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Short amicable sets and Kharaghani type orthogonal designs

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Short amicable sets and Kharaghani type orthogonal designs

C. Koukouvinos\textsuperscript{a} and J. Seberry\textsuperscript{\dagger}

Dedicated to George Szekeres on his 90th Birthday

Abstract

Short amicable sets were introduced recently and have many applications. The construction of short amicable sets had lead to the construction of many orthogonal designs, weighing matrices and Hadamard matrices. In this paper we give some constructions for short amicable sets as well as some multiplication theorems. We also present a table of the short amicable sets known to exist and we construct some infinite families of short amicable sets and orthogonal designs.

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1 Introduction

An orthogonal design of order $n$ and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix $A$ of order $n$ with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_ix_i^2)I_n,$$

where $I_n$ is the identity matrix of order $n$. Let $B_i$, $i = 1, 2, 3, 4$ be circulant matrices of order $n$ with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_iB_i^T = \sum_{i=1}^{u} (s_ix_i^2)I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix}
B_1 & B_2R & B_3R & B_4R \\
-B_3R & B_1 & B_3R & -B_3^T R \\
-B_4R & B_4^T R & B_1 & B_3^T R \\
B_3R & B_3^T R & -B_3^T R & B_1
\end{pmatrix}$$

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where $R$ is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [2] for details.

A pair of matrices $A, B$ is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). To be consistent in the notation of this paper we will also denote these as $2 - SAS(n; s_1, s_2; G)$, where the group $G$ is described below. Following [4] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be amicable if

$$\sum_{i=1}^{n} \left( A_{\sigma(2i-1)}^T A_{\sigma(2i)} - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{1}$$

for some permutation $\sigma$ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left( A_{2i-1}^T A_{2i} - A_{2i} A_{2i-1}^T \right) = 0. \tag{2}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper $R_k$ denotes the back diagonal identity matrix of order $k$.

A set of matrices $\{B_1, B_2, \ldots, B_u\}$ of order $m$ with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{3}$$

Let $\{A_i\}_{i=1}^{S}$ be an amicable set of circulant matrices (or group developed or type 1) of type $(s_1, s_2, \ldots, s_u)$ and order $t$. We denote these by $8 - AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; Z_t)$ (or $8 - AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$ for group developed or type 1). In all cases, the group $G$ of the matrix is such that the extension by Seberry and Whiteman [9] of the group from circulant to type 1 allows the same extension to $R$. Then the Kharaghani array [4]

$$H = \begin{pmatrix}
A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\
A_2 & A_1 & A_4 R_n & -A_3 R_n & A_6 R_n & -A_5 R_n & A_8 R_n & -A_7 R_n \\
-A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_4^2 R_n & A_5^2 R_n & A_8^2 R_n & -A_7^2 R_n \\
-A_3 R_n & A_4 R_n & -A_2 & A_1 & A_4^2 R_n & A_5^2 R_n & -A_8^2 R_n & -A_7^2 R_n \\
-A_6 R_n & -A_5 R_n & A_4^2 R_n & -A_3^2 R_n & A_2 R_n & -A_8 R_n & A_7 R_n & -A_6 R_n \\
-A_5 R_n & A_6 R_n & -A_3^2 R_n & A_4^2 R_n & -A_2 R_n & A_1 R_n & A_8 R_n & -A_7 R_n \\
-A_8 R_n & -A_7 R_n & -A_4^2 R_n & A_3^2 R_n & A_5 R_n & -A_2 R_n & -A_1 R_n & A_6 R_n \\
-A_7 R_n & A_8 R_n & A_4^2 R_n & A_3^2 R_n & -A_5 R_n & -A_2 R_n & A_1 R_n & -A_6 R_n \\
\end{pmatrix}$$

is an $OD(8t; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [1, 3, 4] to obtain infinitely many families of orthogonal designs. Research has yet to be initiated to explore the algebraic restrictions imposed an amicable set by the required constraints.

Short amicable set were defined in [1] as a set of matrices $\{A_i\}_{i=1}^{S}$ of order $m$ and type $(u_1, u_2, u_3, u_4)$, abbreviated as $4 - SAS(m; u_1, u_2, u_3, u_4; G)$, if (2) and (3) are satisfied for $n = 4$ and $u \leq 4$. $4 - SAS(m; u_1, u_2, u_3, u_4; G)$ can be used in either the Goethals-Seidel array or the short Kharaghani array.
to form an $OD(4m; u_1, u_2, u_3, u_4)$. In all cases, the group $G$ of the matrices in the amicable set is such that the extension by Seberry and Whiteman [9] of the group from circulant to type 1 allows the same extension to $R$.

In general a set of $2n$ matrices of order $m$ and type $(s_1, s_2, \ldots, s_n)$ that satisfy equations (2) and (3) will be denoted as $2n - S\lambda(2m; s_1, s_2, \ldots, s_n; G)$. Moreover if these matrices are circulant they will be denoted as $2n - SC\lambda(2m; s_1, s_2, \ldots, s_n; \text{Z}_m)$.

In [1] where all this was first defined was mentioned that:

**Remark 1**

1. If there exists a $2 - S\lambda(n; s_1, s_2; G)$ and a $2 - S\lambda(n; s_3, s_4; G)$ then there exists a $4 - S\lambda(n; s_1, s_2, s_3, s_4; G)$.

2. If there exists a $2 - S\lambda(n; s_1, s_2; G)$, $2 - S\lambda(n; s_3, s_4; G)$, $2 - S\lambda(n; s_5, s_6; G)$ and a $2 - S\lambda(n; s_7, s_8; G)$ there exists an $8 - S\lambda(n; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$.

3. If there exists a $4 - S\lambda(n; s_1, s_2, s_3, s_4; G)$ and a $4 - S\lambda(n; s_5, s_6, s_7, s_8; G)$ there exists an $8 - S\lambda(n; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$.

Thus we can obtain many classes of $4 - S\lambda(n; s_1, s_2, s_3, s_4; G)$ combining together two pairs of the given $2 - S\lambda(n; s_1, s_2; G)$ and $2 - S\lambda(n; s_3, s_4; G)$. Moreover, in Table 2, we give some $4 - S\lambda(m; u_1, u_2, u_3, u_4; \text{Z}_m)$ that cannot be constructed by this method.

Generally, unless we have other information regarding the structure, we are unable to ensure that the matrix $R$ with the desired properties for the Kharaghani, Goethals-Seidel or short Kharaghani arrays exists unless the amicable sets have been generated (circulant or type 1) or constructed from blocks of these kinds. Thus is we have the required matrix $R_i$ for the group $G_i$, $i = 1, 2$ then $R_G = R_1 \times R_2$ will be the required matrix for $G = G_1 \times G_2$, (see [9]).

Let $A_1$ and $A_2$ be matrices of order $m$. We define $\text{circ}(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$. Amicable sets made from $2n$ such block circulant matrices will be called block amicable sets, short block amicable sets or 2-short block amicable sets, $2n - SB\lambda(2m; s_1, s_2, \ldots, s_n; G)$, $n = 1, 2, 4$, where, using $R_t$ for the back-diagonal matrix of order $t$, $G = Z_2 \times Z_m$ and $R_G = R_2 \times R_2$. Here, if $A_1$ and $A_2$ are circulant, then we use the backdiagonal matrix of the same order for $R$ ensuring $A_i(A_j R_i^T = A_i R_i A_j^T$. The required $R_G = R_2 \times R_2$.

A $(1, -1)$ matrix of order $n$ is called a Hadamard matrix if $HH^T = H^T H = n I_n$, where $H^T$ is the transpose of $H$ and $I_n$ is the identity matrix of order $n$. A $(1, -1)$ matrix $A$ of order $n$ is said to be of skew type if $A - I_n$ is skew-symmetric.

A matrix $W = \text{circ}(w_1, \ldots, w_n)$, $w_i \in \{0, \pm 1\}$ which satisfies $WW^T = k I_n$ is called a circulant weighing matrix of order $n$ and weight $k$ or $CW(n, k)$.

We denote the product $Z_p \times Z_p \times \cdots \times Z_p$ ($r$ times) by $EA(p^r)$ the Elementary Abelian group. Moreover $-a$ is denoted by $\bar{a}$.

Throughout this paper we use the symbol $0_m$ to denote the sequence of length $m$ with all elements zero and the symbol $O_t$ to denote the $t \times t$ matrix with all entries zero.

For the undefined terms we refer the reader to the book by Geramita and Seberry [2].

Suppose $C = \text{circ}(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order $n$.  

3
Let

\[ T_n = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & 0 & \ldots & 0
\end{bmatrix} \]

of order \( n \), be the shift matrix. Then we can write \( C = c_0I + c_1T_n + \ldots + c_{n-1}T_n^{n-1} \). Note that \( T_n^n = I \) the identity matrix of order \( n \). We say the Hall polynomial of \( C \) is \( \sum_{i=0}^{n-1} c_i x^i \). The Hall polynomial of \( C^T \) is \( \sum_{i=0}^{n-1} c_i x^{n-i} \).

Given a set of \( \ell \) sequences \( A_j = \{a_{j_1}, a_{j_2}, \ldots, a_{j_n}\}, \ j = 1, \ldots, \ell \), of length \( n \) the non-periodic autocorrelation function, denoted \( NPAF \), \( N_A(s) \) is defined as

\[
N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{j_1} a_{j_2} \ldots a_{j_s}, \quad s = 0, 1, \ldots, n - 1, \quad (4)
\]

If \( A_j(z) = a_{j_1} + a_{j_2} z + \ldots + a_{j_n} z^{n-1} \) is the associated polynomial of the sequence \( A_j \), then

\[
A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{k=1}^{n} a_{j_1} a_{j_2} \ldots a_{j_k} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{i=1}^{n-1} N_A(s) (z^s + z^{-s}), z \neq 0. \quad (5)
\]

Given \( A_i \), as above, of length \( n \) the periodic autocorrelation function, denoted \( PAF \), \( P_A(s) \) is defined, reducing \( i + s \) modulo \( n \), as

\[
P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{j_1} a_{j_2} \ldots a_{j_s}, \quad s = 0, 1, \ldots, n - 1. \quad (6)
\]

We note NPAF sequences imply PAF sequences exist, the NPAF sequences being padded at the end with sufficient zeros to make longer lengths. Hence NPAF sequences can give more general results. If two NPAF sequences have differing lengths then sufficient zeros are added to the end of each to make all the sequences the same length. In all cases NPAF and PAF sequences can be used to make circulant matrices satisfying the additive property (see [3, 4]); if NPAF sequences of lengths \( n_1 \) and \( n_2 \) are used, then by padding, circulant matrices for all orders \( n \geq max(n_1, n_2) \) will exist; if PAF sequences of lengths \( n \) are used, then circulant matrices of order \( n \) exist.

2 Constructions

**Theorem 1** Write \( 0_s \) for the sequence of \( s \) zeros, and let \( a, b, c \) and \( d \) be commuting variables. Use the matrices \( A_1, A_2, A_3 \) and \( A_4 \) given by

\[
A_1 = \text{circ}(0, ba0b0_0), \quad A_2 = \text{circ}(0, c0c0_0), \\
A_3 = \text{circ}(0, 5\bar{d}c0_0), \quad A_4 = \text{circ}(0, \bar{b}0b0b0_0),
\]

can be used in the Goethals-Seidel array to obtain an \( OD(8s + 12; 1, 1, 4, 4) \).
Proof. Observe that

\[
A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T = (a^2 + d^2 + 4b^2 + 4d^2)I_n
\]

and

\[
A_1 A_1^T - A_2 A_2^T + A_3 A_3^T - A_4 A_4^T = 0.
\]

Thus \(A_2, A_2, A_3, A_4\) are a short amicable set and satisfy the additive property (2) so they can be used in the Goethals-Seidel array to obtain an \(OD(8s + 12; 1, 1, 4, 4)\). \(\Box\)

The Melding Construction

Suppose the matrices \(A_1, A_2, A_3\) and \(A_4\) are are short amicable sets, on the set of commuting variables \(\{0, \pm x_1, \pm x_2, \ldots, \pm x_n\}\) or from \(\{0, \pm 1\}\), and satisfy the additive property

\[
\sum_{i=1}^{4} (A_i A_i^T) = \sum_{j=1}^{n} p_j x_j^2 I_n,
\]

and the matrices \(A_5, A_6, A_7\) and \(A_8\) are also short amicable sets, on the set of commuting variables \(\{0, \pm y_1, \pm y_2, \ldots, \pm y_v\}\) or from \(\{0, \pm 1\}\), and satisfy the additive property

\[
\sum_{i=5}^{8} (A_i A_i^T) = \sum_{j=1}^{v} q_j y_j^2 I_n.
\]

Then the eight matrices will form an amicable set so we can use the two together in the Kharaghani array to obtain an \(OD(8n; p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_v)\). \(\Box\)

<table>
<thead>
<tr>
<th>order</th>
<th>type</th>
<th>group</th>
<th>order</th>
<th>type</th>
<th>group</th>
<th>order</th>
<th>type</th>
<th>group</th>
<th>order</th>
<th>type</th>
<th>group</th>
</tr>
</thead>
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<td>1</td>
<td>1</td>
<td>(Z_n)</td>
<td>6</td>
<td>4,4</td>
<td>(Z_6n)</td>
<td>10</td>
<td>4,4</td>
<td>(Z_{10n})</td>
<td>14</td>
<td>8,8</td>
<td>(Z_{14n})</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(Z_{2n})</td>
<td>6</td>
<td>5,5</td>
<td>(Z_5n)</td>
<td>10</td>
<td>9,9</td>
<td>(Z_{10n})</td>
<td>14</td>
<td>10,10</td>
<td>(Z_{14n})</td>
</tr>
<tr>
<td>4</td>
<td>1,4</td>
<td>(Z_{4n})</td>
<td>7</td>
<td>4,4</td>
<td>(Z_{7n})</td>
<td>12</td>
<td>8,8</td>
<td>(Z_{12n})</td>
<td>14</td>
<td>13,13</td>
<td>(Z_{14n})</td>
</tr>
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<td>(Z_{4n})</td>
<td>8</td>
<td>8,8</td>
<td>(Z_{8n})</td>
<td>13</td>
<td>9,9</td>
<td>(Z_{13n})</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table 1: Order and type for small 2-short amicable sets for all \(n \geq 1\).

Using table 2, remark 1 and the above Melding Construction we obtain many 4-short amicable sets and 8-amicable sets.
<table>
<thead>
<tr>
<th>Type</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>ZERO</th>
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<td>b</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>0 - d a d</td>
<td>0 b 0 0</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>a 0</td>
<td>c d</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(1,1,2)</td>
<td>b 0</td>
<td>c - d</td>
<td></td>
<td></td>
<td>2n</td>
</tr>
<tr>
<td>(1,1,4)</td>
<td>0 - c a c</td>
<td>0 - c b - c</td>
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<td></td>
<td>NPAF</td>
</tr>
<tr>
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<td>a 0 a</td>
<td></td>
<td></td>
<td>3n</td>
</tr>
<tr>
<td>(1,1,4)</td>
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<td>c d - c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>- a a a</td>
<td>a 0 a</td>
<td></td>
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<td>NPAF</td>
</tr>
<tr>
<td>(1,1,5)</td>
<td>c 0 0</td>
<td>0 b 0</td>
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<td></td>
<td>4n</td>
</tr>
<tr>
<td>(1,1,5)</td>
<td>- c a c 0</td>
<td>- d b d 0</td>
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</tr>
<tr>
<td>(1,1,5)</td>
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<td>d c d 0</td>
<td></td>
<td></td>
<td>4n</td>
</tr>
<tr>
<td>(1,1,8)</td>
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<td>0 - c d 0 - d c</td>
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<td>NPAF</td>
</tr>
<tr>
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<td>0 - c d b - d c</td>
<td></td>
<td></td>
<td>6n</td>
</tr>
<tr>
<td>(1,2,2)</td>
<td>0 - d a d</td>
<td>0 b 0 c 0</td>
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<td>NPAF</td>
</tr>
<tr>
<td>(1,2,2)</td>
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<td></td>
<td></td>
<td>4n</td>
</tr>
<tr>
<td>(1,4,4)</td>
<td>0 - b a b</td>
<td>0 b 0 b</td>
<td></td>
<td></td>
<td>NPAF</td>
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<tr>
<td>(1,4,4)</td>
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<td>- c d c d</td>
<td></td>
<td></td>
<td>4n</td>
</tr>
<tr>
<td>(2,2,2)</td>
<td>a b</td>
<td>a - b</td>
<td></td>
<td></td>
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</tr>
<tr>
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<td>d c - d c</td>
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<td>6n</td>
</tr>
<tr>
<td>(2,2,8)</td>
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<td>- d c a c d 0</td>
<td></td>
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<td>NPAF</td>
</tr>
<tr>
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<td>- d c a c d 0</td>
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<td>6n</td>
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<tr>
<td>(3,3)</td>
<td>a b</td>
<td>b - a</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(3,3)</td>
<td>a 0</td>
<td>b 0</td>
<td></td>
<td></td>
<td>2n</td>
</tr>
<tr>
<td>(4,4,4)</td>
<td>a a b - b</td>
<td>b b - a</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(4,4,4)</td>
<td>d d c c</td>
<td>c c d - d</td>
<td></td>
<td></td>
<td>4n</td>
</tr>
<tr>
<td>(4,4,8)</td>
<td>d a c c a - d</td>
<td>d b c c b - d</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(4,4,8)</td>
<td>- d b c c b - d</td>
<td>- d a c c a d</td>
<td></td>
<td></td>
<td>6n</td>
</tr>
<tr>
<td>(5,5)</td>
<td>a a - a</td>
<td>a 0 a</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(5,5)</td>
<td>b b - b</td>
<td>b 0 b</td>
<td></td>
<td></td>
<td>3n</td>
</tr>
<tr>
<td>(5,5,5)</td>
<td>- a b a 0 a b</td>
<td>- c d 0 c d</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(5,5,5)</td>
<td>b a - b 0 - a b</td>
<td>d c - d 0 - d c</td>
<td></td>
<td></td>
<td>6n</td>
</tr>
<tr>
<td>(6,6)</td>
<td>a - b a</td>
<td>a a - a</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(6,6)</td>
<td>b a b</td>
<td>b b - b</td>
<td></td>
<td></td>
<td>3n</td>
</tr>
<tr>
<td>(6,6,12)</td>
<td>c c b - c a</td>
<td>c a c - a c</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(6,6,12)</td>
<td>- c b - c a c b</td>
<td>- c b c b c - b</td>
<td></td>
<td></td>
<td>6n</td>
</tr>
<tr>
<td>(8,8)</td>
<td>a a - a a a</td>
<td>b b - b b</td>
<td></td>
<td></td>
<td>NPAF</td>
</tr>
<tr>
<td>(8,8)</td>
<td>b b b b</td>
<td>a a a a</td>
<td></td>
<td></td>
<td>4n</td>
</tr>
</tbody>
</table>

Table 2: Short amicable sets.
3 Some general results

We now consider the use of sequences with zero non-periodic autocorrelation function to make an amicable set of matrices. We refer the reader to [7, 8] for any undefined terms.

Theorem 2 (General construction) Let $X, Y$ be two disjoint $(0, \pm 1)$ sequences with zero non-periodic autocorrelation function of length $n$ and weight $k$, Let $a, b, c, d$ be commuting variables and write $aV$, $bW$ for the circulant (type 1) matrices of order $n$ formed by using the first rows with the elements of $X$ multiplied by $a$ and the elements of $Y$ multiplied by $b$ respectively.

Let $A_i$ be the circulant matrices of order $n$ given by

$$A_1 = aV + bW \quad A_2 = cV + dW \quad A_3 = dV - cW \quad A_4 = bW - aW$$

then $\{A_i\}_{i=1}^4$ is a short amicable set satisfying

$$\sum_{i=1}^{2} \left( A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T \right) = 0,$$  \hspace{1cm} (10)

and the additive property

$$\sum_{i=1}^{4} \left( A_iA_i^T \right) = k(a^2 + b^2 + c^2 + d^2)I_n.$$  \hspace{1cm} (11)

<table>
<thead>
<tr>
<th>Type</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>ZERO</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8,8,8,8)</td>
<td>a a a a b-b-b</td>
<td>b b b-b a-a a</td>
<td>NPAF</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(10,10,10,10)</td>
<td>disjoint</td>
<td>from Golay</td>
<td>NPAF</td>
<td>$8n$</td>
<td></td>
</tr>
<tr>
<td>(13,13,13,13)</td>
<td>from disjoint sequences</td>
<td>NPAF</td>
<td>of length 18 and weight 13</td>
<td>$n \geq 18$</td>
<td></td>
</tr>
<tr>
<td>(16,16,16,16)</td>
<td>disjoint</td>
<td>from Golay</td>
<td>NPAF</td>
<td>$n \geq 16$</td>
<td></td>
</tr>
<tr>
<td>(17,17,17,17)</td>
<td>from disjoint sequences</td>
<td>NPAF</td>
<td>of length 26 and weight 17</td>
<td>$n \geq 26$</td>
<td></td>
</tr>
<tr>
<td>(20,20,20,20)</td>
<td>disjoint</td>
<td>from Golay</td>
<td>NPAF</td>
<td>$n \geq 26$</td>
<td></td>
</tr>
<tr>
<td>(25,25,25,25)</td>
<td>disjoint</td>
<td>from Golay</td>
<td>NPAF</td>
<td>$n \geq 26$</td>
<td></td>
</tr>
<tr>
<td>(26,26,26,26)</td>
<td>disjoint</td>
<td>from Golay</td>
<td>NPAF</td>
<td>$n \geq 36$</td>
<td></td>
</tr>
<tr>
<td>(14,14)</td>
<td>a b b b a a</td>
<td>-b a b a b</td>
<td>NPAF</td>
<td>$7n$</td>
<td></td>
</tr>
<tr>
<td>(17,17)</td>
<td>a a a a a a</td>
<td>c c c c c c</td>
<td>PAF</td>
<td>$9n$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: (continued).
Proof. Now $A_1 = aV + bW$, where $V, W$ are disjoint $(0, \pm 1)$ circulant (type 1 or group developed also suffice) matrices of order $n$ which satisfy $VV^T + WW^T = kI_n$, and similarly for the other $A_j$, $j = 2, 3, 4$.

Then

$$A_1A_1^T = (aV + bW)(aV^T + bW^T) = a^2VV^T + b^2WW^T + ab(VW^T + WV^T).$$

Hence

$$\sum_{i=1}^4 \left( A_iA_i^T \right) = (a^2 + b^2 + c^2 + d^2)(VV^T + WW^T) = k(a^2 + b^2 + c^2 + d^2)I_n.$$

Now

$$A_1A_2^T - A_2A_1^T = (aV + bW)(cV^T + dW^T) - (cV + dW)(aV^T + bW^T)$$

$$= (ad - bc)VV^T + (-ad + bc)WW^T,$$

and

$$A_3A_1^T - A_4A_3^T = (dV - cW)(bV^T - aW^T) - (bV - aW)(dV^T - cW^T)$$

$$= (-ad + cb)VV^T + (ad - cb)WW^T.$$

Thus summing over the four $A_i$ we see they form a short amicable set satisfying the additive property.

\[\-boxed{}\]

Corollary 1 Let $X, Y$ be a pair of disjoint $(0, \pm 1)$ sequences with zero non-periodic autocorrelation function of length $n$ and weight $k$. Then there exists a short amicable set which can be used to form an OD$(An; k, k, k, k)$.

Proof. Use the sequences as in the theorem to form an amicable set with the additive property. Then use this set in the Goethals-Seidel array to obtain the result. \[\-\boxed{}\]

For $a, b, \gamma, \delta, \epsilon, \phi, \psi, \mu, \nu$ non-negative integers, Koukouvinos and Seberry [5, p. 160] show that there exist two disjoint $(0, \pm 1)$ sequences, with zero non-periodic autocorrelation function, of length $\geq n$, $n \in N = \{2 \times 2^a5^b10^c13^d17^e25^f26^g34^h50^i\}$ and weight $k$, $k \in K = \{2^a5^b10^c13^d17^e25^f26^g34^h50^i\}$. These give the results presented in Table 3.

References


\begin{center}
\begin{tabular}{|c|c|}
\hline
Type & ZERO \\
\hline
(1,1,1) & NPAF $n \geq 1$ \\
(2,2,2) & NPAF $n \geq 2$ \\
(4,4,4) & NPAF $n \geq 4$ \\
(5,5,5) & NPAF $n \geq 6$ \\
(8,8,8) & NPAF $n \geq 8$ \\
(10,10,10,10) & NPAF $n \geq 10$ \\
(13,13,13,13) & NPAF $n \geq 18$ \\
(16,16,16,16) & NPAF $n \geq 16$ \\
(17,17,17,17) & NPAF $n \geq 26$ \\
(20,20,20,20) & NPAF $n \geq 20$ \\
(25,25,25,25) & NPAF $n \geq 36$ \\
(26,26,26,26) & NPAF $n \geq 26$ \\
\hline
\end{tabular}
\end{center}

Table 3: Short amicable sets from corollary 1


