June 2003

Some results on Kharaghani type orthogonal designs

S. Georgiou
National Technical University of Athens, Greece

C. Koukouvinos
National Technical University of Athens, Greece

Jennifer Seberry
University of Wollongong, jennie@uow.edu.au

Some results on Kharaghani type orthogonal designs

Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (al, a2), (al, a2, a3, a4) or (al, a2, ..., as) then there exist amicable sequences of length $\ell \equiv 0 \pmod{m}$ and of the same type. We also present a theorem that produces a set of 2v amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type (al, al, a2, a2, ... , av, av) from v pairs of disjoint (0, ±1) amicable sequences. Using these results we can obtain many infinite classes of Kharaghani type orthogonal designs. Actually, if there exists an Kharaghani type orthogonal design of order n and of type (al, a2, ..., av), which is constructed from sequences, then there exists an infinite family of Kharaghani type orthogonal designs of the same type which is constructed from appropriate sequences.

Keywords

Sequences, orthogonal designs, Kharaghani type orthogonal designs, amicable sets, Hall polynomial.

Disciplines

Physical Sciences and Mathematics

Publication Details

This article was originally published as Georgiou, S, Koukouvinos, C and Seberry, J, Some results on Kharaghani type orthogonal designs, Utilitas Mathemeatica, 63, 2003, 43-52.

Some results on Kharaghani type orthogonal designs

S. Georgiou, C. Koukouvinos
Department of Mathematics
National Technical University of Athens
Zografou 15773, Athens, Greece

and

J. Seberry
School of IT and Computer Science
University of Wollongong
Wollongong, NSW, 2522, Australia

Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (a_1, a_2), (a_1, a_2, a_3, a_4) or $(a_1, a_2, \ldots, a_8)$ then there exist amicable sequences of length $\ell \equiv 0 \ (mod\ m)$ and of the same type. We also present a theorem that produces a set of $2v$ amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type $(a_1, a_1, a_2, a_2, \ldots, a_v, a_v)$ from v pairs of disjoint $(0, \pm 1)$ amicable sequences.

Using these results we can obtain many infinite classes of Kharaghani type orthogonal designs. Actually, if there exists an Kharaghani type orthogonal design of order n and of type $(a_1, a_2, \ldots, a_v)$, which is constructed from sequences, then there exists an infinite family of Kharaghani type orthogonal designs of the same type which is constructed from appropriate sequences.

Key words and phrases: Sequences, orthogonal designs, Kharaghani type orthogonal designs, amicable sets, Hall polynomial.

AMS Subject Classification: Primary 05B15, 05B20.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let A_1, A_2 be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2\}$ satisfying $A_1 A_1^T + A_2 A_2^T = (s_1 x_1^2 + s_2 x_2^2) I_n$. Then

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \qquad \text{or} \qquad D = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}. \tag{1}$$

is an $OD(2n; s_1, s_2)$.

Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix} \tag{2}$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [1] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [4] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{3}$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{4}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{5}$$

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices (or type 1) of type $(s_1, s_2, \ldots, s_u)$ of order t. Then the Kharaghani array from [4]

$$H = \begin{pmatrix}
A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\
-A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\
-A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\
-A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\
-A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\
-A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\
-A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\
-A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1
\end{pmatrix} \tag{6}$$

is a Kharaghani type orthogonal design $OD(8m; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [2, 3, 4, 5, 6] to obtain infinitely many families of Kharaghani type orthogonal designs.

A set $\{A_i\}_{i=1}^4$ is said to be a *short amicable set* of length m and type (u_1, u_2, u_3, u_4) if (4) and (5) are satisfied for $n = 4$ and $u \leq 4$. Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix}
A & B & CR & DR \\
-B & A & DR & -CR \\
-CR & -DR & A & B \\
-DR & CR & -B & A
\end{bmatrix} \tag{7}$$

to form an Goethal-Seidel type orthogonal design $OD(4m; u_1, u_2, u_3, u_4)$.

A set of sequences $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, \ldots, 2v$ is said to be a set of $2v$ *amicable sequences* of length m and type $(u_1, u_2, \ldots, u_p)$ if the corresponding circulant matrices which are constructed from these sequences satisfy the equations (4) and (5). On the other hand, it is clear that, if we have a set of circulant amicable matrices then their first rows can be considered as a set of amicable sequences. Therefore, throughout this paper we use either circulant amicable matrices or amicable sequences.

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function (NPAF)* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{8}$$

Given A as above of length n the *periodic autocorrelation function (PAF)* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{9}$$

We define the $NPAF$ (PAF) of a set of sequences the sum of the corresponding $NPAF$ (PAF) of the individual sequences.

Suppose $C = circ(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order n. Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

of order n, be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \ldots + c_{n-1} T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n. We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

2 Multiplication of the length of amicable sets of sequences

Theorem 1 *Let $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, \ldots, 2v$ be a set of $2v$ amicable sequences of length m and type $(u_1, u_2, \ldots, u_p)$. Then there exist a set of $2v$ amicable sequences of length $\ell \equiv 0 \pmod{m} = mi$ for all $i = 1, 2, \ldots$ and type $(u_1, u_2, \ldots, u_p)$.*

Proof. Let i be a constant integer. We use the map T_m^k to define sequences A_k and the map $S_\ell^k = T_m^k$ to define sequences B_k

$$B_k = \sum_{j=0}^{m-1} a_{k,j} S_\ell^j, \ k = 1, 2, \ldots, 2v$$

Now

$$\sum_{k=1}^{2v} A_k A_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} T_m^{j-x} \right) = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_m.$$

Thus we have that

(i) If m is odd then the coefficients of T_m^σ, $\sigma = -(m-1), \dots, -1, 1, \dots, m-1$ is zero, and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum^{m-1} j, x = 0 j - x = \sigma \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} u_k x_k^2 \tag{10}$$

(ii) If m is even, $m = 2n$ then we have that $T_m^n = T_m^{-n}$ and so the coefficients of T_m^σ, $\sigma = -(2n-1), \dots, -(n+1), -(n-1), \dots, -1, 1, \dots, n-1, n+1, \dots, 2n-1$ are zero, the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum^{m-1} j, x = 0 j - x = \sigma \sigma \neq \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0, \sum^{m-1} j, x = 0 j - x = \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \text{ and } \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} \tag{11}$$

Now

$$\sum_{k=1}^{2v} B_k B_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \dots, m-1$, and so using equations (10) or (11) we obtain

$$\sum_{k=1}^{2v} B_k B_k^T = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_{2mi} = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_\ell \tag{12}$$

Moreover

$$\sum_{k=1}^{v} \left(A_{2k-1} A_{2k}^T - A_{2k} A_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) T_m^{j-x} \right)$$
$$= 0$$

and from these we have that

(i) if m odd, then the coefficients of T_m^σ, $\sigma = -(m-1), \dots, m-1$ are zero.That means

$$\sum^{m-1} j, x = 0 j - x = \sigma \sum_{k=1}^{v} (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \tag{13}$$

(ii) if m is even, $m = 2n$ then the coefficients of T_m^σ, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, n-1, n+1, \ldots, 2n-1$ are zero and the coefficient of T_m^n plus the the coefficient of T_m^{-n} is zero. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma \\ \sigma \neq \pm n}}^{m-1} \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) = 0 \text{and} \sum^{m-1} j,x = 0 j - x = \pm n \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a$$

(14)

Now

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \ldots, m-1$ and so using equations (13) or equations (2) we obtain

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = 0 \tag{15}$$

Equations (12) and (15) show that $\{B_k\}_{k=1}^{2v}$ is an amicable set of matrices (sequences) of length $\ell \equiv 0 (mod\ m)$, $\ell = mi$, $i = 1, 2, \ldots$ and type $(u_1, u_2, \ldots, u_p)$. $\square$

Corollary 1 *Let $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2$ be a set of two amicable sequences of length m and type (u_1, u_2). Then there exist a set of two amicable sequences of length $\ell \equiv\ 0\ (\ mod\ m) = mi$ and type (u_1, u_2).*

Proof. Use Theorem 1 with $2v = 2$ and $p = 2$.

Example 1 We have that $A_1 = 0T_4^0 + aT_4^1 + bT_4^2 - aT_4^3$ and $A_2 = 0T_4^0 + aT_4^1 + 0T_4^2 + aT_4^3$ is a set of two amicable matrices (sequences) of length $m = 4$ and type $(1, 4)$. Corollary 1 gives a set of two amicable sequences of length $m = 4i$ and type $(1, 4)$ for all $i = 1, 2, \ldots$.

Corollary 2 *Let $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, 3, 4$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \pm x_3,\ \pm x_4\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, 3, 4$ be a set of four amicable sequences of length m and type (u_1, u_2, u_3, u_4). Then there exist a set of four amicable sequences of length $\ell \equiv\ 0\ (\ mod\ m) = mi$ and type (u_1, u_2, u_3, u_4).*

Proof. Use Theorem 1 with $2v = 4$ and $p = 4$.

Example 2 We have that $A_1 = aT_3^0 - bT_3^1 + aT_3^2$, $A_2 = bT_3^0 + aT_3^1 + bT_3^2$ and $A_3 = aT_3^0 + aT_3^1 - aT_3^2$, $A_4 = bT_3^0 + bT_3^1 + bT_3^2$ is a set of four amicable matrices (sequences) of length $m = 3$ and type $(6,6)$. Corollary 2 gives a set of four amicable sequences of length $m = 3i$ and type $(6,6)$ for all $i = 1, 2, \dots$.

Corollary 3 *Let* $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, 8$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_8\}$, $j = 0, 1, \dots, m-1$ *and* $k = 1, 2, \dots, 8$ *be a set of eight amicable sequences of length m and type* $(u_1, u_2, \dots, u_8)$. *Then there exist a set of eight amicable sequences of length* $\ell \equiv \ 0 \ (mod\ m) = mi$ *and type* $(u_1, u_2, \dots, u_8)$.

Proof. Use Theorem 1 with $2v = 8$ and $p = 8$.

Example 3 We have that $A_1 = -aT_7^0 + aT_7^1 + aT_7^2 + gT_7^3 + aT_7^4 + eT_7^5 + cT_7^6$, $A_2 = -fT_7^0 + fT_7^1 + fT_7^2 - hT_7^3 + fT_7^4 + bT_7^5 - dT_7^6$, $A_3 = -gT_7^0 + gT_7^1 + gT_7^2 - aT_7^3 + gT_7^4 + cT_7^5 - eT_7^6$, $A_4 = -hT_7^0 + hT_7^1 + hT_7^2 + fT_7^3 + hT_7^4 + dT_7^5 + bT_7^6$, $A_5 = -eT_7^0 + eT_7^1 + eT_7^2 - cT_7^3 + eT_7^4 - aT_7^5 + gT_7^6$, $A_6 = -dT_7^0 + dT_7^1 + dT_7^2 - bT_7^3 + dT_7^4 - hT_7^5 + fT_7^6$, $A_7 = -bT_7^0 + bT_7^1 + bT_7^2 + dT_7^3 + bT_7^4 - fT_7^5 - hT_7^6$ and $A_8 = -cT_7^0 + cT_7^1 + cT_7^2 + eT_7^3 + cT_7^4 - gT_7^5 - aT_7^6$ is a set of eight amicable matrices (sequences) of length $m = 7$ and type $(7,7,7,7,7,7,7,7)$. Corollary 3 gives a set of eight amicable sequences of length $m = 7i$ and type $(7,7,7,7,7,7,7,7)$ for all $i = 1, 2, \dots$.

Remark 1 Using Corollaries 1, 2 and 3 as indicated by the examples and using array (1), (2) or (7) and (6) respectively we obtain many infinite classes of orthogonal designs.

3 Construction of amicable sets of sequences from non amicable sets of sequences

Lemma 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m-1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. These are the sequences A_k, $k = 1, 2, \ldots, v_1$ and B_k, $k = 1, 2, \ldots, v_2$ together. □

Corollary 4 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m_1-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m_1 - 1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m_1 *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m_2-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m_2 - 1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m_2 *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* $\ell \cdot i$ *where* $\ell = [m_1, m_2]$ *is the least common multiple (l.c.m.) of* m_1 *and* m_2 *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. Since ℓ is the least common multiple of m_1 and m_2 then $\ell = m_1 \cdot i_1 = m_2 \cdot i_2$. Using theorem 1 we can construct a set of v_1 amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p)$ and a set of v_2 amicable sequences of length ℓ and type $(t_1, t_2, \ldots, t_q)$. Now using Lemma 1 we obtain a set of $v_1 + v_2$ amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Using theorem 1 again in the derived sequences we have the result. □

Example 4 We have that $A_1 = \{e, f\}$, $A_2 = \{e, -f\}$, $A_3 = \{e, 0\}$, $A_4 = \{f, 0\}$ is a short amicable set of length 2 and type $(3, 3)$. We also have that $A_1 = \{a, a, b, -b\}$, $A_2 = \{c, c, d, -d\}$, $A_3 = \{d, d, -c, c\}$, $A_4 = \{b, b, -a, a\}$ is a short amicable set of length 4 and type $(4, 4, 4, 4)$. Now $\ell = [4, 2] = 4$ and thus from corollary 4 we obtain eight amicable sequences of length $\ell \cdot i$ and type $(3, 3, 4, 4, 4, 4)$ for all $i = 1, 2, \ldots$.

Theorem 2 (Doubling the number of sequences) *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots v$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v$ *be* v *sequences with PAF=0 (or NPAF=0) of length* m *and type* $(u_1, u_2, \ldots, u_p)$. *Then there exist a set of* $2v$ *amicable sequences of length* m *and type* $(2u_1, 2u_2, \ldots, 2u_p)$ *with PAF=0 (or NPAF=0).*

Proof. Set $B_{2k-1} = B_{2k} = circ(A_k),\ k = 1, 2, \ldots, v$. Then

$$\sum_{k=1}^{2v} B_k B_k^T = 2 \cdot \sum_{k=1}^{v} A_k A_k^T = \left(\sum_{i=1}^{p} 2u_i x_i^2 \right) I_m$$

and

$$B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T = A_k A_k^T - A_k A_k^T = 0,\ k = 1, 2, \ldots, v.$$

Thus $\{B_k\}_{k=1}^{2v}$ is a set of $2v$ amicable matrices (sequences) of length m and type $(2u_1, 2u_2, \ldots, 2u_p)$. □

4 More Constructions

Theorem 3 *Let $(X_k, Y_k),\ k = 1, 2, \ldots, v$ be v pairs of sequences of lengths m_k with the properties*

$$Z_k Z_k^T + W_k W_k^T = p_k I_{m_k} \tag{16}$$

$$Z_k W_k^T - W_k Z_k^T = 0 \tag{17}$$

$$Z_k * W_k = 0 \tag{18}$$

for all $k = 1, 2, \ldots, v$, where $Z_k = circ(X_k)$ and $W_k = circ(Y_k)$. Then there exist a set of $2v$ amicable sequences of length $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v]$, where $[m_1, m_2, \ldots, m_v]$ is the least common multiple (l.c.m.) of $m_1, m_2, \ldots, m_v$ and of type $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$ on the set $\{a_1, a_2, \ldots, a_{2v}\}$ of commuting variables.

Proof. Set

$$B_k = a_{2k} X_k + a_{2k-1} Y_k, \quad \text{and} \quad C_k = -a_{2k-1} X_k + a_{2k} Y_k, \quad k = 1, 2, \ldots, v$$

Condition (18) gives that $B_k,\ \ k = 1, 2, \ldots, v$ and $C_k,\ k = 1, 2, \ldots, v$ are sequences of lengths $m_k,\ k = 1, 2, \ldots, v$ and type $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$.

For any k and by simple calculations using conditions (16) and (17) we have that

$$B_k B_k^T + C_k C_k^T = (p_k a_{2k-1}^2 + p_k a_{2k}^2) I_{m_k} \quad \text{and} \quad B_k C_k^T - C_k B_k^T = 0$$

Now from theorem 1, there are sequences D_k and E_k of length $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v]$, $k = 1, 2, \ldots, v$, with the desirable properties. By lemma 1 we have the result. □

Example 5 Set $Z_1 = \{1\}$, $W_1 = \{0\}$, $Z_2 = \{1,0\}$, $W_2 = \{0,1\}$, $Z_3 = \{1,1,1,-1\}$, $W_3 = \{0,0,0,0\}$, $Z_4 = \{0,1,0,-1,0,1\}$ and $W_4 = \{0,0,1,0,1,0\}$. These are four pair of sequences of lengths $1,2,4$ and 6 satisfying conditions (16), (17) and (18) with $p_1 = 1$, $p_2 = 2$, $p_3 = 4$ and $p_4 = 5$. We have that $[1,2,4,6] = 12$ and from theorem 3 we obtain eight sequences of length $\ell \equiv 0$ $(mod\ 12)$ and of type $(1,1,2,2,4,4,5,5)$ on the set $\{a_1, a_2, \ldots, a_8\}$ of commuting variables which can be used in the Kharaghani array (6) to obtain an infinite class of Kharaghani type orthogonal designs $OD(8\ell; 1,1,2,2,4,4,5,5)$.

References

[1] A.V.Geramita, and J.Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[2] W.H. Holzmann, and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.*, 22 (2000), 287-299.

[3] W.H. Holzmann, and H. Kharaghani, On the orthogonal designs of order 24, *Discrete Appl. Math.*, 102 (2000), 103-114.

[4] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, 8 (2000), 166-173.

[5] C. Koukouvinos and J. Seberry, Orthogonal designs of Kharaghani type: I, *Ars Combin.*, to appear.

[6] C. Koukouvinos and J. Seberry, Infinite families of orthogonal designs : I, *Bull. Inst. Combin. Appl.*, to appear.