2008

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University of Wollongong
Economics Working Paper Series 2008

http://www.uow.edu.au/commerce/econ/wpapers.html

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WP 08-07

May 2008

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This paper speculates how Ramsey's (1928) model of optimal consumption and saving would have been constructed had Sir Frank Ramsey lived in a period of greater awareness of the environmental damaged caused by production processes. The paper extends Ramsey's model to the case where the instantaneous satisfaction of the representative agent from consumption is spoiled by the degradation of his physical environment and spending on cleaning up and greening operations are therefore engendered.
JEL Classification: O12, Q56
Key Words: Consumption, environmental investment, golden rule

With $E_t \in \mathbb{R}$ indicating the state of the representative agent's environment and $c_t \in \mathbb{R}_+$ his level of consumption at time *t*, the representative agent is said to be environmentally aware if for any hypothetical pair E and E' $(c,E) \succ (c,E')\ \forall c$ when $E > E'$ and $(c,E) \prec (c,E')\ \forall c$ when $E < E'$. An analytically convenient expansion of the instantaneous utility function $u(c_t)$ used by Ramsey that represents this property is:

$$\tilde{u}_t = E_t{}^{\beta} u(c_t) \tag{1}$$

where $\beta \in (0,1)$ is a scalar indicating the representative agent's degree of environmental awareness. Having a continuous and twice differentiable u with $u(0) = 0$, $u' > 0$ and $u'' < 0$, the representative agent is willing to forego a maximal fraction $\varepsilon = 1 - u^{-1}((E'/E)^{\beta} u(c_t))/c_t$ of his current consumption for living in environment E than in the dirtier environment E'.[1] For instance, with $u_t = c_t{}^{\gamma}$ and $0 < \gamma < 1$, $\varepsilon = 1 - (E'_t / E_t)^{\beta/\gamma}$.

As in Ramsey's model, the representative agent is infinitely lived and has time consistent preference with a degree of impatience $\rho > 0$ and a lifetime utility U that is given by an additively separable function. In this extension, $U=\int_0^\infty e^{-\rho t}E(t)^\beta u(c(t))dt$ and the representative agent chooses the joint trajectories of consumption and investment in his environment so as to maximise his lifetime utility subject to the motion equations of his capital stock and state of environment.[2] The representative agent's environment is degraded by the waste generated by his production and improved in a rate g that is concavely increasing in his investment (s) in cleaning up and greening operations:

$$\dot{E}(t)=g(s(t))E(t)-wf(k(t)) \qquad (2)$$

where w is a positive scalar denoting a fixed waste-production coefficient (*waste rate*) and $g'>0, g''<0, g'''=0$. There is a trade-off between environmental investment and consumption. The representative agent's investment in cleaning up and greening operations modifies Ramsey's motion equation of capital as follows:

$$\dot{k}(t)=f(k(t))-c(t)-s(t)-(\delta+n)k(t)\,. \qquad (3)$$

where f is a concave function yielding the per capita income, δ is the capital depreciation rate, and n is the population growth rate.

In addition to the state equations (2) and (3), the solution of the representative agent's problem includes the following motion equations of his consumption and environmental investment:

$$\dot{c}=\frac{[1+w/g'(s)E]f'(k)-(\rho+\delta+n)+\beta[g(s)-wf(k)/E]}{-u''(c)/u'(c)} \qquad (4)$$

$$\dot{s}=\frac{[1+w/g'(s)E]f'(k)-(\delta+n)-\beta g'(s)u(c)/u'(c)-wf(k)/E}{-g''(s)g'(s)}\,. \qquad (5)$$

(See Appendix A for a detailed derivation.)

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[1] $\{E^\beta u((1-\varepsilon)c)=E'^\beta u(c)\}\Rightarrow\{(1-\varepsilon)c=u^{-1}((E/E')^\beta u(c))\}$.

[2] If $E\in(0,1)$, $E(t)^\beta$ can alternatively be interpreted as the probability of living beyond t and U as the expected lifetime utility of the representative agent whose life expectancy is uncertain and endangered by the adverse effect of environmental degradation on his health. (Cf. Levy, 2008)

Equations (4) and (5) constitute an environmentally modified golden rule. In addition to the discrepancy between the marginal product of capital and its user cost, the advocated instantaneous changes in consumption and environmental investment are affected by the full marginal cost of production $[1+w/g'(s)E]$, the elasticity of individual's utility from the state of the environment β and the rate of change of his environment $[g(s)-wf(k)/E]$. The investigation of this set of motion equations leads to the following propositions about the existence and nature of the steady-state combination of consumption and investment in the environment.

PROPOSITION 1 (Existence and uniqueness): If $\rho > (1-\beta)wf(k_{SS})/E_{SS} + \beta g(\bar{s})$, there exists a unique, interior steady-state combination of consumption and investment in the environment (with $s_{SS} = \bar{s}$).

PROPOSITION 2 (Asymptotically stable spiral): From any initial point there is a convergence to the interior steady state with clockwise jointly oscillating consumption and investment in the environment.

Propositions 1 and 2 are proven in Appendix B and depicted by Figure 1. In this extended Ramsey model, it is optimal to have a period of rising consumption and rising environmental investment, followed by a period of increasing consumption and decreasing environmental investment, followed by a period of decreasing consumption and decreasing environmental investment, followed by a period of decreasing consumption and increasing environmental investment, and so on and so forth, but with cycles that are moderated with the passage of time.

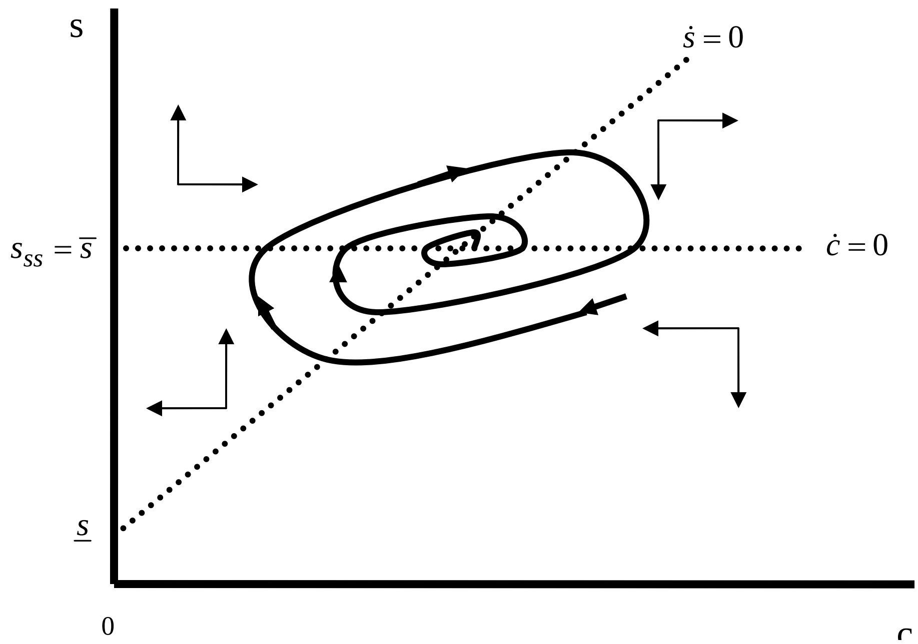

Figure 1. Damped oscillations of consumption and environmental investment

APPENDIX A

The present value Hamiltonian associated with the representative agent's optimal control problem is:

$$H = e^{-\rho t} E^{\beta} u(c) + \lambda_1 [f(k) - c - s - (\delta + n)k] + \lambda_2 [g(s)E - wf(k)] \tag{A1}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are the shadow values of the representative agent's capital and environment, respectively, and where the time index t is omitted for convenience. H is concave in the control variables c and s and in the state variable E. As long as $\lambda_1 > -\lambda_2 w$ H is also concave in k and the Mangasarian's theorem on the sufficiency of the Pontryagin's maximum-principle conditions is valid. These conditions include the adjoint equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial k} = -(\lambda_1 + \lambda_2 w) f'(k) + (\delta + n)\lambda_1 \tag{A2}$$

and

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial E} = -\beta e^{-\rho t} E^{\beta - 1} u(c) - \lambda_2 g(s) \tag{A3}$$

the optimality conditions:

$$\frac{\partial H}{\partial c} = e^{-\rho t} E^{\beta} u'(c) - \lambda_1 = 0 \tag{A4}$$

and

$$\{\frac{\partial H}{\partial s} = -\lambda_1 + \lambda_2 g'(s)E = 0\} \Rightarrow \{\lambda_2 = \lambda_1 / g'(s)E\} \tag{A5}$$

and the transversality condition:

$$\lim_{t \to \infty} H(t) = 0 \,. \tag{A6}$$

By differentiating the optimality condition (A4) with respect to t the following singular control equation is obtained:

$$e^{-\rho t} \beta E^{\beta-1} u'(c)\dot{E} - \rho e^{-\rho t} E^{\beta} u'(c) + e^{-\rho t} E^{\beta} u''(c)\dot{c} - \dot{\lambda}_1 = 0 \tag{A7}$$

and in recalling equations (A2) and (A5):

$$\begin{aligned} &e^{-\rho t} \beta E^{\beta-1} u'(c)\dot{E} - \rho e^{-\rho t} E^{\beta} u'(c) + e^{-\rho t} E^{\beta} u''(c)\dot{c} \\ &+ \lambda_1 \{1 + w / g'(s)E\} f'(k) - (\delta + n)\lambda_1 = 0 \end{aligned} \tag{A8}$$

Recalling (A4):

$$\begin{aligned} &e^{-\rho t} \beta E^{\beta-1} u'(c)\dot{E} - \rho e^{-\rho t} E^{\beta} u'(c) + e^{-\rho t} E^{\beta} u''(c)\dot{c} \\ &+ e^{-\rho t} E^{\beta} u'(c)\{[1 + w / g'(s)E] f'(k) - (\delta + n)\} = 0 \end{aligned} \tag{A9}$$

Dividing by $e^{-\rho t} E^{\beta} u'(c)$:

$$\beta E^{-1} \dot{E} - (\rho + \delta + n) + [u''(c) / u'(c)]\dot{c} + [1 + w / g'(s)E] f'(k) = 0 \,. \tag{A10}$$

Hence,

$$\dot{c} = \frac{[1 + w / g'(s)E] f'(k) - (\rho + \delta + n) + \beta \dot{E} / E}{-u''(c) / u'(c)} \,. \tag{A11}$$

By differentiating the optimality condition (A5) with respect to t the following singular control equation is obtained:

$$-\dot{\lambda}_1 + \dot{\lambda}_2 g'(s)E + \lambda_2 g'(s)\dot{E} + \lambda_2 E g''(s)\dot{s} = 0 \,. \tag{A12}$$

Recalling (A2) and (A3):

$$\begin{aligned} &(\lambda_1 / \lambda_2 + w) f'(k) - (\delta + n)\lambda_1 / \lambda_2 \\ &-[\beta e^{-\rho t} E^{\beta-1} u(c) / \lambda_2 + g(s)] g'(s)E + g'(s)\dot{E} + E g''(s)\dot{s} = 0 \end{aligned} \tag{A13}$$

By (A5) and (A4):

$$\lambda_2 = e^{-\rho t} E^{\beta} u'(c) / g'(s)E \quad \text{(A14)}$$

and

$$\lambda_1 / \lambda_2 = g'(s)E\,. \quad \text{(A15)}$$

Hence,

$$\begin{aligned}&[g'(s)E + w] f'(k) - (\delta + n) g'(s)E \\ &- \{\beta g'(s) u(c) / u'(c) + g(s)\} g'(s)E + g'(s)\dot{E} + Eg''(s)\dot{s} = 0\end{aligned} \quad \text{(A16)}$$

Dividing both sides by $g'(s)E$:

$$\begin{aligned}&[1 + w / g'(s)E] f'(k) - (\delta + n) \\ &- \{\beta g'(s) u(c) / u'(c) + g(s)\} + \dot{E} / E + g''(s) g'(s)\dot{s} = 0\end{aligned} \quad \text{(A17)}$$

Hence,

$$\dot{s} = \frac{[1 + w / g'(s)E] f'(k) - (\delta + n) - \beta g'(s) u(c) / u'(c) - g(s) + \dot{E} / E}{-g''(s) g'(s)}\,. \quad \text{(A18)}$$

From (2),

$$\dot{E} / E = g(s) - wf(k) / E \quad \text{(A19)}$$

and hence:

$$\dot{c} = \frac{[1 + w / g'(s)E] f'(k) - (\rho + \delta + n) + \beta[g(s) - wf(k) / E]}{-u''(c) / u'(c)} \quad \text{(A20)}$$

$$\dot{s} = \frac{[1 + w / g'(s)E] f'(k) - (\delta + n) - \beta g'(s) u(c) / u'(c) - wf(k) / E}{-g''(s) g'(s)}\,. \quad \text{(A21)}$$

APPENDIX B

Proof of Proposition 1

From (4), the isocline $\dot{c} = 0$ is given by:

$$[1 + w / g'(s)E] f'(k) - (\rho + \delta + n) + \beta[g(s) - wf(k) / E] = 0\,. \quad \text{(B1)}$$

This isocline is represented by a horizontal line in the *c*-*s* plane.

From (5), the isocline $\dot{s} = 0$ is given by:

$$[1 + w / g'(s)E] f'(k) - (\delta + n) - \beta g'(s) u(c) / u'(c) - wf(k) / E = 0\,. \quad \text{(B2)}$$

By total differentiation:

$$-[wf'(k) / g'(s)^2 E + \beta u(c) / u'(c)] g''(s) ds - \beta g'(s)[1 - u(c) u''(c) / u'(c)] dc = 0 \quad \text{(B3)}$$

and hence:

$$\frac{ds}{dc}\Big|_{\dot{s}=0} = -\frac{\beta g'(s)[1-u(c)u''(c)/u'(c)]}{[wf'(k)/g'(s)^2 E + \beta u(c)/u'(c)]g''(s)} > 0. \qquad \text{(B4)}$$

Recalling that $u(c)/u'(c) = 0$ for $c = 0$ and (B2), the intercept $(0, \underline{s})$ of the isocline $\dot{s} = 0$ should satisfy:

$$[1 + w/g'(\underline{s})E]f'(k) - (\delta + n) - wf(k)/E = 0 \qquad \text{(B5)}$$

which implies:

$$g'(\underline{s}) = \frac{wf'(k)/E}{(\delta + n) + wf(k)/E - f'(k)}. \qquad \text{(B6)}$$

From (B1), the intercept $(0, \overline{s})$ of the isocline $\dot{c} = 0$ should satisfy:

$$[1 + w/g'(\overline{s})E]f'(k) - (\rho + \delta + n) + \beta[g(\overline{s}) - wf(k)/E] = 0 \qquad \text{(B7)}$$

which implies:

$$g'(\overline{s}) = \frac{wf'(k)/E}{(\rho + \delta + n) + \beta wf(k)/E - \beta g(\overline{s}) - f'(k)}. \qquad \text{(B8)}$$

Recalling the assumptions that $f'(k) > 0$ and $g'(s) > 0$, the denominators on the right-hand sides of (B6) and (B8) are positive. Recalling further that $g''(s) < 0$, $\underline{s} < \overline{s}$ for any given combination of *k* and *E*, most relevant (k_{ss}, E_{ss}), if:

$$(\delta + n) + wf(k_{ss})/E_{ss} - f'(k_{ss}) < (\rho + \delta + n) + \beta wf(k_{ss})/E_{ss} - \beta g(\overline{s}) - f'(k_{ss}) \qquad \text{(B9)}$$

or, equivalently, if:

$$\rho > (1-\beta)wf(k_{ss})/E_{ss} + \beta g(\overline{s}). \qquad \text{(B10)}$$

Recalling that the isocline $\dot{c} = 0$ is horizontal whereas the isocline $\dot{s} = 0$ is upward sloped, if the inequality (B10) holds, the intercept of the isocline $\dot{s} = 0$ is smaller than that of the isocline $\dot{c} = 0$ and hence these isoclines intersect one another only once. Their intersection defines a unique, interior steady state with $s_{ss} = \overline{s}$.

Proof of Proposition 2

By differentiating (4):

$$\frac{d\dot{c}}{ds} = \frac{-wf'(k)g''(s)/g'(s)^2 E + \beta g'(s)}{-u''(c)/u'(c)} > 0 \qquad \text{(B11)}$$

which explains the direction of the horizontal arrows in the four phases in Figure 1.

By differentiating (5):

$$\frac{d\dot{s}}{dc} = \frac{-\beta g'(s)[1-u(c)u''(c)/u'(c)]}{-g''(s)g'(s)} < 0 \qquad \text{(B12)}$$

which explains the direction of the vertical arrows in the four phases displayed in Figure 1. The combinations of the horizontal and vertical arrows reveal that the steady state is either a spiral point or a centre. The particular nature of the steady state can be found by computing the trace of the Jacobian (state-transition) matrix of the linearised (4) and (5) equation system at the steady state. The elements on the main diagonal of this matrix are:

$$J_{11} = \frac{dA}{dc}(ss) = \{1 + w/g'(s_{ss})]f'(k_{ss}) - (\rho + \delta + n)\}\frac{-u''(c_{ss})^2 + u'''(c_{ss})u'(c_{ss})}{[-u''(c_{ss})]^2} \qquad \text{(B13)}$$

$$J_{22} = \frac{dB}{ds} = \{1 + w/g'(s_{ss})]f'(k_{ss}) - (\delta + n)\}\frac{-g''(s_{ss})^2 + g'''(s_{ss})g'(s_{ss})}{[-g''(s_{ss})]^2}. \qquad \text{(B14)}$$

Recalling (A11),

$$[1 + w/g'(s_{ss})\}f'(k_{ss}) - (\rho + \delta + n) = 0 \qquad \text{(B15)}$$

and hence $J_{11} = 0$.

From (B15),

$$[1 + w/g'(s_{ss})\}f'(k_{ss}) - (\delta + n) = \rho > 0 \qquad \text{(B16)}$$

which implies that $J_{22} = -\rho$ as long as $g''' = 0$. Since $trJ = -\rho$ and $\rho > 0$ for a representative agent with a time preference, the joint trajectory of *c* and *s* is a converging spiral. The interior steady state is asymptotically stable and approachable along a clockwise spiral displaying oscillations of consumption and investment in the environment.

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