 13a, 13b)$, $n \geq 14$, if P and Q had $NPAF = 0$. Also, if P and Q had $PAF = 0$, there are two sequences of length $28n$ which are used to form an $OD(56n; 17a, 17b)$. If P and Q had $NPAF = 0$, the two sequences of length $28n$ can be used to form an $OD(2n; 13a, 13b)$, $n \geq 28$. If P and Q are directed A and B are directed. (If P and Q have $NPAF = 0$ then the sequences obtained may be used to construct an $OD(2N; 25a, 25b)$ for all $N \geq 18n$.)*

Theorem 16 *Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two directed sequences of type (a, a) and length n with zero NPAF or PAF. Further suppose that P and Q are sequences of length m with zero NPAF and type (s, t). Then replacing the variables of X and Y by the sequences P and Q gives sequences of type (as, at) of length $N \geq mn$ with zero $NPAF$ or length $N = mn$ with zero PAF.*

Proof. X and Y being directed ensures that the overlaps in the cross correlation in the autocorrelation function by introducing the sequences P and Q are cancelled out. □

Example 7 The directed sequences with zero $NPAF$ `c c-c d 0 d; c 0 c d-d-d` give an $OD(2n; 5, 5)$, $n \geq 6$. Replace c by `b 0 b` and d by `b a-b` to obtain two sequences,

```
b 0 b b 0 b-b 0-b b a-b 0 0 0 b a-b;
b 0 b 0 0 0 b 0 b b a-b-b-a b-b-a b
```

which can be used to form an $OD(2m; 5, 20)$, $m \geq 18$.

The directed sequences with zero $NPAF$ `c-c d d; c c d-d` give an $OD(2m; 4, 4)$ for all $m \geq 4$. Replace c by `b 0 b` and d by `b a-b` to obtain the two sequences with NPAF = 0,

```
b 0 b-b 0-b b a-b b a-b ; b 0 b b 0 b b a-b-b-a b
```

which can be used to form an $OD(2n;4,16)$ for all $n \geq 12$.

See Lemma 9 for applications of this Theorem.

Corollary 9 *An $OD(14n;4,36)$, $n \geq 4$, exists constructed from two circulants. An $OD(154;9,81)$ exists constructed from two circulants.*

Proof. We use the directed sequences which give an $OD(2m;4,4)$ for all $m \geq 4$ from the last example. We then replace the variables by the two sequences of length 7 which give an $OD(14;1,9)$ from Table 5 or the two sequences . The theorem then gives the result.

For the second result we take the directed sequences of length 11 and type $(9,9)$ with zero PAF and the sequences of length 7 and type $(1,9)$ with zero PAF. The theorem then gives the result. □

Example 8 We use the directed PAF (9,9) which exist for lengths 10, 11, 13, 15 and 23 with the NPAF (1,4) of length $n \geq 3$ to get (9,36) of lengths $10n$, $11n$, $13n$, $15n$ and $23n$ for all $n \geq 3$. Similarly we use the directed PAF (9,9) with the NPAF (2,8) of length $n \geq 6$ to get (18,72) of lengths $10n$, $11n$, $13n$, $15n$ and $23n$ for all $n \geq 6$. We use the directed PAF (18,18) which exist for lengths 19 and 21 with the NPAF (1,4) of length $n \geq 3$ to get (18,72) of lengths $19n$ and $21n$ for all $n \geq 3$.

5 Existence of Weighing Matrices

We now review and extend the results of [10] for the existence of $W(2n,k)$, $3 \leq n \leq 21$.

Lemma 8 *Suppose $k \leq 2n-1$ and $k = a^2 + b^2$ where a and b are integers. Then there exists a $W(2n,k)$ for all possible k for $n = 3, 5, 7, 9, 11, 13, 15, 17, 19, 21$. All, except the $W(18,9)$, which cannot be constructed from two circulants and the $W(46,45)$, $W(54,9)$ and $W(54,18)$, which are not known constructed from two circulants, are constructed using two circulant matrices.*

Proof. We consider the orders individually.

Case n = 3: Table 13 gives sequences which can be used to obtain the result.

Case n = 5: The sequences `0 1 - - 1; - 1 1 1 1` give the required result for $k = 9$. Table 13 gives the sequences which can be used to obtain the other results.

Case n = 7: The sequences `0 1 1 - - 1 1; 1 - 1 1 1 1 -` give the required result for $k = 13$. Tables 9 and 13 give the sequences which can be used to obtain the other results.

Case n = 9: A complete search was not able to find a $W(18,9)$ constructed from circulants. We give a $W(18,9)$ in Table 2 for completeness.

Tables 9 and 13 give the sequences which can be used to obtain the other results.

1 0 0 0	1 1 1 1	1 1 1 1	0 0 0	0 0 0
0 1 0 0	1 1 1 1	- - - -	0 0 0	0 0 0
0 0 1 0	1 1 - -	1 1 - -	0 0 0	0 0 0
0 0 0 1	1 1 - -	- - 1 1	0 0 0	0 0 0
1 1 1 1	- 0 0 0	0 0 0 0	0 0 1	1 1 1
1 1 1 1	0 - 0 0	0 0 0 0	0 0 -	- - -
1 1 - -	0 0 - 0	0 0 0 0	0 0 1	1 - -
1 1 - -	0 0 0 -	0 0 0 0	0 0 -	- 1 1
1 - 1 -	0 0 0 0	- 0 0 0	1 1 0	0 1 -
1 - 1 -	0 0 0 0	0 - 0 0	- - 0	0 - 1
1 - - 1	0 0 0 0	0 0 - 0	1 - 1	- 0 0
1 - - 1	0 0 0 0	0 0 0 -	- 1 -	1 0 0
0 0 0 0	0 0 0 0	1 - 1 -	- 0 1	- 1 -
0 0 0 0	0 0 0 0	1 - - 1	0 - -	1 1 -
0 0 0 0	- 1 - 1	0 0 1 -	1 - -	0 0 0
0 0 0 0	- 1 - 1	0 0 - 1	- 1 0	- 0 0
0 0 0 0	- 1 1 -	1 - 0 0	1 1 0	0 - 0
0 0 0 0	- 1 1 -	- 1 0 0	- - 0	0 0 -

Table 2: The $W(18,9)$ from [10, p333].

Case n = 11: Tables 9 and 13 give the sequences which can be used to obtain the result.

Case n = 13: Tables 9 and 13 gives the sequences which can be used to obtain the other results.

Case n = 15: The W(15,9) is (see [10, p325]) found in Table 3.

Tables 9 and 13 give the sequences which can be used to obtain the other results.

Case n = 17: Tables 9 and 13 give the sequences which can be used to obtain the results.

Ohmori and Miyamoto [16] have found the $W(17,9)$ given in Table 4.

Case n = 19: Tables 9 and 13 give the sequences which can be used to obtain the other results.

Case n = 21: The result for $k = 34$ is given by Remark 2. Tables 9 and 13 give the sequences which can be used to obtain the other results. □

In this paper we have for the first time:

Lemma 9 *There exist orthogonal designs constructed from two circulants for orders* $2n$ *as indicated.*

1. $W(2n;13)$, $n \geq 9$,
2. $W(2n;17)$, $n \geq 13$,
3. $W(2n;25)$, $n \geq 18$,
4. $W(2n;34)$, $n \geq 24$,
5. $W(2n;50)$, $n \geq 30$
6. $W(2n;65)$, $n \geq 42$
7. $W(2n;68)$, $n \geq 40$

Proof. Use Table 9 to obtain results 1 to 7. □

0 0 0	1 1 1	0 0 0	1 1 1	- - -
0 0 0	1 1 1	- - -	0 0 0	1 1 1
0 0 0	1 1 1	1 1 1	- - -	0 0 0
1 1 1	0 - 1	0 0 0	0 1 -	0 - 1
1 1 1	- 1 0	0 0 0	1 - 0	- 1 0
1 1 1	1 0 -	0 0 0	- 0 1	1 0 -
0 - 1	0 0 0	1 0 0	- 1 1	- 1 1
0 - 1	0 0 0	0 0 1	1 1 -	1 1 -
0 - 1	0 0 0	0 1 0	1 - 1	1 - 1
1 0 -	0 1 -	- 1 1	0 1 0	0 0 1
1 0 -	1 - 0	1 1 -	1 0 0	0 1 0
1 0 -	- 0 1	1 - 1	0 0 1	1 0 0
- 1 0	0 - 1	- 1 1	0 0 1	0 1 0
- 1 0	- 1 0	1 1 -	0 1 0	1 0 0
- 1 0	1 0 -	1 - 1	1 0 0	0 0 1

Table 3: The W(15,9) from [10, p325].

1	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	1	1	1	1	1	-	-	-	-	0	0	0	0	0	0	0
1	1	1	-	-	0	1	-	0	0	1	1	0	0	0	0	0
1	1	-	0	0	0	1	0	-	0	-	-	1	1	0	0	0
1	1	-	0	0	0	-	0	1	0	0	0	-	0	1	1	1
1	1	0	0	0	-	-	1	0	0	0	0	0	-	-	-	-
1	-	1	0	0	-	0	0	-	0	-	0	0	-	0	1	1
1	-	-	0	0	1	0	0	0	-	0	1	0	0	-	-	1
1	-	0	1	-	0	0	0	0	-	1	-	0	0	1	0	-
1	-	0	-	1	0	-	-	0	1	0	0	0	1	0	0	-
0	0	1	-	0	0	0	1	0	-	-	0	-	1	1	-	0
0	0	1	0	-	0	-	0	1	0	0	-	1	1	-	0	1
0	0	0	1	-	0	0	0	0	0	-	1	-	1	-	1	-
0	0	0	1	0	-	0	-	1	0	-	1	1	0	1	-	0
0	0	0	1	0	-	0	0	-	1	1	0	-	1	0	-	1
0	0	0	0	1	-	0	1	0	-	1	1	1	1	0	1	0
0	0	0	0	1	-	1	-	1	-	0	-	-	0	-	0	0

Table 4: The $W(17, 9)$ found by Ohmori and Miyamoto [16]

Summarizing we have

Lemma 10 *Write S for the set given in Definition 1. There exist $W(2n,k)$ constructed from two circulants for*

1. $n \geq 14,\ k \in \{S, 17\}$;
2. $n \geq 18,\ k \in \{S, 17, 25, 32\}$;
3. $n \geq 24,\ k \in \{S, 17, 25, 32, 34, 40\}$;
4. $n \geq 26,\ k \in \{S, 17, 25, 32, 34, 40, 52\}$;
5. $n \geq 30,\ k \in \{S, 17, 25, 32, 34, 40, 50, 52\}$;
6. $n \geq 42,\ k \in \{S, 17, 25, 32, 34, 40, 50, 52, 64, 65, 80\}$;
7. $n \geq 46,\ k \in \{S, 17, 25, 32, 34, 40, 50, 52, 64, 65, 68, 80\}$;

Proof. The results in Parts 1 to 5 follow immediately for the existence of sequences with the required weight and NPAF = 0 given in Table 9 and Lemma 6. The result in Parts 6 and 7 for $k = 65$ follows from multiplying the directed $OD(2n; 13, 13)$ which exists for all $n \geq 14$ with the two sequences of length 3 and type $(1, 4)$ which given an $OD(2n; 13, 52)$ and a $W(2n; 65)$ constructed from two circulants for all $n \geq 42$. The results in Part 7 follows from the existence of sequences of weight 68 and NPAF = 0 for all lengths ≥ 48 constructed using Lemma 3 and the result for weight 34, NPAF = 0 from Table 9, combined with the existence two sequences of length 23, weight 34 and PAF = 0 from the same Table. □

We also note from Table 9 that $W(2n, 9)$ exist, constructed from two circulants, for 7, 10, 11, 13, 15, 17, 19, 23 and 25 and multiples of these lengths by Lemma 3, Table 9 and [13].

We have

Lemma 11 *Let $k = a^2 + b^2$, $k \leq 2n - 1$. Then there exist $W(2n, k)$ constructed from two circulants except for*

1. $n = 23,\ k \in \{41, 45\}$;
2. $n = 25,\ k \in \{41, 45\}$;
3. $n = 27,\ k \in \{9, 18, 29, 36, 37, 41, 45, 49, 50\}$;
4. $n = 29,\ k \in \{9, 18, 29, 36, 37, 41, 45, 49, 50, 53\}$;
5. $n = 31,\ k \in \{9, 18, 29, 45, 49, 53, 58\}$;
6. $n = 33,\ k \in \{29, 37, 41, 49, 53, 58, 61, 65\}$;
7. $n = 35,\ k \in \{9, 18, 29, 37, 41, 45, 49, 53, 58, 61, 65, 68\}$;
8. $n = 39,\ k \in \{29, 41, 49, 53, 58, 61, 68, 71, 72\}$.

In addition there exist $W(2n, k)$ not constructed from two circulants for

Order $2n$	Non-Existent for two Circulants	Unresolved for two Circulants	Exists but Unresolved for two Circulants	Comments and References
6		None		
10		None		
14		None		
18	9	None		
22		None		
26		None		
30		None		
34		None		
38		None		
42		None		Table 9
46		41,45	45	Table 9
50		41,45		Table 9
54		9,18,W,50	9,18	Table 9
58		9,18,W,50 53,54	9,18	Table 9
62		9,18,W,53 54,58	9,18,29,36,50	Table 9, [3]
66		9,18,W,53 54,58,65	9,18,29,36,50	Table 9, [3]
70		X	29,36,50	Table 9, [3]
74		9,18,X,65,72	9,18,29,36,50	Table 9, [3]
78		X,72,73,74	29,36,37,50,65	Table 9, [3]
82		9,18,X,72 73,74	9,18,29,36 37,50,65,68	Table 9, [3]
86		9,18,Y	9,18,29,36,37 41,50,65,68	Table 9, [3]
90		Y	29,36,37,41,50 65,68	Table 9, [3]
94		9,18,Y,89,90	9,18,29,36,37 41,45,50,65,68	Table 9, [3]
98		Y,89,90	29,36,41,45,37 50,65,68	Table 9, [3]

Table 5: Weighing Matrices Exist for all integers $k = a^2 + b^2$, $k \leq 2n - 1$ unless noted – W, X, Y given in Definition 1.

w	n
2	2...
4	4...
5	6...
8	8...
10	12...
13	14...
16	16...
17	18...
18	20...
20	20...
26	28...
32	32...
34	36...
40	40...
52	52...
80	80...
104	104...
128	128...
130	140...
160	160...

w	n
29	30, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, (54), 56
	(58), 60...
37	38, 42, 44, 46, 48, 50, 52, (54), 56, (58), 60, 62, 64
	(66), 68, (70), 72...
41	42, 44, (46), 48, (50), 52, (54), 56, (58), 60, 62, 64
	(66), 68, (70), 72, (74), 76, (78), 80, (82), 84...
45	46, 48, (50), 52, 54, 56, (58), 60, (62), 64, 66, 68, 70
	72, (74), 76, 78, 80, (82), 84, (86), 88...
50	52, (54), 56, (58), 60...
53	54, 56, (58), 60, (62), 64, (66), 68, (70), 72, (74), 76
	(78), 80, (82), 84, (86), 88, (90), 92, (94), 96, (98)
	100, (102), 104, (106), 108, (110), 112, (114...146)
	148...
58	60, (62), 64, (66), 68, (70), 72, (74), 76, (78), 80
	(82), 84, (86), 88, 90, 92, (94), 96, (98), 100, (102)
	104, (106), 108...
61	62, 64, (66), 68, (70), 72, (74), 76, (78), 80, (82), 84
	(86), 88, (90), 92, (94), 96, (98), 100, (102), 104, (106)
	108, (110), 112, (114), 116, (118), 120, (122), 124...
65	(66), 68, (70), 72, (74), 76, 78...
68	68, (70), 72, (74), 76, (78), 80...
72	72, (74...110), 112, 114, 116, (118...186), 184...
125	126, 128, (130...142), 144, (146...154), 156, (158), 160
	(162...166), 168, (170, 172, 174), 176, (178), 180...

Table 6: Update of Craigen's Table 52.29 [3] Weighing matrices $W(n,w)$, $a^2 \neq w = b^2 + c^2 \leq 72$, $w = 80,\ 104,\ 125,\ 128,\ 130,\ 160$ (even n only)

1. $n = 23$, $k \in \{45\}$;

2. $n = 27,\ 29,\ 31$, $k \in \{9, 18\}$;

3. $n = 31,\ 33,\ 35,\ 39$, $k \in \{29\}$.

Proof. We consider the orders individually.
Case n = 23: The $W(46, 45)$ is given in [23] and [15]: it is not constructed from two circulant matrices. The results, except for 41, are given in Tables 9 and 13.
Case n = 25: The results, except for 41 and 45, are given in Tables 9 and 13.

Case n = 27: The designs known for k = 9,18 are not constructed using two circulants. The results, except for 29,36,37,41,45,50 are given in Tables 9 and 13.

Case n = 29: The designs known for k = 9,18 are not constructed using two circulants. The results, except for 29,36,37,41,45,49,50,53 are given in Tables 9 and 13.

Case n = 31: The designs known for k = 9,18,29 are not constructed using two circulants. The results, except for 37,45,49,50,53,58 are given in Tables 9 and 13. Corollary 3 gives the result for $k = 36, 37$, Remark 2 gives the result for $k = 25, 50$ and the result for $k = 41$ comes from the $W(31, 16)$ and $W(31, 25)$ in Remark 1.

Case n = 33: The designs known for k = 29 are not constructed using two circulants. Remark 2 part 5 gives the result for $k = 25, 50$. The results, except for 36,37,41,49,53,58,61,65 are given in Tables 9 and 13. We use the directed sequences of length 11 and type (9,9), with PAF=0, and replace the variables by the sequences `a b-a ;a 0 a` to obtain sequences of length 33 and type $(9, 36)$. This means we have two sequences which give weights 9, 36 and 45. Furthermore the sequences of weight 9 can be used in Lemma 3 to obtain two PAF sequences of length 33 and weight 18.

Case n = 35: The designs known for k = 9,18,29 are not constructed using two circulants. The results, except for 37,41,45,49,53,58,61,65 are given in Tables 9 and 13.

Case n = 39: The design known for k = 29 is not constructed using two circulants. There is a circulant $W(13, 9)$ and hence a circulant $W(39, 9)$ and an $OD(78; 9, 9)$ constructed from two circulants. This gives the results for 9 and 18. There are two sequences given in Table 13 which are directed, of length 13 and type (9,9) which when used with the two sequences of length 3 and type $(1, 4)$ give an $OD(78; 9, 36)$ and hence a $W(78, 45)$ constructed from two circulants. The results, except for 41,49,53,58,61,71,72 are given in Tables 9 and 13. □

6 Existence of Orthogonal Designs

Theorem 17 *There exist orthogonal designs in order* $2n$ *of type:*

1. $(1, 9)$ *constructed using two circulant matrices exists for lengths* 9, 11, 17, 27, 29, 31, 33, 37, 41, 43, 47 *and* 51 *and their multiples;*

2. $(1, 9)$ *for* $n \geq 6$ *(except possibly for* $n = 9,\ 11$*);*

3. $(4, 9)$ *constructed using two circulant matrices exists for lengths* 19 *and* 21 *and their multiples;*

4. $(4, 9)$ *for* $n \geq 7$ *(except possibly for* $n = 9, 11, 13$*);*

5. $(1, 16)$ *constructed using two circulant matrices exists for lengths* 11, 13, 15, 17, 19 *and* 21 *and their multiples; an* $OD(18 : 1, 9)$*, i.e.* $n = 9$*, does not exist.*

6. $(1, 16)$ *for* $n \geq 10$.

Proof. The results that (1,9) does not exist, constructed from two circulants, for lengths 9, 11, 17, 27, 29, 31, 33, 37, 41, 43, 47 and 51 and does exist for lengths 7, 10, 13, 15, 19, 21, 23, 25, 35, 39 and 45 follows from Table 13 and [13]; the existence of $(1, 9)$ for $2n = 12, 14, 16, 20, 24$, and 30 gives the remainder of Part 2.

The non-existence of the $(4, 9)$, constructed from two circulants, for lengths 7, 9, 10, 11, 13, 15, 17 and existence, constructed from two circulants,for lengths 19 and 21, comes from Table 13 and [13]; the existence of $(4, 9)$ for $2n = 14$ is given explicitly above, and the for $2n = 16, 20, 24$ and 28 gives the remainder of Part 4.

That the $(1, 16)$ does exist, constructed from two circulants, for lengths 11, 13, 15, 17, 19 and 21 and does not exist for lengths 9 follows from Table 13 and [13]; the existence of $(1, 16)$ for $2n = 24, 26, 28, 30, 32, 34, 36$ and 38 gives the remainder of Part 6. □

For reference we now give details of the existence of orthogonal designs in order $2n \equiv 2(mod\ 4)$. The results in Lemma 12 are given in Geramita and Seberry [10, pp329-331], plus the new $OD(22; 1, 16)$, $OD(22; 4, 16)$, $OD(26; 4, 16)$ from Table 13 and the new $OD(26; 1, 9)$ and $OD(26; 2, 18)$ given in Remark 2.

Lemma 12 *The necessary conditions are sufficient for the existence of orthogonal designs* $OD(2n; a, b)$*,* $3 \leq n \leq 13$*, except for possibly* $OD(18; 1, 9)$*,* $OD(18; 4, 9)$*,* $OD(22; 1, 9)$*,* $OD(22; 4, 9)$*,* $OD(22; 9, 9)$*,* $OD(26; 4, 9)$ *and* $OD(26; 5, 20)$*. An* $OD(18; 1, 16)$ *does not exist. The necessary conditions are sufficient for the existence of orthogonal designs* $OD(2n; a, b)$*,* $3 \leq n \leq 13$*, constructed using two circulant matrices except for* $OD(2n; c, d)$ *for* $(c, d) = (1, 9)$ *or* $(4, 9)$*, for* $n = 9$ *or* 11 *and an* $OD(18; 1, 16)$*.*

The $OD(14; 4, 9)$ is given in Table 7 for completeness.

The existence or otherwise of $OD(2n; a, b)$, $15 \leq n \leq 27$, $a + b \leq 2n - 1$ and $a = (x^2 + y^2)z^2$, $b = (x^2 + y^2)w^2$ for x, y, z, w integers, is summarized in Table 12. These results have not been previously published.

Lemma 13 *There exist* $OD(2n; a, b)$ *for the types given in Table 8.*

Proof. $OD(62; 1, 25)$ and $OD(66; 1, 25)$ exist from Remark 2. $OD(2n; 1, 25)$ exist for $2n \geq 88$ from Theorem 17. The existence of all $OD(4m; 1, 25)$, $4m \geq 28$, gives the remainder of the result.

$OD(62; 1, 36)$ exists, from Corollary 3, and given the existence of all $OD(4m; 1, 36)$, $4m \geq 40$ gives the result for $OD(2n; 1, 36)$ for $2n \geq 50$.

0	x	y	y	y	x	x	x	y	y	y	y	y	y
x	0	x	x	x	-y	-y	-y	y	y	y	-y	-y	-y
y	x	0	-y	-y	x	-x	-x	y	y	-y	y	y	-y
y	x	-y	0	-y	-x	x	-x	-y	y	y	-y	y	y
y	x	-y	-y	0	-x	-x	x	y	-y	y	y	-y	y
x	-y	x	-x	-x	0	y	y	-y	y	y	y	-y	-y
x	-y	-x	x	-x	y	0	y	y	-y	y	-y	y	-y
x	-y	-x	-x	x	y	y	0	y	y	-y	-y	-y	y
y	y	y	-y	y	-y	y	y	0	-x	-x	-y	x	-x
y	y	y	y	-y	y	-y	y	-x	0	-x	-x	-y	x
y	y	-y	y	y	y	y	-y	-x	-x	0	x	-x	-y
y	-y	y	-y	y	y	-y	-y	-y	-x	x	0	x	x
y	-y	y	y	-y	-y	y	-y	x	-y	-x	x	0	x
y	-y	-y	y	y	-y	-y	y	-x	x	-y	x	x	0

Table 7: The $OD(14; 4, 9)$ from [10, p331]

1. $(1, 16)$ for $2n = 20, 22, 24, 26, 28, 32, 36;$ and ≥ 40;
2. $(1, 25)$ for $2n = 28, 32, 36, 40, 44, 48, 52, 56, 60, 62, 64, 66$ and ≥ 88;
3. $(1, 36)$ for $2n \geq 100$;
4. $(2, 18)$ for $2n = 20, 24, 26, 28, 32, 36$ and ≥ 40;
5. $(2, 32)$ for $2n \geq 76$;
6. $(2, 50)$ for $2n \geq 112$;
7. $(4, 16)$ for $2n \geq 20$;
8. $(5, 20)$ for $2n = 28, 32$ and ≥ 36;
9. $(9, 9)$ for $2n \geq 20$;
10. $(9, 36)$ for $2n \geq 112$;
11. $(10, 40)$ for $2n = 52, 56$ and ≥ 60;
12. $(16, 25)$ for $2n \geq 104$;
13. $(17, 17)$ for $2n = 38, 38, 40, 42, 44$ and ≥ 48;
14. $(18, 18)$ for $2n \geq 76$;
15. $(25, 25)$ for $2n \geq 60$;
16. $(34, 34)$ for $2n \geq 80$.
17. $(125, 125)$ for $2n \geq 360$.

Table 8: Asymptotic existence of sequences with zero autocorrelation function.

The $OD(26;2,18)$ is given in Remark 2. The existence of $OD(4m;2,18)$ for all $4m \geq 20$ gives the remainder of Part 4.

The $OD(42;2,32)$ is given in Remark 2. The $OD(62;2,32)$ is given by Remark 2 Part 4. The circulant construction for $OD(22;1,16)$ gives a two circulants construction for $OD(66;2,32)$. These, together with the results in orders $4m$ gives us the results of Table 14 for the existence of $OD(2n;2,32)$, $2n \geq 76$.

The $OD(62;2,50)$ is given in Remark 2 Part 4. The $OD(66;2,50)$ is given in Remark 2 Part 5. These together with the existence of $OD(4m;2,50)$ for $4m \geq 52$ gives us the results of Table 8 for $OD(2n;2,50)$.

Examples 7 and 6 give the $OD(2n;5,20)$ for all $2n \geq 36$. The existence of $OD(4m;5,20)$ for $4m = 28, 32$ gives the remainder of the result.

We use the directed $OD(2n;9,9)$ for $n = 10,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21$ from Table 13 to obtain the result for Part 9.

Then we replace the variables of these directed $OD(2n;9,9)$ by `a b-a` and `a 0 a` to obtain the $OD(2n;9,36)$ for $n = 30,\ 33,\ 39,\ 45,\ 51,\ 57,\ 63$. Using these results plus the existence of $OD(4m;9,36)$ in $4m \geq 48$, to obtain $OD(2n;9,36)$ for $2n \geq 112$.

Example 6 and Table 13 give the $OD(2n;10,40)$ for all $2n \geq 30$. The existence of $OD(4m;10,40)$ for $4m = 52, 56$ gives the remainder of the result.

From Theorem 10 and Remark 1 there exists an $OD(62;16,25)$. Hence assuming there is an $OD(4m;16,25)$ for all $4m \geq 44$ we have all $OD(2n;16,25)$ for $2n \geq 104$.

The $OD(42;17,17)$ is given in Remark 2. The existence of $OD(4m;17,17)$ for all $4m \geq 36$ gives the remainder of Part 13.

We use the $OD(2n;18,18)$ for $n = 19,\ 21,\ 23,\ 25$ from Table 13. Also we use the directed $OD(2n;9,9)$ for $n = 11$ from Table 13 and to obtain the $OD(2n;18,18)$ for $n = 33$. Using these results plus the existence of $OD(4m;18,18)$ in $4m \geq 40$, we obtain $OD(2n;18,18)$ for $2n \geq 76$.

The sequence with NPAF = 0 which give (34,34) are given in Table 13. (34,34) also exist in lengths $4m \geq 68$.

We use the directed $OD(2n;25,25)$ for $n \geq 30$ from Table 13 with the directed $OD(2n;5,5)$ for $n \geq 6$ to obtain the result for Part 17.

Lemma 14 *Orthogonal designs $OD(2n;a,b)$, $a+b \leq 2n-1$ and $a = (x^2+y^2)z^2$, $b = (x^2+y^2)w^2$ for x, y, z, w integers, exist for orders $30 \leq 2n \leq 54$, n odd as described in Table 1.*

Proof. The results given here for $OD(2n;a,b)$ for $2n \leq 26$ are summarized in Lemma 12 and given in [10].

We have $OD(2n;9,9)$ for $n \geq 20$ from Theorem 17.

There is an $OD(2n;1,9)$ for $2n = 12,\ 14,\ 16,\ 20,\ 24$. The direct sum of these gives the $OD(2n;1,9)$, for all $n \geq 12$.

The $OD(62;2,32)$, $OD(62;2,50)$ and $OD(62;25,25)$ are given by Remark 2 Part 4. These, together with the results in orders $4m$ gives us the results There is an $OD(2n;4,9)$ for $2n = 14,\ 16,\ 20,\ 24$. The direct sum of these gives the $OD(2n;4,9)$, for all $n \geq 14$.

The $OD(42;1,9)$ is constructed from two circulants. The $OD(42;2,18)$ may be constructed from the two sequences given in Table 13. There is an $OD(2n;2,18)$ for $2n = 20,\ 24,\ 26,\ 28,\ 32,\ 36$. The direct sum of these gives the $OD(2n;2,18)$, for all $n \geq 20$.

We have $OD(2n;1,16)$ for $2n \geq 20$, the $OD(2n;5,20)$ for $2n \geq 36$, the $OD(2n;10,40)$ for $2n \geq 60$ and $OD(2n;1,25)$ for $2n \geq 88$ from Theorem 17. Also from Theorem 17 we note $OD(2n;1,25)$ for $2n = 60,\ 62,\ 64,\ 66$.

The $OD(62;17,17)$ exists from Remark 2. The $OD(2n;17,17)$ arises for $2n \geq 24$ from Lemma 6.

The results for $OD(2n;25,25)$ for $2n \geq 30$ come from Lemma 6.

The remaining results come from Lemma 13. □

Table 9 gives sequences with zero non-periodic and periodic auto-correlation function. The notation PAF or $NPAF$ indicates which type is given.

Length	Weight	Sequences with zero autocorrelation function	Zero
9	13	1 0 - 1 - 0 0 1 1 ;- 0 0 1 1 1 - 1 1	NPAF
9	17	1 - - 1 1 1 1 - - ;0 - 1 - - - - 1 -	PAF
11	9	1 0 0 0 0 0 1 0 - 0 - ; 0 1 0 1 1 0 0 1 0 - 0	PAF
11	13	1 1 1 0 - 1 1 0 - 1 - ; 1 0 1 0 0 0 - 0 0 0 1	NPAF
11	18	0 1 0 1 1 1 - 1 - - 1 ; 0 1 0 1 1 - 1 1 1 - -	PAF
13	17	1 - 1 0 - 0 0 0 1 1 1 0 1;	NPAF
		- 0 - 0 1 1 0 - 0 1 1 - 1	
13	25	0 1 - - - 1 - - 1 - - - 1;	PAF
		1 - 1 1 - - - - - - 1 1 -	
14	17	- 1 0 1 0 0 1 1 0 0 1 0 - 1;	NPAF
		- 1 0 0 - - 0 1 1 1 0 0 1 -	
15	9	0 1 1 0 0 0 0 0 - 1 0 0 0 1;	PAF
		0 0 0 0 1 0 0 0 - 0 0 1 0 - 0	
15	18	0 1 1 0 1 0 0 0 - - 1 1 0 - 1;	PAF
		0 - 0 0 0 1 - 1 - 0 1 1 0 1 1	
15	25	0 1 - 1 0 1 1 1 - - 1 0 - 1 1;	PAF
		1 0 1 1 - - - 1 - 0 1 1 1 1 -	
15	29	0 1 1 - - 1 - - - - 1 - - 1 1;	PAF
		1 - 1 - - - - 1 1 - - - - 1 -	
17	9	0 0 1 1 - 0 0 0 0 0 0 1 0 1 0 0 0;	PAF
		0 - 0 0 0 - 0 1 0 1 0 0 0 0 0 0 0	
17	18	1 1 1 - 1 0 - 0 - 0 1 0 1 0 0 0 0;	PAF
		1 - - 1 1 0 - 0 - 0 - 0 - 0 0 0 0	
17	25	1 - 0 1 0 1 0 1 - 1 1 0 - 0 1 1 -;	PAF
		0 1 1 1 0 0 0 1 - 1 1 - - - - 1 1	
17	29	0 1 1 1 1 - - 1 - 1 1 1 - 0 1 1 -;	PAF
		- 1 0 1 0 1 0 1 1 - 1 - - - - 1 1	
18	25	1 1 - 1 0 1 1 1 - 0 0 0 - - 1 1 0 1;	NPAF
		1 1 - 1 0 1 0 0 0 - 0 - 1 1 - - 0 -	
19	9	0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0 0 0 0;	PAF
		0 0 - 0 0 0 0 0 - 0 0 0 0 1 0 1 0 - 1	
19	18	0 0 - 0 0 0 1 1 - 0 0 0 0 1 1 1 0 - 1;	PAF
		0 0 - 0 0 0 - - - 0 0 0 0 1 - 1 0 - 1	
19	29	0 1 1 1 - 0 0 1 - - 1 1 - 1 1 1 0 1 -;	PAF
		0 - 1 - 0 1 1 1 1 0 - 1 - 1 1 0 0 - -	
19	34	1 1 - 1 - 0 - 1 - - 1 1 1 1 - 0 1 1 1;	PAF
		- 1 1 - 1 0 - 1 1 1 - - - 1 - 0 1 1 1	
19	36	- 1 - - 1 - 1 1 - 0 1 1 1 1 1 1 1 - 1 ;	PAF
		1 - - 1 - 1 1 1 - 0 - - - 1 1 1 - - 1	
19	37	0 - - 1 1 1 1 - 1 1 1 1 - 1 1 1 1 - -;	PAF
		- 1 - 1 1 1 - - 1 - - 1 - - 1 1 1 - 1	

Table 9: Sequences with zero autocorrelation function.

Length	Weight	Sequences with zero autocorrelation function	Zero
21	9	- 0 0 - 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0; 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 - 0 0 1 0	PAF
21	18	0 1 1 0 - 0 1 0 0 0 - 1 0 0 - 0 - 0 1 1 1; 0 1 0 - 1 0 0 1 0 0 0 0 0 1 0 1 - 0 0 0 0	PAF
21	29	- 1 1 1 - 1 1 1 0 1 0 - - 0 - 0 0 1 1 0 1; - 1 - - 0 0 1 1 - - 1 0 1 0 - 1 0 0 0 1 1	PAF
21	36	- 1 - 0 1 1 1 1 - 1 1 - - 1 0 1 1 1 - 0 1; 1 1 - 0 1 1 - - - - 1 - 1 - 0 - - 1 1 0 1	PAF
21	37	0 1 1 - - 1 1 1 1 0 1 - 0 - 1 - 1 1 1 1 -; - - 1 - 1 1 1 1 - 0 1 1 0 1 - - - 1 - - 1	PAF
21	41	0 - 1 - 1 1 - - 1 1 1 1 1 1 - - 1 1 - 1 -; 1 - 1 1 1 1 - 1 1 - - - - 1 1 - 1 1 1 1 -	PAF
23	9	- 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 0 0; 0 0 0 1 1 - 0 0 0 0 0 0 0 0 0 0 - 0 0 0 0 0 0	PAF
23	18	- - 1 0 0 0 0 0 0 0 0 0 - 1 1 1 0 1 0 1 0 0 0; 0 0 1 1 - 0 0 0 0 0 0 0 0 0 0 - 0 0 0 0 0 0 0	PAF
23	29	- 1 0 1 - 1 1 1 0 1 1 - 0 0 - 1 0 0 0 - 0 1 1; 0 1 - 0 1 - - - 0 1 0 1 0 0 1 1 0 1 0 - - 0 1	PAF
23	34	- - - 1 0 1 0 1 - - 1 1 - 1 0 1 1 0 0 1 1 0 1; 0 1 0 1 - 1 1 1 1 - 0 - - 0 0 - 1 0 - 1 1 1 -	PAF
23	36	1 1 1 1 1 1 - - - - 1 0 0 0 0 0 1 1 - 1 1 - 1; 1 - 1 - 1 1 - - - 1 1 0 0 0 0 0 - 1 1 - - 1 -	PAF
23	37	0 1 1 1 0 1 - 1 - 1 - 0 1 1 1 0 - - 1 0 - 1 1; 1 0 - - - 0 1 1 - - 1 1 0 1 1 1 0 - 1 - - 1 -	PAF
24	34	1 - 1 0 - 0 0 0 1 1 1 - 1 - 0 1 1 0 - 0 1 1 - 1; 1 - 1 0 - 0 0 0 1 1 1 1 1 1 0 - - 0 1 0 - - 1 -	NPAF
24	34	1 - 1 1 0 - 0 1 1 0 - 1 - 1 1 1 0 0 0 - 0 1 - 1; 1 - 1 0 - 0 0 0 1 1 1 1 1 1 0 - - 0 1 0 - - 1 -	NPAF
25	9	- 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1 0 0 0 0; 0 0 0 0 - 1 1 0 0 0 0 0 0 0 0 0 0 0 0 - 0 0 0 0 0	PAF
25	18	- 0 0 0 - 0 1 0 0 0 0 0 0 0 0 0 1 0 - 1 0 1 1 1 0; 1 0 0 1 0 - 0 0 0 0 0 0 0 0 - 1 0 1 - 0 0 0 1 1 0	PAF
25	29	- 1 1 1 - 0 - 0 1 0 1 1 0 0 - 1 0 1 0 0 0 - 1 1 0; 1 0 1 1 0 1 1 0 0 - - 0 0 0 - 1 0 0 - 1 - - 1 0 0	PAF
25	36	0 - 0 - 0 0 1 1 1 - 0 1 - 1 1 0 1 - 1 1 0 1 1 1 -; 0 - 0 1 0 0 1 1 - 1 0 1 - - - 0 1 - - - 0 1 1 - 1	PAF
25	37	0 - 0 1 - 1 1 0 0 1 1 1 - 1 - 1 1 0 0 1 - - 1 1 0; - 0 1 - 1 1 1 0 0 - - 1 1 1 0 1 0 1 - 0 - - 1 - -	PAF
25	49	0 - - - 1 - 1 1 - 1 1 1 - - 1 1 1 - 1 1 - 1 - - -; - - 1 1 1 1 1 - 1 - 1 1 - - 1 1 - 1 - 1 1 1 1 1 -	PAF
27	53	0 - 1 - - 1 - 1 1 - - 1 1 1 1 1 1 - - 1 1 - 1 - - 1 -; 1 1 1 - 1 1 1 - - - 1 - 1 1 1 1 - 1 - - - 1 1 1 - 1 1	PAF

Table 9(cont): Sequences with zero autocorrelation function.

Length	Weight	Sequences with zero autocorrelation function	Zero
30	45	1 1 - - 0 - - 0 - 1 0 1 1 1 - 1 0 1 1 0 1 - 0 - 1 1 - 0 0 0; 1 0 1 1 1 - - - 1 - - 1 1 0 1 - - 1 1 1 - 1 1 - 1 0 1 0 0 0	PAF
30	50	1 1 - - - 1 - - 1 1 1 - 1 0 1 1 1 - 1 0 1 1 0 1 1 0 1 - 0 -; 1 1 - - - 1 - - 1 - - 1 1 0 1 - - 1 1 0 1 - 0 - - 0 - 1 0 1	NPAF
31	61	0 1 - - 1 - 1 1 1 1 - - 1 1 - 1 1 - 1 1 - - 1 1 1 1 - 1 - - 1; - - - - 1 1 1 - 1 1 1 - 1 - 1 1 1 1 - 1 - 1 1 1 - 1 1 1 - - -	PAF
34	50	1 1 - 1 0 1 1 1 - 0 0 0 - - 1 1 1 1 1 1 - - 1 0 1 0 0 0 - 0 - 1 - -; 1 1 - 1 0 1 1 1 - 0 0 0 - - 1 1 - 1 - - 1 1 - 0 - 0 0 0 1 0 1 - 1 1	NPAF
35	36	0 1 - 0 1 0 0 1 - 1 0 1 0 0 - 0 1 0 - 0 0 - - - 0 0 0 0 1 1 0 0 - 0 0; 1 1 1 0 1 0 0 - 1 1 0 1 0 0 - - 1 0 1 0 0 - 1 1 0 - 0 0 - 1 - 0 1 0 0	PAF
37	36	1 1 - - 0 - 0 - 1 1 0 1 0 0 1 1 0 1 0 1 0 0 - 1 0 1 0 0 0 - 0 1 0 0 0 0 0; 0 0 0 0 - 0 1 0 0 0 1 0 - - 0 0 - 0 - 0 1 - 0 0 1 0 1 1 - 0 - 0 1 1 - 1 0	PAF
37	72	1 1 - - - - 1 - 1 1 1 1 - - 1 1 - 1 - 1 1 - - 1 1 1 1 1 - - - 1 1 1 - 1 0; 0 - 1 - - 1 1 - 1 - - 1 - 1 - 1 - 1 1 1 1 1 1 1 1 1 - 1 1 - - - 1 - - 1 1	PAF
37	73	0 1 1 1 - 1 1 1 1 - 1 - - 1 - - - 1 1 1 1 - - - 1 - - 1 - 1 1 1 1 - 1 1 1; - - 1 - 1 - 1 1 1 1 - - 1 1 - 1 1 - - - - 1 1 - 1 1 - - 1 1 1 1 - 1 - 1 -	PAF
40	68	1 - 1 1 0 - 0 1 1 0 - 1 - 1 1 1 - 1 - - 1 1 - 1 - - - - - - 0 1 1 0 - 0 1 1 - 1; - 1 - - 0 1 0 - - 0 1 1 1 1 1 1 1 - 1 - - 1 - 1 1 1 1 - 1 - 0 1 1 0 - 0 1 1 - 1	NPAF
42	65	1 1 - 1 1 - 1 1 - 1 0 1 - - 1 1 1 - 1 1 - - 0 - - - 1 1 1 - - - 1 1 0 1 0 0 0 1 0 ; 1 1 - 0 0 0 1 1 - 1 0 1 - 0 - 1 0 1 - - 1 - 0 - - 0 - 1 0 1 1 1 - - 0 - - 0 - - 0 -	NPAF

Table 9(cont): Sequences with zero autocorrelation function.

The results given in Table 10 are previously unpublished.
In Table 11 the "$\surd$" indicates that the result is given here for the first time.

Design	Length	Zero NPAF	Zero PAF	Design	Length	Zero NPAF	Zero PAF
(1,9)	7	No	Yes	(2,18)	10	No	Yes
(4,9)	7	No	No	(4,9)	10	No	No
(1,9)	9	No	No	(9,9)	10	No	Yes
(1,16)	9	No	No	(1,9)	11	No	No
(4,9)	9	No	No	(1,16)	11	No	Yes
(9,9)	9	No	No	(2,18)	11	No	No
(1,9)	10	No	Yes	(17)	11	No	Yes
(1,16)	10	No	Yes	(9,9)	11	No	Yes

Table 10: The results of an exhaustive search for the existence of $OD(2n; a, b)$ constructed via two circulants.

Type	Order		Type	Order		Type	Order	
$(1,1)$	$n \geq 1$		$(4,25)$	$n \geq 248$	$\surd$	(13,13)	$n \geq 14$	$\surd$
$(1,4)$	$n \geq 3$		$(4,36)$	$n \geq 42$	$\surd$	(13,52)	$n \geq 42$	$\surd$
$(1,9)$	$n \geq 12$		$(4,49)$	$n \geq 426$	$\surd$	(16,16)	$n \geq 16$	
$(1,16)$	$n \geq 20$		$(5,5)$	$n \geq 6$		(16,25)	$n \geq 52$	$\surd$
$(1,25)$	$n \geq 44$		$(5,20)$	$n \geq 18$	$\surd$	(17,17)	$n \geq 24$	$\surd$
$(1,36)$	$n \geq 50$	$\surd$	$(8,8)$	$n \geq 8$		(17,68)	$n \geq 72$	$\surd$
$(1,49)$	$n \geq 164$	$\surd$	$(8,18)$	$n \geq 32$	$\surd$	(18,18)	$n \geq 36$	$\surd$
$(2,2)$	$n \geq 2$		$(8,32)$	$n \geq 20$	$\surd$	(20,20)	$n \geq 20$	
$(2,8)$	$n \geq 6$		$(8,50)$	$n \geq 246$	$\surd$	(25,25)	$n \geq 30$	$\surd$
$(2,18)$	$n \geq 18$	$\surd$	$(9,9)$	$n \geq 10$	$\surd$	(26,26)	$n \geq 26$	
$(2,32)$	$n \geq 38$	$\surd$	$(9,16)$	$n \geq 286$	$\surd$	(34,34)	$n \geq 40$	$\surd$
$(2,50)$	$n \geq 56$	$\surd$	$(9,25)$	$n \geq 420$	$\surd$	(50,50)	$n \geq 60$	$\surd$
$(4,4)$	$n \geq 4$		(9,36)	$n \geq 112$	$\surd$	(65,65)	$n \geq 84$	$\surd$
$(4,9)$	$n \geq 14$		(10,10)	$n \geq 10$		(125,125)	$n \geq 180$	$\surd$
$(4,16)$	$n \geq 10$		(10,40)	$n \geq 30$	$\surd$			

Table 11: The existence of $OD(2n; a, b)$

Order	Unsolved	Comment
6	none	
10	none	
14	none	$OD(14;4,9)$ not constructed from two circulants.
18	(1,9),(4,9)	Exhaustive search indicates $OD(18;1,9)$ and $OD(18;4,9)$ cannot be constructed from two circulants. $OD(18;1,16)$ is non-existent.
22	(1,9),(2,18),(4,9)	Exhaustive search indicates $OD(22;1,9)$ $OD(22;2,18)$ and $OD(22;4,9)$ cannot be constructed from two circulants.
26	(4,9)	$OD(26;5,20)$ is non-existent Exhaustive search indicates $OD(26;4,9)$ cannot be constructed from two circulants.
30	(1,25),(2,18),(4,25),(8,18) (9,16),(5,20)	Exhaustive search indicates $OD(30;4,9)$ cannot be constructed from two circulants.
34	(1,25),(2,18),(4,25),(8,18) (8,18),(9,16),(5,20)	Exhaustive search indicates $OD(34;1,9)$ and $OD(34;4,9)$ cannot be constructed from two circulants.
38	(1,25),(1,36),(2,32),(4,25) (9,16)	
42	(1,25),(1,36),(4,25),(4,36) (8,18),(9,16),(9,25),(16,25)	
46	(1,25),(1,36),(2,32),(4,25) (8,18),(9,16),(9,25),(9,36) (16,25),(17,17)	
50	(1,25),(1,36),(2,32),(4,25) (8,18),(9,16),(9,25),(9,36) (16,25)	
54	(1,25),(1,36),(1,49),(2,32) (2,50),(4,25),(4,36),(4,49) (5,45),(8,18),(9,16),(9,25) (9,36),(10,40),(16,25) (16,36),(18,32),(25,25)	Exhaustive search indicates $OD(54;1,9)$ cannot be constructed from two circulants.
58	(1,36),(1,49),(2,32),(2,50) (4,25),(4,36),(4,49),(5,45) (8,18),(9,16),(9,25),(9,36) (10,40),(16,25),(18,32),(25,25)	Exhaustive search indicates $OD(58;1,9)$ cannot be constructed from two circulants.
62	(1,49),(4,25),(4,36),(4,49) (5,45),(8,18),(8,50),(9,16) (9,25),(9,36),(9,49),(16,36) (16,49),(18,32),(18,50),(25,36) (29,29)	Exhaustive search indicates $OD(62;1,9)$ cannot be constructed from two circulants.
66	(1,49),(1,64),(2,32),(2,50) (4,25),(4,36),(4,49),(5,45) (8,50),(9,16),(9,25),(16,25) (16,36),(18,32),(25,36),(29,29)	Exhaustive search indicates $OD(66;1,9)$ cannot be constructed from two circulants.

Table 12: The existence of $OD(2n;a,b)$, $3 \leq n \leq 33$.

Length	Type	Sequences with zero autocorrelation function	Zero
1	(1,1)	a; b	NPAF
2	(2,2)	a b; a -b	NPAF (D)
3	(1,4)	a b -a; a 0 a	NPAF
4	(4,4)	a a b -b; a -a b b	NPAF (D)
6	(2,8)	a b -a a 0 a; a b -a -a 0 -a	NPAF
6	(5,5)	a a -a b 0 b; a 0 a b -b -b	NPAF (D)
7	(1,9)	b -b a b -b 0 0;	PAF
		b -b -b -b -b 0 0	
7	(1,9)	a b b -b b -b -b;	PAF
		0 b b 0 b 0 0	
8	(8,8)	a a -a a b b b -b;	NPAF (D)
		a -a -a -a b -b b b	
9	(5,5)	a b 0 a 0 0 -a b 0;	PAF (D)
		-a -b 0 0 b 0 -a b 0	
10	(1,9)	a 0 -b 0 0 0 0 0 b 0;	PAF
		b -b -b b b 0 b 0 b 0	
10	(1,16)	b b b -b -a b -b -b -b 0;	PAF
		b -b b b 0 b b -b b 0	
10	(9, 9)	a -b -b b a b b -b a 0;	PAF (D)
		b a -a -a b -a a a b 0	
10	(10,10)	a -a -a a b a b b b -b;	NPAF (D)
		a -a -a -a b -a b -b -b b	
10	(2,18)	a b -b b -b -b b b b b;	PAF
		a -b b -b -b b b -b -b -b	
10	(4,16)	a b -a a -a a -a -a b a;	NPAF
		a b -a a a a a a -b -a	
11	(1,16)	a b -b b b b -b -b -b b -b;	PAF
		b -b 0 -b -b -b 0 0 0 -b 0	
11	(9,9)	a b 0 b b 0 a b -a -b -a;	PAF (D)
		b -a b a -b 0 a a 0 a -b	
12	(4,16)	b 0 b -b 0 -b b a -b b a -b;	NPAF
		b 0 b b 0 b b a -b -b -a b	
13	(9,9)	a -a a a -a -a 0 0 a a 0 a 0;	PAF (D)
		b -b b b -b -b 0 0 b b 0 b 0	
13	(1,16)	a b 0 b -b 0 0 0 0 b -b 0 -b;	PAF
		b -b -b 0 0 b b 0 b b b -b b	
14	(13,13)	a a a b -a a a -b -a a -a b 0 b;	NPAF (D)
		a 0 a b -b b -a -b -b b a -b -b -b	
15	(9,9)	0 -b 0 a b 0 -a b 0 -a b 0 a b 0;	PAF (D)
		0 a -b 0 a b 0 a b 0 a -b 0 -a 0	

Table 13: Designs from sequences

In Table 13 the results for (1,1), (2,2), (1,4), (4,4), (2,8), (5,5), (8,8), (4,16), (10,10) and (13,13) are given in [10, Table H.3]. We give them here for completeness. The remaining results are new. Also in Table 13 the notation $NPAF$ (D) or PAF (D) denotes that the sequences we give are directed.

Length	Type	Sequences with zero autocorrelation function	Zero
17	(9,9)	a b b-b a 0-a 0-a 0 b 0 b 0 0 0 0;	PAF
		b-a-a a b 0-b 0-b 0-a 0-a 0 0 0 0	
18	(5,20)	a b-a a 0 a a b-a 0 0 0-a-b a a 0 a;	NPAF
		a b-a a 0 a 0 0 0-a 0-a a b-a-a 0-a	
19	(2,18)	b b b-b b-b a b-b 0 0 0 0 b 0 0 0 0 0;	PAF
		b b b-b b b-a-b-b 0 0 0 0 b 0 0 0 0 0	
19	(4,9)	a a 0 0 b-b-b 0 0 0 0 0 0 0 b 0 0 0 0;	PAF
		a-a b 0 b 0 0 0 0 0 b 0 0 0 0-b 0 0 b	
19	(8,18)	a a b 0 b-b-b-b 0 0 b a-a b b b 0 0 0;	PAF
		a a-b 0 b-b-b b 0 0-b-a a-b b-b 0 0 0	
19	(9,9)	0 0 b 0 0 0-a-a b 0 0 0 0-b-a-b 0 b-b;	PAF
		0 0-a 0 0 0-b-b-a 0 0 0 0 a-b a 0-a a	
19	(17,17)	a b-a a-a 0-b b-a-a a a a b-b 0 b b b;	PAF (D)
		a a a 0-a a-b-b-b b b a-a 0 b-b b a-b	
19	(18,18)	a-a-b a-b a a a-b 0-a-b-b b a b-a-b b;	PAF (D)
		a-a b a-b a-a-a b 0-a-b-b-b-a-b-a b-b	
20	(8,32)	a b-a a-a a-a-a b a a b-a a a a a a-b-a;	NPAF
		a b-a a-a a-a-a b a-a-b a-a-a-a-a-a b a	
21	(1,9)	b 0 0 b 0 0 0 0 0 a 0 0 0 0 0-b 0 0-b 0 0;	PAF
		0 b 0 0 0 0 0 b 0 0 0 0 0 b 0 0-b 0 0 b 0	
21	(2,18)	a b 0 b b 0 b 0 0-b b 0 b 0 0-b 0 0-b 0 0;	PAF
		a-b 0 b-b 0 b 0 0-b-b 0 b 0 0-b 0 0-b 0 0	
21	(4,9)	a b b-b 0-b 0 a 0 0 0 0 0-b 0 0 0 0 b 0 0;	PAF
		a 0 0 0 b 0 0 b 0 0 0 0 0 b-a 0 0 0 0 0 0	
21	(9, 9)	a b 0 a 0 0 0 b 0 0 0 0 0 b 0-a-b 0-a b 0;	PAF (D)
		0 a b 0-a b 0 a 0 0 0 0 0 a 0 0 0-b 0 a-b	
21	(18,18)	b a-a 0 a a-b-b-a-b a-a b-b 0-b-b a b 0 a;	PAF (D)
		b 0-a b a a 0 a-a-b b a-b a a b b 0-b b-a	
23	(2,18)	a b-b 0 b 0 0 0 b b b-b b-b 0 0 0 0 0 0 0 0 0;	PAF
		a-b b 0-b 0 0 0-b-b-b-b b b 0 0 0 0 0 0 0 0 0	
23	(4,36)	a b-b-b b b-b a b b b-b b-b 0 0 0 b b-b-b-b-b;	PAF
		a b-b b b-b b-a b b b-b b-b 0 0 0-b-b b b b b	
23	(9, 9)	a 0 a 0 a b-b-b-a 0 a 0 0 0 0 0 0 b 0 0 0 0 0;	PAF (D)
		0 0 0 0 0 a 0 0 0 0 0 0-b 0 b-a-a a-b 0-b 0-b	
23	(18,18)	a b a b a b-b-b-a-a a 0 0 0 0 0 b b-b a a-a b;	PAF (D)
		a-b a-b a b-b-b-a a a 0 0 0 0 0-b b b-a-a a-b	
24	(17,17)	a-a a 0-a 0 0 0 a a a-b a-b 0 b b 0-b 0 b b-b b;	NPAF (D)
		a-a a a 0-a 0 a a 0-a-b-a-b-b-b 0 0 0 b 0-b b-b	
25	(2,18)	b b-b 0 0 0 0 a b 0 0 0-b 0 0 0 0 0-b 0 b b b 0 0;	PAF
		b b-b 0 0 0 0-a b 0 0 0-b 0 0 0 0 0 b 0 b-b b 0 0	
25	(4,36)	b b-b 0-b b b a b b-b b-b b 0 0 0 0-b-b b b b b-a;	PAF
		b b-b 0 b-b-b a b-b b-b-b-b 0 0 0 0-b b b b b-b a	

Table 13(cont): Designs from sequences

Length	Type	Sequences with zero autocorrelation function	Zero
25	(9,9)	0-a 0 0 0 0 a b 0 0 0 a-b 0 0 0 a-b 0 0 0 a b 0 0; 0-b 0 0 0 0 b-a 0 0 0 b a 0 0 0 b a 0 0 0 b-a 0 0	PAF
25	(18,18)	0-a 0-b 0 0 a b b-a 0 a-b b a 0 a-b b a 0 a b b-a; 0-a 0 b 0 0 a b-b a 0 a-b-b-a 0 a-b-b-a 0 a b-b a	PAF
26	(26,26)	b b b-b-b b b b-b b-b-b-a-b-a a-a a a-a-a a-a -a-a-a b b b b-b b b-b-b b-b b-a b-a-a a-a a a a-a-a a a a	NPAF (D)
30	(10,40)	a b-a-a-b a-a-b a a b-a a 0 a a b-a a 0 a a 0 a a 0 a-a 0-a; a b-a-a-b a-a-b a-a-b a a 0 a-a-b a a 0 a-a 0 -a-a 0-a a 0 a	NPAF
30	(25,25)	a a-a b a b a b-b-b a a-a b 0 b 0 0 0 0-a-a a -b a-b a b-b-b a a-a b a b a b-b-b 0 0 0 0-a 0-a-b b b a a-a b-a b-a-b b b	NPAF (D)
37	(36,36)	a a-a-a-b-a b-a a a b a-b-b a a-b a-b a b-b-a a b a b b-b-a-b a b b-b b 0; 0-a a-a-a b a-b a-a-a b-a b-b a-a b a b a b b a a b-a b b-b-a-b a-b-b b b	PAF (D)
40	(34,34)	a-a a a 0-a 0 a a 0-a a-a a a a-b b-b-a b a-a a -b-b-b-b-b-b 0 b b 0-b 0 b b-b b; -a a-a-a 0 a 0-a-a 0 a a a a a a b-b b-a-b a-a a b b b-b b-b 0 b b 0-b 0 b b-b b	NPAF(D)
42	(13,52)	a c-a a c-a a c-a a 0 a-a-c a a c-a a c-a-a 0-a -a-c a a c-a-a-c a a 0 a 0 0 0 a 0 a a c-a 0 0 0 a c-a a 0 a-a 0-a a 0 a-a-c a-a 0-a -a 0-a a 0 a a c-a-a 0-a-a 0-a-a 0-a	NPAF

Table 13(cont): Designs from sequences

Type	$2n$
(1,1)	2...
(1,4)	6...
(1,9)	12, 14, 16, (18), 20, (22), 24...
(1,16)	20...
(1,25)	28, (30), 32, (34), 36, (38), 40, (42), 44, (46), 48, (50), 52, (54), 56, (58), 60, 62, 64, 66, 68, (70), 72, (74), 76, (78), 80, (82), 84, (86), 88...
(1,36)	40, (42), 44, (46), 48, (50), 52, (54), 56, (58), 60, 62, 64, (66), 68, (70), 72, (74), 76, (78), 80, (82), 84, (86), 88, (90), 92, (94), 96, (98), 100...
(2,2)	4...
(2,8)	12...
(2,18)	20, (22), 24, 26, 28, (30), 32, (34), 36...

Table 14: The existence of selected $OD(2n; a, b)$, $a + b \leq 68$

Type	$2n$
(2,32)	36, (38), 40, 42, 44, (46), 48, (50), 52, (54), 56, (58), 60, 62, 64, 66, 68, (70), 72, (74), 76...
(2,50)	52, (54), 56, (58), 60, 62, 64, 66, 68, (70), 72, (74), 76, 78, 80, (82), 84 (86), 88, 90, 92, (94), 96, (98), 100, 102, 104, (106), 108, (110), 112...
(4,4)	8...
(4,9)	14, 16, (18), 20, (22), 24, (26), 28...
(4,16)	20, (22), 24...
(4,36)	40, (42), 44, 46, 48, 50, 52, (54), 56, (58), 60, (62), 64, (66), 68, 70, 72, (74), 76, 78, 80, (82), 84...
(4,49)	(54), 52, (54...794), 796, 798, 800, (802...850), 852...
(5,5)	12...
(5,20)	28, (30), 32, (34), 36...
(5,45)	52, (54...94), 96, 98, 100, (102...122), 124, 126, 128, (130...146), 148...
(8,8)	16...
(8,18)	28, (30), 32, (34), 36, 38, 40, (42), 44, (46), 48, (50), 52, (54), 56, (58), 60, (62), 64...
(8,32)	40...
(9,9)	20...
(9,16)	(26), 28, (30...542), 544, 546, 548, (550...570), 572...
(9,25)	36, (38), 40, (42...802), 804, 806, 808, (810...838), 840...
(9,36)	48, (50), 52, (54), 56, (58), 60, (62), 64, 66, 68, (70), 72, (74), 76, 78, 80, (82), 84, (86), 88, 90, 92, (94), 96, (98), 100, 102, 104, (106), 108, 110, 112...
(9,81)	92, (94...150), 152, 154, 156, (158...242), 244...
(10,10)	20...
(10,40)	52, (54), 56, (58), 60...
(13,13)	28...
(13,52)	(66), 68, (70), 72, (74), 76, (78), 80, (82), 84...
(16,16)	32...
(16,25)	(42), 44, (46), 48, (50), 52, (54), 56, (58), 60, 62, 64, (66), 68, (70), 72, (74), 76, (78), 80, (82), 84, (86), 88, (90), 92, (94), 96, (98), 100, (102), 104...
(17,17)	36, 38, 40, 42, 44, (46), 48...
(18,18)	36, 38, 40, 42, 44, (46), 48, 50, 52, (54), 56, (58), 60, (62), 64, 66, 68, (70), 72...
(18,72)	92, (94...110), 114, 116, (118), 120, (122), 124, 126, 128, (130...150), 152, 154, 156, (158...186), 188, 190, 192, (194...202), 204...
(20,20)	40...
(25,25)	52, (54), 56, (58), 60...
(26,26)	26...
(34,34)	68, (70), 72, (74), 76, (78), 80...

Table 14(cont): The existence of selected $OD(2n; a, b)$, $a + b \leq 68$.

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