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Solving continuous network design problem with generalized geometric programming approach

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Abstract

To satisfy growing travel demand and reduce traffic congestion, the continuous network design problem (CNDP) is often proposed to optimize road network performance by the expansion of road capacity. In the determination of the equilibrium travel flow pattern, equilibrium principles such as deterministic user equilibrium (DUE) and stochastic user equilibrium (SUE) may be applied to describe travelers' route choice behavior. Because of the different mathematical formulation structures for the CNDP with DUE and SUE principles, most of the existing solution algorithms have been developed to solve the CNDP for either DUE or SUE. In this study, a more general solution method is proposed by applying the generalized geometric programming (GGP) approach to obtain the global optimal solution of the CNDP with both DUE and SUE principles. Specifically, the original CNDP problem is reformulated into a GGP form, and then a successive monomial approximation method is employed to transform the GGP formulation into a standard geometric programming form, which can be cast into an equivalent nonlinear but convex optimization problem whose global optimal solution can be guaranteed and solved by many existing solution algorithms. Numerical experiments are presented to demonstrate the validity and efficiency of the solution method.

Keywords

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Solving Continuous Network Design Problem with Generalized Geometric Programming Approach

Bo Du and David Z. W. Wang

To satisfy growing travel demand and reduce traffic congestion, the continuous network design problem (CNDP) is often proposed to optimize road network performance by the expansion of road capacity. In the determination of the equilibrium travel flow pattern, equilibrium principles such as deterministic user equilibrium (DUE) and stochastic user equilibrium (SUE) may be applied to describe travelers' route choice behavior. Because of the different mathematical formulation structures for the CNDP with DUE and SUE principles, most of the existing solution algorithms have been developed to solve the CNDP for either DUE or SUE. In this study, a more general solution method is proposed by applying the generalized geometric programming (GGP) approach to obtain the global optimal solution of the CNDP with both DUE and SUE principles. Specifically, the original CNDP problem is reformulated into a GGP form, and then a successive monomial approximation method is employed to transform the GGP formulation into a standard geometric programming form, which can be cast into an equivalent nonlinear but convex optimization problem whose global optimal solution can be guaranteed and solved by many existing solution algorithms. Numerical experiments are presented to demonstrate the validity and efficiency of the solution method.

To satisfy rapidly growing travel demand and alleviate traffic congestion, the transportation network is adjusted regularly by link capacity expansion or new link addition within a given investment budget. In the literature on transportation network modeling and optimization, such an adjustment issue is formulated as the network design problem (NDP), which optimizes a specific network performance objective while assuming that travelers' route choice behavior follows certain principles [e.g., deterministic user equilibrium (DUE) or stochastic user equilibrium (SUE)] (*1*). The NDP is known as a nondeterministic polynomial-time hard problem, which can be categorized into the continuous NDP (CNDP), discrete NDP, and mixed NDP (*2*). The CNDP deals with the expansion of the capacity of existing links, whereas the discrete NDP determines the optimal addition of new links, and the mixed NDP handles both of them simultaneously. A large body of literature exists on the NDP; some useful reviews can be found elsewhere (*2–6*).

To date, abundant modeling methods and solution algorithms have been presented to solve the CNDP. Abdulaal and LeBlanc formulated the CNDP as a nonlinear programming problem that could be converted into an unconstrained problem so that it could be solved by a direct search method (*7*). Dantzig et al. adopted the Lagrange multiplier technique and decomposition procedure to handle the CNDP considering a system optimal criterion (*8*). LeBlanc and Abdulaal compared the computational efficiency and results by solving two models for the CNDP with user optimum flow and system optimal flow, respectively (*9*). LeBlanc and Boyce proposed a piecewise bilevel linear programming (BLP) model for the NDP for a middle- and small-sized problem, whereas for larger networks, an equivalent and approximating nonlinear programming problem was transformed from the BLP model and the Frank–Wolfe method was adopted to solve the problem efficiently (*10*). Marcotte used four heuristic procedures to deal with the CNDP and gave a detailed analysis (*11*). Suwansirikul et al. suggested an equilibrium decomposed optimization heuristic method to deal with the CNDP (*12*). Ben-Ayed et al. provided different formulations for the CNDP with different investment functions and gave a more general representation of the travel cost function (*13*). Friesz et al. constructed a mathematical program with variational inequality constraints to describe the CNDP and proposed a simulated annealing algorithm to solve it (*14*). Then Friesz et al. applied a simulated annealing algorithm to solve a multiobjective model of the CNDP (*15*). In Davis's analysis a logit-based SUE principle made the CNDP differentiable and tractable; he used a generalized reduced gradient method and sequential quadratic programming to solve the CNDP with a logit-based SUE principle (*16*). Meng et al. created an equivalent single-level continuously differentiable but still nonconvex optimization formulation for the CNDP by considering the DUE principle and applied a locally convergent augmented Lagrangian method to solve this problem (*5*). Later, Meng and Yang used a penalty function combined with a simulated annealing method to solve a BLP CNDP model with an equity constraint (*17*). Lo and Tung developed a CNDP model with degradable link capacities and used the maximization of a demand multiplier as the objective with reliability constraints (*18*). Chiou exploited a gradient-based descent method to solve the CNDP with corresponding DUE flows following Wardrop's first principle (*19*). Ban et al. proposed a general framework to describe the CNDP as a mathematical program with complementarity constraints, which was converted to a single-level problem and solved by a relaxation scheme (*20*). Josefsson and Patriksson made a sensitivity analysis of separable traffic equilibrium models and used a gradient projection algorithm to solve the CNDP with the DUE principle (*21*). The results showed that the sensitivity analysis was accurate and produced better solutions than previous heuristics. Connors et al. adopted a gradient-based approach to solve the

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NDP with a probit-based SUE principle and elastic demand (*22*). Chiou adopted a conjugate subgradient projection method to solve the CNDP with global convergence, and numerical studies demonstrated the validity and efficiency of the proposed method (*23*). Wang and Lo transformed the CNDP from BLP to mixed integer linear programming by linearization approximation so that the global optimal solution could be guaranteed, and this scheme was also applicable to other types of NDPs (*24*). Li et al. formulated the CNDP as a sequence of single-level concave programs by using the gap function technique and penalty method and solved them by a multicutting plane method (*25*). Although this method was proved to be valid, it consumed significant computational resources. Szeto et al. developed an integrated model to consider a multiobjective time-dependent NDP with land use transportation interaction over time and sustainability (*26*). Wang et al. modeled the CNDP as a mathematical program with equilibrium constraint and solved it by a cut constraint algorithm (*27*). Wang et al. modeled the CNDP as a BLP problem combined with tradable credit to increase road capacity and reduce traffic demand simultaneously; the problem was solved by a relaxation algorithm (*28*). Liu and Wang proposed a global optimization method to handle the CNDP with the SUE principle in which an outer-approximation technique was applied to derive a tight linear programming relaxation; thus, a global solution algorithm could be used based on a range reduction technique (*29*). Wang et al. addressed a novel NDP formulation that aims to determine the optimal new link addition and their optimal capacities simultaneously, and a global optimization solution method was proposed to solve the problem (*30*). A summary of previous solution methods for the CNDP is shown in Table 1.

Much recent research has developed global optimization solution methods to solve the CNDP; however, the proposed methods are only applicable to handling the CNDP with either DUE or SUE constraints but not both, since the methods take advantage of the specific problem formulation structure for the two different routing choice behavioral assumptions when the solution algorithms are designed. In this study, the generalized geometric programming (GGP) approach is employed to solve the global optimal solution of the CNDP. Specifically, the original CNDP problem is reformulated into a GGP form, and then a successive monomial approximation method is used to transform the GGP model into a standard geometric program (GP). The solution of the standard GP has been well studied in the literature, since it can be transferred into an equivalent nonlinear but convex optimization problem whose global optimization solution can be guaranteed and solved by many existing solution algorithms. The solution of the standard GP model has the advantage of being highly efficient and robust; for example, a GP with 1,000 variables and 10,000 constraints can be solved by an interior-point algorithm in less than a minute (*33*). By applying this GGP approach, the global optimal solution for the CNDP with both DUE and SUE constraints can be achieved.

MODEL FORMULATION

Generalized Geometric Programming

It has been shown that geometric programming (GP) can be solved efficiently and reliably by many methods, even for large-scale problems, and GP has been used widely in engineering for resource allocation in communication and network systems, inventory control, and other applications (*34–36*). For more information on the GP and its extensions and applications, some useful studies can be found elsewhere (*37–42*). Although the GP modeling approach has been well investigated and widely used in various applications, it is rarely applied to transportation-related fields. To the authors' best knowledge, no previous scholars have applied the GP to NDP-related problems, and only a few studies can be found on multimodal and trip distribution models in the 1980s. Wong applied the GP to develop a primal-dual relationship between maximum likelihood and entropy maximization formulations of the trip distribution model (*43*). Marín analyzed a multimodal combined model with a more general GP framework (*44*).

The standard GP model is an optimization problem in a special form:

$$\min f_0(x)$$

subject to

$$f_i(x) \leq 1 \qquad i = 1, 2, \ldots, m$$

$$h_j(x) = 1 \qquad j = 1, 2, \ldots, p \tag{1}$$

TABLE 1 General Summary of Previous Solution Methods for CNDP

Algorithm	References
Powell's method and Hooke and Jeeves' method	Abdulaal and LeBlanc (*7*)
Simulated annealing algorithm	Friesz et al. (*14*), Friesz et al. (*15*), Meng and Yang (*17*)
Equilibrium decomposed optimization	Dantzig et al. (*8*), Suwansirikul et al. (*12*)
Gradient-based method	Davis (*16*), Chiou (*19*), Josefsson and Patriksson (*21*)
Conjugate subgradient projection method	Chiou (*23*)
Sensitivity analysis–based method	Connors et al. (*22*), Sumalee et al. (*31*)
Augmented Lagrangian method	Meng et al. (*5*)
Interior point method–based approach	Ukkusuri and Patil (*32*)
Mixed integer linear program	Wang and Lo (*24*)
Multicutting plane method	Li et al. (*25*)
Cut constraint algorithm	Wang et al. (*27*)
Relaxation scheme or algorithm	Ban et al. (*20*), Wang et al. (*28*), Liu and Wang (*29*)

where h_j represents monomial functions (in the form of $cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $c > 0$ and $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$) and f_0, f_i represent posynomial functions in the following form of

$$\sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

where

$c_k > 0$

and

$a_{ik} \in \mathbb{R}$

$i = 1, 2, \ldots, n$;
$k = 1, 2, \ldots, K$; and
x = optimization decision variable vector with components $x_i > 0$.

Since the standard GP model has very strict requirements on the problem format (e.g., the equality constraints must be in the form of a monomial), its practical application is limited. Some relaxation and extension of the standard GP has been made to develop a more general GGP framework, or extended GGP (usually with more relaxations of rules than the GGP), which has a wider application scope. The GGP can be converted to an equivalent GP by using some transformation techniques that can be easily solved as a standard GP (*41*).

To solve a standard GP, it is usually converted to an equivalent nonlinear but convex optimization problem based on the logarithmic transformation technique. Let $y_i = \log x_i$, and minimize $\log f_0$ with inequality constraints $\log f_i < 0$ and equality constraints $\log g_i = 0$; thus the original GP problem can be transformed as follows:

$$\min \log f_0\left(e^y\right)$$

subject to

$$\log f_i\left(e^y\right) \le 0 \qquad i = 1, 2, \ldots, m$$

$$\log h_j\left(e^y\right) = 0 \qquad j = 1, 2, \ldots, p \tag{2}$$

where vector $y = (y_1, y_2, \ldots, y_n)$ contains new variables. This transformed model is indeed convex, with convex objective, convex inequality constraints, and linear equality constraints. For example, given an equality constraint in GP form, $h(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = 1$, with the aforementioned logarithmic transformation technique, the result is $\log g(e^y) = \log c + a_1 \log x_1 + \cdots + a_n \log x_n = 0$, which is an affine function of vector y, $a_1 y_1 + \cdots + a_n y_n = -\log c$. The details of the whole transformation of the GP model and the proof of its convexity can be found elsewhere (*41*). Many existing solution algorithms can be applied to solving the equivalent transformed convex optimization problem, whose solution is indeed the global optimal solution of the problem, and one of the most efficient and commonly used solution algorithms is the interior-point method.

The GGP approach provides a general modeling and solution framework, which can be used to handle various forms of CNDPs. Particularly for CNDP with DUE constraints, the original model formulation can be readily formulated into a GGP form; for CNDP with SUE constraints, the most distinct part from CNDP with DUE is the logit model with exponential function, which is quite suitable to apply the GGP approach since the exponential function could be easily converted to a linear function by using the logarithmic transformation technique. Therefore, the GGP modeling approach can be applied as a unified tool for handling the CNDP with either DUE or SUE constraints. As for the GGP, a variety of model formulations can be developed for different application contexts. In this study, in an attempt to solve the CNDP in a transportation study, a quotient form of the GGP model is given as follows, which is applicable to both DUE- and SUE-based CNDPs:

$$(\mathrm{A})\begin{cases} \min f_0(x) - g_0(x) \\ \text{subject to} \\ f_i(x) - g_i(x) \le 0, \\ \quad i = 1, 2, \ldots, M \\ f_j(x) - g_j(x) = 0, \\ \quad j = 1, 2, \ldots, N \\ 0 < x_k^{\mathrm{LB}} \le x_k, \\ \quad k = 1, 2, \ldots, K \end{cases} \Leftrightarrow (\mathrm{B})\begin{cases} \min x_0 \\ \text{subject to} \\ \dfrac{f_0(x)}{g_0(x) + x_0} \le 1 \\ \dfrac{f_i(x)}{g_i(x)} \le 1, \qquad i = 1, 2, \ldots, M \\ \dfrac{f_j(x)}{g_j(x)} = 1, \qquad j = 1, 2, \ldots, N \\ 0 < x_k^{\mathrm{LB}} \le x_k, \qquad k = 1, 2, \ldots, K \end{cases} \tag{3}$$

where x_k^{LB} are positive lower bounds of variable x_k, $k = 1, 2, \ldots, K$, and f and g are posynomial functions. The left model formulation (A) in Equation 3 is a general form of the GGP, and Formulation B is an equivalent quotient form of the GGP model by introducing an additional variable x_0. The advantage of such a quotient form is that it can be easily and efficiently solved by a condensation technique, which is demonstrated in the section on the solution algorithm. In the subsequent sections, both DUE- and SUE-based CNDPs will be reformulated into this quotient GGP form (B), and more details of this GGP model can be found elsewhere (*45*).

GGP-Based CNDP Model with DUE Constraints

It is straightforward to formulate the CNDP into a bilevel program problem as follows, in which the upper level is to minimize the total cost under the limitation of road capacity enhancement, and the lower level is the standard DUE conditions:

$$\min_y \tau \sum_{a \in A} t_a(x_a, y_a) \cdot x_a + \lambda \sum_{a \in A} g_a(y_a)$$

subject to

$$\underline{y_a} \le y_a - y_a^0 \le \overline{y_a} \qquad \forall a \in A$$

$$\min_f \sum_{a \in A} \int_0^{x_a} t_a(w, y_a)\, dw$$

subject to

$$\sum_{p \in P} f_p^w = q^w \qquad \forall w \in W$$

$$x_a = \sum_{w \in W} \sum_{p \in P} \delta_{ap}^w f_p^w \qquad \forall a \in A$$

$$t_a(x_a, y_a) = \alpha_a + \beta_a x_a^4 y_a^{-4} \qquad \forall a \in A$$

$$f_p^w \ge 0 \qquad \forall w \in W; \forall p \in P$$

$$x_a \geq 0 \qquad \forall a \in A \tag{4}$$

where

x_a = traffic flow on arc a;
y_a^0, y_a = capacity before and after enhancement on arc a, respectively;
$\underline{y_a}, \overline{y_a}$ = lower and upper bounds of capacity enhancement on arc a, respectively;
$t_a(x_a, y_a)$ = travel time through arc a, which is a function of x_a and y_a;
α_a, β_a = parameters of travel time function;
$g_a(y_a)$ = construction cost of arc capacity enhancement, which is a function of y_a, $\forall a \in A$;
A = arc (index) set;
f_p^w = traffic flow on path P between origin–destination (O-D) pair w, $\forall w \in W$;
W = O-D pair (index) set, $\forall p \in P$;
P = path (index) set;
q^w = travel demand between O-D pair w, respectively, $\forall w \in W$;
δ_{ap}^w = link-path incidence factor, which is 1 if arc a is on path p between O-D pair w, 0 otherwise, $\forall a \in A$, $\forall w \in W$, $\forall p \in P$;
τ = value of time; and
λ = relative weight of total capacity enhancement cost in objective function.

To facilitate the application of the GGP approach, the formulation is first converted into an equivalent single-level mathematical program with complementarity constraints (*24*):

$$\min \sum_{w \in W} q^w \pi^w + \lambda \sum_{a \in A} g_a(y_a)$$

subject to

$$\underline{y_a} \leq y_a - y_a^0 \leq \overline{y_a} \qquad \forall a \in A$$

$$\sum_{p \in P} f_p^w = q^w \qquad \forall w \in W$$

$$C_p^w - \pi^w \geq 0 \qquad \forall w \in W; \forall p \in P$$

$$f_p^w \left(C_p^w - \pi^w\right) = 0 \qquad \forall w \in W; \forall p \in P$$

$$C_p^w = \sum_{a \in A} \delta_{ap}^w t_a(x_a, y_a) \qquad \forall w \in W; \forall p \in P$$

$$x_a = \sum_{w \in W} \sum_{p \in P} \delta_{ap}^w f_p^w \qquad \forall a \in A$$

$$t_a(x_a, y_a) = \alpha_a + \beta_a x_a^4 y_a^{-4} \qquad \forall a \in A$$

$$f_p^w \geq 0 \qquad \forall w \in W; \forall p \in P; x_a \geq 0; \forall a \in A \tag{5}$$

where C_p^w denotes the travel time on path P between O-D pair w, $\forall w \in W$, $\forall p \in P$, and π^w represents the equilibrium cost between O-D pair w, $\forall w \in W$.

One can observe that all the equations in Equations 5 are multinomial functions; therefore, the standard GP approach cannot be applied directly. However, the CNDP-DUE model (Equations 5) can be transformed into an equivalent GGP model in the quotient form (B in Equation 3). The GGP model (Equation 3) requires that all variables be strictly positive, whereas the arc traffic volume x_a, as well as the path traffic volume f_p^w, might be zero if the corresponding arcs or paths are not chosen in the equilibrium pattern. In order to meet the requirement of positive variables, $f_p^{w'} = f_p^w + M_p^w$ is used rather than f_p^w in Equations 5, where M_p^w is a sufficiently small positive constant. Similarly, $x_a' = x_a + M_a$ is substituted for x_a with a positive constant:

$$M_a = \sum_{w \in W} \sum_{p \in P} \delta_{ap}^w M_p^w$$

In this study, for simplicity, the construction cost $g_a(y_a)$ is considered as a linear function $g_a(y_a) = d_a(y_a - y_a^0)$, as was done in many previous studies (*5, 14, 24*). By adding a new variable, Z, the objective function can be rewritten as a simple objective function with an inequality constraint as follows:

$$\min \sum_{w \in W} q^w \pi^w + \lambda \sum_{a \in A} d_a\left(y_a - y_a^0\right) \Leftrightarrow \begin{cases} \min Z \\ \text{subject to} \\ \dfrac{\sum_{w \in W} q^w \pi^w + \lambda \sum_{a \in A} d_a y_a}{\lambda \sum_{a \in A} d_a y_a^0 + Z} \leq 1 \end{cases} \tag{6}$$

The value of the original objective function decreases as the value of Z drops, and they both achieve the minimum value simultaneously. Next, the inequality and equality constraints in Equations 5 are rearranged to a quotient form (B), as in the GGP model (Equation 3):

$$f(x) \leq g(x) \Leftrightarrow \frac{f(x)}{g(x)} \leq 1 \qquad f(x) = g(x) \Leftrightarrow \frac{f(x)}{g(x)} = 1$$

For example, to transform the Bureau of Public Roads travel time function in Equation 5, $t_a(x_a, y_a) = \alpha_a + \beta_a x_a^4 y_a^{-4}$, the original nonnegative variable x_a is first replaced with the new positive variable $x_a' = x_a + M_a$ to obtain $t_a(x_a, y_a) = \alpha_a + \beta_a (x_a' - M_a)^4 y_a^{-4}$, which is readily rewritten as the quotient form of

$$\frac{\beta_a x_a'^4 y_a^{-4} + 6\beta_a M_a^2 x_a'^2 y_a^{-4} + \beta_a M_a^4 y_a^{-4} + \alpha_a}{4\beta_a M_a x_a'^3 y_a^{-4} + 4\beta_a M_a^3 x_a' y_a^{-4} + t_a} = 1$$

after the polynomial expansion of $(x_a' - M_a)^4$. Similarly, the rest of the constraints can be written into the quotient form; these simple procedures are not elaborated here because of space limitations. By doing these straightforward transformations, an equivalent GGP model can be obtained in the quotient form (B in Equation 3) and the efficient existing solution methods can be applied to solving this GGP model:

$$\min Z$$

subject to

$$\frac{\sum_{w \in W} q^w \pi^w + \lambda \sum_{a \in A} d_a y_a}{\lambda \sum_{a \in A} d_a y_a^0 + Z} \leq 1$$

$$\frac{1}{\overline{y_a}+y_a^0}y_a \le 1 \qquad \forall a \in A$$

$$\frac{1}{q^w+\sum_{p\in P} M_p^w}\sum_{p\in P} f_p^{w'} = 1 \qquad \forall w \in W$$

$$\pi^w C_p^{w^{-1}} \le 1 \qquad \forall w \in W; \forall p \in P$$

$$\frac{f_p^{w'}C_p^w + M_p^w \pi^w}{f_p^{w'}\pi^w + M_p^w C_p^w} = 1 \qquad \forall w \in W; \forall p \in P$$

$$C_p^{w^{-1}}\sum_{a\in A}\delta_{ap}^w t_a = 1 \qquad \forall w \in W; \forall p \in P$$

$$\frac{\sum_{w\in W}\sum_{p\in P}\delta_{ap}^w f_p^{w'} + M_a}{\sum_{w\in W}\sum_{p\in P}\delta_{ap}^w M_p^w + x_a'} = 1 \qquad \forall a \in A$$

$$\frac{\beta_a x_a'^4 y_a^{-4} + 6\beta_a M_a^2 x_a'^2 y_a^{-4} + \beta_a M_a^4 y_a^{-4} + \alpha_a}{4\beta_a M_a x_a'^3 y_a^{-4} + 4\beta_a M_a^3 x_a' y_a^{-4} + t_a} = 1 \qquad \forall a \in A$$

$$0 < \underline{y_a} + y_a^0 \le y_a \qquad \forall a \in A; 0 < M_p^w \le f_p^w; \forall w \in W,$$
$$\forall p \in P; 0 < M_a \le x_a; \forall a \in A \tag{7}$$

GGP-Based CNDP Model with SUE Constraints

In this section, the travel route choice behavior following the SUE principle is considered. The CNDP model formulation with SUE constraints can be expressed as follows (*16, 46*):

$$\min \sum_{a\in A} t_a(x_a, y_a)\cdot x_a + \lambda\sum_{a\in A} g_a(y_a)$$

subject to

$$\underline{y_a} \le y_a - y_a^0 \le \overline{y_a} \qquad \forall a \in A$$

$$x_a = \sum_{w\in W} q^w \sum_{p\in P}\delta_{ap}^w \rho_p^w(C_p^w) \qquad \forall a \in A$$

$$C_p^w = \sum_{a\in A}\delta_{ap}^w t_a(x_a, y_a) \qquad \forall w \in W; \forall p \in P$$

$$t_a(x_a, y_a) = \alpha_a + \beta_a x_a^4 y_a^{-4} \qquad \forall a \in A$$

$$\rho_p^w(C_p^w) = \frac{e^{-\theta C_p^w}}{\sum_{p'\in P} e^{-\theta C_{p'}^w}} \qquad \forall w \in W; \forall p \in P$$

$$x_a > 0 \qquad \forall a \in A \tag{8}$$

where ρ_p^w is the probability of choosing path p between an O-D pair, which is a function of C_p^w, and θ is a parameter of the logit model. In a similar manner, this CNDP-SUE model can also be transformed into an equivalent GGP model.

First, the objective function is transformed into the quotient GGP form as in part B of Equation 3. Specifically, by substituting the travel time function $t_a(x_a, y_a) = \alpha_a + \beta_a x_a^4 y_a^{-4}$ and construction cost function $g_a(y_a) = d_a(y_a - y_a^0)^2$ into the objective function, the following can be obtained:

$$\sum_{a\in A} t_a(x_a, y_a)\cdot x_a + \lambda\sum_{a\in A} g_a(y_a) = \sum_{a\in A}(\alpha_a + \beta_a x_a^4 y_a^{-4})\cdot x_a$$
$$+ \lambda\sum_{a\in A} d_a(y_a - y_a^0)^2$$

As was done by Davis (*16*), $g_a(y_a) = d_a(y_a - y_a^0)^2$ is applied as the construction cost function. Other forms of construction cost functions can also be used, and the solution method is still applicable. By adding a new additional variable Z, the objective function can be rewritten as a simple objective function with an inequality constraint as demonstrated in the CNDP-DUE model reformulation:

$$\min Z$$

subject to

$$\frac{\sum_{a\in A}\alpha_a x_a + \sum_{a\in A}\beta_a x_a^5 y_a^{-4} + \lambda\sum_{a\in A} d_a y_a^2 + \lambda\sum_{a\in A} d_a y_a^{0^2}}{2\lambda\sum_{a\in A} d_a y_a^0 y_a + Z} \le 1$$

Next the inequality and equality constraints in Equation 8 are handled; the most complicated constraint is the exponential function in the logit model because it cannot be addressed directly by the GGP, and therefore it must be transformed into posynomial form. An additional new variable u_a is used to represent $e^{-\theta t_a} = e^{-\theta(\alpha_a + \beta_a x_a^4 y_a^{-4})}$, and thus

$$\log u_a = -\theta(\alpha_a + \beta_a x_a^4 y_a^{-4}) \qquad \forall a \in A \tag{9}$$

$$e^{-\theta C_p^w} = e^{-\theta\sum_{a\in A}\delta_{ap}^w t_a} = \prod_{a\in A} u_a^{\delta_{ap}^w} \tag{10}$$

$$\rho_p^w(C_p^w) = \frac{\prod_{a\in A} u_a^{\delta_{ap}^w}}{\sum_{p'\in P}\prod_{a\in A} u_a^{\delta_{ap'}^w}} \qquad \forall w \in W; \forall p \in P \tag{11}$$

$$x_a = \sum_{w\in W} q^w \sum_{p\in P}\delta_{ap}^w \frac{\prod_{a\in A} u_a^{\delta_{ap}^w}}{\sum_{p'\in P}\prod_{a\in A} u_a^{\delta_{ap'}^w}} \qquad \forall a \in A \tag{12}$$

To handle the logarithm function in Equation 9, an approximation approach proposed by Boyd et al. (*41*) is applied here:

$$\log u_a \approx W(u_a^{1/W} - 1) \tag{13}$$

where W is a large positive constant. Thus, Equation 9 can be approximated as follows:

$$W(u_a^{1/W} - 1) = -\theta(\alpha_a + \beta_a x_a^4 y_a^{-4}) \tag{14}$$

In the last step, the foregoing inequality and equality constraints are rearranged to a quotient form (B) as in the GGP model (Equation 3). For example, the equality constraint (Equation 14) could be converted into the quotient form:

$$\frac{W}{W-\theta\alpha_a}u_a^{1/W}+\frac{\theta\beta_a}{W-\theta\alpha_a}x_a^4y_a^{-4}=1$$

Similarly, the rest of the constraints can be transformed properly.

To summarize, by adding an additional variable Z, the CNDP-SUE model can be reformulated as the following GGP model:

$$\min Z$$

subject to

$$\frac{\sum_{a\in A}\alpha_a x_a+\sum_{a\in A}\beta_a x_a^5 y_a^{-4}+\lambda\sum_{a\in A}d_a y_a^2+\lambda\sum_{a\in A}d_a y_a^{0^2}}{2\lambda\sum_{a\in A}d_a y_a^0 y_a+Z}\le 1$$

$$\frac{1}{\underline{y_a}+y_a^0}y_a\le 1 \qquad \forall a\in A$$

$$\frac{\sum_{w\in W}q^w\sum_{p\in P}\delta_{ap}^w\prod_{a\in A}u_a^{\delta_{ap}^w}}{x_a\sum_{p'\in P}\prod_{a\in A}u_a^{\delta_{ap'}^w}}=1 \qquad \forall a\in A$$

$$\frac{W}{W-\theta\alpha_a}u_a^{1/W}+\frac{\theta\beta_a}{W-\theta\alpha_a}x_a^4y_a^{-4}=1 \qquad \forall a\in A$$

$$0<\underline{y_a}+y_a^0\le y_a \qquad \forall a\in A;\ 0<x_a;\ \forall a\in A \tag{15}$$

SOLUTION ALGORITHM

First, a condensation procedure is introduced on the basis of a successive monomial approximation technique, which is useful to handle the GGP model. The advantage of this successive monomial approximation technique is that it can handle posynomial equality constraints in an efficient way; this ability makes it applicable to a wide number of fields, even large-scale problems with posynomial equality constraints (*45*).

Given a posynomial function

$$f(x)=\sum_{i=1}^{N}h_i(x)$$

where $h_i(x)$ is monomial, the condensation process is defined as follows:

$$C[f(x),x']=\prod_{i=1}^{N}\left[\frac{h_i(x)}{\delta_i}\right]^{\delta_i} \tag{16}$$

where

$$\delta_i=\frac{h_i(x')}{f(x')} \qquad i=1,2,\ldots,N$$

and $C[f(x), x']$ is a monomial approximation of $f(x)$ such that $C[f(x), x']|_{x=x'}=f(x')$.

On the basis of the successive monomial approximation technique, the quotient GGP model (B in Equation 3) can be transformed into standard GP form, so that a global optimal solution can be obtained by solving a convex programming problem.

The general procedure to solve the quotient GGP model (B in Equation 3) is stated as follows (solution procedure):

Step 1. Choose initial values of the variables as x' and apply the condensation procedure (Equation 16) to the denominators in the constraints of quotient GGP model (B in Equation 3); thus all of the constraints should be posynomial functions.

Step 2. Set new variables $x''=x'$ initially and reapply the condensation procedure to the posynomial constraints; thus all of the constraints should be in the form of monomial functions as follows:

$$\min x_0$$

subject to

$$C_i\prod_{k=1}^{K}x_k^{A_{ik}}\le 1 \qquad i=0,1,\ldots,M$$

$$C_j\prod_{k=1}^{K}x_k^{A_{jk}}=1 \qquad j=1,2,\ldots,N$$

$$0<x_k^{\mathrm{LB}}\le x_k \qquad k=0,1,\ldots,K \tag{17}$$

Step 3. Set new variables,

$$y_k=\log\left[\frac{x_k}{x_x^{\mathrm{LB}}}\right] \qquad k=0,1,\ldots,K$$

and transform the model in Equation 17 into a linear program as follows:

$$\min y_0$$

subject to

$$\log\left[C_i\prod_{k=0}^{K}\left(x_k^{\mathrm{LB}}\right)^{A_{ik}}\right]\le\sum_{k=0}^{K}(-A_{ik}y_k) \qquad i=0,1,\ldots,M$$

$$\log\left[C_j\prod_{k=0}^{K}\left(x_k^{\mathrm{LB}}\right)^{A_{jk}}\right]=\sum_{k=0}^{K}(-A_{jk}y_k) \qquad j=1,2,\ldots,N$$

$$0\le y_k \qquad k=0,1,\ldots,K \tag{18}$$

Step 4. Solve the linear program model (Equation 18) to obtain the solution y''; then the corresponding $x_k''=x_k^{\mathrm{LB}}e^{y_k''}$, $k=0,1,\ldots,K$, can be obtained.

Step 5. Find the most violated inequality posynomial constraints in Step 1 based on the evaluation at x'':

$$V=\max\left(\left.\frac{f_0(x)}{C[g_0(x)+x_0,x']}\right|_{x=x''} \quad \left.\frac{f_i(x)}{C[g_i(x),x']}\right|_{x=x''} \quad i=1,2,\ldots,M\right) \tag{19}$$

Let R represent a small positive tolerance if $V > 1 + R$, and then recondense the most violated posynomial constraint by using the solution x'' and transform it to linear program form based on Step 3. Add this linear constraint to the current linear program model, and then return to Step 4. In certain conditions, this repeated process results in a sequence of GP solutions converging to a Kuhn–Tucker optimum solution (*47*).

If $V \leq 1 + R$, check the convergence criterion by calculating the difference D between x' and x'':

$$D = \sum_{k=0}^{K} \left(\frac{x''_k - x'_k}{x'_k} \right)^2 \tag{20}$$

If D is not sufficiently small, set $x' = x''$ and return to Step 1; otherwise end the procedure with x' as the solution to the quotient GGP model (B in Equation 3).

On the basis of this solution procedure, the GGP-based CNDP-DUE model (Equation 7) and the CNDP-SUE model (Equation 15) can be solved for global optimal solutions.

NUMERICAL EXPERIMENTS

To illustrate the validity and efficiency of the solution algorithm, numerical experiments were conducted with two commonly used example networks, and both DUE and SUE travel patterns were considered. A Dell Precision T3600 workstation (Intel Xeon CPU E5-1650, 3.2 GHz, 16 GB RAM, Windows 7 Professional x64) with MATLAB R2012a was used to conduct the numerical experiments.

Case 1. GGP-Based CNDP with DUE

The benchmark network, shown in Figure 1 and applied by several researchers (*5, 12, 14, 19, 24*), was used. The same data were input. The travel demand from Nodes 1 to 6 is 5, the travel demand from 6 to 1 is 10, the lower and upper bounds of each link capacity expansion are 0 and 10, respectively; the positive constant M_p^w, $\forall w \in W$, $\forall p \in P$, should be set as a sufficiently small positive constant, 1×10^{-3} in this numerical experiment, and the other values of the parameters may be found in work by Friesz et al. (*14*).

TABLE 2 Comparison with Numerical Results by Wang and Lo (*24*)

Method	Partition Schemes	Exact Objective Values	Computational Time
MILP	5×5	202.2289	1.5 min
	10×10	199.7814	6 min
	15×15	199.6261	1.2 h
GGP		200.01	2.1 min

NOTE: MILP = mixed integer linear programming.

The comparison of numerical results with the mixed integer linear programming method in the work by Wang and Lo (*24*) is shown in Table 2. One can observe that the solution of the objective function from the approach discussed here is very close to the global optimal solution obtained by Wang and Lo (*24*), which clearly demonstrates the solution quality of the GGP method proposed in this study. In terms of the solution of optimal road expansion, both methods result in enhancing the road capacity on Links 6 and 16 only; however, the exact solution of the optimal capacity enhancement from the current approach is very different from that of Wang and Lo (*24*) [$y_6 = 4.41$ and $y_{16} = 7.70$ by Wang and Lo (*24*), whereas $y_6 = 4.21$ and $y_{16} = 8.40$ in the current study]. This finding further reflects the nonconvex property of this CNDP. The extremely small gap [(200.01 to 199.6261)/200.01 = 0.19%] between the objective value of the current solution and the solution by Wang and Lo (*24*) may be caused by the approximation process when the GGP is transformed into the standard GP, whose error indeed could be controlled and reduced by setting more accurate approximation parameters up to the requirements of a specific problem in a practical application. However, the computational efficiency of the GGP approach is much higher than that of Wang and Lo (*24*), which can be observed from the much lower computational time needed for the GGP approach as shown in Table 2, even taking into account the higher computational power of the computer used in this numerical test.

Case 2. GGP-Based CNDP with SUE

The test network with six nodes, seven links, and four O-D pairs used by Davis (*16*), shown in Figure 2, is adopted here to demonstrate the

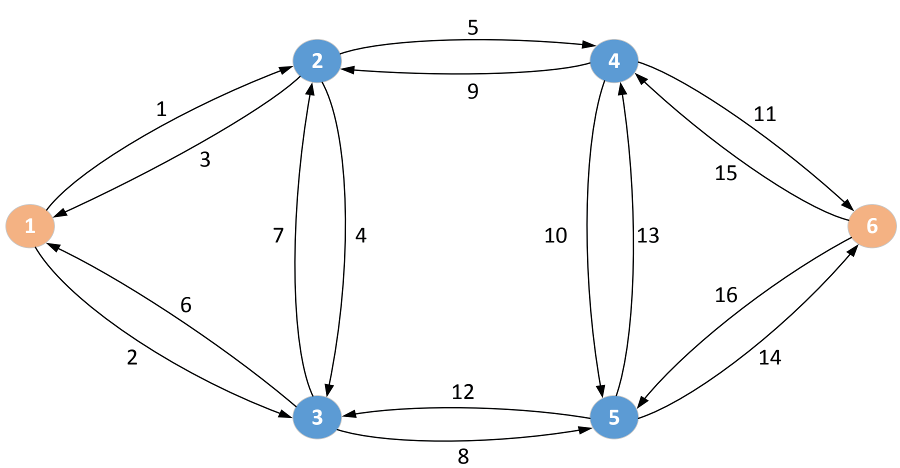

FIGURE 1 Network with 16 arcs.

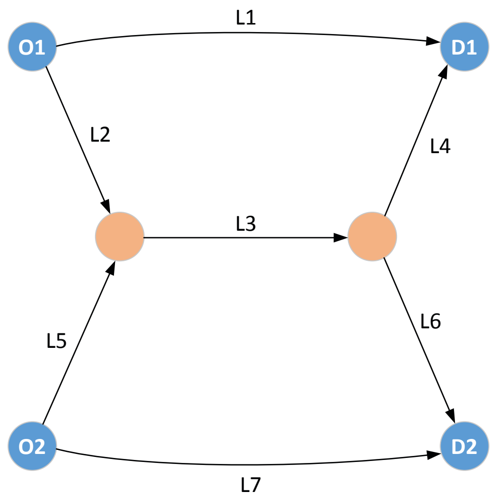

FIGURE 2 Network with seven arcs.

validity of the GGP-based CNDP model with SUE constraints. The input data can be found in the work by Davis (*16*).

Since the values of the parameters λ and θ for the model in Equation 8 are not specified by Davis (*16*), it is not possible to compare the results directly. However, a numerical experiment to test the validity of the model and the solution method can be conducted. In this study, $\lambda = 0.01$ and $\theta = 1$, and the tolerance criterion $R = 1 \times 10^{-5}$ and convergence criterion $D = 1 \times 10^{-4}$ are assumed. The numerical results are shown in Table 3. From the short computational time, one can find that the solution method is very efficient in solving the problem. By comparing the optimal solutions of capacities after expansion with those in Table 3, one can also observe that the solution quality from this method is very high.

CONCLUSIONS

The GP method provides an alternative approach to solve the CNDP in a transportation study. The current researchers developed two GGP-based equivalent single-level CNDP models, with DUE and SUE traffic assignment, respectively. The GGP model requires a strict form and various constraints and thus is difficult to solve directly; a successive monomial approximation technique and logarithmic transformation technique were applied to transform the GGP model into a convex programming problem, which can be solved to its global optimum solution. Numerical experiments of both the DUE and SUE cases were conducted to testify to the validity of the GGP-based models and solution method.

Although much research has been done on the topic of transportation network design and many solution algorithms have been developed, this study proposes the GGP-based modeling and solution approach as an alternative method to solve the global optimization of the CNDP; this approach has obvious advantages: it is easy to implement for the CNDP with both DUE and SUE traffic equilibrium assumptions, and it has high solution efficiency since the original CNDP would eventually be transformed into a nonlinear but convex programming problem. This study attempts to complement the already well-studied topic of solving the CNDP by proposing this GGP-based global optimization solution approach. The developed solution method is very general and can be extended to other research topics in transportation studies, such as the discrete NDP, optimal signal control, routing and scheduling, and location analysis (*48–51*). Furthermore, application of the developed method to solving large-sized network problems in the future is promising.

TABLE 3 Numerical Results

Link	Capacity Before Expansion (pcu/min)	Capacity After Expansion (pcu/min)[a]	SUE Flow (pcu)
1	200	241.8	296.3
2	300	345.6	603.7
3	700	768.4	1,253.1
4	300	345.6	603.7
5	300	357.9	649.4
6	300	357.8	649.4
7	200	261.1	350.6

[a]Total cost = \$3,568.20; CPU seconds = 5.6963; convergence: $D = 7.0731 \times 10^{-5}$.

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