Monge–Ampère type equations and their applications and heat equations driven by irregular terms

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MONGE-AMPÈRE TYPE EQUATIONS AND THEIR APPLICATIONS AND HEAT EQUATIONS DRIVEN BY IRREGULAR TERMS

A Thesis Submitted in Fulfilment of the Requirements for the Award of the Degree of

Doctor of Philosophy

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by

Siyuan Li

School of Mathematics and Applied Statistics
Faculty of Engineering and Information Sciences
2017
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CERTIFICATION

I, Siyuan Li, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, Faculty of Engineering and Information Sciences, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Siyuan Li
5 October 2017
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This work is presented in two parts. In Part I, we focus on the Monge-Ampère type equations and their applications. We investigate the uniqueness of ellipsoid solutions to the $L_p$-Minkowski problem and the Christoffel-Minkowski problem of $L_p$-sum for any $p \in \mathbb{R}$ based on the geometric properties of ellipsoids. Specifically, the ellipsoid solutions are classified into three cases. Based on a form of Alexandrov-Fenchel inequality for some appropriate positive $k$-convex functions, we consider the uniqueness of the $k$-Hessian equation $\sigma_k(u_{ij} + u\delta_{ij}) = fu^{p-1}$ on $\mathbb{S}^n$ for $p > 1$.

In Part II, we achieve regularity results for the Cauchy problems of heat equations driven by a separable inhomogeneous term or a nonseparable general term. As an application, the existence and uniqueness of solutions to the Cauchy problems are arrived. In addition, the results in Part II are applied to obtain pathwise estimates for the heat equation driven by a fractional Brownian sheet.

**KEYWORDS:** Uniqueness; $L_p$-Minkowski Problem; Monge-Ampère Equations; $k$-Hessian Equations; Heat Equations; Regularity; Schauder Estimates; Rough Path
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Chapter 1

Preface and Notation

1.1 Preface

In the first year of my PhD study, I considered Monge-Ampère type equations and their applications. The $L_p$-Minkowski problem is one of the most important applications that are closely related to Monge-Ampère equations. In Part I of this thesis, the uniqueness of the $L_p$-Minkowski problem is obtained under the assumption that the solutions are ellipsoids centred at the origin. As an application, similar methods are utilised to consider the uniqueness of the ellipsoid solutions to the Christoffel-Minkowski problem of $L_p$-sum, which corresponds to the $k$-Hessian equations. Then, without convexity assumptions, we consider the uniqueness of the $k$-Hessian equation

$$\sigma_k(u_{ij} + u\delta_{ij}) = f u^{p-1}$$

on $\mathbb{S}^n$ with $p > 1$ via inequalities directly on $k$-convex functions.

After achieving the uniqueness results mentioned above, I studied some problems closely related to Monge-Ampère equations and their applications. My plan was to study the stochastic partial differential equations (PDEs) of second order. The stochastic case is complicated due not only to the nonlinearity of the Monge-Ampère equations but also to the fact that the driving “noise” — the fractional Brownian motion with a Hurst parameter of $h \neq 1/2$ — is not a semimartingale. To study
nonlinear equations, researchers start with linear ones, and to study PDEs driven by fractional Brownian motions, separable analogues are typical. Therefore, in Part II, a heat equation driven by a separable inhomogeneous term, which is a typical example of second-order linear parabolic PDEs, is considered. As an application, a general nonseparable case is also considered and the existence and uniqueness of solutions to the Cauchy problems of heat equations driven by irregular terms are arrived. In addition, the results for the separable case are applied to obtain a pathwise regularity for the heat equation driven by a fractional Brownian sheet.

The principal objectives of this work are separated into two parts and arranged as follows. Part I addresses Monge-Ampère type equations and their applications. Chapter 2 provides a brief introduction to the Monge-Ampère equations and their motivations. Several models from many fields that are expressed as Monge-Ampère equations are introduced.

Chapter 3 focuses on one of the most important applications of this research. Based on several basic concepts from convex geometry, Chapter 3 presents a literature review of Minkowski-type problems — the Minkowski problem, the $L_p$-Minkowski problem and the Christoffel-Minkowski problem.

Chapter 4 presents the preliminary work for the proof of the main results and summarises the uniqueness results for $p \geq 1$.

The aims of Chapter 5 are to prove the main results of Part I and to present a brief introduction to some problems closely related to the Monge-Ampère equations and their applications.

Part II focuses on the regularity of heat equations driven by irregular inhomogeneous terms. Chapter 6 gives a brief introduction to the motivations and a related type of stochastic process — fractional Brownian motion. In addition, the background of the considered problems is presented.
Chapter 7 presents a literature review of second-order PDEs, from deterministic cases to stochastic cases. In addition, a brief introduction to equations driven by rough paths is provided.

Chapter 8 presents the preliminary work for the proof of the main results.

The aims of Chapter 9 are to prove the main results of Part II and to present several applications.

1.2 Notation

We now summarise some of the notation that is used frequently throughout this thesis.

\[ \mathbb{R} \] : the set of real numbers.

\[ \mathbb{R}^+ \] : the set of positive real numbers.

\[ \mathbb{N} \] : the set of nonnegative integers.

\[ \mathbb{R}^n \] : Euclidean \( n \)-space with \( n \geq 2 \), which is the Cartesian product \( \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \), with points \( x = (x_1, x_2, \cdots, x_n) \), where \( x_i \in \mathbb{R} \) and \( i = 1, 2, \cdots, n \).

\[ \mathbb{S}^n \] : the unit sphere in \( \mathbb{R}^{n+1} \).

\[ C_n^k = \frac{n!}{k!(n-k)!} \] : binomial coefficients.

\[ \langle \cdot, \cdot \rangle_\mathcal{V} \] : the inner product on the space \( \mathcal{V} \) in the context.

\[ \text{a.s.} \] : “almost surely”.

\[ Du = (D_1u, D_2u, \cdots, D_nu) \] : the gradient of function \( u \), where \( D_iu = \partial u / \partial x_i \), \( i = 1, 2, \cdots, n \).

\[ D^2u = (D_{ij}u)_{n \times n} \] : the Hessian matrix of the second derivatives \( D_{ij}u \), where \( D_{ij}u = \partial^2 u / \partial x_i \partial x_j \), \( i, j = 1, 2, \cdots, n \).

\[ C = C(\cdot, \cdots, \cdot) \] : constants that depend only on the quantities appearing in the parentheses. In a given context, the same letter \( C \) may denote different constants depending on the same set of arguments.

The end of each proof is marked with a ‘\( \square \)’.
Part I

Monge-Ampère Equations and Their Applications
Uniqueness is one of the important research topics on PDEs. Part I focuses on uniqueness, namely, the uniqueness of solutions to equations (I.1), (I.2) and (I.3) given below.

Part I starts with the uniqueness of the $L_p$-Minkowski problem. In the smooth category, it is equivalent to considering the uniqueness of the following Monge-Ampère equation:

$$\det(h_{ij} + h\delta_{ij}) = h^{p-1} \quad \text{on} \quad S^n,$$

where $h$ is the support function of an unknown convex body, $h_{ij}$ are the second-order covariant derivatives of $h$ with respect to an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ on the unit sphere $S^n$, $\delta_{ij}$ is the Kronecker delta, and $p \in \mathbb{R}$. From a geometric point of view, we obtain the uniqueness of the ellipsoid solutions (centred at the origin) to (I.1).

As an application, we use a similar method to obtain the uniqueness of ellipsoid solutions to the Christoffel-Minkowski problem of $L_p$-sum. Correspondingly, we consider the following $k$-Hessian equation:

$$\sigma_k(h_{ij} + h\delta_{ij}) = C_k^n h^{p-1} \quad \text{on} \quad S^n,$$

where $\sigma_k$ is the $k$-th elementary symmetric function (defined in Section 3.4), $k \in \{1, 2, \cdots, n\}$, and $C_k^n = \frac{n!}{k!(n-k)!}$ are binomial coefficients. (I.2) reduces to (I.1) if $k = n$.

In addition, based on a form of Alexandrov-Fenchel inequality [62] for certain appropriate positive $k$-convex functions, we consider the uniqueness of solutions to the $k$-Hessian equation

$$\sigma_k(h_{ij} + h\delta_{ij}) = fh^{p-1} \quad \text{on} \quad S^n$$

for $p > 1$, where $f$ is a positive function on $S^n$. 
Chapter 2

Introduction and Motivation

In this chapter, a brief introduction to the Monge-Ampère equations is provided, and several models that are expressed as Monge-Ampère equations are introduced.

2.1 Introduction

A general Monge-Ampère type equation is a special type of fully nonlinear partial differential equation (PDE) of second order in the following form \cite{149}:

$$\det(D^2u - A(x,u,Du)) = f(x,u,Du), \quad \forall \ x \in \Omega \subset \mathbb{R}^n,$$

where $\det(\cdot)$ represents the determinant, $u$ is an unknown function defined on the domain $\Omega \subset \mathbb{R}^n$, $Du = (u_{x_1}, \cdots, u_{x_n})$ is the gradient of $u$, $D^2u = (u_{x_i,x_j})_{n \times n}$ denotes the Hessian matrix of $u$, $A$ is a symmetric matrix, and $f$ is a given function. For the model case

$$\det D^2u = f(x), \quad \forall \ x \in \Omega \subset \mathbb{R}^n,$$  \hspace{1cm} (2.1.1)

we denote $F(M) := \det M$, where $M = (M_{ij})_{n \times n}$ is a real $n \times n$ symmetric matrix. Denote $F_{ij}(M)$ by the cofactor of $M_{ij}$, then we have $F_{ij}(M) = (\det M)M^{ij}$, where
\((M^{ij})_{n \times n}\) is the inverse of \(M\). (2.1.1) is elliptic only for strictly convex functions \(u\) in \(\Omega\), and a necessary condition for (2.1.1) to have an elliptic solution is that \(f\) is positive.

2.2 Motivation

This section introduces several models from various fields of application that are expressed as Monge-Ampère equations.

2.2.1 Prescribing Gauss Curvatures

One of the most important and well-known applications is the problem of prescribing Gauss curvatures, in which the goal is to determine, given a positive function \(K\) on domain \(\Omega \subset \mathbb{R}^n\), whether we can find a hypersurface \(M\) in \(\mathbb{R}^{n+1}\) such that \(M\) is represented by the graph of a function \(u\) over the domain \(\Omega\) and that at each point \((x, u(x)) \in M\), the Gauss curvature is given by \(K(x)\). This yields a PDE as follows:

\[
\det D^2 u - (1 + |Du|^2)^{\frac{n+2}{2}} K = 0 \quad \text{on} \quad \Omega,
\]

which is an equation of the Monge-Ampère type.

2.2.2 Minkowski Problem

The well-known Minkowski problem has been popular for more than 100 years, and it has had a significant impact on mathematics in the 20th century. Let \(S^n\) be the unit sphere in \(\mathbb{R}^{n+1}\). Given a positive function \(f\) on \(S^n\), the classical Minkowski problem (see [139] for its history) considers the existence of a closed convex hypersurface \(M \subset \mathbb{R}^{n+1}\) such that \(K(P) = f(\nu)\) for any point \(P \in M\), where \(\nu \in S^n\) is the normal to \(M\) at \(P\) and \(K(P)\) is the Gauss curvature of \(M\) at \(P\). In the smooth category, this problem
corresponds to the following Monge-Ampère equation:

\[
\det(u_{ij} + u\delta_{ij}) = f^{-1} \quad \text{on} \quad S^n,
\]

where \( u \) is the support function of the solution \( M \), \( u_{ij} \) are the second-order covariant derivatives of \( u \) with respect to an orthonormal frame \( \{e_1, e_2, \ldots, e_n\} \) on the unit sphere \( S^n \), and \( \delta_{ij} \) is the Kronecker delta.

### 2.2.3 Designing a Reflector Antenna

Monge-Ampère type equations also arise in geometric optics [58, 65, 157]. The problem of designing a reflector antenna concerns the existence of a reflector surface that reflects a given input energy (e.g., light or radio signal) into a prescribed output region of the far field with a prescribed distribution density.

In \( \mathbb{R}^3 \), locate a point light source at the origin \( O \), and each direction of the ray is identified as a point on the unit sphere \( S^2 \). The reflector antenna \( \Gamma \) is assumed to be a star-shaped surface with respect to \( O \), and its radial projection onto \( S^2 \) is the domain \( \Omega \); i.e., \( \Gamma \) can be represented as \( \Gamma = \{x \cdot \mu(x) : x \in \Omega\} \). A ray originating from \( O \) in direction \( x \) that is reflected by \( \Gamma \) radially produces a ray in direction \( y = \nu(x) \in S^2 \), and suppose this projection is one-to-one. The target region is assumed to be a prescribed domain \( \Omega^* \subset S^2 \). Thus, we have a mapping \( \nu : \Omega \to \Omega^* \). Let \( f : \Omega \to \mathbb{R} \) be the density of rays originating from \( O \), and let \( g : \Omega^* \to \mathbb{R} \) be the prescribed distribution density in the target region \( \Omega^* \). Both \( f \) and \( g \) are nonnegative and measurable. Suppose that there is no loss of energy in the reflection process; then, by conservation of energy, we have

\[
\int_{\Omega} f(x)dx = \int_{\Omega^*} g(y)dy.
\]

Let \( z(x) \) denote the outer unit normal to \( \Gamma \) at \( x \cdot \mu(x) \). According to the law of reflection, we have \( y = \nu(x) = x - 2(x, z)z \), where \( \langle \ , \ \rangle \) is the inner product in \( \mathbb{R}^3 \),
\[ z = \frac{(\nabla \mu - \mu x)}{\sqrt{\mu^2 + |\nabla \mu|^2}}, \] and \( \nabla \) is the covariant derivative on \( S^2 \). Thus,

\[ y = \nu(x) = \frac{2\mu \nabla \mu + (|\nabla \mu|^2 - \mu^2) x}{\mu^2 + |\nabla \mu|^2}. \]

Suppose that the reflected directions do not overlap; then, the Jacobian of \( \nu(x) \) is equal to \( f(x)/g(\nu(x)) \) for any \( x \in \Omega \). This problem corresponds to the following Monge-Ampère equation:

\[
\frac{\det \left( \nabla_{ij} w + (w - \phi)e_{ij} \right)}{\phi^2 \det(e_{ij})} = \frac{f(x)}{g(\nu(x))}, \quad \forall \ x \in \Omega,
\]

subject to a second boundary condition \( \nu(\Omega) = \Omega^* \), where \( w = 1/\mu \), \( \phi = (|\nabla w|^2 + w^2)/2w \), \( \nabla_{ij} \) denotes the covariant derivatives on \( S^2 \), and \( e \) denotes the first fundamental form of \( S^2 \).

### 2.2.4 Optimal Transport Problem (Monge-Kantorovich Problem)

Another important application of Monge-Ampère type equations is the optimal transport problem, which was proposed by Monge in 1781. The optimal transport problem arises when attempting to lower the total cost of transporting from one location to another.

Assume that \( \Omega, \Omega^* \subset \mathbb{R}^n \) are two bounded domains. \( f \in L^1(\Omega) \) and \( g \in L^1(\Omega^*) \) are nonnegative functions (not identically 0) that represent the mass densities before and after transport, which satisfy \( \int_{\Omega} f(x)dx = \int_{\Omega^*} g(y)dy \). A map \( \phi : \Omega \to \Omega^* \) is said to be a measure-preserving map if \( \phi \) is Borel measurable and for any Borel set \( V \subset \Omega^* \),

\[
\int_{\phi^{-1}(V)} f(x)dx = \int_V g(y)dy.
\]

Let \( S = S(f, g) \) denote all measure-preserving mappings from \( \Omega \) to \( \Omega^* \). Given a cost
function $c : \Omega \times \Omega^* \rightarrow \mathbb{R}$, the total cost $\mathcal{C}$ is defined as

$$\mathcal{C}(s) = \int_{\Omega} c(x, s(x)) f(x) dx, \quad \forall \, s \in \mathcal{S}. \quad (2.2.1)$$

The optimal transport problem considers whether an optimal mapping $t \in \mathcal{S}$ exists such that $\mathcal{C}(t) \leq \mathcal{C}(s)$ for all $s \in \mathcal{S}$, i.e., $t$ minimises the cost functional (2.2.1) in $\mathcal{S}$.

To study the existence of optimal mappings, Kantorovich constructed the following dual functional:

$$J(\phi, \psi) = \int_{\Omega} f(x) \phi(x) dx + \int_{\Omega^*} g(y) \psi(y) dy, \quad \forall \, (\phi, \psi) \in \mathcal{H}, \quad (2.2.2)$$

which linearises Monge’s problem, where $\mathcal{H} = \{(\phi, \psi) \mid \phi(x) + \psi(y) \leq c(x, y), \, \forall \, x \in \Omega, y \in \Omega^*\}$. Therefore, the optimal transport problem is also called the Monge-Kantorovich problem.

The relation between the cost functional $\mathcal{C}$ and the dual functional $J$ is expressed as $\inf_{s \in \mathcal{S}} \mathcal{C}(s) = \sup_{(\phi, \psi) \in \mathcal{H}} J(\phi, \psi)$. The optimal mapping is determined by the maximiser of the dual functional (2.2.2) in $\mathcal{H}$, and such a maximiser of (2.2.2) always exists and is unique up to a constant $19, 52$. Let $(u, v) \in \mathcal{H}$ be a maximiser of (2.2.2), then the potential functions $u$ and $v$ satisfy Monge-Ampère type equations.

If the cost function is given by $c(x, y) = \frac{1}{2} |x - y|^2$ or $c(x, y) = x \cdot y$, then the optimal mapping $t$ is determined by $t = Du$, and $u$ satisfies a standard Monge-Ampère equation (see, e.g., $43$)

$$\det(D^2u) = \frac{f(x)}{g(Du(x))} \quad \text{in} \quad \Omega$$

with a boundary condition $Du(\Omega) = \Omega^*$, where $Du$ is the gradient of $u$ and $D^2u = (D_{ij}u)_{n \times n}$ is the Hessian matrix of $u$. 

Chapter 3

Literature Review

The $L_p$-Minkowski problem is one of the most important problems related to the Monge-Ampère equations and is a generalisation of the classical Minkowski problem. This chapter presents a review of Minkowski-type problems. To understand the background of these problems, several basic concepts are introduced in Section 3.1. In Sections 3.2 to 3.4, a literature review on Minkowski-type problems is presented: the Minkowski problem, the $L_p$-Minkowski problem and the Christoffel-Minkowski problem.

3.1 Basic Concepts in Convex Geometry

This section presents an introduction to several basic concepts in convex geometry and generalisations thereof, such as the generalisations of the Minkowski sum, the surface area measure and quermassintegrals to their respective $L_p$ cases.

3.1.1 Minkowski Sum and Generalisation to the $L_p$ Case

The most important and basic concepts in convex geometry are those of convex bodies, support functions, Minkowski sums and mixed volumes. Their definitions can be found in, for example, [12, 139].
A compact convex subset of $\mathbb{R}^{n+1}$ with a nonempty interior is called a convex body. A support function, which measures the distance from the origin to the support planes of a convex body, is defined as follows.

**Definition 1.** Let $Q \subset \mathbb{R}^{n+1}$ be a convex body with boundary $M$. The support function of $Q$ (or $M$) is defined as

$$h(x) = \max \{(x, y) : y \in Q\}, \quad \forall \ x \in \mathbb{R}^{n+1},$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^{n+1}$.

If $M$ is smooth and strictly convex, it can be represented by the inverse Gauss map $\nu : \mathbb{S}^n \to M$. Then, the support function of $M$ can also be represented by

$$h(x) = \langle x, \nu(x) \rangle, \quad \forall \ x \in \mathbb{S}^n,$$

and will be positively homogeneous of degree 1 after being extended to $\mathbb{R}^{n+1}$ as follows:

$$h(y) = |y| h \left( \frac{y}{|y|} \right), \quad \forall \ y \in \mathbb{R}^{n+1},$$

where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$.

Obviously, the support function of a convex body is convex and positively homogeneous of degree 1; thus, it is completely determined by its value on $\mathbb{S}^n$. Conversely, any continuous function $h$ on $\mathbb{S}^n$ that is convex after being extended to be positively homogeneous of degree 1 on $\mathbb{R}^{n+1}$ determines a convex body

$$Q = \bigcap_{x \in \mathbb{S}^n} \left\{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle \leq h(x) \right\}.$$

Let $\mathcal{S}$ be the set of continuous functions $h$ (as mentioned above) on $\mathbb{S}^n$, and let $\mathcal{K}$ be
the set of all convex bodies in $\mathbb{R}^{n+1}$; then, there exists a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{K}$. Support functions have the following properties.

**Remark 1.** Let $Q \subset \mathbb{R}^{n+1}$ be a convex body with support function $h$, and let the origin lie in the interior of $Q$, then $h$ satisfies the following three conditions [40]:

1. $h(0) = 0$ and $h(y) \geq 0$ for any $y \neq 0$;
2. $h(\mu y) = \mu h(y)$ for any $\mu \geq 0$;
3. $h(x) + h(y) \geq h(x + y)$ for all $x, y \in \mathbb{R}^{n+1}$.

Moreover, the above three conditions are sufficient for a function to be the support function of a unique convex body.

The Minkowski sum of two convex bodies is a linear combination thereof that is defined as follows.

**Definition 2.** Given two convex bodies $Q_1$ and $Q_2$, $Q_1, Q_2 \in \mathcal{K}$, with respective support functions $h_1$ and $h_2$, and given $\lambda, \mu \geq 0$ ($\lambda^2 + \mu^2 > 0$), the Minkowski sum $\lambda Q_1 + \mu Q_2 \in \mathcal{K}$ is defined as the convex body whose support function is $\lambda h_1 + \mu h_2$. 

Figure 3.1: Example: Support function of an ellipsoid $Q$
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Figure 3.2: Example: Minkowski sum of a rectangle $Q$ and a disk $B$

which means

\[
\lambda Q_1 + \mu Q_2 = \bigcap_{x \in \mathbb{S}^n} \{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle \leq \lambda h_1(x) + \mu h_2(x) \}.
\] (3.1.2)

Let $\mathcal{K}_0$ be the set of all convex bodies in $\mathcal{K}$ that contain the origin in their interiors. In 1962, Firey [46] generalised the concept of the Minkowski sum (for $p = 1$) to the $L_p$-sum (for $p \geq 1$) as follows.

**Definition 3.** For $p \geq 1$, given two convex bodies $Q_1$ and $Q_2$, $Q_1, Q_2 \in \mathcal{K}_0$, with respective support functions $h_1$ and $h_2$, and given $\lambda, \mu \geq 0$ ($\lambda^2 + \mu^2 > 0$), the $L_p$-sum $\lambda \circ Q_1 +_p \mu \circ Q_2 \in \mathcal{K}_0$ is the convex body with support function $(\lambda h_1^p + \mu h_2^p)^{\frac{1}{p}}$, which means

\[
\lambda \circ Q_1 +_p \mu \circ Q_2 = \bigcap_{x \in \mathbb{S}^n} \{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle^p \leq \lambda h_1^p(x) + \mu h_2^p(x) \},
\] (3.1.3)

where $+_p$ represents $L_p$ summation and $\circ$ represents $L_p$ multiplication.

Based on the homogeneity of (3.1.3), the relation between classical multiplication
and \(L_p\) multiplication is
\[
\lambda \circ Q = \lambda^{\frac{1}{p}} Q.
\]
If \(p = 1\), then they are equal.

Furthermore, we consider the set of all positive support functions in \(S\), denoted by \(S_0\), i.e., \(S_0 = S \cap \{h > 0\}\). We can further extend the \(L_p\)-sum as given in (3.1.3) to any \(p \in \mathbb{R}\). For \(0 < \lambda < 1\) and \(a, b > 0\), we define
\[
M_p(a, b, \lambda) = \begin{cases} 
\min \{a, b\}, & \text{if } p = -\infty, \\
((1 - \lambda)a^p + \lambda b^p)^{\frac{1}{p}}, & \text{if } p \in (-\infty, 0) \cup (0, \infty), \\
a^{1-\lambda}b^\lambda, & \text{if } p = 0, \\
\max \{a, b\}, & \text{if } p = \infty.
\end{cases}
\] (3.1.4)

\(M_p(a, b, \lambda)\) is increasing with respect to \(p\); namely, if \(-\infty \leq p_1 < p_2 \leq \infty\), then
\[
M_{p_1}(a, b, \lambda) \leq M_{p_2}(a, b, \lambda),
\]
where \(M_{p_1}(a, b, \lambda) = M_{p_2}(a, b, \lambda)\) if and only if \(a = b > 0\). Then, the concept of the \(L_p\)-sum can be generalised as follows.

**Definition 4.** Given two convex bodies \(Q_1\) and \(Q_2\), \(Q_1, Q_2 \in K_0\), with respective support functions \(h_1\) and \(h_2\), \(h_1, h_2 \in S_0\), and given \(\lambda \in (0, 1)\) and \(p \in \mathbb{R}\), the generalised \(L_p\)-sum
\[
(1 - \lambda) \circ Q_1 +_p \lambda \circ Q_2
\]
is defined as
\[
(1 - \lambda) \circ Q_1 +_p \lambda \circ Q_2 = \bigcap_{x \in \mathbb{R}^n} \{y \in \mathbb{R}^{n+1} : \langle x, y \rangle \leq M_p(h_1(x), h_2(x), \lambda)\}. \tag{3.1.5}
\]

Clearly, as an intersection of half-spaces, the generalised \(L_p\)-sum defined in (3.1.5) is a convex body.

**Remark 2.** If \(p \geq 1\), then the convex body defined in (3.1.5) is the \(L_p\)-sum (3.1.3) and
\( M_p(h_1, h_2, \lambda) \) is its exact support function. However, this is not true for \( p < 1 \) because \( M_p(h_1, h_2, \lambda) \) does not satisfy condition (3) in Remark 1 for a function to be a support function when \( p < 1 \).

### 3.1.2 Surface Area Measure and Generalisation to the \( L_p \) Case

**Definition 5.** For \( Q \in \mathcal{K} \), the surface area measure of \( Q \), denoted by \( S(Q, \cdot) \), is a Borel measure defined on \( S^n \) such that

\[
\int_{S^n} h_{Q'}(\omega) S(Q, d\omega) = \lim_{\varepsilon \to 0^+} \frac{\text{Vol} (Q + \varepsilon Q') - \text{Vol} (Q)}{\varepsilon} \tag{3.1.6}
\]

for any convex body \( Q' \in \mathcal{K} \), where \( h_{Q'} \) denotes the support function of \( Q' \), ‘\( \text{Vol} \)’ denotes the volume, and \( Q + \varepsilon Q' \) is the Minkowski sum as defined in Definition 2.

The classical surface area measure was generalised by Lutwak [111] to its \( L_p \) analogue as follows.

**Definition 6.** For \( p \geq 1 \) and \( Q \in \mathcal{K}_0 \), the \( L_p \) surface area measure of \( Q \), denoted by \( S^{(p)}(Q, \cdot) \), is a Borel measure defined on \( S^n \) that satisfies

\[
\frac{1}{p} \int_{S^n} h_{Q'}^p(\omega) S^{(p)}(Q, d\omega) = \lim_{\varepsilon \to 0^+} \frac{\text{Vol} (Q + \varepsilon \circ Q') - \text{Vol} (Q)}{\varepsilon} \tag{3.1.7}
\]

for any convex body \( Q' \in \mathcal{K}_0 \), where \( Q +_p \varepsilon \circ Q' \) is the \( L_p \)-sum that is defined in Definition 3.

If \( p = 1 \), then the \( L_p \) surface area measure defined in (3.1.7) reduces to the classical surface area measure defined in (3.1.6). The relation between the classical and \( L_p \) surface area measures is

\[
S^{(p)}(Q, \cdot) = h_{Q}^{1-p}S(Q, \cdot).
\]
3.1.3 Mixed Volume

The concept of a mixed volume arises naturally from that of the Minkowski sum.

**Definition 7.** Let $Q_1, \cdots, Q_r \subset \mathbb{R}^{n+1}$ be convex bodies, and let $\lambda_1, \cdots, \lambda_r$ be nonnegative real numbers. The volume of their Minkowski sum $\lambda_1 Q_1 + \cdots + \lambda_r Q_r \subset \mathbb{R}^{n+1}$ is a homogeneous polynomial of degree $n+1$ in $\lambda_1, \lambda_2, \cdots, \lambda_r$ and can be represented by

$$\text{Vol} (\lambda_1 Q_1 + \cdots + \lambda_r Q_r) = \sum_{i_1, \cdots, i_{n+1}=1}^r \lambda_{i_1} \cdots \lambda_{i_{n+1}} V(Q_{i_1}, \cdots, Q_{i_{n+1}}),$$

where the function $V$ is symmetric. Then, $V(Q_1, Q_2, \cdots, Q_{n+1})$ is called the mixed volume of $Q_1, Q_2, \cdots, Q_{n+1}$.

As a special case of mixed volumes, we obtain the concept of quermassintegrals.

For $Q \in \mathcal{K}$, the quermassintegrals $[139]$ of $Q$ are denoted by $W_0(Q), W_1(Q), \cdots, W_{n+1}(Q)$ and are defined as follows:

$$W_k(Q) = V(Q, \cdots, Q, B, \cdots, B), \quad k = 0, 1, \cdots, n+1,$$

where $B = \{ x \in \mathbb{R}^{n+1} : x^2 \leq 1 \}$ is the unit ball and $V(\cdot, \cdots, \cdot)$ denotes the mixed volume as defined in Definition 7. Then, $W_0(Q) = \text{Vol}(Q)$ is the volume of $Q$, $(n+1)W_1(Q)$ is the surface area of $Q$, and $W_{n+1}(Q) = \text{Vol}(B)$ is the volume of the unit ball $B$. Surface area measures are local versions of quermassintegrals.

The mixed $p$-quermassintegrals $W_{p,0}(Q_1, Q_2), W_{p,1}(Q_1, Q_2), \cdots, W_{p,n}(Q_1, Q_2)$ are defined as the first variation of the ordinary quermassintegrals with respect to the $L_p$-sum.
Definition 8. For $p \geq 1$, we define
\[
\frac{n+1-k}{p} W_{p,k}(Q_1, Q_2) := \lim_{\varepsilon \to 0^+} \frac{W_k(Q_1 + \varepsilon \circ Q_2) - W_k(Q_1)}{\varepsilon}, \quad k = 0, 1, \ldots, n,
\]
(3.1.8)
for any $Q_1, Q_2 \in \mathcal{K}_0$, where $W_k$, $k = 0, 1, \ldots, n$ are the quermassintegrals, “$+\varepsilon$” represents $L_p$ summation, and “$\circ$” represents $L_p$ multiplication.

Remark 3. When $p = 1$, (3.1.8) is the usual mixed quermassintegral $W_k(Q_1, Q_2)$, namely,
\[
(n+1-k)W_k(Q_1, Q_2) := \lim_{\varepsilon \to 0^+} \frac{W_k(Q_1 + \varepsilon Q_2) - W_k(Q_1)}{\varepsilon}, \quad k = 0, 1, \ldots, n.
\]

According to [111], the mixed $p$-quermassintegral $W_{p,k}$ has the following integral form: For $p \geq 1$, $k = 0, 1, \ldots, n$ and any $Q_1 \in \mathcal{K}_0$,
\[
W_{p,k}(Q_1, Q_2) = \frac{1}{n+1} \int_{\mathbb{S}^n} h_2^p h_1^{1-p} dS_k(Q_1, \cdot), \quad \forall Q_2 \in \mathcal{K}_0,
\]
(3.1.9)
where $h_i$ denotes the support function of $Q_i$ ($i = 1, 2$), and $S_k(Q_1, \cdot)$ is a regular Borel measure on $\mathbb{S}^n$ such that the mixed quermassintegral $W_k$ is represented by
\[
W_k(Q_1, Q_2) = \frac{1}{n+1} \int_{\mathbb{S}^n} h_2 dS_k(Q_1, \cdot), \quad \forall Q_2 \in \mathcal{K}.
\]
(3.1.10)

Remark 4. The existence of $S_k(Q_1, \cdot)$ has been shown in, e.g., [45]. Clearly, when $k = 0$, $S_0(Q, \cdot)$ is the surface area measure of $Q$ (see Definition 3); namely, $S_0(Q, \cdot) = S(Q, \cdot)$. According to (3.1.9) and (3.1.10), the mixed $p$-quermassintegrals and the mixed quermassintegrals are not symmetric with respect to the positions of the two convex bodies.
3.2 Minkowski Problem

The original Minkowski problem is described in the discrete case as follows (see 139 for the history): Given a set of unit vectors \( \{\nu_1, \nu_2, \cdots, \nu_d\} \) on the \( n \)-dimensional unit sphere \( S^n \) and a set of positive real numbers \( \{\alpha_1, \alpha_2, \cdots, \alpha_d\} \), what are the sufficient and necessary conditions to guarantee the existence of a polytope (in \( \mathbb{R}^{n+1} \)) with \( d \) faces such that for each face \( F_i \) (\( i = 1, 2, \cdots, d \)), its outer normal is given by \( \nu_i \) and its area is \( \alpha_i \)? More than a century ago, Minkowski himself solved this discrete problem. He found that when \( \nu_1, \nu_2, \cdots, \nu_d \) do not lie on a closed hemisphere of \( S^n \), the sufficient and necessary condition for the existence and uniqueness (up to a translation) of such a polytope in \( \mathbb{R}^{n+1} \) is

\[
\sum_{i=1}^{d} \alpha_i \nu_i = 0.
\]

Once we have the concept of the surface area measure (see Definition 5), we can consider the non-discrete case of the original Minkowski problem. The classical Minkowski problem considers the existence of a convex body whose surface area measure is prescribed: Given a finite Borel measure \( m \) on \( S^n \), the classical Minkowski problem concerns whether there exists a unique convex body \( Q \subset \mathbb{R}^{n+1} \) such that its surface area measure is given by \( m \). Let \( \mu \) be the surface area measure of \( Q \), then this problem is equivalent to solving

\[
d\mu = dm. \tag{3.2.1}
\]

The existence of solutions to (3.2.1) is determined through approximation [1, 15] (or see 139 for references): if the given measure \( m \) is not concentrated on any great subsphere of \( S^n \), then there exists a unique (up to a translation) convex body \( Q \subset \mathbb{R}^{n+1} \) that satisfies (3.2.1) if and only if

\[
\int_{S^n} x_i dm(x) = 0, \quad \forall \ i = 1, 2, \cdots, n + 1.
\]
In the smooth category, problem (3.2.1) is equivalent to searching for a convex body with a prescribed Gauss curvature, which corresponds to the following Monge-Ampère equation:

\[
det(h_{ij} + h\delta_{ij}) = f \quad \text{on} \quad S^n, \tag{3.2.2}
\]

where \( h \) is the support function of the solution convex body, \( h_{ij} \) is the second-order covariant derivative of \( h \) with respect to an orthonormal frame on \( S^n \), and \( f \) is a given positive continuous function on \( S^n \). (3.2.2) is a fully nonlinear elliptic PDE of second order. This problem has been studied by Lewy [105] and Nirenberg [122] in \( \mathbb{R}^3 \) and by Calabi [21], Cheng and Yau [25], Pogorelov [130] and Caffarelli [18] for general dimensions, as well as by many others. According to their work, for any positive function \( f \in C^2(S^n) \) that satisfies

\[
\int_{S^n} x_i f(x) dx = 0, \quad \forall \quad i = 1, 2, \cdots, n + 1, \tag{3.2.3}
\]

(3.2.2) always has a unique convex solution \( h \in C^4(S^n) \), where a function \( u \in C^2(S^n) \) is called convex if \( (u_{ij} + u\delta_{ij}) > 0 \) on \( S^n \).

**Uniqueness:** The uniqueness of solutions to the classical Minkowski problem is solved by means of the Brunn-Minkowski inequality (Gardner presented some equivalent inequalities in [53]): Let \( Q_1 \) and \( Q_2 \), \( Q_1, Q_2 \in K \), be two convex bodies in \( \mathbb{R}^{n+1} \), and let \( 0 < \lambda < 1 \). Then,

\[
\text{Vol} \left( (1 - \lambda)Q_1 + \lambda Q_2 \right) \geq \text{Vol} \left( Q_1 \right)^{1-\lambda} \text{Vol} \left( Q_2 \right)^{\lambda}, \tag{3.2.4}
\]

where \( \text{Vol}(\cdot) \) denotes the volume of the convex body in the parentheses and ‘+’ denotes the Minkowski sum. The equality in (3.2.4) holds if and only if \( Q_1 \) and \( Q_2 \) are translates.
3.3 \( L_p \)-Minkowski Problem

The \( L_p \)-Minkowski problem, which is an \( L_p \) version of the classical Minkowski problem, was introduced by Lutwak in [111]. Furthermore, the \( L_p \)-Minkowski problem, whose solutions have useful applications related to affine isoperimetric inequalities [30, 70, 71, 113, 164] and certain types of flows (see, e.g., [3, 4, 142, 143]), is an important problem in convex geometric analysis and has been studied by, e.g., [13, 14, 27, 60, 69, 79, 86, 110–112, 114, 142, 143, 165–167], and many others.

In this section, we introduce the formulation of the \( L_p \)-Minkowski problem and collect several results related to the \( L_p \)-Minkowski problem for \( p > 1 \) and \( p < 1 \).

3.3.1 Problem Description

The discrete Minkowski problem can be generalised to its \( L_p \) analogue, which considers the following question: Given a fixed real number \( p \geq 1 \), suppose that \( \{\nu_1, \nu_2, \cdots, \nu_d\} \) is a set of unit vectors on \( S^n \) and that \( \{\alpha_1, \alpha_2, \cdots, \alpha_d\} \) is a set of positive real numbers. Under what conditions does there exist a polytope in \( \mathbb{R}^{n+1} \) with \( d \) faces such that

(i) for each face \( F_i \) \( (i = 1, 2, \cdots, d) \), the outer normal is given by \( \nu_i \), and

(ii) if \( f_i \) is the area of \( F_i \) and \( h_i \) is the support number of \( F_i \) (the distance from the origin to the plane containing \( F_i \)), then

\[
h_i^{1-p} f_i = \alpha_i, \quad \forall \ i = 1, 2, \cdots, d.
\]

Obviously, if \( p = 1 \), then the discrete \( L_p \)-Minkowski problem reduces to the discrete Minkowski problem. In [111], Lutwak solved the discrete \( L_p \)-Minkowski problem with even data for \( p \geq 1 \) and \( p \neq n + 1 \) as follows: Suppose that \( p \geq 1 \) and \( p \neq n + 1 \). If \( \nu_1, \nu_2, \cdots, \nu_d \) do not lie in a great subsphere of \( S^n \), then there exists a convex polytope \( Q \subset \mathbb{R}^{n+1} \) with \( 2d \) faces \( F_i^\pm, i = 1, 2, \cdots, d \), such that \( Q \) is symmetric about the origin,
the outer normal of $F_i^\pm$ is given by $\pm \nu_i$, and if $f_i$ and $h_i$ are the area and support number of $F_i^\pm$, respectively, then $h_i^{1-p}f_i = \alpha_i$ for all $i = 1, 2, \cdots, d$.

In the research article of Firey [46], the Minkowski sum was extended to the general case of $p \geq 1$, which we call the $L_p$-sum (see Definition 3). Firey showed that the properties of the $L_p$-sum are similar to those of the Minkowski sum. The results reported in [46] served as the foundation for subsequent research. Later, in [111], Lutwak generalised the classical surface area measure to the $L_p$ case for $p \geq 1$, which resulted in the $L_p$ surface area measure (see Definition 6). Lutwak [111] defined the mixed $p$-quermassintegrals and first introduced the generalised Minkowski problem, which was thereafter called the $L_p$-Minkowski problem: *Given a finite Borel measure $m$ on $\mathbb{S}^n$, the $L_p$-Minkowski problem concerns whether a unique convex body $Q \subset \mathbb{R}^{n+1}$ exists such that $m$ is the $L_p$ surface area measure of $Q$. Let $\mu$ denote the surface area measure of $Q$, then the $L_p$-Minkowski problem is equivalent to solving the following equation

$$d\mu = h^{p-1}dm,$$

(3.3.1)

where $h$ denotes the support function of the solution convex body $Q$. Obviously, the classical Minkowski problem (3.2.1) is a special case of the $L_p$-Minkowski problem (3.3.1) for $p = 1$.

In the smooth category, suppose that the boundary of the solution $Q$ is smooth and that $f = dm/dx > 0$ is a continuous function defined on $\mathbb{S}^n$. Then, (3.3.1) is equivalent to considering

$$\det(h_{ij} + h\delta_{ij}) = fh^{p-1} \quad \text{on} \quad \mathbb{S}^n.$$

(3.3.2)

When $p = 1$, (3.3.2) reduces to (3.2.2).

Furthermore, if we consider the generalised $L_p$-sum (see Definition 4) for any $p \in \mathbb{R}$, we can drop the restriction $p \geq 1$ from Lutwak’s work in [111] and consider $p$ to be an arbitrary real number.
3.3.2 Cases of $p > 1$

Existence and regularity: Lutwak [111] proved the existence of solutions to (3.3.1) for $p > 1$, except for $p = n + 1$. His result in [111] relies on an evenness assumption. Specifically, he proved the following: if $m$ is an even positive Borel measure on $\mathbb{S}^n$ and is not concentrated on a great sphere of $\mathbb{S}^n$, then for $p > 1$ and $p \neq n + 1$, there exists a unique centred convex body $Q$ that satisfies (3.3.1).

The lack of solutions for the case of $p = n + 1$ in [111] is troubling. In [114], Lutwak, Yang and Zhang showed the existence of solutions to discrete and non-discrete $L_p$-Minkowski problems with a normalised volume for all $p > 1$, still under the evenness assumption. Except for the case of $p = n + 1$, the volume-normalised $L_p$-Minkowski problem is equivalent to the $L_p$-Minkowski problem when $p > 1$. In [112], Lutwak and Oliker obtained a $C^\infty$ solution to the even $L_p$-Minkowski problem for $p > 1$.

Chou and Wang [27] dropped the evenness assumption and solved (3.3.1) with general measures for $p > 1$. In addition, Hug, Lutwak, Yang and Zhang [80] obtained a different proof of the existence of solutions to (3.3.1) with a general measure for $p > n + 1$ and with a discrete measure for $p > 1$.

A $C^{2,\alpha}$ solution to (3.3.2) for $p \geq n + 1$ was given by Chou and Wang in [27]. Guan and Lin independently obtained the existence and regularity ($C^{2,\alpha}$) of the $L_p$-Minkowski problem for $p \geq n + 1$ in [60]. When $1 < p < n + 1$, even if $f > 0$ in (3.3.2) is smooth, the convex body of the solution may have the origin on its boundary (see, e.g., [27, 60, 80]); thus, the solution is not positive. Therefore, when $1 < p < n + 1$, (3.3.2) is degenerate, and the solution to (3.3.2) is not $C^2$ in general. However, for the discrete case, Hug, Lutwak, Yang and Zhang [80] found that the polytope of the solution always has the origin in its interior for $p > 1$ with $p \neq n + 1$.

Uniqueness: The uniqueness of solutions to the $L_p$-Minkowski problem for $p > 1$ and $p \neq n + 1$ is solved by means of the Brunn-Minkowski-Firey inequality [111]. Let
$Q_1$ and $Q_2$, $Q_1, Q_2 \in \mathcal{K}_0$, be two convex bodies that contain the origin in their interiors, and let $p > 1$ and $0 < \lambda < 1$; then,

$$\text{Vol} \left( (1 - \lambda) \circ Q_1 + p \lambda \circ Q_2 \right) \geq \text{Vol} (Q_1)^{1 - \lambda} \text{Vol} (Q_2)^\lambda,$$  \hspace{1cm} (3.3.3)

where ‘$+_p$’ represents $L_p$ summation and ‘$\circ$’ represents $L_p$ multiplication. For $p \in (1, n + 1) \cup (n + 1, \infty)$, the equality holds if and only if $Q_1 = Q_2$. For $p = n + 1$, the uniqueness holds up to a dilation; namely, if $Q \in \mathcal{K}_0$ is a solution to (3.3.1), then \{aQ : a \in \mathbb{R}^+\} are all solutions to (3.3.1). Correspondingly, if $h > 0$ is a solution to (3.3.2), then \{ah : a \in \mathbb{R}^+\} are all solutions to (3.3.2).

3.3.3 Cases of $p < 1$

The cases of $p < 1$ are difficult to settle. Even in $\mathbb{R}^3$, very little is known about the uniqueness of the $L_p$-Minkowski problem for $p < 1$. Chou and Wang \cite{27} obtained a weak solution to (3.3.2) when $-n - 1 < p < n + 1$. Recently, in \cite{167}, Zhu established the existence of the discrete case for $0 < p < 1$. Additionally, several special cases have been considered.

Case of $p = 0$: The $L_0$-Minkowski problem is also called the logarithmic Minkowski problem. In \cite{14}, Böröczky, Lutwak, Yang and Zhang presented a sufficient and necessary condition for the existence of solutions to the even $L_0$-Minkowski problem. Zhu \cite{165} studied the discrete $L_0$-Minkowski problem without the evenness assumption. In $\mathbb{R}^2$, Stancu \cite{142} presented three sufficient conditions for the existence of solutions to the discrete $L_0$-Minkowski problem and a sufficient and necessary condition for the existence of solutions to the even discrete $L_0$-Minkowski problem. In $\mathbb{R}^3$, Firey \cite{49} built a mathematical model to describe the ultimate shape of worn stones in the sense that a centrally symmetric convex stone tends to take on a spherical shape under an idealised wearing process. This is a parabolic problem related to the $L_0$-Minkowski
problem when $f$ is a constant.

*Case of $p = -n - 1$:* The critical case of $p = -n - 1$ is interesting because of its property of invariance under projective transformations on $\mathbb{S}^n$. Chou and Wang studied this case in [27]. In [110], for $p = -n - 1$, Lu and Wang obtained the existence of rotationally symmetric solutions to (3.3.2). Ivaki [84] studied this critical case for $n = 1$ (i.e., the $L_{-2}$-Minkowski problem on $\mathbb{S}^1$), and Zhu [166] studied the discrete case for general dimensions. When $p = -n - 1$ and $f \equiv 1$, all solutions to (3.3.2) are ellipsoids centred at the origin (see [101, 129, 151]).

**Uniqueness:** The uniqueness of the $L_p$-Minkowski problem for $p < 1$ is difficult to resolve and remains open, even in $\mathbb{R}^3$, because the important Brunn-Minkowski-Firey inequality (3.3.3) does not hold for $p < 1$ in general (see Example 1). In [85], Jian, Lu and Wang found that for any $-n - 1 < p < 0$, there exists a positive function $f \in C^\infty(\mathbb{S}^n)$ to guarantee that (3.3.2) has two different solutions, which indicates that additional conditions are necessary for uniqueness. Several special cases have been considered. For example, the uniqueness results for the polygonal $L_0$-Minkowski problem in $\mathbb{R}^2$ were presented by Stancu in [143]. In $\mathbb{R}^3$, Huang, Liu and Xu [79] determined the uniqueness for $-1 \leq p < 1$ when $f \equiv 1$ for $C^4$ smooth convex bodies.

**Example 1.** The following example shows that if $p < 1$, the Brunn-Minkowski-Firey inequality (3.3.3) does not hold in general (see, e.g., [79]).

Suppose that $Q = \{x \in \mathbb{R}^{n+1} : |x_i| \leq \alpha_i, \ \forall \ i = 1, 2, \cdots, n + 1\}$, where $\alpha_i > 0$ are positive constants, and that $Q_\varepsilon = \{x \in \mathbb{R}^{n+1} : |x_1 - \varepsilon| \leq \alpha_1, |x_j| \leq \alpha_j, \ \forall \ j = 2, 3, \cdots, n + 1\}$, where $\varepsilon \in (0, \alpha_1)$ is a small constant. Then, $Q, Q_\varepsilon \in \mathcal{K}_0$. Let $\lambda \in (0, 1)$. 
Then, by Definition 4, we have

\[ \tilde{Q} := (1 - \lambda) \circ Q + p \lambda \circ Q_{\varepsilon} \]

\[ = \{ x \in \mathbb{R}^{n+1} : -M_p(\alpha_1, \alpha_1 - \varepsilon, \lambda) \leq x_1 \leq M_p(\alpha_1, \alpha_1 + \varepsilon, \lambda), \]

\[ |x_j| \leq \alpha_j, \ \forall \ j = 2, 3, \ldots, n + 1 \}. \]

Obviously,

\[ \text{Vol}(Q) = \text{Vol}(Q_{\varepsilon}) = 2^{n+1} \prod_{i=1}^{n+1} \alpha_i, \]

and

\[ \text{Vol}(\tilde{Q}) = 2^n \left( \prod_{j=2}^{n+1} \alpha_j \right) \left( M_p(\alpha_1, \alpha_1 - \varepsilon, \lambda) + M_p(\alpha_1, \alpha_1 + \varepsilon, \lambda) \right), \]

where 'Vol' denotes volume. Define

\[ f(\lambda) = M_p(\alpha_1, \alpha_1 - \varepsilon, \lambda) + M_p(\alpha_1, \alpha_1 + \varepsilon, \lambda), \ \forall \ \lambda \in [0, 1], \]

where \( M_p \) is as defined in (3.1.4). Then, \( f \) is smooth, and \( f(0) = f(1) = 2\alpha_1 \) for all \( p \in \mathbb{R} \). Notice that \( f'' \leq 0 \) if \( p \geq 1 \), whereas \( f'' > 0 \) if \( p < 1 \). Thus, \( f \geq 2\alpha_1 \) if \( p \geq 1 \), and \( f < 2\alpha_1 \) if \( p < 1 \). Therefore, for any \( \lambda \in (0, 1) \), we have

\[ \text{Vol}(\tilde{Q}) \geq \text{Vol}(Q)^{1-\lambda} \text{Vol}(Q_{\varepsilon})^{\lambda}, \ \text{if} \ p \geq 1, \quad (3.3.4) \]

\[ \text{Vol}(\tilde{Q}) < \text{Vol}(Q)^{1-\lambda} \text{Vol}(Q_{\varepsilon})^{\lambda}, \ \text{if} \ p < 1. \quad (3.3.5) \]

(3.3.4) corresponds to the Brunn-Minkowski-Firey inequality (3.3.3), whereas (3.3.5) shows that the Brunn-Minkowski-Firey inequality does not hold in general if \( p < 1 \).
3.4 Christoffel-Minkowski Problem and Its $L_p$ Analogue

The Christoffel-Minkowski problem, which considers the existence of convex bodies with a prescribed surface area measure, arises in the study of surface area functions. For convex bodies with a smooth boundary, the corresponding surface area functions are symmetric with respect to the principal radii of the boundary.

The Christoffel-Minkowski problem considers the existence of a convex body $Q$ whose $k$-th elementary symmetric function of all principal radii of the boundary $M$ is a given function. The given function is defined on the outer normals of $M$. The problem requires a convex solution\(^1\) to the $k$-Hessian equation

$$
\sigma_k(h_{ij} + h\delta_{ij}) = f \quad \text{on} \quad S^n, \quad (3.4.1)
$$

where $1 \leq k \leq n$, $k \in \mathbb{N}$, $h$ is the support function of the solution convex body, $h_{ij}$ denotes the second-order covariant derivatives with respect to an orthonormal frame on $S^n$, $\delta_{ij}$ is the standard Kronecker symbol, and $\sigma_k$ denotes the $k$-th elementary symmetric function, which is defined as follows: for $k \in \{1, 2, \cdots, n\}$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n$,

$$
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.
$$

This definition can be extended to any symmetric matrix $A \in \mathbb{R}^{n \times n}$ as follows:

$$
\sigma_k(A) = \sigma_k(\lambda(A)),
$$

where $\lambda(A) = (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A))$ is the eigenvalue vector of $A$. Additionally, we set $\sigma_0 = 1$ and $\sigma_k = 0$ for all $k > n$.

\(^1\)A function $u \in C^2(S^n)$ is called convex if $(u_{ij} + u\delta_{ij}) > 0$ on $S^n$. 

Pogorelov [130] showed that (3.2.3) is a necessary condition for (3.4.1) to be solvable. As mentioned in Section 3.2, for \( k = n \), (3.2.3) is also sufficient for the existence of convex solutions to the Monge-Ampère equation (3.2.2). However, for \( 1 \leq k < n \), as mentioned by Alexandrov [1], (3.2.3) is not sufficient for the existence of convex solutions to (3.4.1) in general. Alexandrov [1] found an analytic function \( f > 0 \) that satisfies (3.2.3) while (3.4.1) has no convex solution.

If \( k = 1 \), then (3.4.1) is linear and has the following simple form:

\[
\Delta h + nh = f \quad \text{on} \quad S^n, \tag{3.4.2}
\]

which is the equation for the Christoffel problem. This case was addressed by Firey in [47, 48]. The necessary and sufficient conditions for the existence of convex solutions to (3.4.2) were found by Firey [47] using the linear representation formula of Green function. If \( k = n \), (3.4.1) corresponds to the classical Minkowski problem (3.2.2). The intermediate cases of (3.4.1) for \( 1 < k < n \) are complicated and remain open. Guan and Ma [61] presented a sufficient condition for the existence of a unique convex solution to (3.4.1) under a full rank theorem. When \( k \geq 2 \), the \( k \)-Hessian equation (3.4.1) is a fully nonlinear PDE.

The natural class of solutions to (3.4.1) is the class of \( k \)-convex functions defined as follows.

**Definition 9.** For \( k \in \{1, 2, \cdots, n\} \), let \( \Upsilon_k \) be the convex cone in \( \mathbb{R}^n \) that is defined by

\[
\Upsilon_k = \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \sigma_2(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0 \}. 
\]

A function \( u \in C^2(S^n) \) is called \( k \)-convex if \( W(x) = \{ u_{ij}(x) + u(x)\delta_{ij} \} \in \Upsilon_k \) for any \( x \in S^n \).

Clearly, \( u \) is convex on \( S^n \) if \( u \) is \( n \)-convex. In general, \( k \)-convex functions are not convex for \( k < n \).

A function \( u \) is called an admissible solution to (3.4.1) if \( u \) is \( k \)-convex and satisfies
(3.4.1). Guan, Ma and Zhou [63] generalised the full rank theorem and proved that (3.2.3) is sufficient to ensure that (3.4.1) has an admissible solution. Therefore, the difficulty lies in settling the convexity of the admissible solutions.

Similarly, by considering the $L_p$ case of the Christoffel-Minkowski problem, we obtain its $L_p$ analogue, which we call the Christoffel-Minkowski problem of $L_p$-sum. This case is equivalent to studying the following $k$-Hessian equation:

$$\sigma_k(h_{ij} + h\delta_{ij}) = fh^{p-1} \quad \text{on} \quad S^n. \quad (3.4.3)$$

When $p = 1$, (3.4.3) reduces to (3.4.1), and when $k = n$, (3.4.3) reduces to (3.3.2). When $p > k + 1$ and $1 \leq k < n$, under the condition that the function $0 < f \in C^m(S^n)$ ($m \geq 2$) satisfies \( \left( \frac{f^{p-1}}{p+k+1} \right)_{ij} + \delta_{ij}f^{p-1} \geq 0 \) on $S^n$, Hu, Ma and Shen [76] obtained that (3.4.3) has a unique solution $Q \in \mathcal{K}_0$ with a $C^{m+1,\alpha}$ ($0 < \alpha < 1$) boundary (the uniqueness holds up to a dilation when $p = k + 1$).

**Uniqueness:** The uniqueness of the Christoffel-Minkowski problem (corresponding to $p = 1$) was settled as follows by means of the Alexandrov-Fenchel-Jessen theorem in [1] and [45]: If a solution convex body exists, it is unique up to a translation. The uniqueness of the Christoffel-Minkowski problem of $L_p$-sum for $p > 1$ and $p \neq k + 1$ is obtained [111] via the integral representation and inequalities for the mixed $p$-quermassintegrals (the uniqueness holds up to a dilation when $p = k + 1$). However, the uniqueness of the Christoffel-Minkowski problem of $L_p$-sum for $p < 1$ remains an open question.
Chapter 4

Preliminaries

This chapter focuses on the preliminaries for the proof of the main results. To study equations (I.1) and (I.2), some basics of differential geometry are presented in Section 4.1. To study (I.3), some preliminary work reported in [62] is presented in Section 4.2. Section 4.3 summarises the uniqueness results.

4.1 Basics of Differential Geometry

4.1.1 Gauss Curvature

Let \( u \) be a \( C^2 \) function on \( \Omega \subset \mathbb{R}^n \). The graph of \( u \) is the hypersurface

\[
M = \{(x, u(x)) : x = (x_1, x_2, \cdots, x_n) \in \Omega \} \subset \mathbb{R}^{n+1}.
\]

The unit normal at point \( P = (x, u(x)) \in M \) is \( N_P = \left( -\frac{Du(x)}{\sqrt{1 + |Du(x)|^2}}, \frac{1}{\sqrt{1 + |Du(x)|^2}} \right) \), where \( Du \) is the gradient of \( u \). Then, the first and second fundamental forms of \( M \) are

\[
\mathbb{I} = \delta_{ij} + u_i u_j, \quad \mathbb{II} = (1 + |Du|^2)^{-\frac{3}{2}} D^2 u,
\]

(4.1.1)
respectively, where \( \delta_{ij} = 1 \) if \( i = j \) and 0 if \( i \neq j \), \( Du = (u_1, u_2, \cdots, u_n) \) is the gradient of \( u \), and \( D^2 u \) is the Hessian matrix of \( u \). The Gauss curvature of \( M \) is

\[
K = \det (I^2 \Pi) = \frac{\det D^2 u}{(1 + |Du|^2)^{n+2}}. \tag{4.1.2}
\]

### 4.1.2 Gauss Curvature Represented by Support Functions

Assume that \( M \subset \mathbb{R}^{n+1} \) is a smooth, closed, uniformly convex hypersurface that encloses the origin and is parameterised by its inverse Gauss map \( \nu : S^n \to M \). Let \( h \) be the support function of \( M \), then \( h \) is represented by (3.1.1). Let \( \{e_1, e_2, \cdots, e_n\} \) be the local orthonormal frame on \( S^n \), and let \( \nabla_i \) denote covariant differentiation on \( M \) along direction \( e_i \). By differentiating (3.1.1) along \( e_i \), we obtain

\[
\nabla_i h = \langle \nabla_i \nu, x \rangle + \langle \nu, \nabla_i x \rangle = \langle \nu, \nabla_i x \rangle, \tag{4.1.3}
\]

since \( x \) is normal at \( \nu(x) \) and \( \nabla_i \nu(x) \) is tangential at \( \nu(x) \). Differentiating (4.1.3) along \( e_j \) gives

\[
\nabla_{ij} h = \langle \nabla_j \nu, \nabla_i x \rangle + \langle \nu, \nabla_{ij} x \rangle = G_{ij} - h \delta_{ij},
\]

where \( G_{ij} \) is the second fundamental form of \( M \). We omit the details, which can be found in [152]. Hence,

\[
G_{ij} = \nabla_{ij} h + h \delta_{ij}. \tag{4.1.4}
\]

Let \( g_{ij} \) be the metric of \( M \), then according to \( \nabla_i x = G_{ik} g^{km} \nabla_m \nu \), we have

\[
\delta_{ij} = \langle \nabla_i x, \nabla_j x \rangle = G_{ik} g^{km} G_{js} g^{sl} \langle \nabla_m \nu, \nabla_s \nu \rangle = G_{ik} G_{jm} g^{km}.
\]

Thus,

\[
G^{jk} = G_{jm} g^{km},
\]
where we use the summation convention that repeated indices indicate summation from 1 to \( n \). Due to the uniform convexity of \( M \), the Gauss curvature \( K \) of \( M \) is represented by its support function as follows:

\[
K = \det(\mathbb{I}^{-1}\mathbb{I}) = \det(G_{jm}g^{km}) = \det(G^{jk}).
\]

By (4.1.4), we have

\[
\frac{1}{K} = \det(G_{jk}) = \det(\nabla_{jk}h + h\delta_{jk}).
\]  

(4.1.5)

**Remark 5.** The principal radii of \( M \) are the eigenvalues of the matrix \( \{\nabla_{ij}h + h\delta_{ij}\} \).

### 4.2 Preliminaries for the \( k \)-Hessian Equations

**Definition 10.** \( ^{[139]} \) Let \( A_1, A_2, \ldots, A_m \) be real symmetric \( k \times k \) matrices, and let \( \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0 \). The determinant of \( \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m \) is a homogeneous polynomial of degree \( k \) in \( \lambda_1, \lambda_2, \ldots, \lambda_m \) and can be represented as

\[
\det(\lambda_1 A_1 + \cdots + \lambda_m A_m) = \sum_{j_1, \ldots, j_k = 1}^{m} \lambda_{j_1} \cdots \lambda_{j_k} D_k(A_{j_1}, \ldots, A_{j_k}).
\]

In fact, the coefficient of \( \lambda_{j_1} \cdots \lambda_{j_k} \) depends only on \( A_{j_1}, \ldots, A_{j_k} \). The coefficients \( D_k(A_{j_1}, \ldots, A_{j_k}) \) are assumed to be symmetric in their arguments, and thus, they are uniquely determined. Equivalently,

\[
D_k(A_{j_1}, \ldots, A_{j_k}) = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_{j_1} \cdots \partial \lambda_{j_k}} \bigg|_{\lambda_{j_1} = \cdots = \lambda_{j_k} = 0^+} \det(\lambda_1 A_1 + \cdots + \lambda_k A_k).
\]

\( D_k(A_1, \ldots, A_k) \) is called the mixed discriminant of \( A_1, \ldots, A_k \).

For the application presented in Section \( ^{[53]} \) we review several results that have been proven in \( ^{[62]} \). Note that the following lemmas (Lemma 1 to Lemma 4) and
Section 5.3 are presented “directly on the functions and related vector-valued forms on $\mathbb{S}^n$ without convexity assumptions”, as mentioned in [62].

**Lemma 1.** Let $u_1, u_2, \cdots, u_{n+1} \in C^2(\mathbb{S}^n)$ be twice continuously differentiable with respect to an orthonormal frame on $\mathbb{S}^n$, and let the Hessian matrix of $u_m$ ($m = 1, 2, \cdots, n + 1$) with respect to the frame be

$$W_m := \{(u_m)_{ij} + u_m \delta_{ij}\}, \quad m = 1, 2, \cdots, n + 1.$$

Then,

$$V(u_1, u_2, \cdots, u_{n+1}) := \int_{\mathbb{S}^n} u_1 D_n(W_2, W_3, \cdots, W_{n+1}) ds$$

is a symmetric multilinear form on $(C^2(\mathbb{S}^n))^{n+1}$, where $D_n(W_2, W_3, \cdots, W_{n+1})$ is the mixed discriminant of $W_2, W_3, \cdots, W_{n+1}$ (see Definition 10) and $ds$ is the standard area form on $\mathbb{S}^n$.

In the above lemma, if $u_1, u_2, \cdots, u_{n+1}$ are the support functions of convex bodies $Q_1, Q_2, \cdots, Q_{n+1}$, respectively, then $V(u_1, u_2, \cdots, u_{n+1})$ is the mixed volume $V(Q_1, Q_2, \cdots, Q_{n+1})$ (see Definition 7).

**Lemma 2.** Under the assumptions of Lemma 1 for all $1 \leq k \leq n$, let $u_{k+2} = \cdots = u_{n+1} = 1$; then,

$$V_{k+1}(u_1, u_2, \cdots, u_{k+1}) := V(u_1, \cdots, u_{k+1}, 1, \cdots, 1) \quad (4.2.1)$$

$$= \int_{\mathbb{S}^n} u_1 D_k(W_2, W_3, \cdots, W_{k+1}) ds,$$

where $D_k(W_2, W_3, \cdots, W_{k+1})$ is the mixed discriminant of $W_2, W_3, \cdots, W_{k+1}$. Furthermore, if $u_1 = u_2 = \cdots = u_{n+1} = u$, denote $V(u_1, u_2, \cdots, u_{n+1}) := V(u)$ and
\[ V_{k+1}(u_1, u_2, \cdots, u_k) := V_{k+1}(u), \] then

\[ V(u) = \int_{S^n} u \det(u_{ij} + u\delta_{ij}) ds \quad \text{and} \quad V_{k+1}(u) = \int_{S^n} u\sigma_k(u_{ij} + u\delta_{ij}) ds, \]

where \( ds \) is the standard area element on \( S^n \).

**Lemma 3.** For any function \( u \in C^2(S^n) \), \( W = \{u_{ij} + u\delta_{ij}\} \), and \( 1 \leq k < n \), we have

\[ \int_{S^n} u\sigma_k(W) ds = \int_{S^n} \sigma_{k+1}(W) ds, \tag{4.2.2} \]

where \( ds \) is the standard area element on \( S^n \).

In Lemma 3 if \( u \) is the support function of a convex body, then (4.2.2) is a Minkowski-type integral formula.

The following lemma, which comes from [62], is a form of the Alexandrov-Fenchel inequality for positive \( k \)-convex functions. In the cited article, Guan, Ma, Trudinger and Zhu [62] utilised the hyperbolicity of the elementary symmetric functions instead of Alexandrov’s original proof. This replacement guarantees the generalisation of the Alexandrov-Fenchel inequality without the convexity assumption.

**Lemma 4 (Alexandrov-Fenchel Inequality).** If \( u_1, u_2, \cdots, u_k \) are \( k \)-convex, \( u_1 \) is positive, and there exists some \( l \in \{2, 3, \cdots, k\} \) such that \( u_l \geq 0 \) on \( S^n \), then for any \( v \in C^2(S^n) \),

\[ V_{k+1}^2(v, u_1, u_2, \cdots, u_k) \geq V_{k+1}(u_1, u_2, \cdots, u_k) V_{k+1}(v, v, u_2, \cdots, u_k), \tag{4.2.3} \]

where the equality holds if and only if \( v = \alpha u_1 + \sum_{i=1}^{n+1} \alpha_i x_i \) for some constants \( \alpha, \alpha_1, \cdots, \alpha_{n+1} \), where \( V_{k+1} \) is as defined in (4.2.1).
4.3 Uniqueness

Consider (I.2) for $1 \leq k \leq n$ (the case of $k = n$ corresponds to (I.1)). For this case, the uniqueness for $p \geq 1$ has been proven.

**Proposition 1.** Suppose that $1 \leq k \leq n$. If $p \geq 1$, then $h \equiv 1$ is the unique solution to (I.2) (the uniqueness holds up to a translation when $p = 1$ and up to a dilation when $p = k + 1$).

*Proof.* In convex geometry, the uniqueness has been proven by Lutwak in [111] via the integral representation and inequalities for the mixed $p$-quermassintegrals. For example, if $k = n$, (I.2) reduces to (I.1), and the uniqueness of solutions to (I.1) is obtained via the Brunn-Minkowski-Firey inequality (3.3.3).

Earlier, in 1967, Simon [141] presented a number of characterisations of relative spheres and found, in “Satz 6.1 (Theorem 6.1)”, that “for $1 \leq k \leq n$, if the boundary of a convex body $Q$ is smooth and $(C^k_n)^{-1}\sigma_k(\kappa) = g(h)$ with $g \in C^1$ and $g' \geq 0$, where $\kappa = (\kappa_1, \cdots, \kappa_n)$ is the principal curvature vector of the boundary of $Q$, then $Q$ is a ball.” Thus, the proposition holds if $g(h) = h^{p-1}$ and $p \geq 1$.

Alternatively, from the PDE point of view, the uniqueness has been analytically obtained by means of the $C^0$ estimate for the solutions. For (3.3.2), if $p > n + 1$, $0 < f \in C^\alpha(S^n)$, and $h$ is a $C^2$ solution, Chou and Wang [27] obtained the following estimate

$$\frac{1}{\sup f} \leq h^{p-n-1} \leq \frac{1}{\inf f}. \quad (4.3.1)$$

For (3.4.3), if $p > k + 1$ and $h > 0$ is an admissible solution, Hu, Ma and Shen [76] achieved

$$\frac{C^k_n}{\sup f} \leq h^{p-k-1} \leq \frac{C^k_n}{\inf f}. \quad (4.3.2)$$

Therefore, the proposition holds for $p > k + 1$ by taking $f \equiv 1$ in (4.3.1) and (4.3.2). □
Chapter 5

Main Results

The main results of Part I concern the uniqueness of solutions to equations (I.1), (I.2) and (I.3). The uniqueness results concerning (I.1) and (I.2) for \( p \geq 1 \) have previously been obtained (see Proposition 1). As mentioned by Jian, Lu and Wang [85], to consider the uniqueness of solutions to the \( L_p \)-Minkowski problem for \( p < 1 \), additional assumptions are needed. Therefore, in this chapter, under the hypothesis that the solutions are ellipsoids centred at the origin, we reconsider the uniqueness of solutions to (I.1) based on the geometric properties of ellipsoids. As an application, the uniqueness of ellipsoid solutions to (I.2) is also obtained. The following characteristics must be noted:

(i) the ellipsoid assumption is strong, and

(ii) \( f \) is assumed to take certain appropriate constant values in Theorem 1 and Theorem 2 (\( f \equiv 1 \) in the \( L_p \)-Minkowski problem and \( f \equiv C_k^k \) in the Christoffel-Minkowski problem of \( L_p \)-sum).

From the geometric point of view, the uniqueness of solutions to the Christoffel-Minkowski problem of \( L_p \)-sum for \( p > 1 \) and \( p \neq k + 1 \) is derived from [111] for convex bodies that contain the origin in their interiors. Working directly on the functions without convexity assumptions, Guan, Ma, Trudinger, and Zhu [62] achieved a form
of Alexandrov-Fenchel inequality (see Lemma 4) for appropriate \( k \)-convex functions on \( S^n \) and obtained a uniqueness theorem for admissible solutions to (I.3) for \( p = 1 \). Regarding the general assumption of \( p > 1 \), according to Lemma 4, we obtain a generalised Brunn-Minkowski inequality in Proposition 3 and thus we establish the uniqueness results for the admissible solutions to the \( k \)-Hessian equation (I.3) for \( p > 1 \) in Theorem 3.

We now present the main results of Part I.

**Theorem 1.** Suppose that the solution to

\[
det (h_{ij} + h \delta_{ij}) = h^{p-1} \quad \text{on} \quad S^n
\]

(5.0.1)

is an ellipsoid centred at the origin, where \( h \) is the support function of the solution convex body, \( h_{ij} \) are the second-order covariant derivatives of \( h \) with respect to an orthonormal frame on \( S^n \), \( \delta_{ij} \) is the Kronecker delta, and \( p \in \mathbb{R} \). Then, the uniqueness holds for any \( p \in \mathbb{R} \) \( \{ -n - 1 \} \) (the uniqueness holds up to a dilation when \( p = n + 1 \)). When \( p = -n - 1 \), the solutions\footnote{If \( p = -n - 1 \), which corresponds to the Minkowski problem in centroaffine geometry \cite{27}, then all solutions to (5.0.1) are ellipsoids centred at the origin. This case has been considered by \cite{101,129,151} and many others.} to (5.0.1) are all ellipsoids centred at the origin with volume \( \omega_{n+1} \), where \( \omega_{n+1} \) is the volume of the unit ball in \( \mathbb{R}^{n+1} \).

As an application, we consider the uniqueness for the following Christoffel-Minkowski problem of \( L_p \)-sum.

**Theorem 2.** Suppose that the solution to

\[
\sigma_k (h_{ij} + h \delta_{ij}) = C_n^k h^{p-1} \quad \text{on} \quad S^n
\]

(5.0.2)

is an ellipsoid centred at the origin, where \( \sigma_k \) is the \( k \)-th elementary symmetric function, \( 1 \leq k < n, k \in \mathbb{N}, C_n^k = \frac{n!}{k!(n-k)!} \) are the binomial coefficients, and \( p \in \mathbb{R} \). Then, the
uniqueness holds for any \( p \in \mathbb{R} \setminus \{k + 1\} \), and the uniqueness holds up to a dilation when \( p = k + 1 \).

**Remark 6.** Combining the above two theorems, if we consider ellipsoid solutions, then the uniqueness results for the \( L_p \) cases of the Minkowski problem and the Christoffel-Minkowski problem are extended to any \( p \in \mathbb{R} \). Specifically, in the range \( \{1, 2, \ldots, n\} \times \mathbb{R} \) of \( (k, p) \), our results classify the ellipsoid solutions \( Q \) to \((5.0.2)\) into only three cases:

**Case 1:** If \( (k, p) = (n, -n - 1) \), then the product of all the half-axes of \( Q \) is 1, i.e., the volume of \( Q \) is a constant \( \omega_{n+1} \).

**Case 2:** If \( p = k + 1 \), then \( Q \) is an arbitrary ball.

**Case 3:** Otherwise, \( Q \) is a unit ball.

**Theorem 3.** Suppose that \( u \) is a positive admissible solution to

\[
\sigma_k (u_{ij} + u \delta_{ij}) = f u^{p-1} \quad \text{on} \quad \mathbb{S}^n,
\]

where \( 1 \leq k < n \), \( k \in \mathbb{N} \), and \( f \) is a positive function defined on the unit sphere \( \mathbb{S}^n \). If \( p > 1 \) and \( p \neq k + 1 \), the uniqueness of \((5.0.3)\) holds. If \( p = k + 1 \), the uniqueness holds up to a dilation, which means that if \( u \) is a solution to \((5.0.3)\), then \( \{au : \forall a \in \mathbb{R}^+\} \) comprise the entire set of solutions to \((5.0.3)\).

### 5.1 Proof of Theorem 1

We rewrite \( p_0 = p - 1 \) in the following proofs. Without loss of generality, we choose a suitable orthonormal frame on \( \mathbb{R}^{n+1} \) such that the surface \( M \) of the solution ellipsoid \( Q \) to \((5.0.1)\) has the following form:

\[
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} + \frac{x_{n+1}^2}{a_{n+1}^2} = 1 \quad (a_1, a_2, \ldots, a_n, a_{n+1} > 0).
\]
Idea of the Proof: According to (4.1.5), the left-hand side of (5.0.1) is given by the reciprocal of the Gauss curvature $K$; therefore, (5.0.1) is equivalent to

$$\frac{1}{K} = h^{p^0}. \quad (5.1.2)$$

We compute the Gauss curvature $K$ and the support function $h$ of $M$, select specific points on $M$ to obtain the relation of all half-axes $a_i$, and then return to (5.1.2) to obtain the result. We present the proof in the following.

Proof of Theorem. Consider the lower ellipsoidal semi-surface

$$M^- : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}^{n+1}, \quad x \mapsto (x, u^-(x)), \quad (5.1.3)$$

where the domain $\Omega = \{ x \in \mathbb{R}^n : x_i \in [-a_i, a_i], \ i = 1, 2, \cdots, n \}$ and

$$u^-(x) = -a_{n+1} \sqrt{1 - \sum_{i=1}^{n} \frac{x_i^2}{a_i^2}}, \quad \forall \ x = (x_1, x_2, \cdots, x_n) \in \Omega.$$

Namely, $M^-$ is the graph of $u^-$ over $\Omega$. Let $u^+ = -u^-$. When $u^+ \neq 0$, we have

$$u_i^- (x) := \frac{\partial u^-(x)}{\partial x_i} = \frac{a_{n+1}^2}{a_i^2} \frac{x_i}{u^+(x)} \quad (i = 1, 2, \cdots, n), \quad \forall \ x \in \Omega, \quad (5.1.4)$$

and

$$u_{ij}^- (x) := \frac{\partial^2 u^-(x)}{\partial x_i \partial x_j} = \frac{a_{n+1}^2}{a_i^2 a_j^2} \left( \frac{\delta_{ij}}{u^+(x)} + \frac{a_{n+1}^2}{a_j^2} \frac{x_i x_j}{(u^+(x))^3} \right) \quad (i,j = 1, 2, \cdots, n), \quad \forall \ x \in \Omega. \quad (5.1.5)$$
Then,

\[
\det(D^2u^-) = \det \left( \frac{a_{n+1}^2}{a_i^2} \left( \delta_{ij} \frac{\partial}{\partial u^+} + \frac{a_{n+1}^2}{a_j^2} \frac{x_i x_j}{(u^+)^3} \right) \right)
\]

\[
= \frac{a_{n+1}^2}{(u^+)^n} \det \left( E_n + \begin{pmatrix}
\frac{a_{n+1}^2 x_1}{(u^+)^2} \\
\frac{a_{n+1}^2 x_2}{(u^+)^2} \\
\vdots \\
\frac{a_{n+1}^2 x_n}{(u^+)^2}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\right) \prod_{i=1}^{n} \frac{1}{a_i^2}
\]

\[
= \frac{a_{n+1}^2}{(u^+)^n} \det \left( 1 + \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\begin{pmatrix}
\frac{a_{n+1}^2 x_1}{(u^+)^2} \\
\frac{a_{n+1}^2 x_2}{(u^+)^2} \\
\vdots \\
\frac{a_{n+1}^2 x_n}{(u^+)^2}
\end{pmatrix}
\right) \prod_{i=1}^{n} \frac{1}{a_i^2}
\]

\[
= \frac{a_{n+1}^2}{(u^+)^n} \left( 1 + \frac{a_{n+1}^2}{(u^+)^2} \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \right) \prod_{i=1}^{n} \frac{1}{a_i^2}
\]

\[
= \frac{a_{n+1}^2}{(u^+)^{n+2}} \prod_{i=1}^{n} \frac{1}{a_i^2}.
\]

In the third equality above, we have used

\[
\det(\lambda E_m + A_{m \times n} B_{n \times m}) = \lambda^{m-n} \det(\lambda E_n + A_{n \times m} B_{m \times n}),
\]

where \(\lambda \in \mathbb{R}\) is a constant, \(A_{m \times n}\) is a real \(m \times n\) matrix, and \(E_m\) is an \(m\)-order identity matrix. Additionally, we have

\[
\left( 1 + |D^-|^2 \right)^\frac{n+2}{2} = \left( 1 + \frac{a_{n+1}^4}{(u^+)^2} \sum_{i=1}^{n} \frac{x_i^2}{a_i^4} \right)^\frac{n+2}{2}.
\]
By (4.1.2), the Gauss curvature $K^-$ of $M^-$ is

$$K^- = \frac{\det(D^2u^-)}{(1 + |Du^-|^2)^{\frac{n+2}{2}}} = a_{n+1}^2 \left( \prod_{i=1}^n \frac{1}{a_i^2} \right) \left( (u^+)^2 + a_{n+1}^4 \sum_{i=1}^n \frac{x_i^2}{a_i^4} \right)^{-\frac{n+2}{2}}.$$

According to the symmetry of ellipsoids, the Gauss curvature $K$ of (5.1.1) is

$$K(x) = \left( \prod_{i=1}^{n+1} \frac{1}{a_i^2} \right) \left( \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{n+2}{2}}, \quad \forall \ x \in \Omega. \quad (5.1.6)$$

The unit outer normal at an arbitrary point $P = (x_1, x_2, \ldots, x_n, x_{n+1})$ on (5.1.1) is

$$N_P = \left( \frac{x_1}{a_1^2}, \frac{x_2}{a_2^2}, \ldots, \frac{x_n}{a_n^2}, \frac{x_{n+1}}{a_{n+1}^2} \right) \sqrt{\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}};$$

then, the support function $h$ at $P$ is

$$h(P) = \langle P, N_P \rangle = \frac{\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}{\sqrt{\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}} = \frac{1}{\sqrt{\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}}}.$$ 

Thus, the support function $h$ of (5.1.1) is

$$h(x) = \left( \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \right)^{-\frac{1}{2}}, \quad \forall \ x \in \Omega. \quad (5.1.7)$$

Although $u^+$ is present in the denominators of both $\det(D^2u^-)$ and $1 + |Du^-|^2$, the quotient $K^-$ of these quantities avoids this situation. Therefore, we can also use (5.1.6) to obtain the Gauss curvature of $M$ when $u^+ = 0$ because of the continuity of the Gauss curvature of ellipsoids.
By inserting (5.1.6) and (5.1.7) into (5.1.2), we obtain

\[
\left(\prod_{i=1}^{n+1} a_i^2\right) \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}\right)^{\frac{n+2}{2}} = \left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}\right)^{-\frac{p_0}{2}}, \quad \forall x \in \Omega;
\]

thus,

\[
\left(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4}\right)^{\frac{n+2+p_0}{2}} = \prod_{i=1}^{n+1} a_i^{-2}, \quad \forall x \in \Omega. \tag{5.1.8}
\]

To ensure that (5.1.8) is true for all \( P \in M \), we choose \( P_j = (0, \ldots, 0, a_j, 0, \ldots, 0) \) for all \( j \in \{1, 2, \ldots, n+1\} \) in (5.1.8). Then, we have

\[
a_j^{n+2+p_0} = \prod_{i=1}^{n+1} a_i^2, \quad \forall j \in \{1, 2, \ldots, n+1\}. \tag{5.1.9}
\]

We then obtain

\[
\begin{cases}
p_0 = -n - 2 \\
\prod_{i=1}^{n+1} a_i = 1
\end{cases}
\]

or

\[
\begin{cases}
p_0 \neq -n - 2 \\
a_1 = a_j, \quad \forall j
\end{cases}
\]

Case (1): If \( p_0 = -n - 2 \), then the volume of the solution ellipsoid \( Q \) is equal to the volume of the \((n + 1)\)-dimensional unit ball.

Case (2): If \( p_0 \neq -n - 2 \), then by (5.1.9), we have \( a_1^{n+p_0} = a_1^{2n} \); thus,

\[
p_0 = n \quad \text{or} \quad \begin{cases}
p_0 \neq n \\
a_1 = 1
\end{cases}
\]
Hence, the solutions can be classified into the following three cases:

\[
\begin{cases}
    p_0 = -n - 2 \\
    \prod_{i=1}^{n+1} a_i = 1
\end{cases}
\]

or

\[
\begin{cases}
    p_0 = n \\
    a_k = a > 0, \quad \forall \ k = 1, 2, \cdots, n + 1
\end{cases}
\]

or

\[
\begin{cases}
    p_0 \in \mathbb{R} \setminus \{n, -n - 2\} \\
    a_k = 1, \quad \forall \ k = 1, 2, \cdots, n + 1
\end{cases}
\]

Therefore, we have the following results: For all \( p_0 \in \mathbb{R} \setminus \{n, -n - 2\} \), \( M \) is a unit sphere. If \( p_0 = n \), then \( M \) is an arbitrary sphere. If \( p_0 = -n - 2 \), then the product of all the half-axes of \( M \) is 1. This is the end of the proof of Theorem 1. \( \square \)

### 5.2 Proof of Theorem 2

We now prove Theorem 2 for two cases, \( k = 1 \) and \( 1 < k < n \), using a similar method. Without loss of generality, we can assume that the surface \( M \) of the solution ellipsoid \( Q \) to (5.0.2) is represented by (5.1.1).

#### 5.2.1 Case of \( k = 1 \)

If \( k = 1 \), then (5.0.2) becomes the following linear equation:

\[
\sigma_1 (h_{ij} + h\delta_{ij}) = nh^{p_0} \quad \text{on} \quad S^n.
\]  

(Idea of the Proof): \( \sigma_1 (h_{ij} + h\delta_{ij}) \) is the trace of the matrix \( \mathbb{I}^{-1}\mathbb{I} \), where \( \mathbb{I} \) and \( \mathbb{II} \) are the first and second fundamental forms, respectively, of the lower ellipsoidal semi-surface.
By \((4.1.1)\), we have
\[
\sigma_1(h_{ij} + h\delta_{ij}) = \sigma_1(\mathbb{I}^{-1}\mathbb{I})
= \sqrt{1 + |Du^-|^2 \left( \sum_{i=1}^{n} (u^-)^{ii} + \sum_{i,j=1}^{n} u^-_i u^-_j (u^-)^{ij} \right)},
\]
(5.2.2)

where \(Du^-\) and \(((u^-)_{ij})_{n \times n}\) are the gradient and Hessian matrix of \(u\), respectively. \(((u^-)_{ij})_{n \times n}\) is invertible because of the uniform convexity of \(M\), and its inverse matrix is denoted by \(((u^-)^{ij})_{n \times n}\). Similar to the case of \(k = n\), we first calculate the trace of \(\mathbb{I}^{-1}\mathbb{I}\) and then choose specific points on \(M^-\) to obtain the relation of the \(a_i\). In this case \((k = 1)\), we consider the lower semi-surface \(M^-\) and obtain the result due to the symmetry of ellipsoids.

**Proof of Theorem 2** \((k = 1)\). For the lower semi-surface \((5.1.3)\), by \((5.2.2)\), we have
\[
\sigma_1(h_{ij} + h\delta_{ij}) = \sqrt{1 + \sum_{i=1}^{n} \left( \frac{a_{n+1}^2 x_{i}^2}{a_i^4} \right) \left( \sum_{i=1}^{n} (u^-)^{ii} + \sum_{i,j=1}^{n} u^-_i u^-_j (u^-)^{ij} \right)}. \tag{5.2.3}
\]

By substituting \((5.2.3)\) and \((5.1.7)\) into \((5.2.1)\), we obtain
\[
\sum_{i=1}^{n} (u^-)^{ii} + \sum_{i,j=1}^{n} u^-_i u^-_j (u^-)^{ij} = n \left( \frac{|x_{n+1}|}{a_{n+1}^2} \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} \right)^{-\frac{n+1}{2}}. \tag{5.2.4}
\]

Since
\[
\delta_{ij} = \sum_{m=1}^{n} u^-_{im}(u^-)^{mj} = \sum_{m=1}^{n} \frac{a_{n+1}^2 \delta_{im}}{a_i^4} (u^-)^{mj} + \sum_{m=1}^{n} \frac{u^-_i u^-_m (u^-)^{mj}}{u^+} \quad (i, j = 1, 2, \ldots, n),
\]
we obtain
\[
u^+ = \frac{a_{n+1}^2}{a_i^4} (u^-)^{ii} + \sum_{m=1}^{n} u^-_i u^-_m (u^-)^{mi} \quad (i = 1, 2, \ldots, n).
\]
and
\[ nu^+ = a_{n+1}^2 \sum_{i=1}^{n} \frac{(u^-)^{ii}}{a_i^2} + \sum_{i,n=1}^{n} u_i u_m (u^-)^{mi}. \]

Thus, (5.2.4) is equivalent to
\[ na_{n+1} \sqrt{1 - \sum_{i=1}^{n} \frac{x_i^2}{a_i^2}} - a_{n+1}^2 \sum_{i=1}^{n} \frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n} (u^-)^{ii} = n \frac{x_{n+1}}{a_{n+1}} \left( \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} \right)^{-\frac{n_{n+1}}{2}}. \] (5.2.5)

In order for (5.2.5) to be true for any \( P \) on \( M^- \), taking \( P_{n+1} = (0, 0, \ldots, 0, -a_{n+1}) \), we obtain
\[ \left[ na_{n+1} - a_{n+1}^2 \sum_{i=1}^{n} \frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n} (u^-)^{ii} \right] \bigg|_{P_{n+1}} = na_{n+1}^\alpha. \] (5.2.6)

For any fixed \( j \in \{1, 2, \ldots, n\} \), at \( P_j = (0, \ldots, 0, a_j, 0, \ldots, 0) \), we have
\[ \left[ -a_{n+1}^2 \sum_{i=1}^{n} \frac{(u^-)^{ii}}{a_i^2} + \sum_{i=1}^{n} (u^-)^{ii} \right] \bigg|_{P_j} = 0. \] (5.2.7)

Without loss of generality, we assume that
\[ a_{n+1} = \min \{a_1, a_2, \ldots, a_n, a_{n+1}\}. \]

Since \((u^-)^{ii}\) is positive definite on the lower ellipsoidal semi-surface \( M^- \), we have \((u^-)^{ii} > 0 \) \( (i = 1, 2, \ldots, n) \). Then, (5.2.7) shows that
\[ \left[ \sum_{i=1}^{n} \left(1 - \frac{a_{n+1}^2}{a_i^2}\right)(u^-)^{ii} \right] \bigg|_{P_j} = 0; \]
thus,
\[ 1 - \frac{a_{n+1}^2}{a_i^2} = 0 \Rightarrow a_i = a_{n+1} \ (i = 1, 2, \ldots, n). \] (5.2.8)
By substituting (5.2.8) into (5.2.6), we obtain $a_{n+1} = a_{n+1}^{p_0}$. Thus,

$$
\begin{cases}
  p_0 = 1 \\
  a_i = a_{n+1}, \quad i = 1, 2, \ldots, n
\end{cases}
$$

or

$$
\begin{cases}
  p_0 \neq 1 \\
  a_i = 1, \quad i = 1, 2, \ldots, n+1
\end{cases}
$$

Therefore, we have the following results: $M$ is a unit sphere for all $p_0 \in \mathbb{R} \setminus \{1\}$, and $M$ is an arbitrary sphere when $p_0 = 1$.

5.2.2 Case of $1 < k < n$

If $1 < k < n$, (5.0.2) is a fully nonlinear equation. It is complicated to compute the Hessian matrix $(u^-_{ij})_{n \times n}$ and its inverse matrix $((u^-)^{ij})_{n \times n}$ for the intermediate cases of

$$
\sigma_k(h_{ij} + h\delta_{ij}) = C_n^k h^{p_0} \quad \text{on} \quad \mathbb{S}^n, \quad k \in \{2, 3, \ldots, n-1\}. \quad (5.2.9)
$$

Idea of the Proof: In accordance with the discussion for the cases of $k = n$ and $k = 1$, we select several specific points on the surface, calculate the Hessian matrix at these points using (5.1.4) and (5.1.5), and use (5.2.9) to obtain the conclusions. Similar to the case of $k = 1$, we only need to consider the lower semi-surface $M^-$.

Proof of Theorem 2 $(1 < k < n)$. For the lower semi-surface expressed in (5.1.3), according to (5.1.4) and (5.1.5), at point $P_{n+1} = (0, 0, \ldots, 0, -a_{n+1})$, we have

$$
x_i = 0 \quad (i = 1, 2, \ldots, n), \quad x_{n+1} = -a_{n+1}, \quad u^+ = a_{n+1},
$$

$$
u^-_i = 0 \quad (i = 1, 2, \ldots, n), \quad Du^- = 0,
$$

and

$$
du^-_{ij} = 0, \quad du^-_{ij} = 0, \quad (i, j = 1, 2, \ldots, n+1).
$$
\( \bar{u}_{ij} = \frac{a_{n+1}}{a_i^2} \delta_{ij} \quad (i, j = 1, 2, \cdots, n); \)

then,

\[
(u^-)^{ij} = \text{diag} \left( \frac{a_1^2}{a_{n+1}}, \frac{a_2^2}{a_{n+1}}, \cdots, \frac{a_n^2}{a_{n+1}} \right).
\]

Thus,

\[
\sigma_k (h_{ij} + h\delta_{ij}) \bigg|_{P_{n+1}} = \sigma_k \left( \sqrt{1 + |D u^-|^2} \left( (u^-)^{ij} \right) (\delta_{ij} + u_i^- u_j^-) \right) \bigg|_{P_{n+1}}
\]

\[
= \sigma_k \left( \text{diag} \left( \frac{a_1^2}{a_{n+1}}, \frac{a_2^2}{a_{n+1}}, \cdots, \frac{a_n^2}{a_{n+1}} \right) \right)
\]

\[
= \frac{1}{a_{n+1}^k} \sigma_k (a_1^2, a_2^2, \cdots, a_n^2).
\]

Using \(5.2.9\), we obtain

\[
\sigma_k (a_1^2, a_2^2, \cdots, a_n^2) = C_n a_{n+1}^{p_0 + k}.
\]

(5.2.10)

For any fixed \( i = 1, 2, \cdots, n \), and at \( P_i = \left( 0, \cdots, 0, \frac{\sqrt{2}}{2} a_i, 0, \cdots, 0, -\frac{\sqrt{2}}{2} a_{n+1} \right) \), by \(5.1.4\) and \(5.1.5\), we have

\[
x_i = \frac{\sqrt{2}}{2} a_i, \quad x_j = 0 \quad (j = 1, \cdots, i - 1, i + 1, \cdots, n), \quad x_{n+1} = -\frac{\sqrt{2}}{2} a_{n+1},
\]

\[
\bar{u}^+ = \frac{\sqrt{2}}{2} a_{n+1},
\]

\[
\bar{u}_i^- = \frac{a_{n+1}}{a_i}, \quad \bar{u}_j^- = 0 \quad (j = 1, \cdots, i - 1, i + 1, \cdots, n),
\]

\[
Du^- = (0, \cdots, 0, \frac{a_{n+1}}{a_i}, 0, \cdots, 0),
\]

\[
u_i^- = 2\sqrt{2} a_{n+1}^2 a_i^2, \quad u_{mm}^- = \sqrt{2} a_{n+1}^2 \left( m = 1, \cdots, i - 1, i + 1, \cdots, n \right),
\]

\[
u_{mj}^- = 0 \quad (m, j = 1, 2, \cdots, n, m \neq j).
\]
Then,
\[
\left( (u^-)^j \right) = \frac{1}{\sqrt{2}} \text{diag} \left( \frac{a_1^2}{a_{n+1}}, \frac{a_{i-1}^2}{a_{n+1}}, 2a_{n+1}, \frac{a_{i+1}^2}{a_{n+1}}, \cdots, \frac{a_n^2}{a_{n+1}} \right),
\]

\[
(\delta_{ij} + u_i^- u_j^-) = \text{diag} \left( 1, \cdots, 1, 1 + \frac{a_{n+1}^2}{a_i^2}, 1, \cdots \right).
\]

Hence,
\[
\sigma_k \left( h_{ij} + h\delta_{ij} \right) \bigg|_{P_i} = \left. \sigma_k \left( \sqrt{1 + \|Du^-\|^2} \left( (u^-)^j \right) (\delta_{ij} + u_i^- u_j^-) \right) \right|_{P_i} = \frac{(a_i^2 + a_{n+1}^2)^{\frac{k}{2}}}{2^{\frac{k}{2}} a_i a_{n+1}^k} \sigma_k \left( a_1^2, \cdots, a_{i-1}^2, \frac{a_i^2 + a_{n+1}^2}{2}, a_{i+1}^2, \cdots, a_n^2 \right).
\]

Using (5.2.9), we obtain
\[
\sigma_k \left( a_1^2, \cdots, a_{i-1}^2, \frac{a_i^2 + a_{n+1}^2}{2}, a_{i+1}^2, \cdots, a_n^2 \right) = C_n^{k \cdot a_i a_{n+1}^k} \left( a_1^2 + a_{n+1}^2 \right)^{\frac{\mu+\rho}{2}}. \tag{5.2.11}
\]

For \( k \in \{2, 3, \cdots, n-1\} \), denote

\[
\Sigma_1 = \sigma_{k-1} \left( a_1^2, \cdots, a_{i-1}^2, a_{i+1}^2, \cdots, a_n^2 \right), \quad \Sigma_2 = \sigma_k \left( a_1^2, \cdots, a_{i-1}^2, a_{i+1}^2, \cdots, a_n^2 \right),
\]

then, \( \Sigma_1 > 0, \Sigma_2 > 0 \). \( \text{(5.2.10)} \) and \( \text{(5.2.11)} \) reduce to

\[
a_i^2 \Sigma_1 + \Sigma_2 = C_n^{k \cdot a_i a_{n+1}^k}, \tag{5.2.12}
\]

and

\[
\frac{a_i^2 + a_{n+1}^2}{2} \Sigma_1 + \Sigma_2 = C_n^{k \cdot a_i a_{n+1}^k} \left( a_i^2 + a_{n+1}^2 \right)^{\frac{\mu+\rho}{2}}, \tag{5.2.13}
\]

respectively. Next, we prove that \( a_i = a_{n+1} \).
Case (1): \( p_0 + k \geq 0 \). By dividing (5.2.13) by (5.2.12), we obtain

\[
\frac{a_i^2 + a_{n+1}^2 + \sum_2^\infty \Sigma_1^{\infty}}{a_i^2 + \frac{\sum_2^\infty \Sigma_1^{\infty}}{2}} = \frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}}.
\tag{5.2.14}
\]

If \( a_i \geq a_{n+1} \), then the right-hand side of (5.2.14) is

\[
\frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}} \geq \frac{2^{p_0+k}}{(a_i^2)^{k_n+1}} = 1,
\]

and the left-hand side of (5.2.14) is

\[
\frac{a_i^2 + a_{n+1}^2 + \sum_2^\infty \Sigma_1^{\infty}}{a_i^2 + \frac{\sum_2^\infty \Sigma_1^{\infty}}{2}} \leq \frac{a_i^2 + \sum_2^\infty \Sigma_1^{\infty}}{a_i^2 + \frac{\sum_2^\infty \Sigma_1^{\infty}}{2}} = 1.
\]

Then,

\[
\frac{a_i^2 + a_{n+1}^2 + \sum_2^\infty \Sigma_1^{\infty}}{a_i^2 + \frac{\sum_2^\infty \Sigma_1^{\infty}}{2}} = \frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}} = 1 \Rightarrow a_i = a_{n+1}.
\]

Similarly, if \( a_i \leq a_{n+1} \), then \( a_i = a_{n+1} \).

Case (2): \( p_0 + k < 0 \). By subtracting (5.2.13) from (5.2.12), we obtain

\[
\frac{a_i^2 - a_{n+1}^2 + \sum_2^\infty \Sigma_1^{\infty}}{a_i^2 + \frac{\sum_2^\infty \Sigma_1^{\infty}}{2}} = \frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}} \left( 1 - \frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}} \right).
\]

Now, we present the proof by contradiction. If \( a_i \neq a_{n+1} \), then \( \Sigma_1 \) is represented by

\[
\Sigma_1 = \frac{2C_n^{k_n} a_{n+1}^{p_0+k}}{a_i^2 - a_{n+1}^2} \left( 1 - \frac{2^{p_0+k}}{(a_i^2 + a_{n+1}^2)^{k_n+1}} \right).
\]

Note that \( \Sigma_1 = \sigma_{k-1} (a_1^2, \ldots, a_{i-1}^2, a_{i+1}^2, \ldots, a_n^2) \) is independent of \( a_i^2 \); therefore, we have

\[
\frac{\partial \Sigma_1}{\partial (a_i^2)} = 0.
\]
Then,

\[
0 = \frac{\partial \Sigma_1}{\partial (a_i^2)}
\]

\[
= -\frac{2C^k_n a_{p+1}^n}{(a_i^2 - a_{n+1}^2)^2} \left( 1 - \frac{2a_i^{p+k}}{a_i^2 + a_{n+1}^2} \right) - \frac{2C^k_n a_{p+1}^n}{a_i^2 - a_{n+1}^2} \left( a_i^{p+k} - 2 \right),
\]

\[
= -\frac{2C^k_n a_{p+1}^n}{(a_i^2 - a_{n+1}^2)^2} \left( 1 - \frac{2a_i^{p+k}}{a_i^2 + a_{n+1}^2} \right) - \frac{(p_0 + k)2a_i^{p+k}}{(a_i^2 - a_{n+1}^2)} \frac{C^k_n a_{p+1}^n}{a_i^{p+k}}
\]

The above equality is equivalent to

\[
(a_i^2 - a_{n+1}^2) \left( \frac{2a_i^{p+k}}{a_i^2 + a_{n+1}^2} - \frac{a_i^2 + a_{n+1}^2}{a_i^{p+k}} \right) = (p_0 + k)2a_i^{p+k} \frac{C^k_n a_{p+1}^n}{a_i^{p+k}}
\]

The right-hand side of (5.2.15) is positive. If \( a_i > a_{n+1} \), then

\[
2a_i^2 > a_i^2 + a_{n+1}^2 \Rightarrow (2a_i^2) \frac{a_i^{p+k}}{a_i^2 + a_{n+1}^2} < (a_i^2 + a_{n+1}^2) \frac{a_i^{p+k}}{a_i^2 + a_{n+1}^2},
\]

and if \( a_i < a_{n+1} \), then

\[
(2a_i^2) \frac{a_i^{p+k}}{a_i^2 + a_{n+1}^2} > (a_i^2 + a_{n+1}^2) \frac{a_i^{p+k}}{a_i^2 + a_{n+1}^2};
\]

therefore, the left-hand side of (5.2.15) is negative. This is a contradiction.

Hence, by combining Case (1) and Case (2), we have \( a_i = a_{n+1} \) for any \( i = 1, 2, \ldots, n \). Then, by using (5.2.10), we obtain

\[
C^k_n a_{p+1}^n = C^k_n a_{p+1}^n \Rightarrow p_0 = k \quad \text{or} \quad \begin{cases} p_0 \neq k \\ a_{n+1} = 1. \end{cases}
\]

Thus, we have

\[
\begin{cases} p_0 = k \\ a_i = a_{n+1}, \quad i = 1, 2, \ldots, n \end{cases}
\]
or

\[
\begin{cases}
p_0 \neq k \\
a_i = 1, & i = 1, 2, \ldots, n + 1.
\end{cases}
\]

Therefore, we obtain the following results: if the solution to (5.2.9) is an ellipsoidal surface \( M \), then for all \( p_0 \in \mathbb{R} \setminus \{k\} \), \( M \) is a unit sphere, and if \( p_0 = k \), \( M \) is an arbitrary sphere.

We complete the proof of Theorem 2 by combining the above two cases.

### 5.3 Proof of Theorem 3

Based on the Alexandrov-Fenchel equations [62] for \( k \)-convex functions, we consider the uniqueness of the \( k \)-Hessian equation (5.0.3) for \( f > 0 \) and \( p > 1 \). First, we need to prove two important propositions. The methods we use here are adopted from [139].

#### 5.3.1 Two Important Propositions

**Proposition 2.** Suppose that \( u_0 \) and \( u_1 \), \( u_0, u_1 > 0 \), are \( k \)-convex; then,

\[
V_{k+1}^{\frac{1}{k+1}}((1-t)u_0 + tu_1) \geq (1-t)V_{k+1}^{\frac{1}{k+1}}(u_0) + tV_{k+1}^{\frac{1}{k+1}}(u_1), \quad \forall \ t \in [0,1],
\]

(5.3.1)

where the equality holds if and only if \( u_0 = \alpha u_1 + \sum_{i=1}^{n+1} \alpha_i x_i \) for some constants \( \alpha, \alpha_1, \ldots, \alpha_{n+1} \), where \( V_{k+1} \) is as defined in (4.2.1).

**Proof.** We need only prove that \( F(t) := V_{k+1}^{\frac{1}{k+1}}((1-t)u_0 + tu_1) \) is concave on \([0,1]\).

Setting \( u_t = (1-t)u_0 + tu_1 \), \( t \in [0,1] \), we have

\[
F(t) = V_{k+1}^{\frac{1}{k+1}}(u_t, u_t, \ldots, u_t).
\]
By the symmetric multilinear property of $V$, it is obvious that

$$F'(t) = V_{k+1}^{k+1} \left( u_t, \cdots, u_t \right) V_{k+1}(-u_0 + u_1, u_t, \cdots, u_t),$$

and

$$F''(t) = -k V_{k+1}^{k+1} - 2 \left( u_t, \cdots, u_t \right) \left[ V_{k+1}^2(-u_0 + u_1, u_t, \cdots, u_t) - V_{k+1}(u_t, \cdots, u_t) V_{k+1}(-u_0 + u_1, -u_0 + u_1, u_t, \cdots, u_t) \right] \leq 0,$$

where the last inequality uses the Alexandrov-Fenchel inequality \eqref{4.2.3}. Therefore, $F$ is a concave function on $[0,1]$. The equality condition easily follows from Lemma 4.

**Proposition 3** (Generalised Brunn-Minkowski Inequality). Suppose that $u_0$ and $u_1$, $u_0, u_1 > 0$, are $k$-convex; then,

$$\int_{S^n} u_1 \sigma_k ((u_0)_{ij} + u_0 \delta_{ij}) ds \geq V_{k+1}^{1-n} V_{k+1}^{1-n} (u_0),$$

(5.3.2)

where the equality holds if and only if $u_0 = \alpha u_1 + \sum_{i=1}^{n+1} \alpha_i x_i$ for some constants $\alpha, \alpha_1, \cdots, \alpha_{n+1}$.

**Proof.** Set $F(t) = V_{k+1}^{1-n} ((1-t)u_0 + tu_1) - (1-t) V_{k+1}^{1-n} (u_0) - t V_{k+1}^{1-n} (u_1)$; then, $F(0) = F(1) = 0$. By \eqref{5.3.1}, we have $F(t) \geq 0$. Thus, $F'(0) \geq 0$; namely,

$$V_{k+1}^{1-n} (u_0) V_{k+1}(-u_0 + u_1, u_0, \cdots, u_0) + V_{k+1}^{1-n} (u_0) - V_{k+1}^{1-n} (u_1) \geq 0.$$

Then, by Lemma 2

$$V_{k+1}^{1-n} (u_0) \int_{S^n} (-u_0 + u_1) \sigma_k ((u_0)_{ij} + u_0 \delta_{ij}) ds + V_{k+1}^{1-n} (u_0) \geq V_{k+1}^{1-n} (u_1).$$

(5.3.3)
By (4.2.2), (5.3.3) reduces to

\[ V_{k+1}^{1} (u_0) \int_{\mathbb{S}^n} u_1 \sigma_k ((u_0)_{ij} + u_0 \delta_{ij}) ds \geq V_{k+1}^{1}(u_1); \]

then,

\[ \int_{\mathbb{S}^n} u_1 \sigma_k ((u_0)_{ij} + u_0 \delta_{ij}) ds \geq V_{k+1}^{1}(u_1) V_{k+1}^{1} (u_0). \]

The equality condition can be easily obtained by means of Proposition 2.

5.3.2 Proof of Theorem 3

We are now ready to prove Theorem 3. The main methods are adopted from [27] and [139].

Proof of Theorem 3. Suppose that (5.0.3) has two admissible solutions, \( u \) and \( v \), and set \( p_0 := p - 1 \). We consider the equation in the following three cases.

Case 1: \( p_0 > k \). Suppose that \( x_0 \) is the maximum value point of \( G = u/v \); then, at \( x_0 \), we have

\[ 0 = \nabla \ln G = \nabla u/u - \nabla v/v \]

and

\[ 0 \geq \nabla^2 \ln G = \left( \frac{\nabla^2 u}{u} - \frac{(\nabla u)^2}{u^2} \right) - \left( \frac{\nabla^2 v}{v} - \frac{(\nabla v)^2}{v^2} \right) = \frac{\nabla^2 u}{u} - \frac{\nabla^2 v}{v}, \]

i.e., \( \nabla^2 u/u \leq \nabla^2 v/v \). Hence, at \( x_0 \), we have

\[ f u^{p_0}(x_0) = u^k(x_0) \sigma_k \left( \frac{u_{ij}}{u} + \delta_{ij} \right) (x_0) \]

\[ \leq u^k(x_0) \sigma_k \left( \frac{v_{ij}}{v} + \delta_{ij} \right) (x_0) \]

\[ = \frac{u^k(x_0)}{v^k(x_0)} f v^{p_0}(x_0); \]
therefore, \( u^{p_0-k}(x_0) \leq v^{p_0-k}(x_0) \). Thus, at the maximum value point \( x_0 \), we have \( G(x_0) = u(x_0)/v(x_0) \leq 1 \); then, \( u/v \leq 1 \). Similarly, we have \( v/u \leq 1 \). Therefore, \( u \equiv v \).

**Case 2**: \( 0 < p_0 < k \). We have \( u^{-p_0} \sigma_k(u_{ij} + u \delta_{ij}) = v^{-p_0} \sigma_k(v_{ij} + v \delta_{ij}) \); thus,

\[
V_{k+1}(u) = \int_{\mathbb{S}^n} u \sigma_k(u_{ij} + u \delta_{ij}) \, ds \\
= \int_{\mathbb{S}^n} \left( \frac{u}{v} \right)^{p_0+1} v \sigma_k(v_{ij} + v \delta_{ij}) \, ds \\
\geq \left[ \int_{\mathbb{S}^n} u \sigma_k(v_{ij} + v \delta_{ij}) \, ds \right]^{p_0+1} \left[ \int_{\mathbb{S}^n} v \sigma_k(v_{ij} + v \delta_{ij}) \, ds \right]^{-p_0} \\
\geq V_{k+1}^{k+1}(u) V_k^{p_0+k}(v) V_{k+1}(v) \\
= V_{k+1}^{k+1}(u) V_k^{p_0+k+1}(v),
\]

where the Hölder inequality is used in the first inequality and \( \text{(5.3.2)} \) is used in the second. Hence, \( V_{k+1}(u) = V_{k+1}(v) \), which forces both equalities to hold. By the equality condition of the Hölder inequality and \( \text{(5.3.2)} \), there exists a constant \( \alpha \in \mathbb{R} \) such that \( v = \alpha u \). By \( \text{(5.0.3)} \), we know that \( \alpha = 1 \). Therefore, \( u \equiv v \).

**Case 3**: \( p_0 = k \). According to **Case 2**, if \( p_0 = k \), then we have

\[
V_{k+1}(u) = \int_{\mathbb{S}^n} u \sigma_k(u_{ij} + u \delta_{ij}) \, ds \\
= \int_{\mathbb{S}^n} \left( \frac{u}{v} \right)^{k+1} v \sigma_k(v_{ij} + v \delta_{ij}) \, ds \\
\geq \left[ \int_{\mathbb{S}^n} u \sigma_k(v_{ij} + v \delta_{ij}) \, ds \right]^{k+1} \left[ \int_{\mathbb{S}^n} v \sigma_k(v_{ij} + v \delta_{ij}) \, ds \right]^{-k} \\
\geq V_{k+1}(u) V_k^k(v) V_{k+1}^k(v) \\
= V_{k+1}(u).
\]

Therefore, all equalities hold; namely, there exists \( \alpha \in \mathbb{R} \) such that \( v = \alpha u \). Consequently, \( \{ \alpha u : \forall \alpha \in \mathbb{R}^+ \} \) comprise the entire set of solutions to \( \text{(5.0.3)} \).

We thus complete the proof of Theorem 3. \( \square \)
5.4 What’s Next?

After achieving the uniqueness results in Part I, I studied the connection between Monge-Ampère equations (and their applications) and some other relevant problems, such as the close relations between the $L_p$-Minkowski problem (see Section 3.3) and Gauss curvature flows, between the Monge-Kantorovich problem (see Subsection 2.2.4) and martingale optimal transport problem.

5.4.1 Curvature Flows

Curvature flows for hypersurfaces play a significant role in differential geometry. It is known that solutions to the Monge-Ampère equation (3.3.2) describe self-similar solutions for Gauss curvature flows. Thus, the theory of curvature flows is tightly linked to solving geometric problems, such as the $L_p$-Minkowski problem.

As mentioned in the $L_0$-Minkowski problem (see Subsection 3.3.3), Firey [49] proposed a model for the wearing of stones on a beach by water waves, subjected to collisions from all directions with uniform frequency. The model considers the motion of convex surfaces by their Gauss curvature. In the cited article, Firey [49] showed that if a stone is initially convex and centrally symmetric, then the stone becomes spherical in shape during the process and contracts to a point in finite time. Firey conjectured that the results hold in general (without the symmetric assumption), and this was confirmed by Andrews in [4]. Such problems, which study the Gauss curvature flow (and its powers), have been considered by many authors in extensive articles, such as Tso [150], Chow [28, 29], Hamilton [74], Andrews [4, 5], Urbas [152, 154], Guan and Ni [64] and the references therein.

The above problem was generalised as follows to higher dimensions to consider
the motion of hypersurfaces in $\mathbb{R}^{n+1}$ along their normal direction with speed given by a suitable function of the principal radii of curvature (see [152] for details). Let $M_0$ be a smooth, closed, uniformly convex hypersurface given by a smooth embedding $X_0 : S^n \to \mathbb{R}^{n+1}$. Consider the initial value problem

$$\frac{\partial X}{\partial t}(x,t) = \kappa(x,t)N(x,t), \quad X(\cdot,0) = X_0,$$

where $\kappa(\cdot,t)$ is some appropriate curvature function of the hypersurface $M_t$ that is parametrised by $X(\cdot,t) : S^n \to \mathbb{R}^{n+1}$, and $N(\cdot,t)$ is the outer unit normal vector field to $M_t$. The problem (5.4.1) has been studied from different points of view.

§1. Mean Curvature Flow

Huisken [81] studied the case of $\kappa(\cdot,t) = -H(\cdot,t)$ for dimensions $n \geq 2$ (Gage and Hamilton [51] addressed the case of $n = 1$), where $H$ denotes the mean curvature of $M_t$, and discovered that (5.4.1) with $\kappa = -H$ has a unique smooth solution on a maximum finite time interval $[0,T^*)$ such that the hypersurfaces $M_t$ shrink to a point as $t$ tends to $T^*$. Furthermore, the hypersurfaces $\tilde{M}_t$, which are $M_t$ rescaled by a homothetic expansion satisfying $\text{Vol}(\tilde{M}_t) = \text{Vol}(M_0)$, converge to a sphere as $t \to T^*$. This problem is of great geometric interest, and the hypersurface $M_t$ that evolves under the mean curvature obeys a heat-type equation. This is one of the reasons that we would like to study heat equations in the second part of this work.

§2. Gauss Curvature Flow and Its Powers

If $\kappa(\cdot,t) = -K(\cdot,t)$, where $K$ denotes the Gauss curvature of $M_t$, then (5.4.1) (in $\mathbb{R}^3$) is exactly the model proposed by Firey in [49]. In arbitrary dimensions with $\kappa = -K$ for (5.4.1), Tso [150] obtained the same result as that discovered by Huisken, except that one does not know whether or not $\tilde{M}_{T^*}$ is a sphere.

A primary focus of (5.4.1) is on the asymptotic behaviour of the flows. Considering flows where the speed is given by $\kappa = -K^\alpha$, it is known that Firey’s conjecture is
generalised to arbitrary dimensions for this $\alpha$-power Gauss curvature problems with $\alpha > 1/(n+2)$ as follows: The solution hypersurfaces $M_t$ to the problem (5.4.1) with $\kappa = -K^\alpha$, $\alpha > 1/(n+2)$, converge to a round sphere after rescaling. Chow [28] studied the case of $\alpha > 0$ and obtained a result similar to that of Tso. In addition, Chow [28] found that the asymptotic shape is a sphere in the case of $\alpha = 1/n$. Andrews [2] obtained that the flow converges to an ellipsoid if $\alpha = 1/(n+2)$. Very recently, the case of $\alpha \in [1/n, 1 + 1/n]$ was solved by Choi and Daskalopoulos in [26], and Brendle, Choi and Daskalopoulos resolved the case of $\alpha \geq 1/(n+2)$ in [16].

Furthermore, the flow converges to a self-similar solution for any $\alpha \geq 1/(n+2)$ (see Andrews [5] for $1/(n+2) \leq \alpha \leq 1/n$, Kim and Lee [90] for $1/n < \alpha \leq 1$, Guan and Ni [64] for $\alpha = 1$, and Andrews, Guan, and Ni [6] for $\alpha > 1$). Therefore, Firey’s conjecture in high dimensions are reduced to the classification of closed self-similar solutions to the flow, which corresponds to the uniqueness of the $L_p$-Minkowski problem.

§3. Based on the Support Functions

On the other hand, the problem (5.4.1) was reduced to an initial value problem for support functions in [152], the approach of which was similar to that of Tso [150]. Suppose that $X$ solves (5.4.1) and that for any $t \geq 0$, $X(\cdot, t)$ is a parametrisation of $M_t$. The support function of $M_t$ is denoted by $h(\cdot, t)$. Suppose that $\kappa$ in (5.4.1) is expressed as $\kappa(\cdot, t) = f(r_1, r_2, \cdots, r_n)$, where $r_1, r_2, \cdots, r_n$ are the principal radii of curvature of $M_t$, $f > 0$ is a smooth, concave, symmetric function on $\mathbb{R}_+^n := \{ (\mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{R}^n : \mu_i > 0, i = 1, 2, \cdots, n \}$, and $f$ is homogeneous of degree 1 on $\mathbb{R}_+^n$, satisfying $\partial f/\partial \mu_i > 0$. Then, (5.4.1) is equivalent to the following initial value problem

$$
\frac{\partial h}{\partial t} = F(h_{ij} + h\delta_{ij}) \quad \text{on} \quad S^n \times [0, \infty),
$$

$$
h(\cdot, 0) = h_0 \quad \text{on} \quad S^n,
$$

together with the condition that $(h_{ij} + h\delta_{ij}) > 0$ on $S^n \times [0, \infty)$, where $h_0$ is the support
function of $M_0$, $(h_{ij})_{n \times n}$ is the Hessian of $h$ with respect to an orthonormal frame field on $S^n$, and $F(a_{ij}) = f(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of $(a_{ij})_{n \times n}$. In particular, in the case of $\kappa = -K$, Tso $^{150}$ reduced (5.4.1) to the following parabolic equation of Monge-Ampère type:

$$-rac{\partial h}{\partial t} \det(h_{ij} + h\delta_{ij}) = 1 \text{ on } S^n \times [0, \infty),$$

$$h(\cdot, 0) = h_0 \text{ on } S^n.$$

(5.4.2) is a nonlinear parabolic equation of second order. To study nonlinear equations, researchers always begin with linear ones. The classical linear parabolic equations of second order are heat equations. Therefore, in Part II, we study the heat equations.

### 5.4.2 Monge-Kantorovich Problem and Martingale Optimal Transport Problem

From the financial point of view, the close connection between the Monge-Kantorovich problem (see Subsection 2.2.4) and the Martingale optimal transport problem emerge (see, e.g., $^{9, 11, 37, 38}$ and the references therein).

#### 5.4.2.1 Probabilistic Formulation of the Monge-Kantorovich Problem

One can consider the probabilistic formulation of the usual Monge-Kantorovich problem as follows. Given two probability spaces $(\Omega, \mu)$ and $(\Omega^*, \nu)$ (in Subsection 2.2.4 the mass densities are given), we consider the reward problem which asks the way to maximise the total reward from transporting $\mu$ to $\nu$ with a reward function $c : \Omega \times \Omega^* \to \mathbb{R}$ (corresponding to the cost function in Subsection 2.2.4). Namely, consider the
maximisation problem
\[ \sup_{s \in S(\mu, \nu)} \int_{\Omega} c(x, s(x)) \mu(dx), \]
where \( S(\mu, \nu) \) denotes the set of all maps \( s : x \mapsto y = s(x) \) such that \( \nu = \mu \circ s^{-1} \).

Or equivalently, the aim is to find a probability \( P \) on \( \Omega \times \Omega^* \) with marginals \( \mu, \nu \) to maximise the total reward \( \mathbb{E}[c(X, Y)] \), i.e.,

\[ \sup_{P \in \mathbb{P}_{\mu, \nu}} \mathbb{E}^P[c(X, Y)] := \sup_{P \in \mathbb{P}_{\mu, \nu}} \int_{\Omega \times \Omega^*} c(X, Y) dP, \tag{5.4.3} \]

where \( X(x, y) = x, Y(x, y) = y \), the supremum is taken over the set, \( \mathbb{P}_{\mu, \nu} \), of all joint probability measures on \( \Omega \times \Omega^* \) with marginals \( \mu \) and \( \nu \), and \( \mathbb{E}^P \) denotes the expectation with respect to \( P \). The joint measures in \( \mathbb{P}_{\mu, \nu} \) are called transport plans.

Kantorovich's dual plan considers to buy \( \phi(X) \) at price \( \mu(\phi) := \int_{\Omega} \phi d\mu \) and \( \psi(Y) \) at price \( \nu(\psi) := \int_{\Omega^*} \psi d\nu \) such that \( \phi(X) + \psi(Y) \geq c(X, Y) \). Then, for any \( P \in \mathbb{P}_{\mu, \nu} \),

\[ \mathbb{E}^P[c(X, Y)] \leq \mathbb{E}^P[\phi(X) + \psi(Y)] = \mu(\phi) + \nu(\psi). \]

The dual problem considers the following minimisation problem

\[ \inf_{(\phi, \psi) \in \mathbb{K}_{\mu, \nu}} \{ \mu(\phi) + \nu(\psi) \} \tag{5.4.4} \]

with

\[ \mathbb{K}_{\mu, \nu} := \{ (\phi, \psi) \in L^1(\mu) \times L^1(\nu) \mid \phi \oplus \psi \geq c \quad \text{on} \quad \Omega \times \Omega^* \}, \tag{5.4.5} \]

where

\[ \phi \oplus \psi(x, y) := \phi(x) + \psi(y), \quad \forall (x, y) \in \Omega \times \Omega^*. \tag{5.4.6} \]

(5.4.3) and (5.4.4) are equal and the dual optimisers \( (\phi^*, \psi^*) \) exist \[89\].
5.4.2.2 Martingale Optimal Transport Problem

In the martingale optimal transport problem, the transports are required to be martingales\(^3\). Many researchers studied the martingale optimal transport problem, to name but a few, Dolinsky and Soner \(^{37}\) considered the continuous-time cases, and Beiglböck, Henry-Labordère and Penkner \(^9\) and Beiglböck and Juillet \(^10\) considered the discrete-time cases.

§1. One-step Discrete-time Cases

In the following, we introduce a fundamental case, where the transport takes place in a single time step (see, e.g., \(^{11}\) for details). To be specific, fix a European option (see Remark \(^7\)) depending only on the value of a risky asset \(X\) at discrete times \(t_1 < t_2\), and \(\Phi(X_1, X_2)\) denotes its payoff, where \(\Phi\) (corresponding to the reward function \(c\) in the classical Monge-Kantorovich problem) is assumed to be some measurable real function on \(\mathbb{R}^2\) and \(X_i\) denotes the coordinate process:\(^4\)

\[
X_i : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_i, \quad i = 1, 2.
\]

For the sake of simplicity, the interest rate is reduced to zero. Suppose that \(\mu_1, \mu_2\) are two prescribed probability measures on the real line \(\mathbb{R}\). In the arbitrage-free market (which means that “there are never any opportunities to make an instantaneous risk-free profit \(^{159}\)”), one always considers a probability measure \(\mathbb{P}\) on \(\mathbb{R}^2\) such that \(X_i, i = 1, 2,\) is a martingale under \(\mathbb{P}\) in its own filtration. In the one-step case, this martingale

\(^3\) A stochastic process \(M = \{M_t\}_{0 \leq t \leq T}\), where time \(t\) is discrete, i.e., \(t \in \{t_0(= 0), t_1, \ldots, t_n(= T)\}\) or continuous in \([0, T]\), adapted to a filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) is called a martingale \(^{91}\) if the following conditions hold:

(1) For any \(t\), \(\mathbb{E}[|M_t|] < \infty\);

(2) For any \(t_1, t_2\) with \(0 \leq t_1 < t_2 \leq T\), \(\mathbb{E}[M_{t_2}|\mathcal{F}_{t_1}] = M_{t_1}\) a.s..

\(M = \{M_t\}_{0 \leq t < \infty}\) is a martingale on \([0, \infty)\) if the above two conditions hold for any \(0 \leq t_1 < t_2 < \infty\).

\(^4\) Denote \(X_0 = x_0\) by the current spot price.
requirement is the following constraint:

$$E^P[X_2 | X_1] = X_1,$$  (5.4.7)

where $E^P$ denotes the expectation with respect to $P$. Denote $P_{\mu_1, \mu_2}$ by the set of all probability measures $P$ on $\mathbb{R}^2$ with marginals $\mu_1$ and $\mu_2$, i.e., $X_i \sim_P \mu_i$, $i = 1, 2$. Let $M_{\mu_1, \mu_2}$ be the set of all martingale measures in $P_{\mu_1, \mu_2}$, i.e., $M_{\mu_1, \mu_2} = \{P \in P_{\mu_1, \mu_2} : E^P[X_2 | X_1] = X_1\}$. In financial market, the expectation, $E^P[\Phi(X_1, X_2)]$, of the payoff is the fair value of $\Phi$. Under the above conditions, the martingale optimal transport problem considers the following maximisation problem

$$\sup_{P \in M_{\mu_1, \mu_2}} E^P[\Phi(X_1, X_2)],$$  (5.4.8)

which is an analogue of the Monge problem (5.4.3). However, $M_{\mu_1, \mu_2}$ is not nonempty in general. The given marginal measures $\mu_1$, $\mu_2$ need to satisfy additional assumptions to guarantee the existence of martingale transport plans. Strassen [144] found that $M_{\mu_1, \mu_2}$ is nonempty if and only if $\mu_1(f) \leq \mu_2(f)$ for any convex function $f$.

**Remark 7.** [159] A European call option, which is the simplest financial option, is a contract under which the holder of the option may purchase a prescribed asset for a prescribed amount at a prescribed time in the future.

- The word “may” means that the contract is a right instead of an obligation for the holder of the option.
- The prescribed asset is known as the underlying asset.

The Kantorovich’s dual analogue is to construct a semi-static strategy, which consists of two options $\varphi_1(X_1), \varphi_2(X_2)$ (at respective prices $\mu_1(\varphi_1)$ and $\mu_2(\varphi_2)$) and a number $H$ of the underlying asset $X$, to super-hedge\footnote{As defined in [159], “hedging is the reduction of the sensitivity of a portfolio to the movement of} where $\varphi_i : \mathbb{R} \to \mathbb{R}$ is $\mu_i$-measurable,
\[ \mu_i(\varphi_i) = \int \varphi_i d\mu_i, \ i = 1, 2, \] and \( H : \mathbb{R} \to \mathbb{R} \) is assumed to be bounded measurable. The martingale constraint \( (5.4.7) \) is equivalent to \( \mathbb{E}^P[f(X_1)(X_2 - X_1)] = 0 \) for any bounded function \( f : \mathbb{R} \to \mathbb{R} \). Thus, similar to \( (5.4.5) \), the analogue constraint of the dual problem consists of a triplet \((\varphi_1, \varphi_2, H)\) such that

\[ \varphi_1(x_1) + \varphi_2(x_2) + H(x_1)(x_2 - x_1) \geq \Phi(x_1, x_2), \quad \forall (x_1, x_2) \in \mathbb{R}^2. \quad (5.4.9) \]

The financial interpretation of the above inequality is that \( (5.4.9) \) guarantees there are no arbitrage possibilities. The Kantorovich’s dual analogue, which is similar to \( (5.4.4) \), considers the following minimising total price problem:

\[ \inf_{(\varphi_1, \varphi_2, H) \in K_{\mu_1, \mu_2}} \{\mu_1(\varphi_1) + \mu_2(\varphi_2)\} \quad (5.4.10) \]

with \( K_{\mu_1, \mu_2} = \{(\varphi_1, \varphi_2, H) \in L^1(\mu_1) \times L^1(\mu_2) \times L^0 : \varphi_1 \oplus \varphi_2 + H^\otimes \geq \Phi \text{ on } \mathbb{R}^2\} \), where ‘\( \oplus \)’ is as defined in \( (5.4.6) \) and \( H^\otimes(x_1, x_2) := H(x_1)(x_2 - x_1) \) for all \((x_1, x_2) \in \mathbb{R}^2\).

Beiglböck, Henry-Labordère and Penkner \[9\] obtained that \( (5.4.8) \) and \( (5.4.10) \) are equal if \( \Phi \) is upper semi-continuous in addition to some linear growth condition, and they also presented in the cited article that, in general, the dual infimum may not be attained.

**Remark 8.** In the classical Monge-Kantorovich problem, an important cost/reward function is in the form \( c(x, y) = |y - x|^2 \) (see Subsection \[2.2.4\]). However, it plays a different role in the martingale optimal transport problem. Specifically, if \( \Phi(x_1, x_2) = |x_2 - x_1|^2 \), then

\[ \mathbb{E}^P[\Phi(X_1, X_2)] = \mathbb{E}^\mu_2[X_2^2] - \mathbb{E}^\mu_1[X_1^2], \]

which means that the fair value of \( \Phi \) depends only on the marginals instead of on the an underlying asset by taking opposite positions in different financial instruments.”

\(^6\)An additional term \( H_0(x_1 - x_0) \) might be added to the left-hand side of \( (5.4.9) \), where \( H_0 \) is a constant. However, this term is not necessary as it can be subsumed into the term \( \varphi_1(x_1) \).
Main Results, Part I

choice of the martingale transport plan \( \mathbb{P} \in \mathcal{M}_{\mu_1, \mu_2} \). Therefore, some important results for the classical Monge-Kantorovich problem need to be rewritten for the martingale analogue (see, e.g., [10]).

§ 2. Continuous-time Cases

Consider path-dependent European options depending on the value of a risky asset (stock) \( X \) at continuous time \( t \in [0, T] \), where \( T \) is the maturity date, \( X_0 = x_0 > 0 \) is denoted by the initial stock price and \( X = \{X_t, t \in [0, T]\} \) is a continuous process. All possible paths of \( X \) can be viewed as elements of \( C^+[0, T] := \{f : [0, T] \to \mathbb{R}^+ | \text{f is continuous and satisfies } f_0 = x_0\} \). The continuity of the price process \( X \) is the only assumption that is made on the financial market. Denote \( \Psi(X) \) by the payoff of a given European option, where \( \Psi : \mathcal{D}[0, T] \to \mathbb{R} \) is a given deterministic map on the space, \( \mathcal{D}[0, T] \), of all measurable functions \( f : [0, T] \to \mathbb{R} \) equipped with the norm \( \|f\| = \sup_{1 \leq t \leq T} |f_t| \).

Let \( \mu = \delta_{x_0} \) be the initial probability measure of the price process \( X \), where \( \delta_{x_0} \) is the Dirac measure at the point \( x_0 \), i.e., \( X_0 = x_0 \), and \( \nu \), a probability measure on \( \mathbb{R}^+ \), be the final distribution. Consider \( \Theta := C^+[0, T] \) with the canonical process \( Y = \{Y_t, 0 \leq t \leq T\} \), where \( Y_t(\omega) := \omega(t) \) for all \( \omega \in \Theta \), and the canonical filtration \( \mathcal{F}_t := \sigma(Y_{\tau}, 0 \leq \tau \leq t) \) for any \( t \in [0, T] \). The continuous-time martingale optimal transport problem aims to construct a martingale process \( \mathbb{P} \) to maximise the value

\[
\mathbb{E}^\mathbb{P}[\Psi(X)],
\]

over the set, \( \mathcal{M}_{\mu, \nu} \), of all martingale measures \( \mathbb{P} \) on \((\Theta, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})\) such that \( X \) is a martingale with respect to \( \mathbb{P} \), \( X_0 = x_0 \) \( \mathbb{P} \)-a.s., and that the probability distribution of \( X_T \) under \( \mathbb{P} \) is \( \nu \). Namely, consider the following maximising problem

\[
\sup_{\mathbb{P} \in \mathcal{M}_{\mu, \nu}} \mathbb{E}^\mathbb{P}[\Psi(X)],
\]

\[ (5.4.11) \]

\( \text{The payoff functional } \Psi \text{ depends on the whole path instead of simply on the final value which is the case of the classical Monge-Kantorovich problem. Therefore, processes } X = \{X_t, t \in [0, T]\} \text{ are considered instead of the maps } s \in \mathcal{S}(\mu, \nu) \text{ which are considered in the classical case.} \]
where

\[ M_{\mu, \nu} := \{ \mathbb{P} : \text{probability measures on } (\Theta, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}) \text{ such that } X_0 \sim_\mathbb{P} \mu, \]

\[ X_T \sim_\mathbb{P} \nu, \quad \mathbb{E}^\mathbb{P}[X_{\tau_2}|\mathcal{F}_{\tau_1}] = X_{\tau_1} \text{ for all } 0 \leq \tau_1 \leq \tau_2 \leq T \}. \]

(5.4.11) is the continuous-time martingale analogue of the Monge problem.

Now consider the dual problem. Assume that the investor can purchase any call option at time 0 with strike \( a \geq 0 \) for the price \( \int (x - a)^+ d\nu(x) \), where \( \nu \) is a given probability measure on \( \mathbb{R}^+ \); then, the price of a derivative security with the payoff \( G(X_T) \) is given by \( \int G d\nu \), where \( G \) is bounded and \( \nu \)-measurable on \( \mathbb{R}^+ \). The dual problem is to construct a semi-static portfolio, \((G, H)\), which consists of static options that can be exercised at \( t = T \) and a dynamically updated risky asset, where \( G \in L^1(\mathbb{R}^+, \nu) \) and \( H : C^+[0, T] \to D[0, T] \) is of bounded variation and progressively measurable\(^8\).

Under the above assumptions, similar to (5.4.10), the Kantorovich dual analogue of the continuous-time martingale optimal transport problem is to consider the following infimum problem

\[ \inf_{(G, H) \in K_{\mu, \nu}} \{ \nu(G) \}, \tag{5.4.12} \]

where \( \nu(G) = \int G d\nu \) and

\[ K_{\mu, \nu} = \left\{ (G, H) : G(X_T) + \int_0^T H_r(X) dX_r \geq \Psi(X), \quad \forall X \in C^+[0, T] \right\}. \tag{5.4.13} \]

(5.4.13) means that the value of the portfolio \((G, H)\) at \( t = T \) is required to be no less than the payoff \( \Psi(X) \) of the European option for any possible value of the stock process. Dolinsky and Soner \(^{37}\) obtained that (5.4.11) and (5.4.12) are equal if \( \Psi \) satisfy some Lipschitz condition, \( \int x d\nu = x_0 \) and \( \int x^p d\nu < \infty \) for some \( p > 1 \).

**Remark 9.** Some remarks are given as follows for the continuous-time cases of the

\[^8\]I.e., for any \( f^1, f^2 \in C^+[0, T] \), \( f^1_\tau = f^2_\tau, \forall \tau \in [0, t] \) implies \( H(f^1)_t = H(f^2)_t \). \(^{37}\).
martingale optimal transport problem (see [37] for details).

- In the continuous-time cases, the cost/reward function $c$ is replaced by a more general functional $\Psi$ which depends on the whole path of the price process.
- The Dirac measure $\mu = \delta_{x_0}$ corresponds to the Monge-Kantorovich problem. One may also consider other general initial distributions.
- Except for the continuity, there is no other model assumptions on the dynamics of the price process. It is not trivial because there is no priori semi-martingale assumption on the risky asset.
- Without model assumptions on $X$, in order to consider the integration of the form $\int_0^t f_\tau dX_\tau$, pathwise approaches are utilised in [37] and the trading strategies are assumed to be of finite variation. Namely, for any $f : [0, T] \to \mathbb{R}$ which is of finite variation and $X \in C^+[0, T]$, based on the integration by parts, define
  \[
  \int_0^t f_\tau dX_\tau = f_t X_t - f_0 X_0 - \int_0^t X_\tau df_\tau, \quad \forall \, t \in [0, T],
  \]
  where $\int_0^t X_\tau df_\tau$ is the Stieltjes integration.

One can see that the Monge-Ampère equation, a significant type of nonlinear PDE of second order, (and their applications) has been of great importance in geometry, optics (see Subsection 2.2.3), stochastic theory and financial markets (see Subsection 5.4.2), etc. In Part II, we consider the classical parabolic PDE of second order — the heat equation. Instead of stochastic settings, we start with the irregular deterministic analogues, in which the inhomogeneous term is separable in space and time. Thanks to the a priori estimates in Part II, the existence and uniqueness of solutions to the Cauchy problems are arrived. In addition, based on our deterministic results, pathwise a priori estimates for some stochastic cases are obtained.
Part II

Heat Equations Driven by Irregular Terms
In Part II, we study a typical kind of linear parabolic partial differential equation: heat equations. Specifically, we study heat equations driven by irregular terms. We obtain a priori estimates for the following heat equation driven by a separable inhomogeneous term:

\[ du = \Delta u dt + \mathcal{X} d\mathcal{T} \]  

in the whole space \( Q_T := \mathbb{R}^n \times [0, T] \), where \( T > 0 \), \( u \) is an unknown function, and \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \). \( \mathcal{X} : \mathbb{R}^n \to \mathbb{R} \) is \( \delta \)-Hölder continuous with \( \delta \in (0, 1) \), and \( \mathcal{T} : [0, T] \to \mathbb{R} \) is \( \alpha \)-Hölder continuous with \( \alpha \in (1/2, 1] \), i.e.,

\[ [\mathcal{T}]_\alpha = \sup_{t \neq s \in (0, T)} \frac{|\mathcal{T}(t) - \mathcal{T}(s)|}{|t - s|^\alpha} < \infty. \]  

(II.2)

Here, the case of \( \alpha = 1 \), which shows that \( \mathcal{T} \) satisfies a Lipschitz condition, is included for discussion. The results can be applied to stochastic heat equations driven by fractional Brownian motion with a Hurst parameter of \( h \in (1/2, 1) \).

Then, we consider a general case in which the inhomogeneous term depends on both space \( x \) and time \( t \):

\[ du = \Delta u dt + f d\xi_t \]  

(II.3)

in \( Q_T \), where \( f : Q_T \to \mathbb{R} \) is continuously differentiable in the spatial variable \( x \), \( Df \in C^{\delta/2}_{x,t}(Q_T) \), and \( \xi \) is independent of \( x \) and is \( \alpha \)-Hölder continuous in \( t \) with \( \alpha \in (1/2, 1] \), i.e.,

\[ [\xi]_\alpha = \sup_{t \neq s \in (0, T)} \frac{|\xi_t - \xi_s|}{|t - s|^\alpha} < \infty. \]  

(II.4)

As an application, the existence and uniqueness of solutions to the corresponding Cauchy problems are established.

The method used to prove the main results in Part II is mainly based on the perturbation argument presented in [158].
Chapter 6

Introduction and Motivation

In this chapter, we provide a brief introduction to the motivation for heat equations, fractional Brownian motion and the background of the considered problems. Heat equations are of great importance because of their close connections with many other types of equations, such as Burgers’ equation (see Section 6.2) and the Kardar-Parisi-Zhang equation (see Section 6.3), and because of their providing basic methods in some applications, such as in the problem of image denoising (see Section 6.4). In Section 9.3 the a priori estimates of (II.1) are applied to stochastic equations that are driven by fractional Brownian motions. Hence, an introduction to fractional Brownian motions is presented in Section 6.5.

6.1 Heat Equation

The (homogeneous) heat equation is given by the following second-order parabolic PDE:

\[ u_t = \Delta u, \quad (6.1.1) \]

where the function \( u \) describes the temperature of a body (or the whole space), \( u(x,t) \) denotes the temperature at point \( x \in \mathbb{R}^n \) at time \( t \), and \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \). As heat
spreads throughout space, the temperature (the value of $u$) changes over time. \(6.1.1\) describes the distribution of heat in a given region (or the whole space) over time. Consider \(6.1.1\) in the whole space, i.e., $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, with a bounded continuous initial condition $u_0 : \mathbb{R}^n \to \mathbb{R}$. A unique solution to \(6.1.1\) with $u(\cdot, 0) = u_0$ exists in $\mathbb{R}^n$ \(^{[44]}\) and is expressed as

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy, \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. \quad (6.1.2)$$

Now, suppose that the density and heat capacity of the body are constants and that there exists a heat source (e.g., a light bulb) that generates heat per unit volume per unit time (given by a function that depends on space and time). Then, $u$ satisfies the inhomogeneous heat equation

$$u_t = \Delta u + f \quad (6.1.3)$$

in $\mathbb{R}^n \times \mathbb{R}^+$, where $f$ is an additional “forcing term” (a function that depends on $x$ and $t$). Using variation of constants formula \(^{[44]}\), the solution to \(6.1.3\) with an initial condition $u_0 : \mathbb{R}^n \to \mathbb{R}$ is obtained as follows:

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi (t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds.$$

Furthermore, if the heat source (the given function) is perturbed by “noise”, then stochastic heat equations arise. For example, if the source is perturbed by white noise (the derivative of the Brownian motion $\omega$), then we have the following stochastic partial differential equation (SPDE):

$$u_t = \Delta u + \dot{\omega}, \quad (6.1.4)$$

which is reviewed in Subsection 7.2.1. Brownian motion, $\omega = \{\omega_t, \ t \geq 0\}$, is a Gaussian
process with zero mean and a covariance function given by

\[ \text{Cov}(\omega_t, \omega_s) := \mathbb{E}[(\omega_t - \mathbb{E}[\omega_t])(\omega_s - \mathbb{E}[\omega_s])] = \min\{t, s\}, \]

where \( \mathbb{E} \) denotes the mathematical expectation. In general, if we consider fractional Brownian motion (see Section 6.5 for details), we have

\[ u_t = \Delta u + \mathfrak{B}^h, \quad (6.1.5) \]

where \( \mathfrak{B}^h \) is a fractional Brownian motion with a Hurst parameter of \( h \in (0, 1) \).

### 6.2 Burgers’ Equation

The viscous Burgers’ equation provides a simple basic model for the motion of a turbulent fluid and is given by

\[ \frac{\partial u}{\partial t} = \nu \Delta u - (u, \nabla)u, \quad (6.2.1) \]

where \( x \mapsto u(x, t) \) is the velocity field with \( x \in \mathbb{R}^n \), \( \nu \) is the viscosity (or diffusion coefficient), \( \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \), \( \nabla = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n) \) denotes the gradient, and \( (\ , \ ) \) denotes the Euclidean scalar product in \( \mathbb{R}^n \). This equation arises as a special case of the Navier-Stokes system. When \( \nu = 0 \) (i.e., the diffusion term disappears), \( (6.2.1) \) reduces to the inviscid Burgers’ equation, which is a conservation equation.

The original Burgers’ equation is a 1-dimension equation: \( u_t = \nu u_{xx} - uu_x \). Considering the original Burgers’ equation driven by a space-time standard Brownian sheet \( B \), we have

\[ u_t = \nu u_{xx} - uu_x - B_{xt}, \quad (6.2.2) \]
Using the Cole-Hopf transformation, we set \( u = -2\nu \partial (\ln y) / \partial x = -2\nu y_x / y \); then, (6.2.2) is reduced to

\[
-2\nu \frac{\partial}{\partial x} \left( \frac{y_t}{y} - \frac{\nu y_{xx}}{y} - \frac{1}{2\nu} B_t \right) = 0.
\]

By integrating both sides with respect to \( x \), we obtain \( y_t - \nu y_{xx} - \frac{y}{2\nu} B_t = y h'(t) \). Setting \( y = e^{h} v \), we have \( v_t - \nu v_{xx} - \frac{v}{2\nu} B_t = 0 \), i.e.,

\[
dv = \nu v_{xx} dt + \frac{v}{2\nu} dB,
\]

which is a stochastic heat equation with a Brownian random force. We can see that the Cole-Hopf transformation reduces the nonlinear Burgers’ equation (6.2.2) to a heat equation driven by Brownian motion.

On the other hand, if we set \( u = -\partial g / \partial x \) in (6.2.2), then

\[
\frac{\partial^2 g}{\partial t \partial x} = \nu \frac{\partial^3 g}{\partial x^3} + \frac{\partial g}{\partial x} \frac{\partial^2 g}{\partial x^2} + \frac{\partial B_t}{\partial x}.
\]

Integrating both sides with respect to \( x \) yields

\[
\frac{\partial g}{\partial t} = \nu \frac{\partial^2 g}{\partial x^2} + \frac{1}{2} \left( \frac{\partial g}{\partial x} \right)^2 + \frac{\partial B}{\partial t}.
\]

(6.2.3) is equivalent to the Kardar-Parisi-Zhang equation in a field \( g \).

### 6.3 Kardar-Parisi-Zhang Equation

The Kardar-Parisi-Zhang equation [88] is a nonlinear stochastic PDE that describes the long-term behaviour of interface fluctuations and is given by

\[
\frac{\partial u(x, t)}{\partial t} = \nu \Delta u(x, t) + \frac{1}{2} (\nabla u(x, t))^2 + \omega(x, t), \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+.
\]
where \( u(x, t) \) denotes the height of the interface at point \( x \in \mathbb{R}^n \) and time \( t \), \( \Delta \) is the Laplacian, \( \nu \Delta u \) describes the relaxation of the interface by a surface tension \( \nu \), \( \nabla \) is the spatial gradient, and \( \dot{\omega} \) is the space-time white noise, which is a distribution-valued Gaussian field with a mean of zero and a covariance of

\[
\mathbb{E} [ \dot{\omega}(x_1, t_1) \dot{\omega}(x_2, t_2) ] = \delta(x_1 - x_2) \delta(t_1 - t_2).
\]

In 1 dimension, (6.3.1) corresponds to the stochastic version of Burgers’ equation, which is presented in Section 6.2.

### 6.4 Image Denoising Problem

The use of heat equations is one of the naive, linear smoothing approaches to denoise images (see, e.g., [93], [7]). Image smoothing aims to simplify an image by reducing “noise” or useless details while preserving important information. A frequently used type of “noise” is additive and has Gaussian probability distribution with zero-mean and a given variance. For ease of presentation, consider the 2-dimensional gray-scale images.

An image is viewed as a function defined in a domain. Assume that \( u_0 : \mathbb{R}^2 \supset \Omega \to \mathbb{R} \) is a given image which contains noise. The observed image \( u_0 \) depends on the unknown “real” image \( u \) as follows:

\[
u_0 = \mathcal{A}u + \mathcal{N},\]

where \( \mathcal{A} \) is a linear or nonlinear operator and \( \mathcal{N} \) is additive noise that may be caused by the the sensor of a scanner, small imperfections in the equipment, interference in the channel used for transmission, etc. If the noise is purely additive, i.e., \( \mathcal{A} = \text{Id} \), then the most straightforward method to remove the noise is to approximate \( u_0 \) by a mollifier \( \mathcal{U} \). Namely, replace the image function \( u_0 \) by the convolution \( \mathcal{U}^\sigma = G^\sigma * u_0 \),
where ‘∗’ denotes convolution and

\[ G^\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad \forall \ x = (x_1, x_2) \in \mathbb{R}^2. \]

The parameter \( \sigma > 0 \) determines the spatial size of the details that are removed from the initial image: the bigger \( \sigma \) is, the smoother the results are, and the less details are retained. Then, by convolution with \( u_0 = u + \mathcal{N} \),

\[ U^\sigma = G^\sigma \ast u_0 = \int_{\mathbb{R}^2} G^\sigma(x - y)u_0(y)dy. \]

Comparing with the solutions represented in (6.1.2), it is obvious that \( u(x, t) = U^{\sqrt{2t}} = (G^{\sqrt{2t}} \ast u_0)(x) \) satisfies the heat equation

\[ \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t), \quad t \geq 0, \]

\[ u(x, 0) = u_0(x). \]

Therefore, the solution \( u(\cdot, t) \) is the smoothed image (at time \( t \)) of the initial image \( u_0 \). However, this approach does not preserve edges because the heat equation is an isotropic diffusion equation. Thus, nonlinearity is needed to preserve discontinuities.

Several methods have been proposed to avoid the edge blurring. For example, Perona and Malik [128] proposed a model to slow down the diffusion near edges, i.e., a nonlinear anisotropic diffusion equation with a following simple form

\[ \frac{\partial u}{\partial t} = \nabla \cdot \left( \frac{\nabla u}{c + |\nabla u|^2} \right), \quad u(x, 0) = u_0(x), \]

where \( \nabla \) denotes the gradient operator with respect to the space variables and \( c \) is a constant.
6.5 Fractional Brownian Motion

The physical relevance of the considered problems is markedly improved if random phenomena (“noise”) are introduced. In physical systems, the most common model for noise is white noise $\dot{\omega}$ — the derivative of Brownian motion $\omega$.

Brownian motion was named after the botanist Robert Brown for his observing the irregular random motion of a pollen particle suspended in a fluid in 1828. This random motion was proved to be a result of the collision of the particles with the fast-moving molecules in the liquid (by Einstein in 1905). Mathematically, Brownian motion is a continuous stochastic process, whose mathematical foundation was presented by Wiener in 1931. Standard Brownian motion (or Wiener process), $\omega = \{\omega_t, \ t \geq 0\}$, is characterised by the following properties:

1. $\omega_0 = 0$ a.s.;
2. $\omega$ has independent increments: $\omega_{t+s} - \omega_t$ ($s \geq 0$) is independent of the past, $\omega_\tau$ ($0 \leq \tau \leq t$);
3. $\omega$ has Gaussian increments: $\omega_{t+s} - \omega_t$ ($s \geq 0$) is normally distributed with mean 0 and variance $s$, i.e., $\omega_{t+s} - \omega_t \sim \mathcal{N}(0, s)$;
4. $\omega$ has continuous paths: $\omega_t$ is continuous in $t$ with probability 1.

Stochastic equations arise when the formal derivative $\dot{\omega}$ is taken as the noise in dynamic systems. Moreover, $\dot{\omega}_t$ and $\dot{\omega}_s$ are independent if $t \neq s$.

However, in many situations, correlations exist at different times. For example, Hurst statistically analysis the yearly water runoffs of Nile river. Suppose that $\xi_1, \xi_2, \ldots, \xi_m$ are the values of $m$ successive yearly water runoffs, and denote $\Xi_m = \sum_{i=1}^{m} \xi_i$. The empirical mean deviation is $\Psi_m = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \tilde{\Xi}_i^2}$, where the deviation

$\tilde{\Xi}_i := \Xi_i - \frac{1}{m} \Xi_m, \ i = 1, 2, \ldots, m$, and consider the following amplitude of $\tilde{\Xi}_i$:

$$\Phi_m = \max_{1 \leq i \leq m} \tilde{\Xi}_i - \min_{1 \leq i \leq m} \tilde{\Xi}_i.$$
Hurst found that $\Phi_m/\Psi_m$ behaves as $cm^h$ and that $\Xi_m$ and $m^h\xi_1$ have approximately the same distribution with a parameter $1/2 < h < 1$. Therefore, $\xi_1, \xi_2, \ldots, \xi_m$ cannot be assumed to be values of independent and identically distributed random variables, and alternative models are required. Fractional Brownian motion, whose increments need not be independent, is a natural correlated Gaussian model that should be investigated. Thus, one may assume that $\xi_1, \xi_2, \ldots, \xi_m$ are the increments of a fractional Brownian motion. Due to Hurst’s work, Mandelbrot gave the parameter $h$ of fractional Brownian motion a name, the Hurst parameter.

A fractional Brownian motion, $\mathcal{B}^h = \{\mathcal{B}^h_t, 0 \leq t < \infty\}$, with a Hurst parameter of $h \in (0, 1)$ on a given probability space $(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a zero-mean Gaussian process with the covariance function

$$\mathbb{E}(\mathcal{B}^h_{t}\mathcal{B}^h_{s}) = \frac{1}{2} \left(t^{2h} + s^{2h} - |t - s|^{2h}\right). \quad (6.5.1)$$

Such processes were first introduced by Kolmogorov [94] and were later studied by Levy [104], Mandelbrot and Van Ness [119], Lin [107], Dai and Heyde [33] and many others. Fractional Brownian motion $\mathcal{B}^h = \{\mathcal{B}^h_t, t \geq 0\}$ has the following properties (see, e.g., [123] for details):

1. **Self-similarity**: For any constant $\mu > 0$, the processes $\{\mu^{-h}\mathcal{B}^h_{\mu t}, t \geq 0\}$ and $\{\mathcal{B}^h_t, t \geq 0\}$ have the same distribution because the covariance function (6.5.1) is homogeneous of degree $2h$.

2. **Long-range dependence**: The covariance function of $\mathcal{B}^h_{t+\tau} - \mathcal{B}^h_t$ and $\mathcal{B}^h_{s+\tau} - \mathcal{B}^h_s$...
Introduction and Motivation, Part II

(setting $t - s = n\tau \geq \tau$) is

$$g^h(n) = \text{Cov} \left( \mathfrak{B}^h_{t+\tau} - \mathfrak{B}^h_t, \mathfrak{B}^h_{s+\tau} - \mathfrak{B}^h_s \right)$$

$$= \frac{1}{2} \tau^{2h} \left( (n + 1)^{2h} - 2n^{2h} + (n - 1)^{2h} \right)$$

$$\approx \tau^{2h} h(2h - 1)n^{2h-2} \to 0, \quad n \to \infty.$$

- If $h > 1/2$, then $g^h(n) > 0$ for sufficiently large $n$ and $\sum_{n=1}^{\infty} g^h(n) = \infty$. In this case, the increments $\mathfrak{B}^h_{t+\tau} - \mathfrak{B}^h_t$ and $\mathfrak{B}^h_{t+2\tau} - \mathfrak{B}^h_{t+\tau}$ are positively correlated, and the process $\mathfrak{B}^h$ with $h > 1/2$ exhibits long-range dependence.

- If $h < 1/2$, then $g^h(n) < 0$ for sufficiently large $n$ and $\sum_{n=1}^{\infty} |g^h(n)| < \infty$. In this case, the increments $\mathfrak{B}^h_{t+\tau} - \mathfrak{B}^h_t$ and $\mathfrak{B}^h_{t+2\tau} - \mathfrak{B}^h_{t+\tau}$ are negatively correlated.

(3) Stationary increments: According to (6.5.1), the increment of $\mathfrak{B}^h$ in an interval $[s, t]$ is given by

$$\mathbb{E}|\mathfrak{B}^h_t - \mathfrak{B}^h_s|^2 = |t - s|^{2h}. \quad (6.5.2)$$

Thus, $\mathfrak{B}^h$ is stated to have stationary increments.

(4) Hölder continuous modification: Based on (6.5.2) and the Kolmogorov continuity theorem\textsuperscript{1}, there exists a version of $\mathfrak{B}^h$ with continuous paths. Specifically, $\mathfrak{B}^h$ has $\beta$-Hölder continuous paths for all $\beta \in (0, h)$; namely, the Hurst parameter $h$ controls the regularity of the paths. Furthermore, based on the Garsia-Rodemich-Rumsey inequality\textsuperscript{56}, for any $\varepsilon \in (0, h)$ and any $T > 0$, there exists a nonnegative random

\textsuperscript{1}Kolmogorov continuity theorem\textsuperscript{126}: Let $Z = \{Z_t, t \geq 0\}$ be a stochastic process defined on a probability space $(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Assume that for all $T > 0$, there exist positive constants $\lambda, \mu$ and $C$ such that

$$\mathbb{E}\left[|Z_t - Z_s|^\lambda\right] \leq C|t - s|^{1+\mu}, \quad \forall s, t \in [0, T].$$

Then, there exists a continuous modification of $Z$, i.e., a process $\tilde{Z} = \{\tilde{Z}_t, t \geq 0\}$ whose paths are almost surely continuous, and for any time $t \geq 0$, $\mathbb{P}(Z_t = \tilde{Z}_t) = 1$. Furthermore, the paths of $\tilde{Z}$ are almost surely locally $\nu$-Hölder continuous for every $\nu \in (0, \mu/\lambda)$.\textsuperscript{56}
variable $\zeta_{\epsilon,T}$ with $\mathbb{E} \left[ |\zeta_{\epsilon,T}|^\mu \right] < \infty$ for all $\mu \geq 1$ such that

$$|\mathfrak{S}_t^h - \mathfrak{S}_s^h| \leq \zeta_{\epsilon,T} |t - s|^{h - \epsilon} \quad \text{a.s.}$$

(6.5.3)

for any $0 \leq s, t \leq T$.

The long-range dependence and self-similarity of fractional Brownian motions make them highly useful in various applications [119, 123], such as describing the long-term storage capacity of reservoirs, modelling telecommunication traffic, and describing the behaviour of stock market prices.

If $h = 1/2$, then the right-hand side of (6.5.1) is equal to $\min\{t, s\}$, which means that the process $\mathfrak{B}^h$ with $h = 1/2$ is a standard Brownian motion. Itô calculus, a stochastic calculus with respect to Brownian motion, was developed by Itô to solve stochastic differential equations (SDEs) (see, e.g., [91] for details). However, if $h \neq 1/2$, then $\mathfrak{B}^h$ has no independent increments and is not a semimartingale. Consequently, the classical Itô calculus cannot be utilised to describe integrations with respect to fractional Brownian motion with a Hurst parameter of $h \neq 1/2$. Therefore, an alternative theory is required to address equations driven by fractional Brownian motions.

### 6.6 Objectives of Part II

Part II focuses on the regularity of solutions to heat equations with irregular inhomogeneous terms. We now roughly summarise the previous work on regularities; details are given in Chapter 7. According to the classical Schauder theory, for the inhomogeneous heat equation (6.1.3), if $f$ is Hölder continuous in $x$ and $t$, then so are $D^2u$ and $u_t$. Symbolically, we have the following conclusion:

$$\text{in (6.1.3), } f \in C^{\delta,\delta/2}_{x,t} \text{ implies } u \in C^{2+\delta,1+\delta/2}_{x,t}. $$
In numerical analysis for nonlinear parabolic equations, curvature and gradient flows, it is of importance to study the case in which the inhomogeneous term of a parabolic equation is discontinuous in $t$. As to the linear equations, due to Brandt \cite{15} and Kerr \cite{92}, if $f$ is $\delta$-Hölder continuous in $x$ uniformly in $t$, then the second spatial derivative $D^2u$ of the solutions to (6.1.3) is also Hölder continuous in $x$ and $t$.

Symbolically, we have the following conclusion:

\[
\text{in (6.1.3), } f \in C^{\delta,0}_{x,t} \implies u \in C^{2+\delta,\delta/2}_{x,t}.
\]

Furthermore, (6.1.3) can be written in the following form:

\[
du = \Delta u dt + dF, \quad (6.6.1)
\]

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a function of both $x$ and $t$. Roughly speaking, if $F$ is differentiable with respect to $t$ and $\partial F/\partial t \in C^{\delta,0}_{x,t}$, then $D^2u$ is Hölder continuous in $x$ and $t$. This case corresponds to the Lipschitz condition for $F$ with respect to $t$.

Symbolically, we have the following conclusion:

\[
\text{in (6.6.1), } F \in C^{\delta,1}_{x,t} \implies D^2u \in C^{\delta,\delta/2}_{x,t}.
\]

On the other hand, equation (6.6.1) corresponds to equations (6.1.4) and (6.1.5) if (6.6.1) is considered in the stochastic case; namely, $u$ and $F$ are real-valued random fields on some filtered complete probability space. Comparing the above three conclusions, we are interested in studying, using the theory of deterministic PDEs, the following: under what assumptions regarding $F$ on space and time (weaker than the Lipschitz condition), can the Hölder continuity of $D^2u$ still be guaranteed for (6.6.1)? Correspondingly, we can obtain pathwise estimates for stochastic heat equations. Pathwise means that we consider and obtain solutions to stochastic equations path by path, without constructing
stochastic integrals in stochastic theory. Therefore, we start with considering the separable case (II.1) and obtain the following conclusion (symbolically) in Section 9.1:

\[
\text{in (II.1), } \mathcal{X} \mathcal{X} \in C_{x,t}^{\delta,\alpha} (\delta + 2\alpha > 2) \implies D^2 u \in C_{x,t}^{\gamma,\gamma/2} (\gamma = \delta + 2\alpha - 2).
\]

(6.6.2)

Considering the assumptions and results related to (II.1), the regularity of (6.6.1) is similarly obtained (see Subsection 9.3.1 for details). The conclusion (6.6.2) shows that the Hölder continuities of space and time are mutually complementary.

With going deep into studying, it shows that equations (6.6.1) and (II.1) have good connections with the stochastic/rough equations driven by fractional Brownian motions. Specifically, the assumption regarding the Hölder exponent of \( T \) in (II.1) is \( \alpha \in (1/2, 1] \), which corresponds to fractional Brownian motion with a Hurst parameter in (1/2, 1). Therefore, a review of stochastic/rough equations is presented in Sections 7.2 and 7.3 and we consider the pathwise regularity for equations driven by fractional Brownian sheet in Subsection 9.3.2.

However, (II.1) is a special model because, in general, the inhomogeneous term is not separable and sometimes even depends on the unknown function \( u \), such as Burgers’ equation or the Kardar-Parisi-Zhang equation. Thus, we consider a general case (II.3) based on the Young integral inequality, although the results we obtain for (II.3) are not as good as those for (II.1) because \( f \) in (II.3) need be differentiable with respect to \( x \) and \( D f \in C_{x,t}^{\delta/2} \). Symbolically, we have the following conclusion in Section 9.2:

\[
\text{in (II.3), } f \in C_{x,t}^{1+\delta,\delta} \text{ and } \xi \in C_{t}^{\alpha} (\delta + 2\alpha > 2) \implies D^3 u \in C_{x,t}^{\gamma,\gamma/2} (\gamma = \delta + 2\alpha - 2).
\]

For more general and complicated cases, if \( f \) depends on the unknown function \( u \), then
there arises an equation driven by rough paths, which has the following model form:

\[ du = \Delta u dt + f(u) d\xi_t. \]  \hspace{1cm} (6.6.3)

Clearly, the rough PDE (6.6.3) reduces to (II.3) if \( f \) is independent of \( u \). Rough path theory has been popular for decades since it was first investigated by Lyons and co-authors (see [115, 116, 118]). A brief introduction is provided in Chapter 7.

To the best of my knowledge, the advantages of the results presented in Part II are as follows:

(1) The solutions are in a quasi-classical sense (see Definition 12) instead of in a mild sense, in contrast to the solutions presented in many other articles on rough PDEs.

(2) The existence and uniqueness of solutions to heat equations driven by irregular terms are arrived based on the a priori estimates of Part II (see Theorems 6 and 7).

(3) The regularity of solutions to (II.1) depends on the Hölder exponents of space with \( \delta \) and time with \( \alpha \). The mutually complementary roles of space and time are useful for considering (II.3) and equations driven by fractional Brownian sheets path by path.
Chapter 7

Literature Review

This chapter focuses on literature reviews and is arranged as follows. Section 7.1 presents a review of the second-order deterministic PDEs. Because an application to stochastic equations driven by fractional Brownian motions is presented in Subsection 9.3.2, we give a brief review of stochastic equations in Section 7.2, from equations driven by Brownian motions to equations driven by fractional Brownian motions (with Hurst parameters greater than 1/2). Furthermore, a brief introduction to equations driven by rough paths (correspondingly, driven by fractional Brownian motions with Hurst parameters less than 1/2) is provided in Section 7.3.

7.1 Deterministic PDEs

(II.1) and (II.3) are linear parabolic PDEs of second order. The theory of parabolic PDEs closely follows that of elliptic PDEs. Among elliptic cases, the Laplace equation is typical. Proofs of the Schauder estimates for the Laplace equation have traditionally been developed using Newton potential theory [137, 138]. Alternative proofs have been presented based on Campanato spaces and a perturbation method [22, 23], the convolution of functions [127], the mollification of functions [148], and a blow-up
argument \[140\], among other approaches. Furthermore, some arguments concerning fully nonlinear uniformly elliptic equations have been applied to the Laplace equation; see \[17\ 20\ 135\ 136\]. Wang presented an elementary and simple perturbation argument to prove the classical Schauder estimates for linear elliptic and parabolic equations in \[158\]; this argument also applies to fully nonlinear equations, including the Monge-Ampère equations \[87\ 158\]. Tian and Wang \[145\] established a partial and anisotropic Schauder estimate for linear and nonlinear elliptic equations based on the perturbation argument presented in \[158\].

Consider the classical linear parabolic equation

\[v_t = a^{ij} v_{ij} + b^i v_i + cv + f,\] (7.1.1)

where we use the summation convention that repeated indices indicate summation from 1 to \(n\). According to the classical Schauder theory, if \(a^{ij}, b^i, c\) and \(f\) are Hölder continuous in \(x\) and \(t\), then so are \(D^2v\) and \(v_t\). The Schauder estimates for (7.1.1) were first obtained by Ciliberto in \[31\] and have been generalised and simplified by many authors; see \[24\ 140\ 155\ 156\].

Regarding the numerical analysis of nonlinear parabolic equations, curvature and gradient flows, it is necessary to study the case in which the inhomogeneous term of a parabolic equation is discontinuous in \(t\). Brandt \[15\] was the first to show that for the linear equation (7.1.1), when \(a^{ij}, b^i, c\) and \(f\) are Hölder continuous in \(x\) uniformly in \(t\) (corresponding to (II.1) with \(\alpha = 1\)), \(D^2v\) is Hölder continuous in space. Later, Knerr \[92\] improved upon Brandt’s result and demonstrated that \(D^2v\) is also Hölder continuous in time. Both Brandt and Knerr utilised a maximum principle. An alternative proof was presented by Lieberman in \[109\] by means of Campanato space theory; see also \[98\ 99\ 109\]. Recently, Tian and Wang \[146\] obtained the Hölder continuity for the second-order derivatives \(D^2v\) of solutions to the fully nonlinear uniformly parabolic
equation \[ v_t - F(D^2 v, Dv, v, x, t) = f(x, t), \] for the case in which \( f \) is Hölder continuous in \( x \) and bounded and measurable in \( t \).

**Remark 10.** The case of \( \alpha = 1 \) in Theorem 4 corresponds to the equations that were considered by Brandt [15] and Knerr [92] as well as many authors mentioned above (see Remark 14 for details), i.e., the equations in which the coefficients and inhomogeneous term are Hölder continuous in space uniformly in time.

### 7.2 Stochastic Equations

Stochastic equations arise when random phenomena (“noise”) are introduced into the problems under consideration. The most commonly considered type of noise is white noise — the derivative of Brownian motion.

#### 7.2.1 Stochastic PDEs Driven by Brownian Motion \((h = 1/2)\)

For the case of \( h = 1/2 \) in (6.5.1), which corresponds to Brownian motion, the following general stochastic PDE of the Itô type has been studied by many researchers:

\[
du = (a^{ij}u_{ij} + b^i u_i + cu + f)dt + (\sigma^{ik}u_i + \nu^k u + g^k)d\omega^k_t, \tag{7.2.1}
\]

where \( \{\omega^k\}, k = 1, 2, \cdots, \) are countable independent Brownian motions in a filtered complete probability space \( (\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \); \( a = (a^{ij}) \) is a symmetric matrix; and \( \sigma^i, v \) and \( g \) take values in \( \ell^2 \). A uniform parabolic condition is always assumed, i.e., there exists a positive constant \( \mu > 0 \) such that

\[
2a^{ij} - \sigma^{ik}\sigma^{jk} \geq \mu \delta_{ij} \quad \text{on} \quad \Theta \times \mathbb{R}^n \times \mathbb{R}^+, \]
where $\delta_{ij}$ is the Kronecker delta. One important example of (7.2.1) is the Zakai equation [163] that arises in filtering theory.

The regularity of weak solutions to (7.2.1) in Sobolev spaces has been studied by many authors. A complete theory in the Sobolev space $W^{m,2}(\mathbb{R}^n)$ has been presented by Krylov and Rozovskiĭ [100], Rozovskiĭ [133], Da Prato and Zabczyk [32] and the references therein, where $W^{m,2}(\mathbb{R}^n)$ consists of all generalised functions on $\mathbb{R}^n$ whose derivatives up to (and including) the $m$-th order are in $L^2(\mathbb{R}^n)$. Krylov established an $L^p$ theory for (7.2.1) in [96] and [97]. For Hölder classes, using methods related to deterministic PDEs, Mikulevicius [121] obtained the existence and uniqueness of a strong solution to the Cauchy problem of (7.2.1), with $a^{ij}$ being deterministic and $\sigma^{ik} \equiv 0$. Mikulevicius [121] improved the results that were first considered by Rozovskiĭ in [132], as Rozovskiĭ’s results were not sharp. Recently, Du and Liu [39, 40] established a sharp Schauder estimate for the Cauchy problem of (7.2.1) over the whole space. In [40], based on a perturbation argument presented by Wang [158], Du and Liu considered the general case of (7.2.1), with all coefficients being random and $\sigma^i$ being not identically 0; namely, the derivatives $u_i$ of the unknown function $u$ were considered in the stochastic term. Furthermore, Du and Liu [39, 40] obtained the time continuity of $D^2u$ for (7.2.1).

### 7.2.2 Stochastic Equations Driven by Fractional Brownian Motions with Hurst Parameters of $h > 1/2$

Brownian motion is not a suitable means of representing noise if long-range dependence exists. Instead, the fractional Brownian motion is considered, which exhibits long-range dependence if the Hurst parameter $h > 1/2$.

Several approaches have been proposed for defining stochastic integrals with respect to fractional Brownian motions. Lin [107] introduced a stochastic integral as the limit of
Riemann sums for $h > 1/2$; see also [33]. In [33], Dai and Heyde obtained an Itô formula for fractional Brownian motions. Their definitions gave rise to a stochastic integral of the Stratonovich type. A pathwise stochastic integral with respect to fractional Brownian motions with Hurst parameters of $h \in (0, 1)$ was introduced by Zähle in [161], and Ruzmaikina established an extension of the Riemann-Stieltjes integral in [134]. Malliavin calculus was first used by Decreusefond and Üstünel [35] to consider a stochastic calculus for fractional Brownian motions, which was developed by many others, such as Duncan, Hu and Pasik-Duncan [41] for $h > 1/2$ and Nualart [124] and the references therein.

7.2.2.1 SDEs Driven by Fractional Brownian Motions with $h > 1/2$

Pathwise integrals with respect to fractional Brownian motions with Hurst parameters of $h > 1/2$ can be defined using the results of Young [160] or based on fractional calculus [161]. Consider the following SDE on $\mathbb{R}^n$ driven by a fractional Brownian motion $\mathcal{B}^h$ with a Hurst parameter of $h \in (1/2, 1)$:

$$du = b(u, t)dt + \sigma(u, t)\, d\mathcal{B}^h, \quad t \in [0, T] \quad (7.2.2)$$

with an initial condition of $u_0$ (an $n$-dimensional random variable). Using Young’s results and the notion of $p$-variation (see Definition [11]), Lyons [115] obtained the existence and uniqueness of solutions to (7.2.2) with $b \equiv 0$ and $\sigma(u, t) \equiv \sigma(u)$. Specifically, if $\mathcal{B}^h$ has locally bounded $p$-variation paths for $p > 1/h$ and $\sigma$ has a Hölder continuous derivative of exponent $\beta > 1/h - 1$, then a unique solution to $du = \sigma(u)\, d\mathcal{B}^h$ exists. Thereafter, Lyons developed the theory of rough paths in collaboration with his co-authors [116-118]. The alternative approach developed by Zähle [161] was based on fractional calculus. Using the approach of Zähle [161], Nualart and Răşcanu [125] established the existence and uniqueness of solutions to (7.2.2) for a Hurst parameter $h \in (1/2, 1)$.
of $h > 1/2$.

**Definition 11.** A function $f : [0, T] \to \mathbb{R}^n$ is said to be of finite $p$-variation for some $p \geq 1$ if

$$
\|f\|_{p-\text{var};[0,T]} := \left( \sup_{k \geq 1} \sum_{i=0}^{k-1} \|f(\tau_{i+1}) - f(\tau_i)\|^p \right)^{1/p} < \infty,
$$

where $\{\tau_0, \tau_1, \ldots, \tau_k\}$ are partitions of $[0, T]$. $C^{p-\text{var}}([0,T];\mathbb{R}^n)$ denotes the space of continuous functions $f : [0, T] \to \mathbb{R}^n$ that are of finite $p$-variation.

**Remark 11.** If $1 \leq p_1 < p_2$, then $f \in C^{p_1-\text{var}}([0,T];\mathbb{R}^n)$ implies $f \in C^{p_2-\text{var}}([0,T];\mathbb{R}^n)$.

### 7.2.2.2 SPDEs Driven by Fractional Brownian Motions with $h > 1/2$

SPDEs driven by fractional Brownian motions have been considered in several articles. Hu [77] considered the stochastic heat equation $du = \frac{1}{2} \Delta u dt + u d\mathbb{B}_h$ in the whole space $\mathbb{R}^n \times (0, \infty)$ driven by either time-independent fractional white noise or time-dependent fractional white noise. Based on the Fourier transform (with respect to $t$), Balan and Tudor [8] studied the existence of solutions to a stochastic heat equation driven by a Gaussian noise that is coloured in space and fractional in time in $\mathbb{R}^n \times (0, T)$ with a zero initial condition. Using the Galerkin method, Grecksch and Anh [59] proved the existence and uniqueness of solutions to a quasilinear stochastic evolution equation driven by Hilbert-space-valued fractional Brownian motions.

In Hilbert space, Duncan, Pasik-Duncan and Maslowski [42] investigated the existence and uniqueness of mild solutions to stochastic evolution equations with standard cylindrical fractional Brownian motions with Hurst parameters of $h \in (1/2, 1)$. Tindel, Tudor and Viens [147] studied the function-valued solutions for the general case of $h \in (0, 1/2) \cup (1/2, 1)$. In [147], Tindel, Tudor and Viens presented some optimal conditions on the spatial regularity. Also in Hilbert space, using a fractional calculus
approach and semigroup estimates, Maslowski and Nualart \cite{120} obtained the existence and uniqueness of mild solutions to nonlinear stochastic evolution equations driven by cylindrical fractional Brownian motion with a Hurst parameter of \( h > 1/2 \).

**Remark 12.** The results in Part II correspond to the equations driven by fractional Brownian motions with Hurst parameters \( h > 1/2 \). Our results are good in the sense that the solutions are in a quasi-classical sense instead of the \( L^\lambda (\lambda \geq 2) \) sense which was introduced in the stochastic theory.

### 7.3 Rough Equations

The theory of rough paths was developed to study equations driven by paths that are even more irregular, or rougher, such as those driven by fractional Brownian motion with a Hurst parameter of \( h \leq 1/2 \). In particular, a path of Brownian motion \((h = 1/2)\) is a significant example of a rough path. The rough path theory has been applied to consider SDEs driven by Brownian motions that are traditionally considered using Itô’s theory. However, Itô’s calculus is not defined in a pathwise manner; it is only defined in the sense of \( L^2 \). Rough path theory introduces a pathwise definition of SDEs, and it has been under development by Lyons and his co-authors \cite{116, 118} since the 1990s and has been studied by various authors \cite{66, 67, 102} in recent years. This theory is a deterministic version of Itô’s theory of SDEs and extends Itô’s theory far beyond the semimartingale setting.

#### 7.3.1 What Are Rough Paths?

Rough path theory focuses on capturing and accurately determining the interplay between highly oscillatory paths and non-linear systems. For example (see \cite{118}), consider a case in which engineers need to test the response of a suspension bridge to
earthquakes after building a computer model of the bridge. Data must be collected before the simulation. Data collection is always expensive, and therefore, the engineers must decide what data to collect (e.g., the surface height) and how to minimise the size of each data set to ensure its adequacy and non-repititiveness.

A continuous function $Z$ defined on an interval $[0, T]$ with values in a Banach space $\mathcal{V}$ is called a path. Lyons achieved a breakthrough in [116] by considering iterated integrals of a path together with the path itself. This breakthrough makes it possible to conduct a pathwise study of SDEs. Strictly speaking, as defined in [50], for any $2 \leq p < 3$, “a rough path (of roughness $p$) on an interval $[0, T]$ with values in a Banach space $\mathcal{V}$ consists of a continuous function $Z : [0, T] \to \mathcal{V}$ and a continuous ‘second-order process’ $Z : [0, T]^2 \to \mathcal{V} \otimes \mathcal{V}$, subject to certain algebraic and analytical conditions.” Let $Z = (Z, Z)$, $Z_{\tau_1, \tau_2} = Z_{\tau_2} - Z_{\tau_1}$ and $Z_{\tau_1, \tau_2} = Z_{\tau_2} - Z_{\tau_1}$. A rough path $Z = (Z, Z)$ (of roughness $p$) is assumed to satisfy the following two conditions [83]:

(i) (Chen’s identity) For any $0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq T$,

$$Z_{\tau_1, \tau_2} = Z_{\tau_1, \tau_3} + Z_{\tau_3, \tau_2}, \quad Z_{\tau_1, \tau_2} = Z_{\tau_1, \tau_3} + Z_{\tau_3, \tau_2} + Z_{\tau_1, \tau_3} \otimes Z_{\tau_3, \tau_2};$$

(ii) (Finite $p$-variation) $Z \in C^{p-var}([0, T]; \mathcal{V})$ and $Z \in C^{p/2-var}([0, T]^2; \mathcal{V} \otimes \mathcal{V})$.

### 7.3.2 Rough Differential Equations

Rough path theory concerns controlled differential equations and allows for pathwise solutions to stochastic or rough differential equations (SDEs/RDEs) as follows:

$$dW_t = f(W_t)dZ_t$$

(7.3.1)

with a given initial condition $W_0 = 0$, where $Z : [0, T] \to \mathbb{R}^n$ is a Hölder continuous path in $\mathbb{R}^n$ of exponent $\beta$ and $f$ is a map from $\mathbb{R}^m$ to the set of $m \times n$ real matrices.
In most articles on rough paths, the notion of “$\beta$-Hölder continuity” is replaced with that of “finite $p$-variation” (see Definition 11) with $p = 1/\beta$. Therefore, the integer $[p] = \max\{m \in \mathbb{N} : m \leq p\}$ plays a significant role in the theory of rough paths. In probability theory, most stochastic processes, such as Brownian motion, are not of finite variation. However, a path of Brownian motion is almost surely $\beta$-Hölder continuous for all $\beta < 1/2$ (or, equivalently, of finite $p$-variation for all $p > 2$). Therefore, rough path theory can be applied to SDEs driven by Brownian motions (without using the properties of martingale, isometry, Markov processes, filtration, etc.).

The existence and uniqueness of solutions to (7.3.1) driven by an irregular path that is Hölder continuous with an exponent greater than $1/2$ have been determined via Young integration (see, e.g., [102, 103, 115, 118]) or fractional calculus (see, e.g., [161, 162]). In [102], Lejay also considered (7.3.1) driven by a Hölder continuous path with an exponent in $(1/3, 1/2]$. By constructing Euler approximations, Davie [34] developed an alternative approach for considering (7.3.1) driven by rough paths that are of finite $p$-variation with $p < 2$ or $2 \leq p < 3$. For a comprehensive introduction, we refer to the excellent books of Lyons and Qian [118] and Friz and Hairer [50].

As an application, consider the example presented at the beginning of Subsection 7.3.1 which serves as a model of a controlled system. It is a complex dynamic system (the bridge) dominated by an external control (the movement of the ground). The model can be described by a differential equation in the form of (7.3.1) with some precise function $f$. However, the external control is not always smooth; indeed, it is typically much rougher.
7.3.3 Rough Partial Differential Equations

A natural extension of the research discussed above is to study stochastic/rough PDEs driven by irregular paths. Correspondingly, consider the following evolution problem:

\[ dW_t = AW_t dt + \Phi(W_t)dZ_t \]  

(7.3.2)

with a given initial condition, where \( Z \) is a \( \beta \)-Hölder function in \( t \), \( \beta \in [0, 1] \), that takes values in some Banach space \( \mathcal{V} \); \( A \) is an operator defined on a subspace of \( \mathcal{V} \); and \( \Phi \) is a map from \( \mathcal{V} \) to some appropriate operator space \( \mathcal{L}(\mathcal{V}; \mathcal{V}') \) that satisfies certain given conditions, e.g., some Lipschitz-type conditions \([67]\). Obviously, (7.3.2) reduces to (II.1) and (II.3) when \( A = \Delta \) and \( \Phi \) is independent of \( W_t \).

The theories concerning rough PDEs are only partially treated. Hairer constructed a robust solution theory for the Kardar-Parisi-Zhang equation using rough paths in \([72]\), and later, he proposed a theory of regularity structures \([73]\), for which he was awarded a Fields medal in 2014. Since that time, rough path theory has garnered increasing attention.

Gubinelli, Lejay and Tindel \([67]\) studied the nonlinear case of (7.3.2) with values in a distribution space and obtained mild solutions through Young integration. In \([68]\), Gubinelli and Tindel utilised a rough path method to consider mild solutions to (7.3.2) under an unnatural condition that the noise is generated by a \( \beta \)-Hölder process with \( \beta > 5/6 \). Later, Deya, Gubinelli and Tindel \([36]\) investigated the nonlinear rough heat equation

\[ dW_t = \Delta W_t dt + dZ_t(W_t) \]  

(7.3.3)

in its mild form and obtained some existence and uniqueness results for rough cases with exponents greater than 1/4. In \([36]\), they considered (7.3.3) with a nonlinear fractional perturbation and improved the results for heat equations with polynomial
perturbations obtained in the work of Gubinelli and Tindel [68]. Recently, in [55], Garrido-Atienza, Lu and Schmalfuss established the existence of a global mild solution to (7.3.2) in a separable Hilbert space, improving on their local results presented in [54].

Remark 13. Stochastic/rough PDEs driven by fractional Brownian motions with Hurst parameters less than $1/2$ are difficult to settle due not only to the high oscillation of paths but also to the nonlinearity of the equations, and solutions to them are always in the mild sense in related articles. In Part II, our results are narrow in the sense that the unknown function $u$ does not appear in the stochastic/rough term of equations (II.1) and (II.3). On the other hand, the results in Part II are good to the extent that the solutions are in a quasi-classical sense (see Definition 12); therefore, the results are applied to obtain pathwise estimates for some SPDE (see Subsection 9.3.2).
Chapter 8

Preliminaries

This chapter is organised as follows. Section 8.1 presents some notation that is used frequently throughout Part II. Section 8.2 focuses on several preliminary results for classical heat equations, which are useful for the proof presented in Chapter 9. In Section 8.3, a brief introduction to Young’s integral is provided.

8.1 Notation

Let $\Omega \subset \mathbb{R}^n$ be a domain, $I \subset \mathbb{R}$ be an interval, and $Q = \Omega \times I$. For a function $w : \Omega \to \mathbb{R}$, we define the supremum norm $\|w\|_{0, \Omega}$ (or $\|w\|_{L^\infty(\Omega)}$) := $\sup_{x \in \Omega} |w(x)|$, and for any $\beta \in (0, 1)$, we define

$$[w]_{\beta; \Omega} := \sup_{x \neq y \in \Omega} \frac{|w(x) - w(y)|}{|x - y|^{\beta}}.$$

Let $C^\beta(\Omega)$, $\beta \in (0, 1)$, denote the space of functions $w$ defined on $\Omega$ that satisfy $[ \cdot ]_{\beta; \Omega} < \infty$ equipped with the finite Hölder norm

$$\|w\|_{\beta, \Omega} := \|w\|_{0, \Omega} + [w]_{\beta; \Omega}.$$  \hspace{1cm} (8.1.1)
For any points \( X = (x, t) \) and \( Y = (y, s) \), \( X, Y \in \mathbb{R}^n \times \mathbb{R} \), the parabolic distance between \( X \) and \( Y \) is defined as \(|X - Y|_p = |x - y| + \sqrt{|t - s|}\). For a function \( v : Q \to \mathbb{R} \), we define the supremum norm \( \|v\|_{0;Q} := \sup_{X \in Q} |v(X)| \), and for \( \beta \in (0, 1) \), we define

\[
[v]_{\beta, \beta/2;Q} := \sup_{X \neq Y \in Q} \frac{|v(X) - v(Y)|}{|X - Y|_p}.
\]

Let \( m \) be a nonnegative integer. \( C_{x,t}^{m+\beta,\beta/2}(Q) \) denotes the space of functions \( v \) defined on \( Q \) with the finite norm

\[
\|v\|_{m+\beta,\beta/2;Q} := \sum_{i=0}^{m} \|D^i v\|_{0;Q} + [D^m v]_{\beta,\beta/2;Q}.
\]

For any nonnegative integers \( k \) and \( l \), let \( C_{x,t}^{k,l}(Q) \) denote the set of functions (defined on \( Q \)) that are continuously differentiable with respect to the spatial variable \( x \) up to the \( k \)-th order and with respect to the time variable \( t \) up to the \( l \)-th order. Define

\[
[v]_{2+\beta,1+\beta/2;Q} := [D^2 v]_{\beta,\beta/2;Q} + [v_t]_{\beta,\beta/2;Q},
\]

and \( C_{x,t}^{2+\beta,1+\beta/2}(Q) \) represents the functions defined on \( Q \) with the finite norm

\[
\|v\|_{2+\beta,1+\beta/2;Q} := \sum_{i=0}^{2} \|D^i v\|_{\beta,\beta/2;Q} + \|v_t\|_{\beta,\beta/2;Q}.
\]

Throughout Part II, we use \( D^i v, i = 0, 1, 2, \cdots, \) to denote the \( i \)-th order derivative of a function \( v \) with respect to \( x \), for example, \( Dv = (D_1 v, \cdots, D_n v) \) and \( D^2 v = (D_{ij} v : i, j = 1, \cdots, n) \), where

\[
D_i v = D_{x_i} v, \quad D_{ij} v = D_i D_j v = D_{x_i x_j} v,
\]

and \( v_t \) denotes the derivative of \( v \) with respect to \( t \). \( C = C(\cdot, \cdot, \cdot, \cdot) \) denotes a constant
that depends only on the quantities that appear in the parentheses.

8.2 Interior Estimates for Classical Heat Equations

For any point \( X = (x, t) \in \mathbb{R}^n \times \mathbb{R} \), denote

\[
B_r(x) = \{ y \in \mathbb{R}^n : |y - x| < r \}, \quad I_r(t) = \{ s \in \mathbb{R} : t - r^2 < s \leq t \},
\]

\[
Q_r(X) = Q_r(x, t) = B_r(x) \times I_r(t).
\]

For simplicity, we write \( B_r, I_r \) and \( Q_r \) at point \( X = 0 = (0, 0) \).

**Lemma 5.** Let \( Q = \Omega \times I \subset \mathbb{R}^n \times \mathbb{R}, R > 0, \) and \( X_0 = (x_0, t_0) \in Q, Q_R(X_0) \subset Q, \) and let \( v \) be a solution to

\[
v_t = \Delta v + g \tag{8.2.1}
\]

in \( Q \). Then, for any \( 0 < \rho < R \),

\[
\sup_{I_r(t_0)} \int_{B_r(x_0)} v^2 dx + \iint_{Q_r(X_0)} |Dv|^2 dx dt \\
\leq C \left( \frac{1}{(R - \rho)^2} \iint_{Q_R(X_0)} v^2 dx dt + (R - \rho)^2 \iint_{Q_R(X_0)} g^2 dx dt \right), \tag{8.2.2}
\]

where \( C = C(n) > 0 \) depends only on \( n \). Furthermore, for any \( m = 0, 1, 2, \ldots, \)

\[
\sup_{I_{R/2}(t_0)} \int_{B_{R/2}(x_0)} |D^m v|^2 dx + \iint_{Q_{R/2}(X_0)} |D^{m+1} v|^2 dx dt \\
\leq C \left( \frac{1}{R^{2m+2}} \iint_{Q_R(X_0)} |v|^2 dx dt + \sum_{i=0}^{m} \frac{1}{R^{2i-2}} \iint_{Q_R(X_0)} |D^{m-i} g|^2 dx dt \right), \tag{8.2.3}
\]

where \( C = C(m, n) > 0 \).

**Proof.** \([8.2.3]\) is easily obtained through induction on \( m \). Thus, it is sufficient to prove
(8.2.3) for \( m = 0 \).

For any \( 0 < \rho < R \), let \( \eta \in C_0^\infty(\mathbb{R}^n) \), \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) in \( B_\rho(x_0) \), and \( |D\eta| \leq C/(R - \rho) \). Let \( \zeta \in C_0^\infty(\mathbb{R}) \), \( 0 \leq \zeta \leq 1 \), \( \zeta(t) \equiv 0 \) if \( t \leq t_0 - R^2 \), \( \zeta(t) \equiv 1 \) if \( t \geq t_0 - \rho^2 \), and \( 0 \leq \zeta' \leq C/(R - \rho)^2 \). Multiply both sides of (8.2.1) by \( \eta^2 \zeta^2 v \) and integrate over \( Q_R(x_0) := B_R(x_0) \times (t_0 - R^2, s) \), \( \forall \ s \in I_R(t_0) \). Then,

\[
\iint_{Q_R(x_0)} \eta^2 \zeta^2 v dv = \iint_{Q_R(x_0)} \eta^2 \zeta^2 v \Delta v dx dt + \iint_{Q_R(x_0)} \eta^2 \zeta^2 v g dx dt.
\]

Based on the Cauchy inequality with \( \varepsilon \) and via integration by parts, we obtain

\[
\begin{align*}
\frac{1}{2} \int_{B_R(x_0)} \eta^2 \zeta^2 v^2 dx &+ \frac{1}{2} \int_{Q_R(x_0)} \eta^2 \zeta^2 |Dv|^2 dx dt \\
\leq & \int_{Q_R(x_0)} \eta^2 \zeta \zeta' v^2 dx dt + 2 \int_{Q_R(x_0)} \zeta^2 v^2 |D\eta|^2 dx dt + \frac{1}{2} \varepsilon \int_{Q_R(x_0)} \eta^2 \zeta^2 v^2 dx dt \\
&+ \frac{\varepsilon}{2} \int_{Q_R(x_0)} \eta^2 \zeta^2 g^2 dx dt \\
\leq & \frac{C}{(R - \rho)^2} \int_{Q_R(x_0)} v^2 dx dt + \frac{(R - \rho)^2}{C} \int_{Q_R(x_0)} g^2 dx dt.
\end{align*}
\]

Then, for any \( s \in I_\rho(t_0) \),

\[
\begin{align*}
\int_{B_\rho(x_0)} v^2 dx &+ \int_{Q_\rho(x_0)} |Dv|^2 dx dt \\
\leq & \int_{B_R(x_0)} \eta^2 \zeta^2 v^2 dx |_s + \int_{Q_R(x_0)} \eta^2 \zeta^2 |Dv|^2 dx dt \\
\leq & \frac{C}{(R - \rho)^2} \int_{Q_R(x_0)} v^2 dx dt + \frac{(R - \rho)^2}{C} \int_{Q_R(x_0)} g^2 dx dt.
\end{align*}
\]
Hence,

\[
\sup_{t_\rho(t_0)} \int_{B_\rho(x_0)} v^2 dx + \iint_{Q_\rho(x_0)} |Dv|^2 dt \\
\leq C \left( \frac{1}{(R - \rho)^2} \iint_{Q_R(x_0)} v^2 dx dt + (R - \rho)^2 \iint_{Q_R(x_0)} g^2 dx dt \right),
\]

where \( C = C(n) \). Thus, we obtain (8.2.2). Then, by choosing \( \rho = R/2 \), we obtain (8.2.3) with \( m = 0 \).

**Lemma 6.** Under the hypotheses of Lemma 5 and for \( m \geq 0 \), if \( v \) is a solution to (8.2.1) in \( Q \), then for \( 2(m - |\nu|) > n \),

\[
\sup_{Q_{R/2}(x_0)} |D^\nu v|^2 \leq C \left( \frac{1}{R^{n+2|\nu|+2}} \iint_{Q_R(x_0)} |v|^2 dx dt \\
+ \sum_{i=0}^{m} \frac{1}{R^{n-2m+2|\nu|+2i-2}} \iint_{Q_R(x_0)} |D^{m-i}g|^2 dx dt \right),
\]

(8.2.4)

where \( \nu = (\nu_1, \nu_2, \cdots, \nu_n) \) is a multi-index, \( |\nu| = \sum_{i=1}^{n} \nu_i \) and \( C = C(m, n) \).

**Proof.** (8.2.4) directly follows from (8.2.3) by Sobolev’s embedding theorem.

**Corollary 1.** It follows from Lemma 6 that if \( g \equiv 0 \), then for \( 2(m - |\nu|) > n \),

\[
\sup_{Q_{R/2}(x_0)} |D^\nu v|^2 \leq C \frac{1}{R^{n+2|\nu|+2}} \iint_{Q_R(x_0)} |v|^2 dx dt,
\]

(8.2.5)

where \( C = C(m, n) \).

**Lemma 7.** \([95]\) Let \( Q_T := \mathbb{R}^n \times [0, T] \) and \( g \in C^{\beta, \beta/2}(Q_T) \) with \( \beta \in (0, 1) \). Then, there exists a unique function \( v \in C^{2+\beta, 1+\beta/2}_{x,t}(Q_T) \) such that \( v_t = \Delta v + g \) in \( Q_T \) and
v(\cdot, 0) = 0 \text{ in } \mathbb{R}^n. \text{ Moreover, there exists a positive constant } C = C(n, \beta) \text{ such that }

\begin{align*}
[v]_{2+\beta, 1+\beta/2; Q_T} + T^{-\beta/2} \left( \|D^2 v\|_{0; Q_T} + \|v_t\|_{0; Q_T} \right) + T^{-(1+\beta)/2} \|Dv\|_{0; Q_T} \\
+ T^{-(2+\beta)/2} \|v\|_{0; Q_T} & \leq C \left( T^{-\beta/2} \|g\|_{0; Q_T} + [g]_{\beta, \beta/2; Q_T} \right). \tag{8.2.6}
\end{align*}

### 8.3 Young Integral Inequality

For two paths \( W \) and \( Z \) defined on \([0, T]\), Young’s integral is defined as

\[
\int_0^T W_t dZ_t = \lim_{|\mathcal{P}| \to 0} \sum_{[\tau_1, \tau_2] \in \mathcal{P}} W_{\tau_1} (Z_{\tau_2} - Z_{\tau_1}),
\tag{8.3.1}
\]

where \( \mathcal{P} \) denotes a partition of \([0, T]\) and \(|\mathcal{P}|\) denotes the length of the largest element of \( \mathcal{P} \). \( C^\beta([0, T]; \mathcal{V}), 0 < \beta \leq 1 \), denotes the set of \( \beta \)-Hölder continuous paths that are defined on \([0, T]\) with values in \( \mathcal{V} \). For a path \( W \in C^\beta([0, T]; \mathcal{V}), 0 < \beta \leq 1 \), the usual \( \beta \)-Hölder semi-norm is defined as

\[
\|W\|_\beta := \sup_{s, t \in [0, T]} \frac{|W_t - W_s|}{|t - s|^{\beta}} < \infty.
\]

If equipped with the norm \( W \mapsto |W_0| + \|W\|_\beta, C^\beta([0, T]; \mathcal{V}) \) is a Banach space. The term \( |W_0| \) can be omitted for paths that start at the origin, i.e., we can work directly on \( \|\cdot\|_\beta \). Young’s integral was studied in detail by Young in \([160]\), where it was shown that the Riemann-Stieltjes sum \((8.3.1)\) converges if \( Z \in \mathcal{C}^{\beta_1} \) and \( W \in \mathcal{C}^{\beta_2} \) with \( \beta_1 + \beta_2 > 1 \).

In particular, this result is sharp because there exist sequences of smooth functions \( Z^n \to 0 \) and \( W^n \to 0 \) in \( C^{1/2}([0, T]; \mathbb{R}) \) such that \( \int_0^T W^n dZ^n \to \infty \). Specifically, we have the following lemma.

**Lemma 8.** \([160]\) If \( W \in \mathcal{C}^{\beta_1}([0, T]; \mathcal{V}) \) and \( Z \in \mathcal{C}^{\beta_2}([0, T]; \mathcal{V}) \), provided that \( \beta_1 + \beta_2 > 1 \), then as a consequence of Young’s inequality \([160]\), there exists a positive constant \( C \)
depending only on $\beta_1 + \beta_2$ such that for any $0 \leq s \leq t \leq T$,

\[ \left| \int_s^t W_\tau dZ_\tau - W_s(Z_t - Z_s) \right| \leq C \| W \|_{\beta_1} \| Z \|_{\beta_2} |t - s|^{{\beta_1} + {\beta_2}}. \quad (8.3.2) \]

In other words, we have the following equivalent results in terms of “p-variation”.

**Lemma 9.** Let $Z \in C^{p-\text{var}}([0,T];\mathbb{R}^n)$ and $W \in C^{q-\text{var}}([0,T];\mathbb{R}^{m \times n})$ with $p, q \geq 1$ and $\mu = \frac{1}{p} + \frac{1}{q} > 1$. Then, the following estimate holds: for $0 \leq s \leq t \leq T$,

\[ \left| \int_s^t W_\tau dZ_\tau - W_s(Z_t - Z_s) \right| \leq \frac{1}{1 - 2^{1-\mu}} \| Z \|_{p-\text{var};[s,t]} \| W \|_{q-\text{var};[s,t]}, \quad (8.3.3) \]

where $C^{p-\text{var}}([0,T];\mathbb{R}^n)$ and $\| \cdot \|_{p-\text{var};[s,t]}$ are as defined in Definition 11.
Chapter 9

Main Results

In this chapter, we present the proofs of the regularities for solutions to the separable case (II.1) and the general case (II.3) in Sections 9.1 and 9.2, respectively. Then, in Section 9.3 several applications are presented. The approach is mainly based on the perturbation argument presented in [158].

To state the results, we introduce the following notion of solutions.

Definition 12. $u$ is called a solution to (II.3) if

(i) for any $t \in (0, T)$, $u$ is twice strongly differentiable in $\mathbb{R}^n$, and

(ii) $u$ satisfies

$$u(x, t) - u(x, 0) = \int_0^t \Delta u(x, s) ds + \int_0^t f(x, s) d\xi_s, \quad \forall (x, t) \in Q_T = \mathbb{R}^n \times [0, T],$$

where the last integral is understood to be Young’s integral (see Section 8.3).

Clearly, $u$ is a solution to (II.1) if $u$ satisfies (i) in the above definition and

$$u(x, t) - u(x, 0) = \int_0^t \Delta u(x, s) ds + \mathcal{X}(x) (T(t) - T(0)), \quad \forall (x, t) \in Q_T.$$

The main results of Part II are the following global a priori estimates.

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Theorem 4. Let $u \in C^{2,0}_{x,t}(Q_T)$ be a solution to (II.1) in $Q_T$ with a zero initial condition, where $\mathcal{X} : \mathbb{R}^n \to \mathbb{R}$ is $\delta$-Hölder continuous with $\delta \in (0, 1)$, $\mathcal{T} : [0, T] \to \mathbb{R}$ is $\alpha$-Hölder continuous with $\alpha \in (1/2, 1]$, and $0 < \gamma := \delta + 2\alpha - 2 < 1$. Then,

$$
\|u\|_{2+\gamma,\gamma/2; Q_T} \leq C \|\mathcal{X}\|_{\delta; \mathbb{R}^n} [\mathcal{T}]_\alpha (1 + 2T^\alpha + T^{1+\gamma/2}),
$$

(9.0.1)

where $\|\cdot\|_{\delta; \mathbb{R}^n}$ is as defined in (8.1.1), $[\mathcal{T}]_\alpha$ is as defined in (II.2) and $C$ is a positive constant that depends only on $n$ and $\gamma$.

Theorem 5. Let $u \in C^{3,0}_{x,t}(Q_T)$ be a solution to (II.3) in $Q_T$ with a zero initial condition, where $f \in C^{1+\delta,\delta/2}_{x,t}(Q_T)$, $\delta \in (0, 1)$, $\xi$ is $\alpha$-Hölder continuous in $(0, T)$, $\alpha \in (1/2, 1]$, and $0 < \gamma := \delta + 2\alpha - 2 < 1$. Then,

$$
\|u\|_{3+\gamma,\gamma/2; Q_T} \leq C \|f\|_{1+\delta,\delta/2; Q_T} [\xi]_\alpha \left(1 + (T^\alpha + C_1T^{\gamma/2+1})(T^{\gamma/2} + 2)\right),
$$

(9.0.2)

where $\|\cdot\|_{1+\delta,\delta/2; Q_T}$ is as defined in (8.1.2), $[\xi]_\alpha$ is as defined in (II.4), $C = C(n, \gamma)$ and $C_1 = C_1(\delta + 2\alpha)$.

Thanks to the above a priori estimates, the solvability of (II.1) and (II.3) yields as follows.

Theorem 6. Assume that $\mathcal{X} : \mathbb{R}^n \to \mathbb{R}$ is $\delta$-Hölder continuous with $\delta \in (0, 1)$, $\mathcal{T} : [0, T] \to \mathbb{R}$ is $\alpha$-Hölder continuous with $\alpha \in (1/2, 1]$, and $0 < \gamma := \delta + 2\alpha - 2 < 1$. Then, equation (II.1) with a zero initial condition admits a unique solution in $C^{2+\gamma,\gamma/2}_{x,t}(Q_T)$.

Proof. Notice that the existence holds based on the classical theory [95] if $\mathcal{X}\mathcal{T} \in C^\infty_0(\mathbb{R}^{n+1})$. By using the standard mollifier on $\mathbb{R}^n$ (resp. $\mathbb{R}$), for any given $\mathcal{X} \in C^\delta(\mathbb{R}^n)$ (resp. $\mathcal{T} \in C^\alpha([0,T]))$, there exist a sequence of functions $\mathcal{X}_m \in C^\infty_0(\mathbb{R}^n)$ (resp. $\mathcal{T}_m \in C^\infty_0(\mathbb{R})$) such that $\mathcal{X}_m \to \mathcal{X}$ (resp. $\mathcal{T}_m \to \mathcal{T}$) as $m \to \infty$ at any point and that the $C^\delta(\mathbb{R}^n)$-norms of $\mathcal{X}_m$ (resp. the $C^\alpha(\mathbb{R})$-norms of $\mathcal{T}_m$) are uniformly bounded. Let $u_m$
be the corresponding solutions. Based on the a priori estimate \([9.0.1]\), the \(C^{2+\gamma/2}_{x,t}(Q_T)\)-norms of \(u_m\) are bounded. Then, a subsequence \(\{u_{m_k}\}\) exists and converges to a function \(u \in C^{2+\gamma/2}_{x,t}(Q_T)\) as \(k \to \infty\) which solves \((\text{II}.1)\). The existence is arrived.

The uniqueness is obtained easily from the estimate \([9.1.18]\) which is presented in the proof of Theorem 4.

**Theorem 7.** Assume that \(f \in C^{1+\delta/2}_{x,t}(Q_T)\) with \(\delta \in (0, 1)\), \(\xi\) is \(\alpha\)-Hölder continuous in \((0, T)\) with \(\alpha \in (1/2, 1]\), and \(0 < \gamma := \delta + 2\alpha - 2 < 1\). Then, equation \((\text{II}.3)\) with a zero initial condition admits a unique solution in \(C^{3+\gamma/2}_{x,t}(Q_T)\).

The proof of Theorem 7 is similar to that of Theorem 6 and is omitted here.

Our main results in Theorem 4 (resp. Theorem 5) exclude the case of \(\alpha = 1/2\) (namely, \(1/2\)-Hölder continuity in time). In the proof of Theorem 4 (resp. Theorem 5), \(1/2\)-Hölder continuity in time is found to be insufficient to obtain the estimates for \(D^2u\) (resp. \(D^3u\)) unless the inhomogeneous term admits a spatial Hölder continuity with an exponent greater than 1 (resp. 2).

We now present the proof of Theorem 4 and Theorem 5 in Section 9.1 and Section 9.2, respectively.

### 9.1 Proof of Theorem 4

#### 9.1.1 Auxiliary Results

In this subsection, we present several auxiliary estimates. Specifically, we study the following two problems:

\[
\begin{cases}
  v_t = \Delta v + g_1 & \text{in } Q_T, \\
  v = 0 & \text{on } \partial_p Q_T,
\end{cases}
\]

(9.1.1)
with
\[ g_1(x, t) = (\mathcal{X}(x) - \mathcal{X}(0)) \left( T(t) - T(-r^2) \right), \quad \forall (x, t) \in Q_r, \quad (9.1.2) \]
and
\[
\begin{cases}
  v_t = \Delta v + g_2 & \text{in } Q_T, \\
  v(\cdot, 0) = 0 & \text{in } \mathbb{R}^n,
\end{cases}
\quad (9.1.3)
\]
with
\[ g_2(x, t) = \mathcal{X}(x) \left( T(t) - T(0) \right), \quad \forall (x, t) \in Q_T, \quad (9.1.4) \]

where \( Q_r \) is as defined at the beginning of Section 8.2, \( \partial_p \) denotes the parabolic boundary, \( Q_T = \mathbb{R}^n \times [0, T] \) with \( T > 0 \), \( \mathcal{X} \in C^\delta(\mathbb{R}^n) \) with \( \delta \in (0, 1) \), and \( T \in C^\alpha(\mathbb{R}) \) with \( \alpha \in (1/2, 1] \).

**Proposition 4.** Under the above hypotheses, we have \( g_1 \in C^\delta,\delta/2_{x,t}(Q_r) \) and \( g_2 \in C^\delta,\delta/2_{x,t}(Q_T) \). To be specific, we have the following estimates:

\[
\|g_1\|_{0;Q_r} \leq [\mathcal{X}]_{\delta;\mathbb{R}^n}[T]_{\alpha;\mathbb{R}}r^{\delta+2\alpha},
\]

\[
[g_1]_{\delta,\delta/2;Q_r} \leq 2[\mathcal{X}]_{\delta;\mathbb{R}^n}[T]_{\alpha;\mathbb{R}}r^{2\alpha},
\]

\[
\|g_2\|_{0;Q_T} \leq \|\mathcal{X}\|_{0;\mathbb{R}^n}[T]_{\alpha;\mathbb{R}}T^\alpha,
\]

\[
[g_2]_{\delta,\delta/2;Q_T} \leq \|\mathcal{X}\|_{\delta;\mathbb{R}^n}[T]_{\alpha;\mathbb{R}}(T^\alpha + T^{\alpha-\delta/2}).
\]

We then have the following estimates for the first derivatives \( v_t \) of solutions \( v \) to (9.1.1) in the bounded domain \( Q_r \) and to (9.1.3) in the unbounded domain \( Q_T \).

**Proposition 5.** If \( v \in C^2_{x,t}(Q_r) \) is a solution to (9.1.1), then

\[
\|v_t\|_{0;Q_r} \leq C[\mathcal{X}]_{\delta;\mathbb{R}^n}[T]_{\alpha;\mathbb{R}}r^{\delta+2\alpha},
\]

where \( C = C(n) \).
Proof. Based on the global estimate for the inhomogeneous heat equation, we have

\[ \|v_t\|_{0;Q_r} \leq \|D^2v\|_{0;Q_r} + \|g_1\|_{0;Q_r} \]
\[ \leq r^\delta [D^2v]_{\delta,\delta/2;Q_r} + \frac{C}{r^2} \|v\|_{0;Q_r} + \|g_1\|_{0;Q_r} \]
\[ \leq C r^\delta \left( \frac{1}{r^{2+\delta}} \|v\|_{0;Q_r} + \frac{1}{r^2} \|g_1\|_{0;Q_r} + [g_1]_{\delta,\delta/2;Q_r} \right) + \frac{C}{r^2} \|v\|_{0;Q_r} + \|g_1\|_{0;Q_r} \]
\[ \leq C (\|g_1\|_{0;Q_r} + r^\delta [g_1]_{\delta,\delta/2;Q_r}) \]
\[ \leq C [\mathcal{X}]_{\delta;\mathbb{R}^n}[T]_\alpha \|T\|_{\alpha;\mathbb{R}^{\delta+2\alpha}} \]

with \( C = C(n) > 0 \), where we have used the equation that \( v \) satisfies in the first inequality, the interpolation inequality \([57]\) in the second inequality, global estimates for \( D^2v \) \([95]\) in the third inequality, and the maximum principle in the fourth inequality. We then use \([9.1.5]\).

Proposition 6. If \( v \in C^{2,1}_{x,t}(Q_T) \) is a solution to \([9.1.3]\), then

\[ \|v_t\|_{0;Q_T} \leq C \|\mathcal{X}\|_{\delta;\mathbb{R}^n}[T]_\alpha (2T^\alpha + T^{\alpha+\delta/2}), \] \hspace{1cm} (9.1.8)

where \( C = C(n, \delta) \).

Proof. According to Lemma \([7]\), we have

\[ \|v_t\|_{0;Q_T} \leq C (\|g_2\|_{0;Q_T} + T^{\delta/2}[g_2]_{\delta,\delta/2;Q_T}) \]
\[ \leq C (\|\mathcal{X}\|_{0;\mathbb{R}^n}[T]_\alpha T^\alpha + \|\mathcal{X}\|_{\delta;\mathbb{R}^n}[T]_\alpha (T^\alpha + T^{\alpha+\delta/2})) \]
\[ \leq C \|\mathcal{X}\|_{\delta;\mathbb{R}^n}[T]_\alpha (2T^\alpha + T^{\alpha+\delta/2}) \]

with \( C = C(n, \delta) \), where we use \([8.2.6]\) in the first inequality and then use \([9.1.6]\). \( \square \)
9.1.2 Interior Estimates for (II.1) in $Q_1$

In this subsection, we prove the a priori interior Hölder estimates for (II.1) in $Q_1 = B_1 \times I_1$. To be general, we assume that $X \in C^\delta(\mathbb{R}^n)$ with $\delta \in (0, 1)$, $T \in C^\alpha(\mathbb{R})$ with $\alpha \in (1/2, 1]$, and $0 < \gamma := \delta + 2\alpha - 2 < 1$.

**Lemma 10.** Let $u \in C_{x,t}^{2,0}$ be a solution to (II.1). Under the above assumptions, for any $X, Y \in Q_{1/4}$,

$$|D^2u(X) - D^2u(Y)| \leq C \left[ p \left( \|u\|_{0,Q_1} + \|X\|_{\delta}[T]_\alpha \right) + p^\gamma [X]_\delta [T]_\alpha \right], \quad (9.1.9)$$

where $C = C(n, \gamma)$ and $p = |X - Y|_p$ is the parabolic distance between $X$ and $Y$. Furthermore,

$$[D^2u]_{\gamma,\gamma/2;Q_{1/4}} \leq C \left( \|u\|_{0,Q_1} + \|X\|_{\delta}[T]_\alpha \right), \quad (9.1.10)$$

where $C = C(n, \gamma) > 0$.

Before the proof of Lemma 10 we present several remarks.

**Remark 14.** We identify three circumstances for Lemma 10.

1. If $\alpha = 1$, i.e., $T$ satisfies a Lipschitz condition, then $T$ is differentiable almost everywhere in $I_1$, and $T'$ is bounded. Then, for any $\delta \in (0, 1)$ and $X \in C^\delta(B_1)$, (9.1.10) yields

$$[D^2u]_{\delta,\delta/2;Q_{1/4}} \leq C \left( \|u\|_{0,Q_1} + \|X\|_{\delta}[T]_1 \right),$$

where $[T]_1$ is as defined in (II.2) and $C = C(n, \delta)$. This case corresponds to the heat equation whose inhomogeneous term is Hölder continuous in $x$ and bounded in $t$; see, e.g., [146] for details.

2. If $\delta = 1$, i.e., $X$ satisfies a Lipschitz condition, then from the proof of Lemma
for any $\alpha \in (1/2, 1)$ (excluding $\alpha = 1$), (9.1.10) also holds; namely,

$$[D^2 u]_{2 \alpha - 1, (2 \alpha - 1)/2; Q_{1/4}} \leq C \left( \|u\|_{0; Q_1} + \|\mathcal{X}\|_{Lips} [T]_{\alpha} \right),$$

where $C = C(n, \alpha)$ and

$$\|\mathcal{X}\|_{Lips} = \|\mathcal{X}\|_0 + \sup_{x \neq y} \frac{|\mathcal{X}(x) - \mathcal{X}(y)|}{|x - y|}.$$

(3) If $\delta = \alpha = 1$, then the following log-Lipschitz continuity for $D^2 u$ is obtained:

$$|D^2 u(X) - D^2 u(Y)| \leq C p \left( \|u\|_{0, Q_1} + \|\mathcal{X}\|_{Lips} [T]_1 (1 + |\ln p|) \right),$$

(9.1.11)

where $C = C(n)$ and $p = |X - Y|_p$. (9.1.11) can be easily obtained from the proof of Lemma 10 by considering $\delta = \alpha = 1$ in the estimates for $A_1, A_2$ and $A_3$. The details of proof for (9.1.11) are thus omitted here.

We are now ready to prove Lemma 10.

Proof of Lemma 10. Using the technique of mollification, we first suppose that $u, \mathcal{X}$ and $T$ are smooth and that $u$ is a solution to (II.1) in the classical sense. Without loss of generality, suppose that $Y = 0 = (0, 0)$ and that $X = X_0 = (x_0, t_0)$ is an arbitrary point near the origin $0$. Let $\rho = 1/2$ and

$$Q_j = Q_{\rho^j}(0, 0) = B_{\rho^j}(0) \times I_{\rho^j}(0) := B_j \times I_j, \quad j = 0, 1, 2, \ldots .$$

Suppose that $u^j$ is a solution to the following problem:

$$\begin{align*}
&dw^j = \Delta w^j dt + \mathcal{X}(0) dT \quad \text{in} \quad Q_j, \\
&u^j = u \quad \text{on} \quad \partial_p Q_j,
\end{align*}$$

\text{in} \quad Q_j.$
where $\partial_p$ denotes the parabolic boundary. Set
\[ v^j(x, t) = \int_{-\rho^2 j}^t (u(x, s) - u^j(x, s)) \, ds, \]
then
\[ v^j_t = u - u^j, \tag{9.1.12} \]
and $v^j$ satisfies
\[
\begin{cases}
    v^j_t = \Delta v^j + g^j & \text{in } Q_j, \\
    v^j = 0 & \text{on } \partial_p Q_j,
\end{cases}
\]
where
\[ g^j(x, t) = (\mathcal{X}(x) - \mathcal{X}(0)) \left( T(t) - T(-\rho^2 j) \right), \quad \forall (x, t) \in Q_j. \]

By (9.1.12), we have
\[ \|v^j_t\|_{0; Q_j} \leq C K H \rho^\alpha_j, \tag{9.1.13} \]
where $C = C(n) > 0$ and for simplicity, we adopt the notation $K = [T]_{a; \mathbb{R}}$ and $H = [\mathcal{X}]_{a; \mathbb{R}^n}$ in the proof. Then, according to (9.1.12) and (9.1.13), we have \[ \|u - u^j\|_{L^\infty(Q_j)} \leq C K H \rho^\alpha_j. \] Thus,
\[
\|u^j - u^{j+1}\|_{L^\infty(Q_{j+1})} \leq C K H \rho^\alpha_j.
\]

Note that $u^j - u^{j+1}$ satisfies $d(u^j - u^{j+1}) = \Delta (u^j - u^{j+1}) \, dt$ in $Q_{j+1}$, which is homogeneous; then, from the interior estimate for the homogeneous heat equation \[ [44], \]
we have
\[
\sup_{Q_{j+2}} |D^i (u^j - u^{j+1})| \leq \frac{C}{\rho^{(j+1)}_i} \|u^j - u^{j+1}\|_{L^\infty(Q_{j+1})} \leq C K H \rho^\alpha (j+1)^{-i}, \quad \forall i = 0, 1, 2, \ldots. \tag{9.1.14}
\]
For any $X_0 = (x_0, t_0)$ near the origin, there exists $k \in \mathbb{N}$ such that $\rho^{k+4} \leq |X_0|_p = |x_0| + \sqrt{|t_0|} \leq \rho^{k+3}$. Now, we estimate $|D^2 u(X_0) - D^2 u(0)|$. Denote

$$|D^2 u(X_0) - D^2 u(0)| \leq |D^2 u(X_0) - D^2 u^k(X_0)| + |D^2 u^k(X_0) - D^2 u^k(0)| + |D^2 u^k(0) - D^2 u(0)|$$

$$:= A_1 + A_2 + A_3.$$

**Step 1.** To estimate $A_3$, \([9.1.14]\) with $i = 2$ gives

$$\sup_{Q_{j+2}} |D^2(u^j - u^{j+1})| \leq CH \rho^{(\delta + 2\alpha - 2)j}.$$

Since $\gamma := \delta + 2\alpha - 2 \in (0, 1)$,

$$\sum_{j=k}^{\infty} \|D^2 (u^j - u^{j+1})\|_{L^\infty(Q_{j+2})} \leq CH \sum_{j=k}^{\infty} \rho^{\gamma j}$$

$$= CH \frac{\rho^{\gamma k}}{1 - \rho^{\gamma}},$$

which implies that $\{D^2 u_j(0)\}_{j=1}^{\infty}$ converges. We **claim** that the limit is $D^2 u(0)$. Then,

$$A_3 \leq \sum_{j=k}^{\infty} \|D^2 (u^j - u^{j+1})\|_{L^\infty(Q_{j+2})} \leq CH |X_0|^{\gamma},$$

where $C = C(n, \gamma)$.

**Step 2.** To estimate $A_2$, denote $h^j = u^j - u^{j-1}$, $j = 1, 2, \cdots$. We then have

$$A_2 \leq |D^2 u^{k-1}(X_0) - D^2 u^{k-1}(0)| + |D^2 h^k(X_0) - D^2 h^k(0)|$$

$$\leq |D^2 u^0(X_0) - D^2 u^0(0)| + \sum_{j=1}^{k} |D^2 h^j(X_0) - D^2 h^j(0)|$$

$$:= A_{21} + A_{22}.$$
For the term $A_{22}$, using (9.1.14) with $i = 3$,

$$
|D^2 h^j(X_0) - D^2 h^j(0)| \leq |X_0|_p \sup_{Q_{j+1}} |D^3 h^j|
$$

$$
\leq C K H \rho^{-\delta-2\alpha} |X_0|_p \rho^{(\delta+2\alpha-3)j};
$$

then,

$$
A_{22} \leq \frac{C K H}{\rho^{\delta+2\alpha}} |X_0|_p \sum_{j=1}^{k} \rho^{(\delta+2\alpha-3)j}
$$

$$
= \frac{C K H}{\rho^{3}} |X_0|_p \rho^{(\delta+2\alpha-3)k-1} \frac{1}{\rho^{\delta+2\alpha-3} - 1}
$$

$$
\leq C K H |X_0|_p^\gamma;
$$

where $C = C(n, \gamma)$.

For the term $A_{21}$, note that $D u^0$ satisfies the homogeneous heat equation in $Q_0$; by (8.2.5), we have

$$
A_{21} = |D^2 u^0(X_0) - D^2 u^0(0)|
$$

$$
\leq |X_0|_p \sup_{Q_2} |D^3 u^0|
$$

$$
\leq C |X_0|_p \|D u^0\|_{L^2(Q_1)}.
$$

Denote $u^0 = u_1^0 + u_2^0$, where

$$
\begin{cases}
  du_1^0 = \Delta u_1^0 dt & \text{in } Q_0, \\
  u_1^0 = u & \text{on } \partial \rho Q_0,
\end{cases}
$$
and

\[
\begin{align*}
\begin{cases}
    du_2^0 = \Delta u_2^0 dt + \mathcal{X}(0)dT & \text{in } Q_0, \\
    u_2^0 = 0 & \text{on } \partial_p Q_0.
\end{cases}
\end{align*}
\]

Then, based on (8.2.5) and the maximum principle,

\[
\|Du_2^0\|_{L^2(Q_1)} \leq C\|u_2^0\|_{L^2(Q_0)} \leq C\|u\|_{L^\infty(Q_0)}.
\]

For \( u_2^0 \), we consider \( w = \int_{t-1}^t u_2^0(x,s)ds \), then \( w_t = u_2^0 \) and

\[
\begin{align*}
\begin{cases}
    w_t = \Delta w + \mathcal{X}(0)(\mathcal{T}(t) - \mathcal{T}(-1)) & \text{in } Q_0, \\
    w = 0 & \text{on } \partial_p Q_0,
\end{cases}
\end{align*}
\]

Hence, by (8.2.3),

\[
\begin{align*}
\|Du_2^0\|_{L^2(Q_1)} & \leq \|Dw_t\|_{L^2(Q_1)} \leq \|D^3w\|_{L^2(Q_1)} \\
& \leq C \left( \|w\|_{L^2(Q_0)}^2 + \|\mathcal{X}(0)(\mathcal{T}(t) - \mathcal{T}(-1))\|_{L^2(Q_0)}^2 \right)^{1/2} \\
& \leq C \|\mathcal{X}(0)(\mathcal{T}(t) - \mathcal{T}(-1))\|_{L^\infty(Q_0)} \\
& \leq C K \|\mathcal{X}\|_{L^\infty(B_0)}.
\end{align*}
\]

We now have

\[
A_{21} \leq C|X_0|_{p} \left( \|u\|_{L^\infty(Q_1)} + K\|\mathcal{X}\|_{L^\infty(B_1)} \right),
\]

where \( C = C(n) \).

**Step 3.** For \( A_1 \), we similarly define

\[
\begin{align*}
\begin{cases}
    dw^j = \Delta w^j dt + \mathcal{X}(x_0)dT & \text{in } Q_j(X_0), \\
    w^j = u & \text{on } \partial_p Q_j(X_0),
\end{cases}
\end{align*}
\]
where

\[ Q_j(X_0) = Q_{\rho^j}(x_0, t_0) = B_{\rho^j}(x_0) \times I_{\rho^j}(t_0) := B_j(x_0) \times I_j(t_0), \quad j = 0, 1, 2, \cdots; \]

then,

\[ A_1 \leq |D^2u(X_0) - D^2w^k(X_0)| + |D^2w^k(X_0) - D^2u^k(X_0)| \]

\[ := A_{11} + A_{12}. \]

For \( A_{11} \), similar to the case for \( A_3 \), we have

\[ A_{11} \leq \sum_{j=k}^{\infty} \|D^2 (w^j - w^{j+1})\|_{L^\infty(Q_{j+2}(X_0))} \leq CKH|X_0|^\gamma, \]

where \( C = C(n, \gamma) \). For \( A_{12} \), \( w^k - u^k \) satisfies

\[
\begin{cases}
  d(w^k - u^k) = \Delta(w^k - u^k)dt + [\mathcal{X}(x_0) - \mathcal{X}(0)]d\mathcal{T} & \text{in} \quad Q_k \cap Q_k(X_0), \\
  w^k - u^k = u - u^k & \text{on} \quad \partial_p(Q_k \cap Q_k(X_0)) \cap Q_k, \\
  w^k - u^k = u^k - u & \text{on} \quad \partial_p(Q_k \cap Q_k(X_0)) \cap Q_k(X_0).
\end{cases}
\]

We write \( w^k - u^k := U_1^k + U_2^k \), where

\[
\begin{cases}
  dU_1^k = \Delta U_1^k dt & \text{in} \quad Q_k \cap Q_k(X_0), \\
  U_1^k = u - u^k & \text{on} \quad \partial_p(Q_k \cap Q_k(X_0)) \cap Q_k, \\
  U_1^k = w^k - u & \text{on} \quad \partial_p(Q_k \cap Q_k(X_0)) \cap Q_k(X_0),
\end{cases}
\]

and

\[
\begin{cases}
  dU_2^k = \Delta U_2^k dt + [\mathcal{X}(x_0) - \mathcal{X}(0)]d\mathcal{T} & \text{in} \quad Q_k \cap Q_k(X_0), \\
  U_2^k = 0 & \text{on} \quad \partial_p(Q_k \cap Q_k(X_0)).
\end{cases}
\]
Then,
\[ A_{12} \leq \sup_{Q_{k+2}(x_0/2,t_0)} \| D^2 U^k_1 \| + \sup_{Q_{k+2}(x_0/2,t_0)} \| D^2 U^k_2 \|. \]  
(9.1.15)

The first term on the right-hand side of (9.1.15) is dominated by
\[
\sup_{Q_{k+2}(x_0/2,t_0)} \| D^2 U^k_1 \| \leq C \rho^{2(k+1)} \sup_{Q_{k+1}(x_0/2,t_0)} \| U^k_t \| \]
\[
\leq C \rho^{2(k+1)} \max \left\{ \sup_{\partial_p(Q_k \cap Q_k(x_0)_c \cap Q_k)} \| u - u^k \|, \sup_{\partial_p(Q_k \cap Q_k(x_0)_c \cap Q_k(x_0))} \| w^k - u \| \right\} \]
\[
\leq CKH \rho^{(\delta+2\alpha-2)k}.
\]

For the second term, we consider
\[ U^k = \int_{-\rho^2k}^t U^k_2(x,s)ds; \]
then, \[ U^k_t = U^k_2 \]
and
\[
\begin{aligned}
U^k_t &= \Delta U^k + \bar{g}^k \quad \text{in} \quad Q_k \cap Q_k(x_0), \\
U^k &= 0 \quad \text{on} \quad \partial_p(Q_k \cap Q_k(x_0)),
\end{aligned}
\]
where \( \bar{g}^k = (X(x_0) - X(0)) \left( T(t) - T(-\rho^2k) \right). \) Hence,
\[
\sup_{Q_{k+2}(x_0/2,t_0)} \| D^2 U^k_2 \|
\leq \sup_{Q_{k+2}(x_0/2,t_0)} \| D^2 U^k_t \|
\leq \sup_{Q_{k+2}(x_0/2,t_0)} \| D^4 U^k \|
\leq C \left( \frac{1}{\rho^{(n+10)(k+1)}} \| U^k \|_{L^2(Q_{k+1}(x_0/2,t_0))}^2 + \frac{1}{\rho^{(k+1)(n+6)}} \| g^k \|_{L^2(Q_{k+1}(x_0/2,t_0))}^2 \right)^{1/2}
\leq C \rho^{2(k+1)} \| g^k \|_{L^\infty(Q_{k+1}(x_0/2,t_0))}
\leq CKH \rho^{(\delta+2\alpha-2)k},
\]
where (8.2.4) is used in the third inequality. Thus, we have \( A_{12} \leq CKH \rho^{(\delta+2\alpha-2)k}. \) As
a result, 

\[ A_1 \leq CKH|X_0|^p. \]

**Step 4.** We now prove the remaining claim 

\[ \lim_{j \to \infty} D^2 u^j(0) = D^2 u(0). \]

Based on the interior estimate from Lemma 6, we have 

\[
|D^2 u(0) - D^2 u^j(0)| \leq \left\| D^2 u^j \right\|_{L^\infty(Q_{j+1})} \\
\leq \left\| D^4 v^j \right\|_{L^\infty(Q_{j+1})} + \left\| D^2 g^j \right\|_{L^\infty(Q_{j+1})} \\
\leq C \left( \frac{2\omega_n}{\rho^4} \left\| (\mathcal{X} - \mathcal{X}(0))(T - T(-\rho^2)) \right\|_{L^\infty(Q_j)}^2 \\
+ \sum_{i=0}^{m-1} \frac{\omega_n}{\rho(-2m+2i+4)j} \left\| D^{m-i} \mathcal{X} \cdot (T - T(-\rho^2)) \right\|_{L^\infty(Q_j)}^2 \right)^{1/2} \\
+ \left\| D^2 \mathcal{X} \cdot (T - T(-\rho^2)) \right\|_{L^\infty(Q_j)} \\
:= C(M_1 + \sum_{i=0}^{m-1} M_2^i)^{1/2} + M_3, \quad (9.1.16)
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \) and [8.2.4] is used in the third inequality. We then have 

\[
M_1 \leq \frac{2\omega_n}{\rho^{4j}} \sup_{Q_j} (H|x|^\delta K \rho^{2\alpha j})^2 \\
\leq C \rho^{2(\delta+2\alpha-2)j} \to 0, \quad j \to \infty,
\]

\[
M_2^i \leq \frac{\omega_n}{\rho(-2m+2i+4)j} \left\| D^{m-i} \mathcal{X} \right\|_{L^\infty(B_j)}^2 K^2 \rho^{4\alpha j} \\
\leq C \rho^{2(m-2i-4+4\alpha)j} \to 0, \quad j \to \infty, \quad \forall i = 0, 1, \cdots, m-1,
\]
and

\[ M_3 \leq C \rho^{2\alpha j} \to 0, \quad j \to \infty. \]

From the assumption that \( X \) is smooth, it is obvious that all constants \( C \) are independent of \( j \), and the right-hand side of (9.1.16) tends to 0 when \( j \to \infty \).

By combining \( A_1 \), \( A_2 \) and \( A_3 \), we obtain

\[ |D^2 u(X_0) - D^2 u(0)| \leq C \left[ |X_0|_p \left( \|u\|_{\partial Q_1} + \|\mathcal{X}\|_{\partial [T]_\alpha} \right) + |X_0|_p [\mathcal{X}]_{\delta [T]_\alpha} \right], \]

where \( \gamma = \delta + 2\alpha - 2 \in (0, 1) \) and \( C = C(n, \gamma) > 0 \). Thus, we complete the proof of (9.1.9).

(9.1.10) follows immediately from (9.1.9), and thus, we complete the proof of Lemma 10.

Let \( Q_T = \mathbb{R}^n \times [0, T] \) and \( Q_r = B_r \times [0, T] \) for \( r, T > 0 \). The following corollary is obtained immediately from Lemma 10.

**Corollary 2.** Let \( u \in C^{2,0}_{x,t}(Q_T) \) be a solution to (II.1). Suppose that \( u \) and \( T \) vanish when \( t \leq 0 \), \( \mathcal{X} \in C^\delta(\mathbb{R}^n) \), \( \delta \in (0, 1) \), \( T \in C^\alpha(\mathbb{R}) \), \( \alpha \in (1/2, 1] \), and \( 0 < \gamma := \delta + 2\alpha - 2 < 1 \). Then, there exists a positive constant \( C = C(n, \gamma) \) such that

\[ [D^2 u]_{\gamma, \gamma/2; Q_T} \leq C \left( \|u\|_{0; Q_T} + \|\mathcal{X}\|_{\delta; \mathbb{R}^n [T]_\alpha} \right). \]  

**Proof.** It follows from (9.1.10) that for any \( X = (x, t) \in \mathbb{R}^{n+1} \),

\[ [D^2 u]_{\gamma, \gamma/2; Q_{1/4}(X)} \leq C \left( \|u\|_{0; Q_{1/4}(X)} + \|\mathcal{X}\|_{\delta [T]_\alpha} \right). \]
Setting \( x = 0 \) and letting \( t \) run over \([0, T]\), we obtain

\[
[D^2u]_{\gamma/2; Q_T^{1/4}} \leq C \left( \|u\|_{0; Q_T^1} + \|\mathcal{X}\|_{\delta[T]_\alpha} \right).
\]

We now move the centre of the domain to any point \( x \in \mathbb{R}^n \), from which (9.1.17) follows.

\[ \square \]

### 9.1.3 Global Estimates for (II.1) in \( Q_T \)

We are now ready to prove Theorem 4.

**Proof of Theorem 4.** Let \( u \in C^{2,0}_{x,t}(Q_T) \) be a solution to (II.1) with \( u(\cdot, 0) = 0 \) in \( \mathbb{R}^n \).

Set \( v(x, t) = \int_0^t u(x, s) ds \), \( \forall (x, t) \in Q_T \). Then, \( v_t = u \), and

\[
\begin{cases}
  v_t = \Delta v + \mathcal{X}(T - T(0)) & \text{in } Q_T, \\
  v(\cdot, 0) = 0 & \text{in } \mathbb{R}^n.
\end{cases}
\]

According to (9.1.8), we have

\[
\|u\|_{0; Q_T} \leq C\|\mathcal{X}\|_{\delta[T]_\alpha}(2T^\alpha + T^{\alpha+\delta/2}). \tag{9.1.18}
\]

Thus, with (9.1.17) and (9.1.18), we complete the proof of (9.0.1). \[ \square \]

**Remark 15.** If \( 1 < \delta + 2\alpha < 2 \), we can obtain a priori estimates only for \( Du \).

### 9.2 Proof of Theorem 5

In this section, we discuss the general case (II.3) in which \( f \) depends on space and time. The main tool is the Young integral inequality. However, directly utilising the Young integral inequality results in a loss of Hölder continuity in space. Therefore, we
require that \( f \) is differentiable with respect to \( x \) and that \( Df \) is in \( C_{x,t}^{\delta,\delta/2} \). Therefore, the results for (II.3) are not as good as those for (II.1). The proof of Theorem 5 is similar to that of Theorem 4.

### 9.2.1 Auxiliary Results

We now present several auxiliary estimates for the heat equation whose inhomogeneous term is in the form of a Young integral. Specifically, we study the following two problems:

\[
\begin{cases}
 v_t = \Delta v + g_1 & \text{in } Q_r, \\
 v = 0 & \text{on } \partial_p Q_r,
\end{cases}
\]

(9.2.1)

with

\[ g_1(x,t) = \int_{-r^2}^t [f(x,\tau) - f(0,\tau) - Df(0,\tau) \cdot x] d\xi, \quad \forall \ (x,t) \in Q_r, \]

and

\[
\begin{cases}
 v_t = \Delta v + g_2 & \text{in } Q_T, \\
 v(\cdot,0) = 0 & \text{in } \mathbb{R}^n,
\end{cases}
\]

(9.2.2)

with

\[ g_2(x,t) = \int_0^t f(x,\tau) d\xi, \quad \forall \ (x,t) \in Q_T, \]

where \( f \in C_{x,t}^{1+\delta/2}(\mathbb{R}^n \times \mathbb{R}) \) with \( \delta \in (0,1) \), \( \xi \) is \( \alpha \)-Hölder continuous in \( \mathbb{R} \) with \( \alpha \in (1/2,1] \), \( \partial_p \) denotes the parabolic boundary, and \( Q_T = \mathbb{R}^n \times [0,T] \) with \( T > 0 \).

**Proposition 7.** Under the above hypotheses, we have \( g_1 \in C_{x,t}^{\delta,\delta/2}(Q_r) \) and \( g_2 \in C_{x,t}^{\delta,\delta/2}(Q_T) \) for any \( \delta \in (0,1) \).

**Proof.** The Hölder continuity of \( g_i \) (\( i = 1,2 \)) is easily obtained from the Young integral inequality; we present only the following proof for \( g_1 \). For any \( x \in B_r \) and \( s,t \in I_r \), we
have

\[
|g_1(x, t) - g_1(x, s)| = \left| \int_s^t (Df(\hat{x}, \tau) - Df(0, \tau)) \, d\xi \cdot x \right|
\]

\[
\leq |Df(\hat{x}, s) - Df(0, s)| |\xi_t - \xi_s||x| + C_1 \sup_x |(Df(\hat{x}, \cdot) - Df(0, \cdot))|_{\delta/2} |x| |\xi| \cdot t - s|^{\delta/2 + \alpha}
\]

\[
\leq \|Df\|_{\delta, \delta/2} |\hat{\xi}|_{\alpha} |t - s|^\alpha |x| + 2C_1 \|Df\|_{\delta, \delta/2} r |\xi|_{\alpha} |t - s|^{\delta/2 + \alpha}
\]

\[
\leq (1 + 2C_1) \|Df\|_{\delta, \delta/2} |\xi|_{\alpha} r^{1+\delta} |t - s|^\alpha,
\]

where we use the mean value theorem in the first equality and then use the Young integral inequality, and \(C_1 = C_1(\delta/2 + \alpha) > 0\) as in Lemma 8. Similarly,

\[
|g_1(x, t) - g_1(y, t)| \leq (1 + 2C_1) \|Df\|_{\delta, \delta/2} |\xi|_{\alpha} r^{1+\delta} |t - s|^\alpha,
\]

\(\forall x, y \in B_r, \ t \in I_r.\)

Then, for any \(\beta \in (0, 1),\)

\[
[g_1]_{\beta, \beta/2; Q_r} \leq 3(1 + 2C_1) \|Df\|_{\delta, \delta/2} |\xi|_{\alpha} r^{1+\delta+2\alpha-\beta},
\]

thus, \(g_1 \in C^{\beta, \beta/2}_{x,t}(Q_r).\) Similarly, for \(g_2,\) for any \(\beta \in (0, 1),\) we have

\[
[g_2]_{\beta, \beta/2; Q_T} \leq \|f\|_{1+\delta, \delta/2} |\xi|_{\alpha} (1 + C_1 T^{\delta/2} + T^{\alpha-\beta/2}),
\]

where \(C_1 = C_1(\delta + 2\alpha).\)

We now have the following estimates for the first derivatives \(v_t\) of solutions \(v\) to (9.2.1) in the bounded domain \(Q_r\) and to (9.2.2) in the unbounded domain \(Q_T.\) The proofs are similar to those for Proposition 5 and Proposition 6 and are omitted here.

**Proposition 8.** If \(v \in C^{2,1}_{x,t}(Q_r)\) is a solution to (9.2.1), then

\[
\|v_t\|_{0; Q_r} \leq C \|Df\|_{\delta, \delta/2} |\xi|_{\alpha} r^{1+\delta+2\alpha},
\]

(9.2.3)
where \( C = C(n, \delta + 2\alpha) > 0 \).

**Proposition 9.** If \( v \in C^{2,1}_{x,t}(Q_T) \) is a solution to \([9.2.2]\), then for any \( \beta \in (0, 1) \),

\[
\|v_t\|_{0;Q_T} \leq C \left( \|f\|_{1+\delta,\delta/2}[\xi] \|f\|_{1+T^{\delta/2}}(T^{\alpha+\beta/2} + 2T^\alpha) \right),
\]

(9.2.4)

where \( C = C(n, \beta) \) and \( C_1 = C_1(\delta + 2\alpha) \).

### 9.2.2 Interior Estimates for (II.3) in \( Q_1 \)

In this subsection, we prove the a priori interior Hölder estimates for (II.3) in \( Q_1 = B_1 \times I_1 \). To be general, we assume that \( f \in C^{1+\delta,\delta/2}(\mathbb{R}^n \times \mathbb{R}) \) with \( \delta \in (0, 1) \), \( \xi \) is \( \alpha \)-Hölder continuous in \( \mathbb{R} \) with \( \alpha \in (1/2, 1] \), and \( 0 < \gamma := \delta + 2\alpha - 2 < 1 \).

**Lemma 11.** Let \( u \in C^{3,0}_{x,t}(Q_1) \) be a solution to (II.3). Under the above assumptions, we have

\[
[D^3u]_{\gamma/2:Q_1/4} \leq C \left( \|u\|_{0;Q_1} + [\xi]_\alpha \|f\|_{1+\delta,\delta/2} \right),
\]

(9.2.5)

where \( C = C(n, \gamma) > 0 \).

We note two circumstances before presenting the proof of Lemma 11.

**Remark 16.** (1) If \( \delta = 1 \), i.e., \( Df \) satisfies a Lipschitz condition with respect to \( x \) and is \( 1/2 \)-Hölder continuous with respect to \( t \), then from a step-by-step check of the proof of Lemma 11, it is clear that for any \( \alpha \in (1/2, 1) \) (excluding \( \alpha = 1 \), \( 9.2.5 \) also holds; namely,

\[
[D^3u]_{2\alpha-1,(2\alpha-1)/2:Q_1/4} \leq C \left( \|u\|_{0;Q_1} + [\xi]_\alpha \|f\|_{1+Lips,1/2} \right),
\]

where \( C = C(n, \alpha) \), \( [\xi]_1 \) is as defined in (II.4), and the following notation is adopted:

\[
\|f\|_{1+Lips,1/2;Q_1} := \|f\|_{0;Q_1} + \|Df\|_{0;Q_1} + \sup_{X \neq Y \in Q_1} \frac{|Df(X) - Df(Y)|}{|X - Y|_p}.
\]

(9.2.6)
(2) If $\delta = \alpha = 1$, then the following log-Lipschitz continuity for $D^3u$ is obtained:

$$|D^3u(X) - D^3u(Y)| \leq C p \left( \|u\|_{0,Q_1} + \|f\|_{1+\text{Lips},1/2,Q_1} [\xi]_1 (1 + |\ln p|) \right),$$

where $C = C(n)$, $p = |X - Y|_p$, and the notation $\|f\|_{1+\text{Lips},1/2,Q_1}$ is as defined in (9.2.6).

We now present the proof of Lemma 11. Because this proof is similar to that of Lemma 10, we omit some redundant procedures.

**Proof of Lemma 11.** Using the technique of mollification, we first suppose that $u$, $f$ and $\xi$ are smooth and that $u$ is a solution to $(\text{II.3})$ in the classical sense. Without loss of generality, suppose that $X_0 = (0,0)$ is an arbitrary point near the origin $0$. Let $\rho = 1/2$ and $Q_j = Q_{\rho j}(0,0) = B_{\rho j}(0) \times I_{\rho j}(0) := B_j \times I_j, j = 0,1,2,\cdots$. Suppose that $u^j$ is a solution to the following problem:

$$dw^j = \Delta u^j dt + (f(0,t) + Df(0,t) \cdot x) d\xi t \text{ in } Q_j,$$

$$u^j = u \text{ on } \partial_p Q_j,$$

where $\partial_p$ denotes the parabolic boundary. Set $v^j(x,t) := \int_{-\rho^2}^t (u(x,s) - u^j(x,s)) ds$. Then,

$$v_t^j = u^j - u^j$$

and $v^j$ satisfies

$$\begin{cases}
  v_t^j = \Delta v^j + g^j \text{ in } Q_j, \\
  v^j = 0 \text{ on } \partial_p Q_j,
\end{cases}$$

where

$$g^j(x,t) = \int_{-\rho^2}^t (f(x,s) - f(0,s) - Df(0,s) \cdot x) d\xi s, \quad \forall (x,t) \in Q_j.$$
Based on (9.2.3), we have

$$\| v_j \|_{0, Q_j} \leq C \| Df \|_{\delta, \delta/2} [\xi]_\alpha \rho^{(1+\delta+2\alpha)j},$$

(9.2.8)

where $C = C(n, \delta + 2\alpha) > 0$. Then, according to (9.2.7) and (9.2.8), we have

$$\| u - u^j \|_{L^\infty(Q_j)} \leq CH[\xi]_\alpha \rho^{(1+\delta+2\alpha)j},$$

where we write $H = \| Df \|_{\delta, \delta/2}$ for simplicity and $C = C(n, \delta + 2\alpha)$. Then,

$$\| u^j - u^{j+1} \|_{L^\infty(Q_{j+1})} \leq CH[\xi]_\alpha \rho^{(1+\delta+2\alpha)j}.$$

From the interior estimate for the homogeneous heat equation \[44\], we have

$$\sup_{Q_{j+2}} |D^i(u^j - u^{j+1})| \leq \frac{C}{\rho^{(j+1)}} \| u^j - u^{j+1} \|_{L^\infty(Q_{j+1})}$$

$$\leq CH[\xi]_\alpha \rho^{(1+\delta+2\alpha-\delta-i)j-i}, \quad \forall \ i = 0, 1, 2, \ldots.$$  

(9.2.9)

Then, for $X_0 = (x_0, t_0)$ near the origin, there exists $k \in \mathbb{N}$ such that $\rho^{k+4} \leq |X_0|_p \leq \rho^{k+3}$. We now estimate $|D^3 u(X_0) - D^3 u(0)|$. Denote

$$|D^3 u(X_0) - D^3 u(0)| \leq |D^3 u(X_0) - D^3 u^k(X_0)| + |D^3 u^k(X_0) - D^3 u^k(0)|$$

$$+ |D^3 u^k(0) - D^3 u(0)|$$

$$:= A_1 + A_2 + A_3.$$
To estimate $A_3$, using (9.2.9) with $i = 3$, we obtain

$$
\sum_{j=k}^{\infty} \| D^3 (u^j - u^{j+1}) \|_{L^\infty(Q_{j+2})} \leq CH[\xi]_{\alpha} \sum_{j=k}^{\infty} \rho^{(\delta+2\alpha-2)j} \\
= CH[\xi]_{\alpha} \rho^{(\delta+2\alpha-2)k} \frac{1}{1 - \rho^{\delta+2\alpha-2}},
$$

which implies that $\{D^3 u^j(0)\}_{j=1}^{\infty}$ converges. It is sufficient to show that $D^3 u^j(0) \to D^3 u(0)$ as $j \to \infty$, which can be easily obtained from the estimates for $\| D^3 u_l^j \|_{L^\infty(Q_{j+1})}$.

Then, we have

$$A_3 \leq \sum_{j=k}^{\infty} \| D^3 (u^j - u^{j+1}) \|_{L^\infty(Q_{j+2})} \leq CH[\xi]_{\alpha} \| X_0 \|_{p}^\gamma,$$

where $\gamma = \delta + 2\alpha - 2$ and $C = C(n, \gamma)$. Similarly, we can obtain the estimate for $A_1$ by shifting the centre of the domains. The method is similar to the Step 3 of the proof for Lemma 10 and is omitted here. To estimate $A_2$, denote $h^j = u^j - u^{j-1}$, $j = 1, 2, \cdots$; then,

$$A_2 \leq \| D^3 u^0(X_0) - D^3 u^0(0) \| + \sum_{j=1}^{k} \| D^3 h^j(X_0) - D^3 h^j(0) \| \\
:= A_{21} + A_{22}.$$

Similar to the proof of Lemma 10, for the term $A_{22}$, we have $A_{22} \leq CH[\xi]_{\alpha} \| X_0 \|_{p}$, where $C = C(n, \gamma)$. For the term $A_{21}$, we have

$$A_{21} \leq C \| X_0 \|_{p} \left( \| u \|_{L^\infty(Q_{0})} + [\xi]_{\alpha} \| f \|_{1+\delta,\delta/2} \right),$$
where \( C = C(n, \gamma) > 0 \). By combining \( A_1, A_2 \) and \( A_3 \), we obtain

\[
|D^3u(X_0) - D^3u(0)| \leq C \left[ |X_0|_p \left( \|u\|_{0; Q_1} + \|f\|_{1+\delta, \delta/2} \right) + \|Df\|_{\delta, \delta/2} \right].
\]

where \( C = C(n, \gamma) > 0 \). From this, (9.2.5) easily follows; thus, we complete the proof of Lemma 11.

Let \( Q_T = \mathbb{R}^n \times [0, T] \) and \( Q^r_T = B_r \times [0, T] \) for \( r, T > 0 \). The following corollary is obtained immediately from Lemma 11.

**Corollary 3.** Let \( u \in C_{x,t}^{3,0}(Q_T) \) be a solution to \((\text{II.3})\). Suppose that \( u \) and \( f \) vanish when \( t \leq 0 \), \( f \in C_{x,t}^{1+\delta, \delta/2}(\mathbb{R}^n \times \mathbb{R}) \), \( \delta \in (0, 1) \), \( \xi \) is \( \alpha \)-Hölder continuous in \( \mathbb{R} \) with \( \alpha \in (1/2, 1] \), and \( 0 < \gamma := \delta + 2\alpha - 2 < 1 \). Then, there exists a positive constant \( C = C(n, \gamma) \) such that

\[
|D^3u|_{\gamma, \gamma/2; Q_T} \leq C \left( \|u\|_{0; Q_T} + [\xi]_\alpha \|f\|_{1+\delta, \delta/2; Q_T} \right).
\]

### 9.2.3 Global Estimates for \((\text{II.3})\) in \( Q_T \)

We are now ready to prove Theorem 5.

**Proof of Theorem 5.** Let \( u \in C_{x,t}^{3,0}(Q_T) \) be a solution to \((\text{II.3})\) with \( u(\cdot, 0) = 0 \) in \( \mathbb{R}^n \). Set \( v(x,t) = \int_0^t u(x,s)ds \), \( \forall (x,t) \in Q_T \); then, \( v_t = u \) and

\[
\begin{cases}
  v_t = \Delta v + \int_0^t f d\xi_s & \text{in } \mathbb{R}^n \times [0, T], \\
  v(\cdot, 0) = 0 & \text{in } \mathbb{R}^n.
\end{cases}
\]

According to (9.2.4), we have

\[
\|u\|_{0; Q_T} \leq C \left( \|f\|_{1+\delta, \delta/2} \right) \left( 1 + C_1 T^{\delta/2} (T^{\alpha+\beta/2} + 2T^\alpha) \right), \quad \forall \beta \in (0, 1).
\]
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Select \( \beta = \gamma = \delta + 2\alpha - 2; \) then,

\[
\|u\|_{0;Q_T} \leq C \left( \|f\|_{1+\delta,\delta/2}[\xi]_\alpha (T^{\alpha} + C_1 T^{\gamma/2 + 1})(T^{\gamma/2} + 2) \right).
\]

Hence, (9.0.2) is obtained.

\[\square\]

9.3 Applications

This section presents two applications of the main results of Part II.

9.3.1 Application to the Irregular Case (6.6.1)

Here, we determine the regularity of solutions to (6.6.1). For the irregular heat equation (6.6.1), if \( F \) is \( \alpha \)-Hölder continuous in \( t \) uniformly in \( x \), i.e.,

\[
\sup_{x \in \mathbb{R}^n} \sup_{t \neq s} \frac{|F(x,t) - F(x,s)|}{|t - s|^\alpha} \leq M, \quad \text{(9.3.1)}
\]

and

\[
|F(x,t) - F(y,t) - F(x,s) + F(y,s)| \leq M|x - y|^\delta|t - s|^\alpha \quad \text{(9.3.2)}
\]

for any \( x, y \in \mathbb{R}^n \) and \( t, s \in (0,T) \), where \( M \) is a positive constant, \( \delta \in (0,1) \), \( \alpha \in (1/2,1] \), and \( 0 < \gamma := \delta + 2\alpha - 2 < 1 \), then

\[
\|u\|_{2+\gamma,\gamma/2;Q_T} \leq C(1 + 2T^\alpha + T^{1+\gamma/2}),
\]

where \( C \) is a positive constant that depends on \( M, n \) and \( \gamma \).

We note that (9.3.2) is the essential assumption, which is obviously satisfied in the separable case. Interestingly, condition (9.3.2) is similar to the condition used by Hu and Le [78] and Hinz [75] for the increment of a two-parameter fractional Brownian
field along a rectangle. For the general case (II.3), it is difficult to directly derive a condition such as (9.3.2) because the use of the Young integral inequality results in a loss of Hölder continuity in space.

9.3.2 Application to Fractional Brownian Motions

As an application, consider the following heat equation:

\[ du = \Delta u dt + dB \] (9.3.3)

with a zero initial condition over the entire space \( \Theta \times \mathbb{R}^n \times \mathbb{R}^+ \), where \( \mathcal{B} = \mathcal{B}^{p,q} = \{ \mathcal{B}^{p,q}(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}^+ \} \) denotes a fractional Brownian sheet defined on a given probability space \((\Theta, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})\) with Hurst parameters of \( 0 < p, q < 1 \). Specifically, as defined in [75], \( \mathcal{B} \) is a random field on \( \Theta \times \mathbb{R}^n \times \mathbb{R}^+ \) to \( \mathbb{R} \) such that \( \mathcal{B} \) is a centred Gaussian process with the covariance function

\[ \mathbb{E} [\mathcal{B}^{p,q}(x,t)\mathcal{B}^{p,q}(y,s)] = \frac{1}{2} (|x|^{2p} + |y|^{2p} - |x-y|^{2p}) \frac{1}{2} (|t|^{2q} + |s|^{2q} - |t-s|^{2q}) \] (9.3.4)

for any \( x, y \in \mathbb{R}^n \) and \( t, s \in \mathbb{R}^+ \). For any fixed \( x \), \( \mathcal{B}(x, \cdot) \) is a 1-dimensional fractional Brownian motion with a Hurst parameter of \( q \). For any fixed \( t \), \( \mathcal{B}(\cdot, t) \) is a fractional Brownian field with a Hurst parameter of \( p \) in the sense of [108], up to a constant: A fractional Brownian field with a Hurst parameter of \( p \) is a random field \( \mathbb{B}^p : \Theta \times \mathbb{R}^n \to \mathbb{R} \) such that

1. \( \mathbb{B}^p(0) = 0 \ a.s.; \)
2. for all \( z_1, z_2, \cdots, z_d \in \mathbb{R}^n \), the random vector \( (\mathbb{B}^p(z_1), \mathbb{B}^p(z_2), \cdots, \mathbb{B}^p(z_d)) \) is Gaussian with zero mean;
3. for any \( z_1, z_2 \in \mathbb{R}^n \), \( \mathbb{E} [(\mathbb{B}^p(z_1) - \mathbb{B}^p(z_2))^2] = |z_1 - z_2|^{2p} ; \) and
4. \( z \to \mathbb{B}^p(\omega, z) \) is continuous for almost all \( \omega \in \Theta \).
In the following, we abbreviate $\mathfrak{B}^{p,q}$ as $\mathfrak{B}$. Let the rectangular increments of $\mathfrak{B}$ be denoted by
\[
\Box \mathfrak{B}_{(x,t), (y,s)} = \mathfrak{B}(x, t) - \mathfrak{B}(x, s) - \mathfrak{B}(y, t) + \mathfrak{B}(y, s).
\]
From (9.3.4), we can easily obtain
\[
\mathbb{E}[\Box \mathfrak{B}_{(x,t), (y,s)}]^2 = |x - y|^p |t - s|^q
\]  
for any $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}^+$. Similar to (6.5.3), using the two-parameter Garsia-Rodemich-Rumsey inequality [78, 131], we obtain the following proposition.

**Proposition 10.** Let $\mathfrak{B} = \mathfrak{B}^{p,q} = \{\mathfrak{B}^{p,q}(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}^+\}$ be a fractional Brownian sheet with Hurst parameters of $0 < p, q < 1$. Then, for any $0 < \varepsilon < \min\{p, q\}$ and any $T > 0$, there exists a positive random variable $\zeta_{\varepsilon, T}$ with $\mathbb{E}[\zeta_{\varepsilon, T}^\mu] < \infty$ for any $\mu \geq 1$ such that
\[
|\Box \mathfrak{B}_{(x,t), (y,s)}| \leq \zeta_{\varepsilon, T}|x - y|^{p-\varepsilon}|t - s|^{q-\varepsilon} \text{ a.s.,}
\]
\[
|\mathfrak{B}(x, t) - \mathfrak{B}(x, s)| \leq \zeta_{\varepsilon, T}|t - s|^{q-\varepsilon} \text{ a.s.,}
\]
for any $x, y \in \mathbb{R}^n$ and $0 \leq s, t \leq T$.

If $0 < p, q < 1$ and $p + 2q > 2$, then we can claim that there exist $\delta_0 \in (0, p)$ and $\alpha_0 \in (1/2, q)$ that satisfy $\delta_0 + 2\alpha_0 > 2$. Therefore, from Proposition 10, a path of fractional Brownian motion $\mathfrak{B}^{p,q}$ with Hurst parameters of $0 < p, q < 1$ almost surely satisfies
\[
|\Box \mathfrak{B}_{(x,t), (y,s)}| \leq C(\delta_0, \alpha_0, T)|x - y|^{\delta_0}|t - s|^{\alpha_0}
\]  
and
\[
|\mathfrak{B}(x, t) - \mathfrak{B}(x, s)| \leq C(\delta_0, \alpha_0, T)|t - s|^{\alpha_0}
\]  
for any $x, y \in \mathbb{R}^n$ and $s, t \in [0, T]$, where $C(\delta_0, \alpha_0, T)$ is a positive constant that depends
on $\delta_0, \alpha_0$ and $T$. It is clear that (9.3.6) and (9.3.7) correspond exactly to conditions (9.3.2) and (9.3.1). Then, by Theorem 4, we obtain a pathwise estimate for solutions to (9.3.3) in $\mathbb{R}^n \times \mathbb{R}^+$, namely, $u \in C^{2+\gamma_0,\gamma_0/2}_{x,t}$ with $\gamma_0 = \delta_0 + 2\alpha_0 - 2$ almost surely.

Now it remains to prove the above claim. Suppose that $p + 2q = 2 + \varepsilon$ for $\varepsilon > 0$. The claim is easily satisfied by selecting $\delta_0 = p - \varepsilon/4$ and $\alpha_0 = q - \varepsilon/4$ because

\begin{align}
0 &< p - \varepsilon/4 < p, \quad \tag{9.3.8} \\
1/2 &< q - \varepsilon/4 < q, \quad \tag{9.3.9}
\end{align}

and

\[
\delta_0 + 2\alpha_0 = p + 2q - 3\varepsilon/4 = 2 + \varepsilon/4 > 2.
\]

We prove (9.3.8) and (9.3.9) by contradiction: If $p - \varepsilon/4 \leq 0$, then

\[
\varepsilon/4 + 2q \geq p + 2q = 2 + \varepsilon \quad \text{implies} \quad q \geq 1 + 3\varepsilon/8 > 1,
\]

which is a contradiction. If $q - \varepsilon/4 \leq 1/2$, then

\[
p + 1 + \varepsilon/2 \geq p + 2q = 2 + \varepsilon \quad \text{implies} \quad p \geq 1 + \varepsilon/2 > 1,
\]

which is also a contradiction. Therefore, the claim is obtained.
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