December 2000

On amicable sequences and orthogonal designs

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Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (al, a2), (al, a2, a3, a4) or (al, a2, ... , a8) then there exist amicable sequences of length $\ell \equiv 0 \pmod{m}$ and of the same type. We also present a theorem that produces a set of 2v amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type (al, al, a2, a2, ... , av, av) from v pairs of disjoint (0, ±1) amicable sequences.

Using these results we can obtain many infinite classes of orthogonal designs. Actually, if there exists an orthogonal design of order n and of type (al, a2,... , av), which is constructed from sequences, then there exists an infinite family of orthogonal designs of the same type which is constructed from appropriate sequences.

Keywords

Construction, sequences, orthogonal designs, amicable sets, AMS Subject Classification: Primary 05B15, 05B20.

Disciplines

Physical Sciences and Mathematics

Publication Details

This conference was originally published as Georgiou, S, Koukouvinos, C and Seberry, J, On amicable sequences and orthogonal designs, Proceedings of the 23rd Australasian Conference on Combinatorial Mathematics and Combinatorial Computing, Christchurch, New Zealand, 4-7 December 2000.

On amicable sequences and orthogonal designs

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November 27, 2000

Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (a_1, a_2), (a_1, a_2, a_3, a_4) or $(a_1, a_2, \ldots, a_8)$ then there exist amicable sequences of length $\ell \equiv 0 \ (mod\ m)$ and of the same type. We also present a theorem that produces a set of $2v$ amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type $(a_1, a_1, a_2, a_2, \ldots, a_v, a_v)$ from v pairs of disjoint $(0, \pm 1)$ amicable sequences.

Using these results we can obtain many infinite classes of orthogonal designs. Actually, if there exists an orthogonal design of order n and of type $(a_1, a_2, \ldots, a_v)$, which is constructed from sequences, then there exists an infinite family of orthogonal designs of the same type which is constructed from appropriate sequences.

Key words and phrases: Construction, sequences, orthogonal designs, amicable sets.

AMS Subject Classification: Primary 05B15, 05B20.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let A_1, A_2 be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2\}$ satisfying $A_1 A_1^T + A_2 A_2^T = (s_1 x_1^2 + s_1 x_1^2) I_n$. Then

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}. \tag{1}$$

is an $OD(2n; s_1, s_2)$.

Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

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Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^TR & -B_3^TR \\ -B_3R & -B_4^TR & B_1 & B_2^TR \\ -B_4R & B_3^TR & -B_2^TR & B_1 \end{pmatrix} \tag{2}$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \dots, s_u)$. See page 107 of [1] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [5] a set $\{A_1, A_2, \dots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{3}$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{4}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \dots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \dots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{5}$$

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices (or group developed of type 1) of type $(s_1, s_2, \dots, s_u)$ of order t. Then the Kharaghani array from [5]

$$H = \begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\ -A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\ -A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\ -A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1 \end{pmatrix} \tag{6}$$

is an $OD(8m; s_1, s_2, \dots, s_u)$.

The Kharaghani array has been used in a number of papers [2, 3, 5, 6, 7] to obtain infinitely many families of orthogonal designs.

A set $\{A_i\}_{i=1}^4$ is said to be a *short amicable set* of length m and type (u_1, u_2, u_3, u_4) if (4) and (5) are satisfied for $n = 4$ and $u \leq 4$. Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{bmatrix} \tag{7}$$

to form an $OD(4m; u_1, u_2, u_3, u_4)$.

Short amicable sets were introduced in [4] where they are used to find infinite families of orthogonal designs.

A set of sequences $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, \ldots, 2v$ is said to be a set of $2v$ *amicable sequences* of length m and type $(u_1, u_2, \ldots, u_p)$ if the circulant matrices which are constructed from these satisfy the equations (4) and (5).

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function (NPAF)* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \;\; s = 0, 1, \ldots, n-1, \tag{8}$$

Given A as above of length n the *periodic autocorrelation function (PAF)* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \;\; s = 0, 1, \ldots, n-1. \tag{9}$$

We define the $NPAF$ (PAF) of a set of sequences the sum of the corresponding $NPAF$ (PAF) of the individual sequences.

Suppose $C = circ(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order n.
Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

of order n, be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \ldots + c_{n-1} T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n. We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

2 Multiplication of the length of amicable sets of sequences

Theorem 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, 2v$ *be a set of* $2v$ *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p)$. *Then there exist a set of* $2v$ *amicable sequences of length* $\ell \equiv (0 \bmod m) = mi$ *for all* $i = 1, 2, \ldots$ *and type* $(u_1, u_2, \ldots, u_p)$.

Proof. Let i be a constant integer. We use the map T_m^k to define sequences A_k and the map $S_\ell^k = T_m^k$ to define sequences B_k

$$B_k = \sum_{j=0}^{m-1} a_{k,j} S_\ell^j, \ k = 1, 2, \ldots, 2v$$

Now

$$\sum_{k=1}^{2v} A_k A_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} T_m^{j-x} \right) = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_m.$$

Thus we have that

(i) If m is odd then the coefficients of T_m^σ, $\sigma = -(m-1), \ldots, -1, 1, \ldots, m-1$ is zero, and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} u_k x_k^2 \tag{10}$$

(ii) If m is even, $m = 2n$ then we have that $T_m^n = T_m^{-n}$ and so the coefficients of T_m^σ, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, -1, 1, \ldots, n-1, n+1, \ldots, 2n-1$ are zero, the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma \\ \sigma\neq\pm n}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0, \quad \sum_{\substack{j,x=0 \\ j-x=\pm n}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} u_k x_k^2 \tag{11}$$

Now

$$\sum_{k=1}^{2v} B_k B_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \ldots, m-1$, and so using equations (10) or (11) we obtain

$$\sum_{k=1}^{2v} B_k B_k^T = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_{2mi} = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_\ell \tag{12}$$

Moreover

$$\sum_{k=1}^{v} \left(A_{2k-1} A_{2k}^T - A_{2k} A_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) T_m^{j-x} \right) = 0$$

and from these we have that

(i) if m odd, then the coefficients of $T_m^\sigma,\ \sigma = -(m-1), \ldots, m-1$ are zero.That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma}}^{m-1} \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) = 0 \tag{13}$$

(ii) if m is even, $m = 2n$ then the coefficients of $T_m^\sigma,\ \sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, n-1, n+1, \ldots, 2n-1$ are zero and the coefficient of T_m^n plus the the coefficient of T_m^{-n} is zero. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma \\ \sigma\neq\pm n}}^{m-1} \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) = 0 \ \textit{and} \sum_{\substack{j,x=0 \\ j-x=\pm n}}^{m-1} \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) = 0 \tag{14}$$

Now

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x})S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \ldots, m-1$ and so using equations (13) or equations (14) we obtain

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = 0 \tag{15}$$

Equations (12) and (15) show that $\{B_k\}_{k=1}^{2v}$ is an amicable set of sequences of length $\ell \equiv 0(mod\ m),\ \ell = mi,\ i = 1, 2, \ldots$ and type $(u_1, u_2, \ldots, u_p)$. Λ

Corollary 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\},\ k = 1, 2$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2\},\ j = 0, 1, \ldots, m-1$ *and* $k = 1, 2$ *be a set of two amicable sequences of length* m *and type* (u_1, u_2)*. Then there exist a set of two amicable sequences of length* $\ell \equiv (0\ mod\ m) = mi$ *and type* (u_1, u_2)*.*

Proof. Use Theorem 1 with $2v = 2$ and $p = 2$.

Example 1 We have that $A_1 = 0T_4^0 + aT_4^1 + bT_4^2 - aT_4^3$ and $A_2 = 0T_4^0 + aT_4^1 + 0T_4^2 + aT_4^3$ is a set of two amicable sequences of length $m = 4$ and type $(1, 4)$. Corollary 1 gives a set of two amicable sequences of length $m = 4i$ and type $(1, 4)$ for all $i = 1, 2, \ldots$.

Corollary 2 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\},\ k = 1, 2, 3, 4$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\},\ j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, 3, 4$ *be a set of four amicable sequences of length* m *and type* (u_1, u_2, u_3, u_4)*. Then there exist a set of four amicable sequences of length* $\ell \equiv (0\ mod\ m) = mi$ *and type* (u_1, u_2, u_3, u_4)*.*

Proof. Use Theorem 1 with $2v = 4$ and $p = 4$.

Example 2 We have that $A_1 = aT_3^0 - bT_3^1 + aT_3^2,\ A_2 = bT_3^0 + aT_3^1 + bT_3^2$ and $A_3 = aT_3^0 + aT_3^1 - aT_3^2,\ A_4 = bT_3^0 + bT_3^1 + bT_3^2$ is a set of four amicable sequences of length $m = 3$ and type $(6, 6)$. Corollary 2 gives a set of four amicable sequences of length $m = 3i$ and type $(6, 6)$ for all $i = 1, 2, \ldots$.

Corollary 3 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 8$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_8\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, 8$ *be a set of eight amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_8)$. *Then there exist a set of eight amicable sequences of length* $\ell \equiv 0 \ (mod\ m) = mi$ *and type* $(u_1, u_2, \ldots, u_8)$.

Proof. Use Theorem 1 with $2v = 8$ and $p = 8$.

Example 3 We have that $A_1 = -aT_7^0 + aT_7^1 + aT_7^2 + gT_7^3 + aT_7^4 + eT_7^5 + cT_7^6$, $A_2 = -fT_7^0 + fT_7^1 + fT_7^2 - hT_7^3 + fT_7^4 + bT_7^5 - dT_7^6$, $A_3 = -gT_7^0 + gT_7^1 + gT_7^2 - aT_7^3 + gT_7^4 + cT_7^5 - eT_7^6$, $A_4 = -hT_7^0 + hT_7^1 + hT_7^2 + fT_7^3 + hT_7^4 + dT_7^5 + bT_7^6$, $A_5 = -eT_7^0 + eT_7^1 + eT_7^2 - cT_7^3 + eT_7^4 - aT_7^5 + gT_7^6$, $A_6 = -dT_7^0 + dT_7^1 + dT_7^2 - bT_7^3 + dT_7^4 - hT_7^5 + fT_7^6$, $A_7 = -bT_7^0 + bT_7^1 + bT_7^2 + dT_7^3 + bT_7^4 - fT_7^5 - hT_7^6$ and $A_8 = -cT_7^0 + cT_7^1 + cT_7^2 + eT_7^3 + cT_7^4 - gT_7^5 - aT_7^6$ is a set of eight amicable sequences of length $m = 7$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$. Corollary 3 gives a set of eight amicable sequences of length $m = 7i$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$ for all $i = 1, 2, \ldots$.

Remark 1 Using Corollaries 1, 2 and 3 as indicated by the examples and using array (1), (2) or (7) and (6) respectively we obtain many infinite classes of orthogonal designs.

3 Construction of amicable sets of sequences from non amicable sets of sequences

Lemma 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m-1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. These are the sequences A_k, $k = 1, 2, \ldots, v_1$ and B_k, $k = 1, 2, \ldots, v_2$ together. Λ

Corollary 4 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m_1-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m_1 - 1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m_1 *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m_2-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m_2 - 1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m_2 *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* $\ell \cdot i$ *where* $\ell = [m_1, m_2]$ *is the least common multiple (l.c.m.) of* m_1 *and* m_2 *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. Since ℓ is the least common multiple of m_1 and m_2 then $\ell = m_1 \cdot i_1 = m_2 \cdot i_2$. Using theorem 1 we can construct a set of v_1 amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p)$ and a set of v_2 amicable sequences of length ℓ and type $(t_1, t_2, \ldots, t_q)$. Now using Lemma 1 we obtain a set of $v_1 + v_2$ amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Using theorem 1 again in the derived sequences we have the result. Λ

Example 4 We have that $A_1 = \{e, f\}$, $A_2 = \{e, -f\}$, $A_3 = \{e, 0\}$, $A_4 = \{f, 0\}$ is a short amicable sets of length 2 and type $(3, 3)$. We also have that $A_1 = \{a, a, b, -b\}$, $A_2 =$

$\{c, c, d, -d\}$, $A_3 = \{d, d, -c, c\}$, $A_4 = \{b, b, -a, a\}$ is a short amicable set of length 4 and type $(4, 4, 4, 4)$. Now $\ell = [4, 2] = 4$ and thus from corollary 4 we obtain eight amicable sequences of length $\ell \cdot i$ and type $(3, 3, 4, 4, 4, 4)$ for all $i = 1, 2, \ldots$.

Theorem 2 (Doubling the number of sequences) *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots v$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v$ *be* v *sequences with PAF=0 (or NPAF=0) of length* m *and type* $(u_1, u_2, \ldots, u_p)$. *Then there exist a set of* $2v$ *amicable sequences of length* m *and type* $(2u_1, 2u_2, \ldots, 2u_p)$ *with PAF=0 (or NPAF=0).*

Proof. Set $B_{2k-1} = B_{2k} = circ(A_k)$, $k = 1, 2, \ldots, v$. Then

$$\sum_{k=1}^{2v} B_k B_k^T = 2 \cdot \sum_{k=1}^{v} A_k A_k^T = \left(\sum_{i=1}^{p} 2u_i x_i^2 \right) I_m$$

and

$$B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T = A_k A_k^T - A_k A_k^T = 0, \ k = 1, 2, \ldots, v.$$

Thus $\{B_k\}_{k=1}^{2v}$ is a set of $2v$ amicable sequences of length m and type $(2u_1, 2u_2, \ldots, 2u_p)$. Λ

4 More Constructions

Theorem 3 *Let* (X_k, Y_k), $k = 1, 2, \ldots, v$ *be* v *pairs of sequences of lengths* m_k *with the properties*

$$Z_k Z_k^T + W_k W_k^T = p_k I_{m_k} \quad (16)$$

$$Z_k W_k^T - W_k Z_k^T = 0 \quad (17)$$

$$Z_k * W_k = 0 \quad (18)$$

for all $k = 1, 2, \ldots, v$, *where* $Z_k = circ(X_k)$ *and* $W_k = circ(Y_k)$. *Then there exist a set of* $2v$ *amicable sequences of length* $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v]$, *where* $[m_1, m_2, \ldots, m_v]$ *is the least common multiple (l.c.m.) of* $m_1, m_2, \ldots, m_v$ *and of type* $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$ *on the set* $\{a_1, a_2, \ldots, a_{2v}\}$ *of commuting variables.*

Proof. Set

$$B_k = a_{2k} X_k + a_{2k-1} Y_k, \quad \text{and} \quad C_k = -a_{2k-1} X_k + a_{2k} Y_k, \quad k = 1, 2, \ldots v$$

Condition (18) gives that B_k, $k = 1, 2, \ldots, v$ and C_k, $k = 1, 2, \ldots, v$ are sequences of lengths m_k, $k = 1, 2, \ldots, v$ and type $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$.

For any k and by simple calculations using conditions (16) and (17) we have that

$$B_k B_k^T + C_k C_k^T = (p_k a_{2k-1}^2 + p_k a_{2k}^2) I_{m_k} \quad \text{and} \quad B_k C_k^T - C_k B_k^T = 0$$

Now from theorem 1, for all k=1,2, ... ,v, there are sequences D_k and E_k of length $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v]$ and $k = 1, 2, \ldots, v$ with the desirable properties. By lemma 1 we have the result. Λ

Example 5 Set $Z_1 = \{1\}$, $W_1 = \{0\}$, $Z_2 = \{1, 0\}$, $W_2 = \{0, 1\}$, $Z_3 = \{1, 1, 1, -1\}$, $W_3 = \{0, 0, 0, 0\}$, $Z_4 = \{0, 1, 0, -1, 0, 1\}$ and $W_4 = \{0, 0, 1, 0, 1, 0\}$. These are four pair of sequences of lengths $1, 2, 4$ and 6 satisfying conditions (16), (17) and (18) with $p_1 = 1$, $p_2 = 2$, $p_3 = 4$ and $p_4 = 5$. We have that $[1, 2, 4, 6] = 12$ and from theorem 3 we obtain eight sequences of length $\ell \equiv 0 \ (mod\ 12)$ and of type $(1, 1, 2, 2, 4, 4, 5, 5)$ on the set $\{a_1, a_2, \ldots, a_8\}$ of commuting variables and can be used in the Kharaghani array (6) to obtain an infinite class of orthogonal designs $OD(8\ell; 1, 1, 2, 2, 4, 4, 5, 5)$.

References

[1] A.V.Geramita, and J.Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[2] W.H. Holzmann, and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.*, to appear.

[3] W.H. Holzmann, and H. Kharaghani, On the orthogonal designs of order 24, *Discrete Appl. Math.*, 102 (2000), 103-114.

[4] W.H. Holzmann, H. Kharaghani, C. Koukouvinos and Jennifer Seberry, Infinite families of orthogonal designs III: Short amicable sets, (in preparation).

[5] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, to appear.

[6] C. Koukouvinos and Jennifer Seberry, An infinite family of Plotkin type arrays, (submitted).

[7] C. Koukouvinos and Jennifer Seberry, Infinite families of orthogonal designs : I, *Bull. Inst. Combin. Appl.*, to appear