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Abstract

Texture is a phenomenon in image data that continues to receive wide-spread interest due to its broad range of applications. The paper focuses on but one of several ways to model textures, namely, the class of stochastic texture models. the authors introduce a new spatial stochastic model called partially ordered Markov models, or POMMs. They show how POMMs are a generalization of a class of models called Markov mesh models, or MMMs, that allow an explicit closed form of the joint probability, just as do MMMs. While POMMs are a type of Markov random field model (MRF), the general MRFs do not have such an explicit closed form of the joint probability. The authors present results on texture synthesis and texture classification, introducing a very fast one-pass texture synthesis algorithm, and show that parameter estimation of natural textures can give quite satisfactory results. They remark that, while the theory underlying POMMs has been applied only to texture analysis, in their most general form, POMMs have the potential to be applied to such diverse areas outside of imaging as probabilistic expert systems, Bayesian hierarchical modeling, influence diagrams, and random graphs and networks.

Keywords

texture, analysis, markov, partially, models, ordered

Disciplines

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TEXTURE ANALYSIS USING PARTIALLY ORDERED MARKOV MODELS¹

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1. ABSTRACT

Texture is a phenomenon in image data that continues to receive wide-spread interest due to its broad range of applications, including remotely sensed data, medical imaging, and military applications, to name a few. This paper focuses on but one of several ways to model textures, namely, the class of stochastic texture models. We introduce a new spatial stochastic model called *partially ordered Markov models*, or POMMs. We show how POMMs are a generalization of a class of models called *Markov mesh models*, or MMMs, that allow an explicit closed form of the joint probability, just as do MMMs. While POMMs are a type of Markov random field model (MRF), the general MRFs do not have such an explicit closed form of the joint probability. We present results on texture synthesis and texture classification, introducing a very fast one-pass texture synthesis algorithm, and show that parameter estimation of natural textures can give quite satisfactory results. We remark that, while the theory underlying POMMs has been applied only to texture analysis, in their most general form, POMMs have the potential to be applied to such diverse areas outside of imaging as probabilistic expert systems, Bayesian hierarchical modeling, influence diagrams, and random graphs and networks.

2. STOCHASTIC MODELS: MMMs, POMMs, AND MRFs

The Markov mesh models (MMM) were [1] introduced as a way of generalizing the order implicit in a one-dimensional Markov chain to two dimensions. Abend et al. [1] defined their model in terms of conditional probabilities in a local neighborhood of a pixel. If Z_{ij} and X_{ij} are defined as in Figure 1, and $U_{ij} \subset X_{ij} \setminus \{i, j\}$, then a MMM must satisfy:

$$P(a_{ij}|Z_{ij}) = P(a_{ij}|U_{ij}),$$

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where $A = \{a_{ij} : 1 \leq i \leq M, 1 \leq j \leq N\}$ is the $M \times N$ rectangular array of random variables (r.v.s.) As a result, Abend et al. provided the following two theorems:

Figure 1. (a) Z_{ij} , and (b) X_{ij} in Abend's MMM.

Theorem 1.

$$P(A) = \prod_{(i,j)} P(a_{ij}|U_{ij}), \quad (1)$$

for $U_{ij} = \{a_{i-1,j}, a_{i-1,j-1}, a_{i,j-1}\}$ for interior r.v.s, and simply a truncated version of the boundary r.v.s.; and

Theorem 2.

$$P(a_{ij}|A \setminus \{a_{ij}\}) = P \begin{pmatrix} a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} \\ a_{ij} & a_{i,j-1} & a_{i,j+1} \\ a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} \end{pmatrix} \quad (2)$$

Note that Theorem 1 states that the joint probability density function (pdf) can be written in closed form as a product of local, conditional probabilities. This is not the general case for a MRF. It is well known that a MRF can be expressed as a Gibbs distribution (see for example [7]), and the joint probability density function (pdf) can be represented in the following way [4]: the joint pdf is $P(A) = \frac{1}{Z_0} \exp(-U_0(A))$, with energy function $U_0(A) = \sum_{c \in C} V_c(A)$ and set of *cliques* C , and Z_0 is the partition function or normalizing constant defined by $Z_0 = \sum_{A \in \Omega} \exp(-U_0(A))$. Here, Ω is the set of all possible realizations of the random image A . When implementing algorithms such as the maximum likelihood estimator (MLE) or maximum a posteriori estimator (MAP) where the joint pdf must be calculated, typically only approximations of the normalizing constant Z_0 are available. This

has the potential to severely limit the efficacy of the MRF in practical situations. Thus, if its use is appropriate, a MMM model allows exact calculation of the joint pdf.

As seen by Theorem 2, MMMs are a subclass of MRFs. While Equation 2 corresponds to a well-utilized MRF, there is no MMM corresponding to the popular MRF displayed in Figure 2 [8]. This model is also known as the Ising model. Thus, while MMMs have an attractive computational advantage of closed-form pdf, they cannot give modelers as much selection as do MRFs.

Figure 2. Von Neumann neighborhood

MMMs do not have a sequential ordering of pixel locations as do Markov chains. Thus, if $\{a(i) : i = 0, 1, 2, \dots\}$ is a Markov chain, the indexing set $\{0, 1, 2, \dots\}$ is a *totally ordered set* [10]. It is well known that there is no canonical ordering of Z^2 , where Z = set of integers, although the lexicographical ordering on Z^2 is often used in imaging processing. However, close inspection of the third order MMM given in Equation 1 shows that there is indeed some type of ordering of the sites. In fact, in [5] we show that this model has a *partial ordering* [10] of pixel sites. Informally, a partial ordering of a set is a relation existing between (not necessarily all) pairs of elements in the set. In a totally ordered set, every pair of elements must be related. An example of a totally ordered set is the set of integers under the familiar relation of "less than or equal to."

POMMs are a generalization of MMMs that allow an arbitrary partial order to underlie the set of r.v.s. Figure 3 depicts the relationship between MRFs, MMMs, and POMMs. As classes of models, we have the following relationship: $\text{MMMs} \subset \text{POMMs} \subset \text{MRFs}$.

Figure 3. Relationship between MRFs, POMMs, and MMMs.

We next describe formally a POMM. First, if $(A, <)$ is a poset of r.v.s. with underlying partial order $<$ on the set of pixel locations, and if $x \in A$, then:

$$\begin{aligned} \text{cone } x &= \{y : y < x\} \\ \overline{\text{cone}} x &= \{x\} \cup \text{cone } x \\ \text{adj}_{<} x &= \{y : y < x \text{ and } y \text{ is adjacent to } x\}, \text{ and} \\ \overline{\text{adj}}_{<} x &= \{x\} \cup \text{adj}_{<} x. \end{aligned}$$

Finally, due to the isomorphism between an acyclic directed graph and a partially ordered set [2], we can use concepts in both areas interchangeably. Thus, we define $L^0 \equiv$ set of minimal elements of A , and $L^{\text{max}} \equiv$ set of maximal elements of A .

A partially ordered Markov model is defined as follows: For $x \in A$, where $(A, <)$ is a poset of r.v.s., define $Y_x = \{z : z \text{ and } x \text{ are not related}\}$. Then the random field A is called a partially ordered Markov model (POMM) if for any subset $W_x \subset Y_x$, we have:

$$P(x | \text{cone } x, W_x) = P(x | \text{adj}_{<} x). \quad (3)$$

Theorem 3 is a generalization of Theorem 1:

Theorem 3:

$$P(A) = P(\overline{\text{cone}}(L^{\text{max}})) = P(L^0) \prod_{x \in A \setminus L^0} P(x | \text{adj}_{<} x). \quad (4)$$

Theorem 4 is a generalization of Theorem 2:

Theorem 4:

$$P(x | A \setminus \{x\}) = P(x | \text{dil}^* x) = \frac{\prod_{y \in Q_x} P(y | \text{adj}_{<} y)}{\sum_{\text{All values } x \text{ can assume}} \prod_{y \in Q_x} P(y | \text{adj}_{<} y)}. \quad (5)$$

Here,

$$\text{dil}^* x = \left[\bigcup_{\{z : z \in \overline{\text{adj}}_{<} z\}} \overline{\text{adj}}_{<} z \right] \setminus \{x\};$$

Indeed, the set $\text{dil } x = \text{dil}^* x \cup \{x\}$ is simply the mathematical morphology operation of dilation [9] of the set $\overline{\text{adj}}_{<} x$ with itself. Further results and more general forms of Theorems 3 and 4 can be found in [5].

Note that Equation 4 gives an exact expression for the joint pdf. In the next section we use this fact to derive a fast one-pass algorithm that generates an image from this expression. Also note that Equation 5 shows that a POMM is indeed a MRF. Finally, we remark that the specification of parameters for the POMM is straightforward, through the use of Equation 4 and the specific form of conditional pdf used.

3. POMMs FOR TEXTURE SYNTHESIS AND CLASSIFICATION

POMMs can be represented graphically using their digraph form. In Figure 4(a), the basic neighborhood $\text{adj}_{<} a_{ij}$ used for both texture synthesis and classification is depicted using a digraph representation, where the arrow

from $a_{i-1,j}$ to a_{ij} means $a_{i-1,j} \prec a_{ij}$ under the partial order. This neighborhood relation holds for all (i,j) in an $M \times N$ rectangular array with appropriate modifications on the boundary. This particular relation was chosen for the representation of the dependence on the horizontal and vertical direction, and on the two diagonal directions. We chose the conditional binomial distribution to represent the spatial interaction between pixels:

$$P(a_{ij} = m | adj_{\prec} a_{ij}) = \binom{G-1}{m} \theta^m (1-\theta)^{G-1-m},$$

$$m = 0, 1, \dots, G-1, \text{ where } \theta = \frac{e^T}{1 + e^T}, \text{ and}$$

$$T = \alpha + \beta a_{i-1,j} + \gamma a_{i-1,j-1} + \delta a_{i,j-1} + \epsilon a_{i+1,j-1}. \quad (6)$$

While POMMs are MRFs and textures can be synthesized using Flinn's spin-exchange algorithm [6], this is most certainly a waste of CPU time. In [5], we give a method called the level-set algorithm that generates an

Figure 4. (a) The set $adj_{\prec} a_{ij}$ used for texture generation. (b) The parameters used in Eq. (6).

image after visiting each site in the image once. The general idea of the level set algorithm is to follow a sequence of special antichains of the poset and generate a value for each site according to the specific expression of the condition probability of the model (Equation 3). In Figure 5, we present a limited selection of synthetic textures, six images generated using the level-set algorithm. The parameters for each are listed to the left.

Figure 5. Six different textures generated using the level-set algorithm.

When using algorithms that require knowledge of the joint probability, such as maximum likelihood estimation, POMMs offer a particularly nice advantage over MRFs. The POMMs have a tractable closed form expression for the joint probability that, in general, MRFs do not have. Parameter estimation for both synthetic and natural texture

estimation using POMMs was performed using the maximum log-likelihood estimator and the POMM model in Figure 4(b). A 5-dimensional iterative Newton-Raphson method was used to search the parameter space $(\alpha, \beta, \gamma, \delta, \epsilon)$.

The advantage of estimating parameters using known synthetic data is that we know what the true model and

its parameters are. We applied the parameter-estimation algorithm to each of the six synthetic images in Figure 5, images generated by the level-set algorithm. In Table 1, we give these results; many estimates were quite close.

Table 1 Results of parameter estimation on 6 level-set generated textures from Fig. 5.

Image		α	β	γ	δ	ϵ
Figure 5(a)	True Values	-2.2	1.1	1.1	1.1	1.1
	Estimated Values	-2.166	0.963	1.254	1.178	0.986
Figure 5(b)	True Values	-1.0	1.6	-1.0	-1.0	2.3
	Estimated Values	-0.919	1.530	-0.890	-1.096	2.304
Figure 5(c)	True Values	-3.0	0.0	3.1	0.0	3.1
	Estimated Values	-2.884	-0.161	3.100	-0.856	3.101
Figure 5(d)	True Values	0.6	-5.0	2.5	2.5	0.1
	Estimated Values	0.821	-4.851	2.096	2.150	0.129
Figure 5(e)	True Values	0.90	-2.5	1.0	2.0	-2.5
	Estimated Values	0.702	-2.142	0.917	1.910	-2.367
Figure 5(f)	True Values	-6.0	0.0	8.0	0.0	8.0
	Estimated Values	-5.406	1.503	5.431	-1.351	5.640

We also estimated parameters of natural textures by fitting them with POMMs. We used several natural Brodatz textures [3]. It was observed that maximum likelihood parameter estimation gave better results when the sample size was larger, or equivalently, when the image array size was bigger. The estimation results on the Brodatz images of a cloud, water, and tree bark are shown in Figure 6. The first column shows the original image, the second column shows the values of the estimated parameters, and the third column shows a texture generated from the estimated parameters. Note that the synthetic tree and water textures were especially similar to the original textures.

This particular homogeneous POMM did better in certain situations than others: if the texture was homogeneous and small scale, the POMM did a better job, such as on Brodatz textures of water and tree bark. If there were large scale structures or spatially varying (heterogeneous) structures across the image, such as in the Brodatz cloud image, then the homogeneous POMM we used did not reproduce the texture well. Specifying POMMs with different underlying partial orders, and introducing heterogeneity, will undoubtedly allow an even wider variety of textures to be synthesized as well as classified.

4. CONCLUSIONS

In this paper we have demonstrated the POMMs' usefulness in both synthesizing data and estimating parameters for a specific model. Clearly, there are two main advantages a POMM has over a general MRF: 1) a closed form for the joint probability is available for the POMM, which is useful in implementing computer simulations of statistical algorithms that require the joint probability; and 2) due to its closed form joint distribution, computer synthesis of data can be more quickly calculated using the level-set algorithm than using Flinn's algorithm. The full impact of these models in image analysis will likely be felt after more thorough investigations into its theoretical properties and applications have been completed.

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(a) Cloud

-0.07 0.22
0.11 ↘
0.009 ↗
 $\alpha = -2.62$

(b) Estimated parameters from (a) plus synthesized image.

(c) Water

-0.11 0.29
0.06 ↘
0.1 ↗
 $\alpha = -2.45$

(d) Estimated parameters from (c) plus synthesized image.

(e) Tree bark

-0.13 0.18
0.21 ↘
0.07 ↗
 $\alpha = -2.54$

(f) Estimated parameters from (e) plus synthesized image.

Figure 6. Original 128x128 Brodatz textures and corresponding textures synthesized with estimated parameters.