

# GRAPHS OF $C^*$ -CORRESPONDENCES AND FELL BUNDLES

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ABSTRACT. We define the notion of a  $\Lambda$ -system of  $C^*$ -correspondences associated to a higher-rank graph  $\Lambda$ . Roughly speaking, such a system assigns to each vertex of  $\Lambda$  a  $C^*$ -algebra, and to each path in  $\Lambda$  a  $C^*$ -correspondence in a way which carries compositions of paths to balanced tensor products of  $C^*$ -correspondences. Under some simplifying assumptions, we use Fowler's technology of Cuntz-Pimsner algebras for product systems of  $C^*$ -correspondences to associate a  $C^*$ -algebra to each  $\Lambda$ -system. We then construct a Fell bundle over the path groupoid  $\mathcal{G}_\Lambda$  and show that the  $C^*$ -algebra of the  $\Lambda$ -system coincides with the reduced cross-sectional algebra of the Fell bundle. We conclude by discussing several examples of our construction arising in the literature.

## 1. INTRODUCTION

The Cuntz-Krieger algebras introduced in [7] in 1980 were considered the  $C^*$ -analogues of type III factors, and formed a bridge between symbolic dynamics and operator algebras. These algebras have since been generalised in various ways including discrete graph algebras [33], Cuntz-Pimsner algebras [45], topological graph algebras [26], and higher rank graph algebras [31]. Between them, these generalisations include large classes of classifiable  $C^*$ -algebras, like AF-algebras [11], AT-algebras [42], crossed products by  $\mathbb{Z}^k$  [18, 32], and Kirchberg algebras [54, 60].

In the current paper, we will be interested in a further generalisation of this theory based on structures which we call  $\Lambda$ -systems of  $C^*$ -correspondences, and on the relationship between this construction and Fell bundles over groupoids. Our construction contains elements of both higher-rank graph  $C^*$ -algebras and of Cuntz-Pimsner algebras, so we must digress a little to discuss both before describing our results.

Building on the theory of graph  $C^*$ -algebras introduced in [13, 33], higher-rank graphs, or  $k$ -graphs, and their  $C^*$ -algebras were developed in [31] to provide a graph-based model for the Cuntz-Krieger algebras of Robertson and Steger [54]. They have subsequently attracted widespread research interest (see, for example, [8, 14, 17, 22, 29, 47, 48, 57]).

A  $k$ -graph is a kind of  $k$ -dimensional graph, which one visualises as a collection  $\Lambda^0$  of vertices together with  $k$  collections of edges  $\Lambda^{e_1}, \dots, \Lambda^{e_k}$  which we think of as lying in  $k$  different dimensions. As an aid to visualisation, we distinguish the different types of edges using  $k$  different colours. The higher-dimensional nature of a  $k$ -graph is encoded by the factorisation property which implies that each path consisting of two edges of different colours can be re-factorised in a unique way with the order of the colours reversed. When  $k = 1$ , we obtain an ordinary directed graph, and the definition of the higher-rank graph

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$C^*$ -algebra given by Kumjian and Pask then reduces to that of the graph  $C^*$ -algebra [33]. However, for  $k \geq 2$ , there are many  $k$ -graph algebras which do not arise as graph algebras. For example, the original work of Robertson and Steger on higher-rank Cuntz-Krieger algebras describes numerous 2-graphs  $\Lambda$  for which  $C^*(\Lambda)$  is a Kirchberg algebra and  $K_1(C^*(\Lambda))$  contains torsion.

Cuntz-Pimsner algebras, originally introduced by Pimsner in [45] and since studied by many authors (see, for example, [1, 4, 5, 10, 20, 21, 27, 35, 40, 55]) are a common generalisation of graph  $C^*$ -algebras and of crossed products by  $\mathbb{Z}$ . Let  $B$  be a  $C^*$ -algebra and let  $X$  be a right Hilbert  $B$ -module. If a second  $C^*$ -algebra  $A$  acts by adjointable operators on  $X$ , we refer to  $X$  as a  $C^*$ -correspondence from  $A$  to  $B$ . The prototype arises from a  $C^*$ -homomorphism  $\phi : A \rightarrow B$ ; there is then a  $C^*$ -correspondence  ${}_{\phi}B$  from  $A$  to  $B$  which is equal to  $B$  as a Banach space, has inner product  $\langle x, y \rangle_B := x^*y$  for  $x, y \in {}_{\phi}B$ , and has operations given by  $a \cdot x \cdot b = \phi(a)xb$  for  $a \in A$ ,  $x \in {}_{\phi}B$ , and  $b \in B$ . A  $C^*$ -correspondence from  $A$  to itself is called a  $C^*$ -correspondence over  $A$ . Pimsner's construction associates to each  $C^*$ -correspondence over  $A$  a  $C^*$ -algebra  $\mathcal{O}_X$  which (under mild hypotheses) contains an isomorphic copy  $i_A(A)$  of  $A$  and an isometric copy  $i_X(X)$  of  $X$ , and in which the operations in  $X$  are implemented  $C^*$ -algebraically.

In this paper we seek to combine elements of these two constructions. Our model is the situation of  $\Gamma$ -systems of  $k$ -morphs introduced in [34] to unify various constructions of higher rank graphs. A  $k$ -morph is the analogue at the level of  $k$ -graphs of a  $C^*$ -correspondence between  $C^*$ -algebras. For a precise definition, see Example 3.1.6(ii).

One of the main results of [34] is that there is a category  $\mathcal{M}$  whose objects are  $k$ -graphs and whose morphisms are isomorphism classes of  $k$ -morphs, and there is a functor  $(\Lambda \mapsto C^*(\Lambda), [W] \mapsto [\mathcal{H}(W)])$  from  $\mathcal{M}$  to the category  $\mathcal{C}$  whose objects are  $C^*$ -algebras and whose morphisms are isomorphism classes of  $C^*$ -correspondences. Given an  $\ell$ -graph  $\Gamma$ , a  $\Gamma$ -system of  $k$ -morphs is a collection  $\{\Lambda_v : v \in \Gamma^0\}$  of  $k$ -graphs connected by  $k$ -morphs  $\{W_\gamma : \gamma \in \Gamma\}$  satisfying appropriate compatibility conditions. This gives rise to a family indexed by  $\gamma \in \Gamma$  of  $C^*(\Lambda_{r(\gamma)})$ - $C^*(\Lambda_{s(\gamma)})$   $C^*$ -correspondences  $\mathcal{H}(W_\gamma)$ . The  $\Gamma$ -system also gives rise to a  $(k+l)$ -graph  $\Sigma$  called the  $\Gamma$ -bundle for the system, and this  $\Sigma$  encodes all the information in the  $\Gamma$ -system. The  $C^*$ -algebra  $C^*(\Sigma)$  contains isomorphic and mutually orthogonal copies of the  $C^*(\Lambda_v)$  and isometric copies of the  $\mathcal{H}(W_\gamma)$ , so it has the flavour of a Cuntz-Pimsner algebra for the system of correspondences arising from the  $\Gamma$ -system. Indeed, when  $\Gamma$  is the graph consisting of just one vertex  $v$  and one loop  $e$ , Theorem 6.8 of [34] shows that  $C^*(\Sigma) \cong \mathcal{O}_{\mathcal{H}(W_e)}$ .

In this work, we generalize this idea, defining a system  $(A, X, \chi)$  of  $C^*$ -correspondences over a  $k$ -graph  $\Lambda$ , by associating a  $C^*$ -algebra  $A_v$  to each vertex  $v \in \Lambda^0$ , a  $A_{r(\lambda)}$ - $A_{s(\lambda)}$   $C^*$ -correspondence  $X_\lambda$  to each  $\lambda \in \Lambda$ , and a compatibility isomorphism  $\chi_{\lambda, \mu} : X_\lambda \otimes_{A_{s(\lambda)}} X_\mu \rightarrow X_{\lambda\mu}$  to each composable pair  $\lambda, \mu$  of paths. The case  $k = 1$  is the easiest, since we may always take each  $X_\lambda$  to be  $X_{\lambda_1} \otimes \cdots \otimes X_{\lambda_n}$ , where  $\lambda = \lambda_1 \cdots \lambda_n$  with  $d(\lambda_i) = 1$ , and the compatibility isomorphisms  $\chi_{\lambda, \mu}$  to be the canonical ones. Under a number of simplifying assumptions (see Definition 3.1.2), we define a  $C^*$ -algebra  $C^*(A, X, \chi)$  by first constructing from  $(A, X, \chi)$  a product system  $Y$  of  $C^*$ -correspondences over  $\mathbb{N}^k$ , and then tapping into Fowler's theory of Cuntz-Pimsner algebras for such product systems [19]. Under the same simplifying hypotheses, we then construct a Fell bundle  $E_X$  over the graph groupoid  $\mathcal{G}_\Lambda$  developed in [31] such that  $C_r^*(\mathcal{G}_\Lambda, E_X)$  and  $C^*(A, X, \chi)$  are isomorphic. Our main results are: a version of the gauge-invariant uniqueness theorem for  $C^*(A, X, \chi)$  (Theorem 3.3.1); the construction of the Fell bundle  $E_X$  itself (Theorem 4.3.1); and the

isomorphism between  $C^*(A, X, \chi)$  and the reduced cross-sectional algebra  $C_r^*(\mathcal{G}_\Lambda, E_X)$  (Theorem 4.3.6).

The relationship between product systems of  $C^*$ -correspondences and  $k$ -graphs was previously investigated in [51]. There the authors show that given a  $k$ -graph  $\Lambda$  there is a product system  $Y_\Lambda$  over  $\mathbb{N}^k$  whose coefficient algebra is  $C_0(\Lambda^0)$  and whose fibre over  $n$  is the completion of  $C_c(\Lambda^n)$  under an appropriate  $C_0(\Lambda^0)$ -valued inner product (the  $\Lambda^n$  are given the discrete topology). This relates to  $\Lambda$ -systems as follows. Let  $X$  be the  $\Lambda$ -system with  $X_\lambda = \mathbb{C}$  for all  $\lambda$  and with the  $\chi_{\lambda, \mu}$  determined by multiplication. Then the product system  $Y$  described in the preceding paragraph is precisely the  $Y_\Lambda$  arising in [51]; in particular, it follows from Theorem 4.2 of [51] that our  $\mathcal{O}_Y$  coincides with  $C^*(\Lambda)$  under our regularity hypotheses, which in this situation just boil down to the requirement that  $\Lambda$  is row-finite and has no sources.

Our construction is also consistent with the example of  $\Gamma$ -systems of  $k$ -morphs. Given an  $\ell$ -graph  $\Gamma$  and a  $\Gamma$ -system  $W$  of  $k$ -morphs in the sense of [34], there is a  $\Gamma$ -system  $X = (A_v, X_\gamma, \chi)$  of  $C^*$ -correspondences, where  $A_v = C^*(\Lambda_v)$  and  $X_\gamma$  is equal to the  $C^*$ -correspondence  $\mathcal{H}(W_\gamma)$  constructed in [34, Proposition 6.4]. Moreover,  $C^*(A, X, \chi) \cong C^*(\Sigma)$  where  $\Sigma$  is the  $\Gamma$ -bundle described above associated to the  $\Gamma$ -system of  $k$ -morphs  $W_\gamma$ .

Our construction is quite general, and includes a number of diverse situations studied by a variety of authors in recent papers. We will briefly discuss here how our work relates to four such situations; we present the details of these examples as well as a number of others in Section 5.

The first situation covered by our construction which we mention here is Cuntz's study of twisted tensor products [6]. Cuntz considers a  $C^*$ -algebra  $A \times_{\mathcal{U}} \mathcal{O}_n$ , where  $A$  is a  $C^*$ -algebra,  $\mathcal{O}_n$  is the Cuntz algebra, and  $\mathcal{U} = (U_1, \dots, U_n)$  is a family of unitaries implementing automorphisms  $\alpha_i$  of  $A$ . This defines a system of  $C^*$ -correspondences over the 1-graph  $B_n$  with one vertex  $v$  and  $n$  loop-edges  $e_1, \dots, e_n$  based at  $v$ , whose  $C^*$ -algebra  $C^*(B_n)$  is canonically isomorphic to  $\mathcal{O}_n$ . The  $C^*$ -algebra which we associate to this  $B_n$ -system coincides with Cuntz's twisted tensor product.

Our construction also generalises the situation considered in Section 5.3 of [46]. There Pinzari, Watatani and Yonetani study KMS states on a  $C^*$ -algebra constructed from a family of compatible  $C^*$ -correspondences  $X_{i,j}$ , one for each non-zero entry in a finite  $\{0, 1\}$ -matrix  $\Sigma = (\sigma_{i,j}) \in M_n(\{0, 1\})$ . The  $C^*$ -algebra they associate to these data is the Cuntz-Pimsner algebra of  $\bigoplus_{\sigma_{i,j}=1} X_{i,j}$ . Let  $\Lambda$  be the 1-graph with a vertex  $v_i$  for each  $1 \leq i \leq n$  and an edge  $e_{i,j}$  from  $v_j$  to  $v_i$  if and only if  $\sigma_{i,j} = 1$ . Then  $X_{e_{i,j}} := X_{i,j}$  determines a  $\Lambda$ -system of  $C^*$ -correspondences  $(A, X, \chi)$ . By construction, our  $C^*(A, X, \chi)$  coincides with the Cuntz-Pimsner algebra studied by Pinzari-Watatani-Yonetani.

A third situation related to our work is that of Ionescu, Ionescu-Watatani, and Quigg [23, 24, 25, 49] on  $C^*$ -algebras associated to Mauldin-Williams graphs. In the setting studied by Ionescu [24], a Mauldin-Williams graph consists of a directed graph  $\Lambda$ , compact metric spaces  $T_v$  associated to the vertices of  $\Lambda$ , and strict contractions  $\varphi_e : T_{s(e)} \rightarrow T_{r(e)}$  associated to the edges. Let  $A_v := C(T_v)$  for each vertex  $v$ . For each edge  $e$ , the induced homomorphism  $\varphi_e^* : C(T_{r(e)}) \rightarrow C(T_{s(e)})$  determines a  $C^*$ -correspondence  $X_e := \varphi_e^* A_{s(e)}$  from  $A_{r(e)}$  to  $A_{s(e)}$ . These correspondences determine a  $\Lambda$ -system  $(A, X, \chi)$ . Quigg [49] considers a more general situation.

A fourth connection between our construction and the literature arises in Katsura's realisation of the Kirchberg algebras using topological graph  $C^*$ -algebras. In [28, section 3], Katsura uses a family of topological graphs which are fibered over discrete graphs to construct all nonunital Kirchberg algebras. We can interpret his construction in terms of systems of  $C^*$ -correspondences over 1-graphs, where each vertex algebra is  $C(\mathbb{T})$  and each  $C^*$ -correspondence is constructed using two covering maps (see also [9]).

Our paper is structured as follows. In Section 2 we collect basic facts about  $k$ -graphs,  $C^*$ -correspondences, product systems and Fell bundles, and we establish notation. In Section 3 we define  $\Lambda$ -systems  $(A, X, \chi)$  of  $C^*$ -correspondences and, for systems satisfying a number of simplifying hypotheses which we refer to collectively as regularity (see below), the associated  $C^*$ -algebras  $C^*(A, X, \chi)$ . In Section 4 we construct from each regular  $\Lambda$ -system  $(A, X, \chi)$  a Fell bundle  $E_X$  over the graph groupoid  $\mathcal{G}_\Lambda$  and prove that there is an isomorphism  $C_r^*(\mathcal{G}_\Lambda, E_X) \cong C^*(A, X, \chi)$ . Section 5 is devoted to a discussion of the examples outlined above amongst others.

As already mentioned, for all the  $C^*$ -algebraic results, we restrict our attention to the *regular*  $\Lambda$ -systems such that the  $k$ -graph  $\Lambda$  is row-finite and has no sources (that is, for any degree in  $\mathbb{N}^k$  and any vertex  $v$  of  $\Lambda$ , the set of paths with range  $v$  and degree  $n$  is finite and nonempty), that the  $C^*$ -correspondences  $X_\gamma$  are full and nondegenerate, and that left action of each  $A_{r(\gamma)}$  on  $X_\gamma$  is implemented by an injective homomorphism into the compacts. One good reason for this is that if  $\Lambda = T_k$  is the  $k$ -graph with one vertex and one path of each degree (that is,  $\Lambda$  is isomorphic as a category to  $\mathbb{N}^k$ ), then  $\Lambda$ -systems are precisely the product systems of Hilbert bimodules over  $\mathbb{N}^k$  considered in [19, 56]; in particular, since this special case is not yet well understood, it seems bootless to worry overmuch about non-regular  $\Lambda$ -systems at this juncture.

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## 2. PRELIMINARIES

**2.1. Higher-rank graphs.** We will adopt the conventions of [31, 41] for  $k$ -graphs. Given a nonnegative integer  $k$ , a  $k$ -graph is a nonempty countable small category  $\Lambda$  equipped with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  satisfying the *factorisation property*: for all  $\lambda \in \Lambda$  and  $m, n \in \mathbb{N}^k$  such that  $d(\lambda) = m + n$  there exist unique  $\mu, \nu \in \Lambda$  such that  $d(\mu) = m$ ,  $d(\nu) = n$ , and  $\lambda = \mu\nu$ . When  $d(\lambda) = n$  we say  $\lambda$  has *degree*  $n$ . We will use  $d$  to denote the degree functor in every  $k$ -graph in this paper; the domain of  $d$  is always clear from context.

For  $k \geq 1$ , the standard generators of  $\mathbb{N}^k$  are denoted  $e_1, \dots, e_k$ , and for  $n \in \mathbb{N}^k$  and  $1 \leq i \leq k$  we write  $n_i$  for the  $i^{\text{th}}$  coordinate of  $n$ .

Given a  $k$ -graph  $\Lambda$ , for  $n \in \mathbb{N}^k$ , we write  $\Lambda^n$  for  $d^{-1}(n)$ . The *vertices* of  $\Lambda$  are the elements of  $\Lambda^0$ . The factorisation property implies that  $o \mapsto \text{id}_o$  is a bijection from the objects of  $\Lambda$  to  $\Lambda^0$ . We will frequently use this bijection to silently identify  $\text{Obj}(\Lambda)$  with  $\Lambda^0$ . The domain and codomain maps in the category  $\Lambda$  therefore become maps  $s, r : \Lambda \rightarrow \Lambda^0$ . More precisely, for  $\alpha \in \Lambda$ , the *source*  $s(\alpha)$  is the identity morphism associated with the object  $\text{dom}(\alpha)$  and similarly,  $r(\alpha) = \text{id}_{\text{cod}(\alpha)}$ .

Note that a 0-graph is then a countable category whose only morphisms are the identity morphisms; we think of a 0-graph as a collection of isolated vertices.

For  $u, v \in \Lambda^0$  and  $E \subset \Lambda$ , we write  $uE$  for  $E \cap r^{-1}(u)$  and  $Ev$  for  $E \cap s^{-1}(v)$ . We say that  $\Lambda$  is *row-finite* and has *no sources* if  $v\Lambda^n$  is finite and nonempty for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ .

Given  $\mu, \nu \in \Lambda$ , we say  $\lambda$  is a *common extension* of  $\mu$  and  $\nu$  if  $\lambda = \mu\alpha = \nu\beta$  for some  $\alpha, \beta \in \Lambda$ . This forces  $d(\lambda) \geq d(\mu) \vee d(\nu)$ . We say  $\lambda$  is *minimal* if  $d(\lambda) = d(\mu) \vee d(\nu)$ . We define

$$\Lambda^{\min}(\mu, \nu) = \{(\alpha, \beta) : \mu\alpha = \nu\beta \text{ is a minimal common extension of } \mu \text{ and } \nu\}.$$

Two important examples of  $k$ -graphs are the following. (1) For  $k \geq 1$  let  $\Omega_k$  be the small category with objects  $\text{Obj}(\Omega_k) = \mathbb{N}^k$ , and morphisms  $\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$ ; the range and source maps are given by  $r(m, n) = m$ ,  $s(m, n) = n$ . Define  $d : \Omega_k \rightarrow \mathbb{N}^k$  by  $d(m, n) = n - m$ . Then  $\Omega_k$  is a  $k$ -graph. (2) Let  $T = T_k$  be the semigroup  $\mathbb{N}^k$  viewed as a small category. If  $d : T \rightarrow \mathbb{N}^k$  is the identity map, then  $(T, d)$  is also a  $k$ -graph.

**2.2. The path groupoid of a higher rank graph.** In this section we summarise the construction and properties of the path groupoid associated to a row-finite higher-rank graph with no sources. For further details, see [31, 43].

By a  *$k$ -graph morphism* from a  $k$ -graph  $\Lambda$  to a  $k$ -graph  $\Gamma$ , we mean a functor  $f : \Lambda \rightarrow \Gamma$  which respects the degree maps.

**Definition 2.2.1.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. We define the infinite path space of  $\Lambda$  by

$$\Lambda^\infty = \{x : \Omega_k \rightarrow \Lambda : x \text{ is a } k\text{-graph morphism}\}.$$

For each  $p \in \mathbb{N}^k$  define  $\sigma^p : \Lambda^\infty \rightarrow \Lambda^\infty$  by  $\sigma^p(x)(m, n) = x(m + p, n + p)$  for all  $x \in \Lambda^\infty$  and  $(m, n) \in \Omega_k$ .

We define  $r : \Lambda^\infty \rightarrow \Lambda^0$  by  $r(x) = x(0, 0)$ . Our identification of vertices and objects in  $k$ -graphs identifies  $x(0, 0)$  with  $x(0)$ , and each  $x(n, n)$  with  $x(n)$ . For  $v \in \Lambda^0$ , set

$$v\Lambda^\infty := \{x \in \Lambda^\infty : r(x) = v\}.$$

For  $\lambda \in \Lambda$  and  $z \in s(\lambda)\Lambda^\infty$ , there is a unique  $x \in \Lambda^\infty$  such that  $x(0, d(\lambda)) = \lambda$  and  $\sigma^{d(\lambda)}(x) = z$ . We denote this infinite path  $x$  by  $\lambda z$ . The cylinder sets

$$Z(\lambda) := \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\},$$

where  $\lambda \in \Lambda$  are a basis of compact open sets for a Hausdorff topology on  $\Lambda^\infty$ . Note that  $v\Lambda^\infty$  is just  $Z(v)$ .

**Definition 2.2.2.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. Let

$$\mathcal{G}_\Lambda := \{(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \sigma^\ell(x) = \sigma^m(y), n = \ell - m\}.$$

Define the range and source maps  $r, s : \mathcal{G}_\Lambda \rightarrow \Lambda^\infty$  by  $r(x, n, y) = x$ ,  $s(x, n, y) = y$ . For  $(x, n, y), (y, \ell, z) \in \mathcal{G}_\Lambda$  set  $(x, n, y)(y, \ell, z) := (x, n + \ell, z)$ , and  $(x, n, y)^{-1} := (y, -n, x)$ . Then  $\mathcal{G}_\Lambda$  is a groupoid called the path groupoid of  $\Lambda$ .

For  $\lambda, \mu \in \Lambda$  with  $s(\lambda) = s(\mu)$ , we define

$$Z(\lambda, \mu) = \{(\lambda z, d(\lambda) - d(\mu), \mu z) \in \mathcal{G}_\Lambda : z \in s(\lambda)\Lambda^\infty\}.$$

The family  $\mathcal{U}_\Lambda := \{Z(\lambda, \mu) : s(\lambda) = s(\mu)\}$  is a basis of compact open bisections for a topology under which  $\mathcal{G}_\Lambda$  becomes a Hausdorff étale groupoid. Moreover, for every  $g = (x, n, y) \in \mathcal{G}_\Lambda$ , the collection of sets  $Z(\lambda, \mu)$  such that there is  $z \in \Lambda^\infty$  with  $r(z) = s(\lambda)$  satisfying  $x = \lambda z$ ,  $y = \mu z$  and  $n = d(\lambda) - d(\mu)$ , constitutes a neighborhood basis for  $g$ .

Fix  $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \Lambda$  with  $s(\lambda_i) = s(\mu_i)$ . Suppose that  $(x, n, y) \in Z(\lambda_1, \mu_1) \cap Z(\lambda_2, \mu_2)$ . Then  $x = \lambda_1 z_1 = \lambda_2 z_2$  and  $y = \mu_1 z_1 = \mu_2 z_2$  for some  $z_1, z_2 \in \Lambda^\infty$ . The factorisation property forces  $z_1 = \alpha z$  and  $z_2 = \beta z$  for some  $(\alpha, \beta) \in \Lambda^{\min}(\lambda_1, \lambda_2)$ . We then have

$$(2.2.1) \quad \mu_1 \alpha z = \mu_1 z_1 = y = \mu_2 z_2 = \mu_2 \beta z$$

so that  $\mu_1 \alpha = \mu_2 \beta$  is a common extension of  $\mu_1$  and  $\mu_2$ . Moreover that  $(x, n, y) \in Z(\lambda_1, \mu_1) \cap Z(\lambda_2, \mu_2)$  forces  $d(\lambda_1) - d(\mu_1) = n = d(\lambda_2) - d(\mu_2)$ , so

$$\begin{aligned} d(\alpha) &= (d(\lambda_1) \vee d(\lambda_2)) - d(\lambda_1) = ((d(\mu_1) + n) \vee (d(\mu_2) + n)) - (d(\mu_1) + n) = \\ &= (d(\mu_1) \vee d(\mu_2)) - d(\mu_1), \end{aligned}$$

and similarly

$$d(\beta) = (d(\mu_1) \vee d(\mu_2)) - d(\mu_2).$$

In particular  $d(\mu_1 \alpha) = d(\mu_1) \vee d(\mu_2) = d(\mu_2 \beta)$ , and combined with (2.2.1) this forces  $(\alpha, \beta) \in \Lambda^{\min}(\mu_1, \mu_2)$ . It is easy to check that for any  $(\alpha, \beta) \in \Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)$  and any  $z \in s(\alpha)\Lambda^\infty$ , we have  $(\lambda_1 \alpha z, d(\lambda_1) - d(\mu_1), \mu_1 \alpha z) \in Z(\lambda_1, \mu_1) \cap Z(\lambda_2, \mu_2)$ . It is likewise easy to check that if  $(\alpha, \beta), (\alpha', \beta')$  are distinct elements of  $\Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)$ , then  $\Lambda^{\min}(\lambda_1 \alpha, \lambda_2 \alpha') = \emptyset$ . We conclude that

$$(2.2.2) \quad \begin{aligned} Z(\lambda_1, \mu_1) \cap Z(\lambda_2, \mu_2) &= \bigsqcup_{(\alpha, \beta) \in \Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)} Z(\lambda_1 \alpha, \mu_1 \alpha) \\ &= \bigsqcup_{(\alpha, \beta) \in \Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)} Z(\lambda_2 \beta, \mu_2 \beta) \end{aligned}$$

Let  $p := (d(\lambda_1) \vee d(\lambda_2)) - d(\lambda_1)$ . Since

$$Z(\lambda_1, \mu_1) = \bigsqcup_{\alpha \in s(\lambda_1)\Lambda^p} Z(\lambda_1 \alpha, \mu_1 \alpha),$$

we also have

$$(2.2.3) \quad Z(\lambda_1, \mu_1) \setminus Z(\lambda_2, \mu_2) = \bigsqcup \{Z(\lambda_1 \alpha, \mu_1 \alpha) : \alpha \in s(\lambda_1)\Lambda^p, \nexists \beta \text{ such that } (\alpha, \beta) \in \Lambda^{\min}(\lambda_1, \lambda_2) \cap \Lambda^{\min}(\mu_1, \mu_2)\}.$$

**Lemma 2.2.3.** *Any finite union  $\bigcup_{i=1}^m Z(\lambda_i, \mu_i)$  of elements of  $\mathcal{U}_\Lambda$  can be expressed as a finite disjoint union  $\bigsqcup_{j=1}^n Z(\sigma_j, \tau_j)$  of elements of  $\mathcal{U}_\Lambda$  in such a way that each  $(\sigma_j, \tau_j)$  has the form  $(\sigma_j, \tau_j) = (\lambda_{i(j)} \nu_j, \mu_{i(j)} \nu_j)$  for some  $i(j) \leq m$  and  $\nu_j \in s(\lambda_{i(j)})\Lambda^0$ .*

*Proof.* Fix a finite collection of pairs  $\{(\lambda_i, \mu_i) : i = 1, \dots, n\}$  such that each  $s(\lambda_i) = s(\mu_i)$ . Then we may write  $\bigcup_{i=1}^n Z(\lambda_i, \mu_i)$  as the disjoint union

$$\bigcup_{i=1}^n Z(\lambda_i, \mu_i) = \bigsqcup_{i=1}^n \left( \bigcap_{j=1}^{i-1} (Z(\lambda_i, \mu_i) \setminus Z(\lambda_j, \mu_j)) \right).$$

Since intersection distributes over unions, it therefore suffices to show that given basis sets  $Z(\lambda, \mu)$  and  $Z(\sigma, \tau)$  in  $\mathcal{U}_\Lambda$ , the intersection  $Z(\lambda, \mu) \cap Z(\sigma, \tau)$  and the relative complement  $Z(\lambda, \mu) \setminus Z(\sigma, \tau)$  can both be written as finite disjoint unions of elements of  $\mathcal{U}_\Lambda$  of the desired form. These statements follow from (2.2.2) and (2.2.3).  $\square$

**2.3.  $C^*$ -algebras associated to higher-rank graphs.** Given a row-finite  $k$ -graph  $\Lambda$  with no sources, a Cuntz-Krieger  $\Lambda$ -family is a collection  $\{t_\lambda : \lambda \in \Lambda\}$  of partial isometries satisfying the Cuntz-Krieger relations:

- $\{t_v : v \in \Lambda^0\}$  is a collection of mutually orthogonal projections;
- $t_\lambda t_\mu = t_{\lambda\mu}$  whenever  $s(\lambda) = r(\mu)$ ;
- $t_\lambda^* t_\lambda = t_{s(\lambda)}$  for all  $\lambda \in \Lambda$ ; and
- $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$  for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ .

In [51], a family satisfying only the first three of these relations is called a Toeplitz-Cuntz-Krieger  $\Lambda$ -family. The  $k$ -graph  $C^*$ -algebra  $C^*(\Lambda)$  is the universal  $C^*$ -algebra generated by a Cuntz-Krieger  $\Lambda$ -family  $\{s_\lambda : \lambda \in \Lambda\}$ . That is, for every Cuntz-Krieger  $\Lambda$ -family  $\{t_\lambda : \lambda \in \Lambda\}$  there is a homomorphism  $\pi_t$  of  $C^*(\Lambda)$  satisfying  $\pi_t(s_\lambda) = t_\lambda$  for all  $\lambda \in \Lambda$ .

By [52, Theorem 3.15], the generators  $s_\lambda$  of  $C^*(\Lambda)$  are all nonzero.

If  $\Lambda$  is a 0-graph, then it trivially has no sources, and the last three Cuntz-Krieger relations follow from the first one. So  $C^*(\Lambda)$  is the universal  $C^*$ -algebra generated by mutually orthogonal projections  $\{s_v : v \in \Lambda^0\}$ ; that is  $C^*(\Lambda) \cong c_0(\Lambda^0)$ .

Let  $\Lambda$  be a  $k$ -graph. A standard argument (see, for example, [50, Proposition 2.1]) using the universal property shows that there is a strongly continuous action  $\gamma$  of  $\mathbb{T}^k$  on  $C^*(\Lambda)$ , called the *gauge action*, such that  $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$  for all  $z \in \mathbb{T}^k$  and  $\lambda \in \Lambda$ .

**2.4.  $C^*$ -correspondences.** We define Hilbert modules following [36] and [2, §II.7]. Let  $B$  be a  $C^*$ -algebra and let  $\mathcal{H}$  be a right  $B$ -module. Then a  $B$ -valued inner product on  $\mathcal{H}$  is a function  $\langle \cdot, \cdot \rangle_B : \mathcal{H} \times \mathcal{H} \rightarrow B$  satisfying the following conditions for all  $\xi, \eta, \zeta \in \mathcal{H}$ ,  $b \in B$  and  $\alpha, \beta \in \mathbb{C}$ :

- $\langle \xi, \alpha\eta + \beta\zeta \rangle_B = \alpha\langle \xi, \eta \rangle_B + \beta\langle \xi, \zeta \rangle_B$ ,
- $\langle \xi, \eta b \rangle_B = \langle \xi, \eta \rangle_B \cdot b$ ,
- $\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*$ ,
- $\langle \xi, \xi \rangle_B \geq 0$ , and  $\langle \xi, \xi \rangle_B = 0$  if and only if  $\xi = 0$ .

If  $\mathcal{H}$  is complete with respect to the norm  $\|\xi\|^2 := \|\langle \xi, \xi \rangle_B\|$ , then  $\mathcal{H}$  is said to be a (right-) *Hilbert  $B$ -module*. If the range of the inner product is not contained in any proper ideal in  $B$ ,  $\mathcal{H}$  is said to be *full*. Note that  $B$  may be endowed with the structure of a full Hilbert  $B$ -module by taking  $\langle \xi, \eta \rangle_B = \xi^* \eta$  for all  $\xi, \eta \in B$ . Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert  $B$ -modules; a map  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is an *adjointable operator* if there is a map  $T^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  such that  $\langle T\xi, \eta \rangle_B = \langle \xi, T^*\eta \rangle_B$  for all  $\xi, \eta \in \mathcal{H}$ . Such an operator is necessarily linear and bounded. We denote the collection of all adjointable operators by  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ . For  $\xi_i \in \mathcal{H}_i$  there is a rank-one adjointable operator  $\theta_{\xi_2, \xi_1} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  defined by  $\theta_{\xi_2, \xi_1}(\eta) = \xi_2 \cdot \langle \xi_1, \eta \rangle_B$ . Note that  $\theta_{\xi_2, \xi_1}^* = \theta_{\xi_1, \xi_2}$ . The closure of the span of such operators in  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  is

denoted  $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$  (the space of compact operators). For a Hilbert  $B$ -module  $\mathcal{H}$ , both  $\mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{H}, \mathcal{H})$  and  $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$  are  $C^*$ -algebras. Moreover,  $\mathcal{K}(\mathcal{H})$  is an essential ideal in  $\mathcal{L}(\mathcal{H})$ , and  $\mathcal{L}(\mathcal{H})$  may be identified with the multiplier algebra of  $\mathcal{K}(\mathcal{H})$ . There is a canonical identification  $\mathcal{H} \equiv \mathcal{K}(B, \mathcal{H})$  which identifies  $\xi \in \mathcal{H}$  with the operator  $b \mapsto \xi b$ . With this identification we have the factorization  $\theta_{\xi_2, \xi_1} = \xi_2 \xi_1^*$ . Set  $\mathcal{H}^* := \mathcal{K}(\mathcal{H}, B)$ . We may regard  $\mathcal{H}^*$  as a left-Hilbert  $B$ -module.

Let  $A$  and  $B$  be  $C^*$ -algebras; then a  $C^*$ -correspondence from  $A$  to  $B$  or more briefly an  $A$ - $B$   $C^*$ -correspondence is a Hilbert  $B$ -module  $\mathcal{H}$  together with a  $*$ -homomorphism  $\phi : A \rightarrow \mathcal{L}(\mathcal{H})$ . We often suppress  $\phi$ , writing  $a \cdot \xi$  for  $\phi(a)\xi$ . A homomorphism  $\phi : A \rightarrow B$  becomes a homomorphism from  $A$  to  $\mathcal{K}(B) = B$  when  $B$  is regarded as a Hilbert  $B$  module as above, and therefore gives  $B$  the structure of an  $A$ - $B$   $C^*$ -correspondence denoted  ${}_{\phi}B$ . So it is natural to think of an  $A$ - $B$   $C^*$ -correspondence as a generalised homomorphism from  $A$  to  $B$ .

A  $C^*$ -correspondence  $\mathcal{H}$  is said to be *nondegenerate* if  $\overline{\text{span}} \{ \phi(a)\xi : a \in A, \xi \in \mathcal{H} \} = \mathcal{H}$  (some authors have also called such  $C^*$ -correspondences *essential*). The above  $C^*$ -correspondence  ${}_{\phi}B$  is nondegenerate if and only if  $\phi : A \rightarrow B$  is *approximately unital*, that is,  $\phi$  maps approximate units into approximate units (note that by [2, Theorem II. 7.3.9] this condition implies that  $\phi$  extends to a unital map between the multiplier algebras).

As discussed in [2, 12, 37, 55], there is a category  $\mathcal{C}$  such that  $\text{Obj}(\mathcal{C})$  is the class of  $C^*$ -algebras, and  $\text{Hom}_{\mathcal{C}}(A, B)$  consists of all isomorphism classes of  $A$ - $B$   $C^*$ -correspondences (with identity morphisms  $[A]$ ). Composition

$$\text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

is defined by  $([\mathcal{H}_1], [\mathcal{H}_2]) \mapsto [\mathcal{H}_2 \otimes_B \mathcal{H}_1]$  where  $\mathcal{H}_2 \otimes_B \mathcal{H}_1$  denotes the balanced tensor product of  $C^*$ -correspondences. This  $\mathcal{H}_2 \otimes_B \mathcal{H}_1$  is called the *internal tensor product* of  $\mathcal{H}_2$  and  $\mathcal{H}_1$  by Blackadar and the *interior tensor product* by Lance (see [2, II.7.4.1] and [36, Prop. 4.5] and the following discussion).

Observe that any full right-Hilbert  $B$ -module  $\mathcal{H}$  is a nondegenerate  $\mathcal{K}(\mathcal{H})$ - $B$   $C^*$ -correspondence ( $\phi$  is the inclusion map); this is the basic example of an imprimitivity bimodule between  $\mathcal{K}(\mathcal{H})$  and  $B$ . In this case,  $\mathcal{K}(\mathcal{H})$  and  $B$  are said to be Morita-Rieffel equivalent. Moreover,  $\mathcal{H}^*$  may also be viewed as a  $B$ - $\mathcal{K}(\mathcal{H})$  imprimitivity bimodule. Given two Hilbert  $B$ -modules  $\mathcal{H}_1, \mathcal{H}_2$ , there is a natural isomorphism  $\mathcal{K}(\mathcal{H}_2, \mathcal{H}_1) \rightarrow \mathcal{H}_1 \otimes_B \mathcal{H}_2^*$  such that  $\theta_{\xi_1, \xi_2} \mapsto \xi_1 \otimes \xi_2^*$ .

**2.5. Representations of  $C^*$ -correspondences.** Let  $\mathcal{H}$  be an  $A$ - $A$   $C^*$ -correspondence. Recall from [27, 45], that a representation of  $\mathcal{H}$  in a  $C^*$ -algebra  $B$  is a pair  $(t, \pi)$  where  $\pi : A \rightarrow B$  is a homomorphism,  $t : \mathcal{H} \rightarrow B$  is linear, and such that for all  $a \in A$  and  $\xi, \eta \in \mathcal{H}$ , we have  $t(a \cdot \xi) = \pi(a)t(\xi)$ ,  $t(\xi \cdot a) = t(\xi)\pi(a)$ , and  $\pi(\langle \xi, \eta \rangle_A) = t(\xi)^*t(\eta)$ .

Given a  $C^*$ -correspondence  $\mathcal{H}$  over  $A$  and a representation  $(t, \pi)$  of  $\mathcal{H}$  in  $B$ , there is a homomorphism  $t^{(1)} : \mathcal{K}(\mathcal{H}) \rightarrow B$  satisfying  $t^{(1)}(\theta_{\xi, \eta}) = t(\xi)t(\eta)^*$  for all  $\xi, \eta \in \mathcal{H}$  (Pimsner denotes this homomorphism  $\pi^{(1)}$  in [45]). The pair  $(t, \pi)$  is said to be *Cuntz-Pimsner covariant* if  $t^{(1)}(\phi(a)) = \pi(a)$  for all  $a \in \phi^{-1}(\mathcal{K}(\mathcal{H})) \cap (\ker \phi)^\perp$  (see [27, Definition 3.4]).

In the cases of interest later in this paper,  $\phi$  is injective and  $\phi(A) \subset \mathcal{K}(\mathcal{H})$ , so  $t^{(1)} \circ \phi$  is a homomorphism from  $A$  to  $B$ . Then the pair  $(t, \pi)$  is Cuntz-Pimsner covariant if  $t^{(1)} \circ \phi = \pi$ .



Given a  $C^*$ -correspondence  $\mathcal{H}$  over  $A$ , there is a Cuntz-Pimsner covariant representation  $(j_{\mathcal{H}}, j_A)$  in a  $C^*$ -algebra  $\mathcal{O}_{\mathcal{H}}$  which is universal in the sense that given another Cuntz-Pimsner covariant representation  $(t, \pi)$  of  $\mathcal{H}$  in  $B$  there is a unique homomorphism  $t \times \pi : \mathcal{O}_{\mathcal{H}} \rightarrow B$  satisfying  $(t \times \pi) \circ j_{\mathcal{H}} = t$  and  $(t \times \pi) \circ j_A = \pi$ . Both  $j_{\mathcal{H}}$  and  $j_A$  are isometric. Moreover,  $\mathcal{O}_{\mathcal{H}}$  is unique up to canonical isomorphism. There is a strongly continuous gauge action  $\gamma : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_{\mathcal{H}})$  such that  $\gamma_z(j_{\mathcal{H}}(\xi)) = zj_{\mathcal{H}}(\xi)$  for  $\xi \in \mathcal{H}$  and  $\gamma_z(j_A(a)) = j_A(a)$  for  $a \in A$ .

**2.6. Product systems and representations.** Let  $(P, \cdot)$  be a discrete semigroup with identity  $e$  and let  $A$  be a  $C^*$ -algebra. A *product system* of  $A$ - $A$   $C^*$ -correspondences over  $P$  is a semigroup  $Y = \bigsqcup_{p \in P} Y_p$  such that

- for each  $p \in P$ ,  $Y_p \subset Y$  is a  $A$ - $A$   $C^*$ -correspondence with inner product  $\langle \cdot, \cdot \rangle_A^p$ ;
- the identity fiber  $Y_e$  is the  $A$ - $A$   $C^*$ -correspondence  ${}_{id}A$ ;
- for  $p, q \in P \setminus \{e\}$  there is an isomorphism  $\Theta_{p,q} : Y_p \otimes_A Y_q \rightarrow Y_{pq}$  satisfying  $\Theta_{p,q}(x \otimes_A y) = xy$  for all  $x \in Y_p$  and  $y \in Y_q$ ;
- multiplication in  $Y$  by elements of  $Y_e = A$  implements the right and left actions of  $A$  on each  $Y_p$ .

The homomorphism of  $A$  into  $\mathcal{K}(X_p)$  implementing the left action on  $Y_p$  is denoted  $\phi_p$ . The product system  $Y$  is said to be *nondegenerate* if each  $Y_p$  is a nondegenerate correspondence.

Let  $B$  be a  $C^*$ -algebra, and let  $Y$  be a nondegenerate product system such that each  $\phi_p$  is an injection into  $\mathcal{K}(X_p)$ . A map  $\psi : Y \rightarrow B$  is called a representation of  $Y$  if, writing  $\psi_p$  for  $\psi|_{Y_p}$ , we have

- each  $(\psi_p, \psi_e)$  is a representation of  $Y_p$ ; and
- $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$  for all  $p, q \in P, x \in Y_p, y \in Y_q$ .

There is a  $C^*$ -algebra  $\mathcal{T}_Y$  (called the Toeplitz algebra) and a representation  $i_Y : Y \rightarrow \mathcal{T}_Y$  which is universal in the following sense:  $\mathcal{T}_Y$  is generated by  $i_Y(Y)$  and for any representation  $\psi : Y \rightarrow B$  there is a homomorphism  $\psi_* : \mathcal{T}_Y \rightarrow B$  such that  $\psi_* \circ i_Y = \psi$ . The representation  $i_Y$  is isometric and unique up to isomorphism.

For each  $p \in P$  we write  $\psi^{(p)}$  for the homomorphism  $(\psi_p)^{(1)}$  discussed in the preceding section. The representation  $\psi$  is *Cuntz-Pimsner covariant* if each  $(\psi_p, \psi_e)$  is Cuntz-Pimsner covariant, that is if  $\psi^{(p)} \circ \phi_p = \psi_e$  for all  $p \in P$ .

There is a  $C^*$ -algebra  $\mathcal{O}_Y$  and a Cuntz-Pimsner covariant representation  $j_Y : Y \rightarrow \mathcal{O}_Y$  which is universal in the following sense: for any Cuntz-Pimsner covariant representation  $\psi : Y \rightarrow B$  there is a unique homomorphism  $\psi_* : \mathcal{O}_Y \rightarrow B$  such that  $\psi_* \circ j_Y = \psi$ . The pair  $(\mathcal{O}_Y, j_Y)$  is unique up to canonical isomorphism and  $j_Y$  is isometric. For more details about product systems, see [19].

For  $P = \mathbb{N}^k$ , universality allows us to define strongly continuous gauge actions  $\gamma : \mathbb{T}^k \rightarrow \text{Aut}(\mathcal{O}_Y)$  and  $\tilde{\gamma} : \mathbb{T}^k \rightarrow \text{Aut}(\mathcal{T}_Y)$  such that  $\gamma_z(j_Y(y)) = z^n j_Y(y)$  and  $\tilde{\gamma}_z(i_Y(y)) = z^n i_Y(y)$  for  $y \in Y_n$ . If  $Y$  is nondegenerate and each  $\phi_n$  is an injection into  $\mathcal{K}(Y_n)$ , the fixed point algebra  $\mathcal{O}_Y^\gamma$  is isomorphic to the inductive limit

$$\varinjlim_{n \in \mathbb{N}^k} \mathcal{K}(Y_n).$$

We will be mostly interested in nondegenerate product systems over  $P = \mathbb{N}^k$  (under addition) such that each  $\phi_n$  is an injection into  $\mathcal{K}(Y_n)$ .

**2.7. Fell bundles over groupoids.** Fell bundles over groupoids were introduced in [61] and subsequently studied in [30]. The notion of Fell bundle over a locally compact groupoid generalizes both the notion of  $C^*$ -algebraic bundle over a group (see [16, §11]) and that of  $C^*$ -algebra bundle over a space. Actions of groupoids on  $C^*$ -algebra bundles yield Fell bundles but not all Fell bundles arise in this way.

We assume familiarity with groupoids; we direct the reader to [53] or [44] for the necessary background or to [31] for details on groupoids associated to higher-rank graphs.

**Definition 2.7.1** ([30, Definition 2.1]). Let  $\mathcal{G}$  be a locally compact Hausdorff groupoid and let  $\pi : E \rightarrow \mathcal{G}$  be a Banach bundle. Define

$$E^{(2)} = \{(e_1, e_2) \in E \times E : (\pi(e_1), \pi(e_2)) \in \mathcal{G}^{(2)}\}.$$

A *multiplication* on  $E$  is a continuous map  $(e_1, e_2) \mapsto e_1 e_2$  from  $E^{(2)}$  to  $E$  which satisfies:

- (i)  $\pi(e_1 e_2) = \pi(e_1) \pi(e_2)$  for all  $(e_1, e_2) \in E^{(2)}$
- (ii) the induced map  $E_{g_1} \times E_{g_2} \rightarrow E_{g_1 g_2}$  is bilinear for all  $(g_1, g_2) \in \mathcal{G}^{(2)}$
- (iii)  $(e_1 e_2) e_3 = e_1 (e_2 e_3)$  whenever the multiplication is defined
- (iv)  $\|e_1 e_2\| \leq \|e_1\| \|e_2\|$  for all  $(e_1, e_2) \in E^{(2)}$ .

An *involution* on  $E$  is a continuous map  $e \mapsto e^*$  from  $E$  to  $E$  which satisfies:

- (v)  $\pi(e^*) = \pi(e)^{-1}$  for all  $e \in E$
- (vi) the restriction of the involution to  $E_g$  is conjugate linear for all  $g \in \mathcal{G}$
- (vii)  $e^{**} = e$  for all  $e \in E$ .

Finally, the bundle  $E$  together with the structure maps is said to be a *Fell bundle* if in addition the following conditions hold:

- (viii)  $(e_1 e_2)^* = e_2^* e_1^*$  for all  $(e_1, e_2) \in E^{(2)}$
- (ix)  $\|e^* e\| = \|e\|^2$  for all  $e \in E$
- (x) for each  $e \in E$ ,  $e^* e$  is positive as an element of  $E_{s(\pi(e))}$  (which is a  $C^*$ -algebra by (i)-(ix)).

The Fell bundle  $E$  is said to be *saturated* if  $E_{g_1} \cdot E_{g_2}$  is total in  $E_{g_1 g_2}$  for all  $(g_1, g_2) \in \mathcal{G}^{(2)}$ .

*Remark 2.7.2.* It follows that  $\|e^*\| = \|e\|$  for all  $e \in E$ .

Note that  $E_g$  is a right-Hilbert  $E_{s(g)}$ -module with right action implemented by multiplication and inner product given by

$$\langle e_1, e_2 \rangle_{E_{s(g)}} = e_1^* e_2.$$

Similarly,  $E_g$  can be regarded as a left Hilbert  $E_{r(g)}$ -module. Observe that  $E$  is saturated if and only if  $E_g$  is full as a right Hilbert module for all  $g$ . Moreover,  $E$  is saturated if and only if  $E_g$  is an  $E_{r(g)}-E_{s(g)}$  imprimitivity bimodule for all  $g$ . In this case, if  $s(g_1) = x = r(g_2)$ , there is an isomorphism  $E_{g_1} \otimes_{E_x} E_{g_2} \cong E_{g_1 g_2}$  such that  $e_1 \otimes e_2 \mapsto e_1 e_2$ .

We denote by  $E^{(0)}$  the restriction of the bundle  $E$  to  $\mathcal{G}^{(0)}$ , and observe that  $E^{(0)}$  is a  $C^*$ -bundle. We denote the corresponding  $C^*$ -algebra of sections vanishing at infinity by  $C_0(\mathcal{G}^{(0)}, E^{(0)})$ .

Given a Fell bundle  $E$  over an étale groupoid  $\mathcal{G}$  we construct the  $C^*$ -algebra  $C_r^*(\mathcal{G}, E)$  as a completion of  $C_c(\mathcal{G}, E)$  in the following way. First we use the groupoid structure of  $\mathcal{G}$  to define a product and involution on  $C_c(\mathcal{G}, E)$ :

$$(f_1 f_2)(g) = \sum_{g=g_1 g_2} f_1(g_1) f_2(g_2), \quad f^*(g) = f(g^{-1})^*.$$

Then regard  $C_c(\mathcal{G}, E)$  as a pre-Hilbert right  $C_0(\mathcal{G}^{(0)}, E^{(0)})$ -module under pointwise operations, and form the completion  $L^2(\mathcal{G}, E)$ . Left multiplication by elements of  $C_c(\mathcal{G}, E)$  induces an embedding into  $\mathcal{L}(L^2(\mathcal{G}, E))$ . We define  $C_r^*(\mathcal{G}, E)$  to be the completion of the image of  $C_c(\mathcal{G}, E)$  in the operator norm. For more details see [30, §3].

### 3. $\Lambda$ -SYSTEMS OF $C^*$ -CORRESPONDENCES AND REPRESENTATIONS

**3.1.  $\Lambda$ -systems of  $C^*$ -correspondences.** Given a  $k$ -graph  $\Lambda$ , we define a  $\Lambda$ -system of  $C^*$ -correspondences by associating a  $C^*$ -algebra to each vertex, and a  $C^*$ -correspondence to each path, as follows.

**Definition 3.1.1.** Let  $\Lambda$  be a  $k$ -graph. Fix

- for each vertex  $v \in \Lambda^0$  a  $C^*$ -algebra  $A_v$ ;
- for each  $\lambda \in \Lambda$  an  $A_{r(\lambda)}-A_{s(\lambda)}$   $C^*$ -correspondence  $X_\lambda$ ; and
- for each composable pair  $\alpha, \beta$  in  $\Lambda$  a map of  $A_{r(\alpha)}-A_{s(\beta)}$   $C^*$ -correspondences:  $\chi_{\alpha, \beta} : X_\alpha \otimes_{A_{s(\alpha)}} X_\beta \rightarrow X_{\alpha\beta}$  such that if  $\alpha \notin \Lambda^0$ , then  $\chi_{\alpha, \beta}$  is an isomorphism.

Suppose that the  $A_v$ , the  $X_\lambda$  and the  $\chi_{\alpha, \beta}$  have the following properties:

- (1) for each  $v \in \Lambda^0$ ,  $X_v = id A_v$  (the identity correspondence over  $A_v$ );
- (2) for each  $\lambda \in \Lambda$ , the maps  $\chi_{r(\lambda), \lambda}$  and  $\chi_{\lambda, s(\lambda)}$  are given by

$$\chi_{r(\lambda), \lambda}(a \otimes_{A_{r(\lambda)}} x) = \phi_\lambda(a)x \quad \text{and} \quad \chi_{\lambda, s(\lambda)}(x \otimes_{A_{s(\lambda)}} a) = x \cdot a.$$

- (3) for each composable triple  $\alpha, \beta, \gamma \in \Lambda$ , the following diagram commutes.

$$\begin{array}{ccc} X_\alpha \otimes_{A_{s(\alpha)}} X_\beta \otimes_{A_{s(\beta)}} X_\gamma & \xrightarrow{\chi_{\alpha, \beta} \otimes id_{X_\gamma}} & X_{\alpha\beta} \otimes_{A_{s(\beta)}} X_\gamma \\ id_{X_\alpha} \otimes \chi_{\beta, \gamma} \downarrow & & \chi_{\alpha\beta, \gamma} \downarrow \\ X_\alpha \otimes_{A_{s(\alpha)}} X_{\beta\gamma} & \xrightarrow{\chi_{\alpha, \beta\gamma}} & X_{\alpha\beta\gamma} \end{array}$$

Then we say that  $(A, X, \chi)$  is a  $\Lambda$ -system of  $C^*$ -correspondences. By the usual abuse of notation, we will frequently just say that  $X$  is a  $\Lambda$ -system of  $C^*$ -correspondences.

**Definition 3.1.2.** We shall say that a system  $X$  of  $C^*$ -correspondences is *regular* if it satisfies all of the following assumptions:

- $\Lambda$  is row-finite and has no sources;
- each  $X_\lambda$  is nondegenerate and full; and
- each  $\phi_\lambda : A_{r(\lambda)} \rightarrow \mathcal{K}(X_\lambda)$  is injective and takes values in  $\mathcal{K}(X_\lambda)$ .

*Remark 3.1.3.* Suppose that  $X$  is a regular  $\Lambda$ -system of  $C^*$ -correspondences. Note that each map  $\phi_\lambda : A_{r(\lambda)} \rightarrow \mathcal{K}(X_\lambda)$  is approximately unital. Furthermore, for every  $\lambda$ , since  $X_\lambda$  is nondegenerate,  $\chi_{r(\lambda), \lambda} : X_{r(\lambda)} \otimes_{A_{r(\lambda)}} X_\lambda \rightarrow X_\lambda$  is an isomorphism. Hence by the third bullet-point of Definition 3.1.1,  $\chi_{\alpha, \beta}$  is an isomorphism for every composable pair  $\alpha, \beta$ .

*Remark 3.1.4.* Fix a  $\Lambda$ -system  $X$  of  $C^*$ -correspondences. Recall that  $\mathcal{C}$  denotes the category whose objects are  $C^*$ -algebras and whose morphisms are isomorphism classes of  $C^*$ -correspondences. There is a contravariant functor  $F_X$  from  $\Lambda$  to  $\mathcal{C}$  determined by  $F_X(\lambda) = [X_\lambda]$ ; in particular, the object map satisfies  $F_X^0(v) = A_v$ . As with systems of  $k$ -morphs (see [34]), more than one system may determine the same functor, and there are functors which cannot be obtained in this way from any system.

*Remark 3.1.5.* To specify a  $\Lambda$ -system when  $\Lambda$  is a 1-graph it suffices to give a  $C^*$ -algebra  $A_v$  for each vertex  $v \in \Lambda^0$  and a  $C^*$ -correspondence  $X_\lambda$  for each edge  $\lambda \in \Lambda^1$ . For  $\lambda = \lambda_1 \cdots \lambda_n \in \Lambda^n$ , with  $n \geq 2$  and  $\lambda_i \in \Lambda^1$ , define

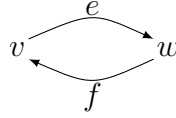
$$X_\lambda = X_{\lambda_1} \otimes_{A_{s(\lambda_1)}} X_{\lambda_2} \otimes_{A_{s(\lambda_2)}} \cdots \otimes_{A_{s(\lambda_{n-1})}} X_{\lambda_n};$$

the maps  $\chi_{\alpha,\beta}$  are given by the canonical isomorphisms.

When  $\Lambda$  is a 0-graph, a  $\Lambda$ -system of  $C^*$ -correspondences simply consists of a  $C^*$ -algebra  $A_v$  for each vertex  $v \in \Lambda^0$ .

*Examples 3.1.6.* We pause to mention a number of examples from the literature which can be regarded as  $\Lambda$ -systems. We will indicate how each example relates to a  $\Lambda$ -system, but will postpone detailed discussions of these and a number of other examples until Section 5.

- (i) In [39, section 3], the authors consider two  $C^*$ -algebras  $A$  and  $B$ , an  $A$ - $B$   $C^*$ -correspondence  $R$ , and a  $B$ - $A$   $C^*$ -correspondence  $S$ . Suppose  $R$  and  $S$  are nondegenerate with both left actions injective and given by compacts. Then they prove that the Cuntz-Pimsner algebras  $\mathcal{O}_{R \otimes_B S}$  and  $\mathcal{O}_{S \otimes_A R}$  are Morita-Rieffel equivalent. Let  $\Lambda$  be the path category of the directed graph pictured below.



If we let  $A_v := A$ ,  $A_w := B$ ,  $X_e = R$  and  $X_f = S$ , then we obtain a  $\Lambda$ -system  $X$  of correspondences as in Remark 3.1.5. If we assume in addition that  $R$  and  $S$  are full, then the  $\Lambda$ -system is regular.

- (ii) Recall from [34] that, given  $k$ -graphs  $\Lambda$  and  $\Gamma$ , a  $\Lambda$ - $\Gamma$   $k$ -morph  $X$  is a set  $X$  together with range and source maps  $r : X \rightarrow \Lambda^0$  and  $s : X \rightarrow \Gamma^0$  and a bijection

$$\phi : \{(x, \gamma) \in X \times \Gamma : s(x) = r(\gamma)\} \rightarrow \{(\lambda, y) \in \Lambda \times X : s(\lambda) = r(y)\}$$

such that: whenever  $\phi(x, \gamma) = (\lambda, y)$ , we have  $r(x) = r(\lambda)$ ,  $s(y) = s(\gamma)$ , and  $d(\lambda) = d(\gamma)$ ; and whenever  $\phi(x, \gamma) = (\lambda, y)$  and  $\phi(y, \sigma) = (\mu, z)$ , we have  $\phi(x, \gamma\sigma) = (\lambda\mu, z)$ . If  $\Gamma$  is an  $\ell$ -graph, then a  $\Gamma$ -system  $W$  of  $k$ -morphs consists, roughly speaking, of  $k$ -graphs  $\Lambda_v$  associated to the vertices  $v \in \Gamma^0$ ,  $k$ -morphs  $W_\gamma$  associated to the paths  $\gamma \in \Gamma$ , and compatible isomorphisms  $\theta_{\alpha,\beta} : W_\alpha *_{\Lambda_{s(\alpha)}} W_\beta \rightarrow W_{\alpha\beta}$ . Under the technical hypothesis  $(\spadesuit)$ , we have by [34, Proposition 6.4] that each  $k$ -morph  $W_\gamma$  gives rise to a full nondegenerate  $C^*$ -correspondence  $\mathcal{H}(W_\gamma)$  whose left action is implemented by an injective homomorphism into the compacts and it also follows implicitly from the proof of [34, Theorem 6.6] that the  $\theta_{\alpha,\beta}$  determine isomorphisms  $\chi(\theta_{\alpha,\beta}) : \mathcal{H}(W_\alpha) \otimes_{C^*(\Lambda_{s(\alpha)})} \mathcal{H}(W_\beta) \rightarrow \mathcal{H}(W_{\alpha\beta})$ . In particular, if  $\Gamma$  is row-finite with no sources, and each  $W_\gamma$  satisfies  $(\spadesuit)$ , then the assignments  $A_v := C^*(\Lambda_v)$  and  $X_\gamma := \mathcal{H}(W_\gamma)$  and the isomorphisms  $\chi(\theta_{\alpha,\beta}) : X_\alpha \otimes_{A_{s(\alpha)}} X_\beta \rightarrow X_{\alpha\beta}$  determine a regular  $\Gamma$ -system of  $C^*$ -correspondences.

- (iii) Every product system of  $C^*$ -correspondences over  $\mathbb{N}^k$  can be regarded as a  $T_k$ -system of  $C^*$ -correspondences where  $T_k$  is the  $k$ -graph isomorphic as a category to  $\mathbb{N}^k$  (see section 2.1).
- (iv) Given a  $C^*$ -algebra  $A$ , let  $\text{End}_1(A)$  denote the semigroup of approximately unital endomorphisms of  $A$ , regarded as a category with one object  $A$ . Let  $\Lambda$  be a  $k$ -graph. Then each contravariant functor  $\varphi : \Lambda \rightarrow \text{End}_1(A)$  determines a  $\Lambda$ -system

of  $C^*$ -correspondences. Specifically,  $A_v := A$  for all  $v \in \Lambda^0$ ,  $X_\lambda := \varphi(\lambda)A$  for all  $\lambda \in \Lambda$ , and  $\chi_{\alpha,\beta}$  is defined by  $\chi_{\alpha,\beta}(a \otimes b) := \phi_\beta(a)b$  for all  $a \in X_\alpha = \varphi(\alpha)A$  and  $b \in X_\beta = \varphi(\beta)A$ . Conditions (1) and (2) of Definition 3.1.1 are clearly satisfied, and condition (3) boils down to the identity

$$\varphi_\gamma(\varphi_\beta(a)b)c = \varphi_{\beta\gamma}(a)\varphi_\gamma(b)c.$$

The system is regular precisely when  $\Lambda$  is row-finite with no sources, and each  $\varphi_\lambda$  is injective. This example may easily be generalized by replacing the target category  $\text{End}_1(A)$  with the category of  $C^*$ -algebras and approximately unital homomorphisms.

Fix a  $\Lambda$ -system  $X$  of  $C^*$ -correspondences. Let  $A$  denote the  $c_0$  direct sum

$$A = \bigoplus_{v \in \Lambda^0} A_v.$$

We can regard the  $X_\lambda$  as  $A$ - $A$  correspondences in the obvious way. Hence, for  $n \in \mathbb{N}^k$  we may define a  $C^*$ -correspondence  $Y_n$  over  $A$  by

$$Y_n = \bigoplus_{\lambda \in \Lambda^n} X_\lambda$$

(this time, we are taking an  $\ell^2$  direct sum). For  $\alpha \in \Lambda$ , let  $\iota_\alpha : X_\alpha \rightarrow Y_{d(\alpha)}$  denote the inclusion map.

By checking that it preserves inner-products, one can see that for  $m, n \in \mathbb{N}^k \setminus \{0\}$ , the formula

$$\Theta_{m,n}(\iota_\alpha(x) \otimes_A \iota_\beta(y)) := \begin{cases} \iota_{\alpha,\beta}(\chi_{\alpha,\beta}(x \otimes_{A_{s(\alpha)}} y)) & \text{if } s(\alpha) = r(\beta) \\ 0_{Y_{m+n}} & \text{otherwise} \end{cases}$$

determines an isomorphism  $\Theta_{m,n} : Y_m \otimes_A Y_n \rightarrow Y_{m+n}$ .

**Proposition 3.1.7.** *With notation as above,*

$$Y = Y_X := \bigsqcup_{n \in \mathbb{N}^k} Y_n$$

*is a product system over  $\mathbb{N}^k$ . If  $X$  is regular (this entails, in particular, that  $\Lambda$  is row-finite and has no sources), then  $Y_X$  is nondegenerate, and the left action of  $A$  on each fibre  $Y_n$  of  $Y_X$  is implemented by an injection of  $A$  into  $\mathcal{K}(Y_n)$ .*

*Proof.* Fix  $l, m, n \in \mathbb{N}^k$ . Then  $Y_l \otimes_A Y_m \otimes_A Y_n$  is spanned by the subspaces

$$\{\iota_\lambda(X_\lambda) \otimes_{A_{s(\lambda)}} \iota_\mu(X_\mu) \otimes_{A_{s(\mu)}} \iota_\nu(X_\nu) : \lambda \in \Lambda^l, \mu \in \Lambda^m, \nu \in \Lambda^n, s(\lambda) = r(\mu), s(\mu) = r(\nu)\}.$$

Fix  $x \in X_\mu, y \in X_\nu$ , and  $z \in X_\nu$ , and for convenience, write  $\bar{x}$  for  $\iota_\lambda(x) \in Y_l$  and similarly for  $\mu$  and  $\nu$ . Then the associativity condition Definition 3.1.1(3) ensures that

$$\begin{aligned} (\bar{x}\bar{y})\bar{z} &= \theta_{lm,n}(\theta_{l,m} \otimes 1)(\bar{x} \otimes \bar{y} \otimes \bar{z}) \\ &= \iota_{\lambda\mu\nu}(\chi_{\lambda\mu,\nu}(\chi_{\lambda,\mu} \otimes 1)(x \otimes y \otimes z)) = \iota_{\lambda\mu\nu}(\chi_{\lambda,\mu\nu}(1 \otimes \chi_{\mu,\nu})(x \otimes y \otimes z)) = \bar{x}(\bar{y}\bar{z}). \end{aligned}$$

Hence  $Y$  is a product system.

Now suppose that  $X$  is regular. It is immediate that  $Y$  is nondegenerate because each  $X_\lambda$  is. To see that each  $\phi_n$  is injective, fix  $n \in \mathbb{N}$  and  $a \in A \setminus \{0\}$ . Then there is some  $v \in \Lambda^0$  such that the component  $a_v$  of  $a$  in  $A_v$  is nonzero. Since  $\Lambda$  has no sources, there exists  $\lambda \in v\Lambda^n$ . Since  $\phi_\lambda$  is injective, the direct summand  $\phi_\lambda(a_v)$  of  $\phi_n(a)$  is nonzero, and hence  $\phi_n(a)$  is itself nonzero. Finally, to see that  $\phi_n$  takes values in  $\mathcal{K}(Y_n)$ , observe that the  $A_v$  span a dense subspace of  $A$  and that for a fixed  $v$  and  $a \in A_v$ , the operator

$\phi_n(a) = \bigoplus_{\lambda \in v\Lambda^n} \phi_\lambda(a)$  belongs to  $\mathcal{K}(Y_n)$  because each  $\phi_\lambda(a) \in \mathcal{K}(X_\lambda)$  and because  $v\Lambda^n$  is finite.  $\square$

The construction of the product system  $Y$  from the  $\Lambda$ -system  $X$  is the analogue for systems of correspondences of the  $\Gamma$ -bundle construction from [34].

Fix a  $\Lambda$ -system  $X$ , and paths  $\alpha, \beta \in \Lambda$  with  $s(\alpha) = r(\beta)$ . As on [36, page 42], there is a homomorphism  $(\phi_\beta)_* : \mathcal{L}(X_\alpha) \rightarrow \mathcal{L}(X_\alpha \otimes_{A_{s(\alpha)}} X_\beta)$  characterised by

$$(\phi_\beta)_*(T)(x \otimes y) = T(x) \otimes y.$$

By [36, Proposition 4.7], if  $\phi_\beta(A_{s(\alpha)}) \subset \mathcal{K}(X_\beta)$ , then  $(\phi_\beta)_*(\mathcal{K}(X_\alpha)) \subset \mathcal{K}(X_\alpha \otimes_{A_{s(\alpha)}} X_\beta)$ , and  $(\phi_\beta)_*$  is injective if  $\phi_\beta$  is injective, and surjective if  $\phi_\beta$  is surjective. We define a homomorphism  $i_\alpha^{\alpha\beta} : \mathcal{L}(X_\alpha) \rightarrow \mathcal{L}(X_{\alpha\beta})$  by

$$(3.1.1) \quad i_\alpha^{\alpha\beta}(S) := \chi_{\alpha,\beta} \circ (\phi_\beta)_*(S) \circ \chi_{\alpha,\beta}^{-1}.$$

Hence, if  $\phi_\beta : A_{r(\beta)} \rightarrow \mathcal{L}(X_\beta)$  takes values in  $\mathcal{K}(X_\beta)$ , then  $i_\alpha^{\alpha\beta}$  restricts to a homomorphism from  $\mathcal{K}(X_\alpha)$  to  $\mathcal{K}(X_{\alpha\beta})$ , which is injective if  $\phi_\beta$  is.

The maps  $i_\alpha^{\alpha\beta}$  are compatible with composition in  $\Lambda$  in the sense that for a composable triple  $\alpha, \beta, \gamma$  of  $\Lambda$ , we have

$$i_{\alpha\beta}^{\alpha\beta\gamma} \circ i_\alpha^{\alpha\beta} = i_\alpha^{\alpha\beta\gamma}.$$

To see this, fix  $S \in \mathcal{L}(X_\alpha)$ , apply each of  $(i_{\alpha\beta}^{\alpha\beta\gamma} \circ i_\alpha^{\alpha\beta})(S)$  and  $i_\alpha^{\alpha\beta\gamma}(S)$  to the image of an elementary tensor  $x \otimes y \otimes z$  and use condition (3) of Definition 3.1.1 to see that they agree.

Similarly, if  $Y$  is a product system over  $\mathbb{N}^k$  and  $m, n \in \mathbb{N}^k$ , let  $i_m^{m+n} : \mathcal{L}(Y_m) \rightarrow \mathcal{L}(Y_{m+n})$  denote the map obtained as in (3.1.1) from the isomorphisms  $\Theta_{m,n} : Y_m \otimes_A Y_n \rightarrow Y_{m+n}$ . If  $\phi_n(A) \subset \mathcal{K}(Y_n)$  we reuse the symbol  $i_m^{m+n}$  to denote the restriction of  $i_m^{m+n}$  to a homomorphism from  $\mathcal{K}(Y_m)$  to  $\mathcal{K}(Y_{m+n})$ .

### 3.2. Representations of $\Lambda$ -systems of $C^*$ -correspondences.

**Definition 3.2.1.** Let  $(A, X, \chi)$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences. A *representation* of  $X$  in a  $C^*$ -algebra  $B$  is a pair  $(\rho, \pi)$  consisting of

- linear maps  $\rho_\lambda : X_\lambda \rightarrow B$ , and
- homomorphisms  $\pi_v : A_v \rightarrow B$

which satisfy the following conditions.

- (1) For each  $v \in \Lambda^0$ ,  $\rho_v = \pi_v$ .
- (2) For  $\alpha, \beta \in \Lambda$ ,  $x \in X_\alpha$  and  $y \in X_\beta$ ,

$$\rho_\alpha(x)\rho_\beta(y) = \begin{cases} \rho_{\alpha\beta}(\chi_{\alpha,\beta}(x \otimes_{A_{s(\alpha)}} y)) & \text{if } s(\alpha) = r(\beta) \\ 0_B & \text{if } s(\alpha) \neq r(\beta). \end{cases}$$

- (3) For all  $\alpha, \beta \in \Lambda$  with  $d(\alpha) = d(\beta)$ , and all  $x \in X_\alpha$  and  $y \in X_\beta$ ,

$$\rho_\alpha(x)^* \rho_\beta(y) = \begin{cases} \pi_{s(\alpha)}(\langle x, y \rangle_{A_{s(\alpha)}}) & \text{if } \alpha = \beta \\ 0_B & \text{otherwise.} \end{cases}$$

We say that a representation  $(\rho, \pi)$  of  $X$  in  $B$  is *Cuntz-Pimsner covariant* if

(4) for all  $v \in \Lambda^0$ , all  $n \in \mathbb{N}^k$  and all  $a \in A_v$ , we have

$$\pi_{r(\lambda)}(a) = \sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)}(\phi_\lambda(a)),$$

where  $\rho^{(\lambda)} = \rho_\lambda^{(1)}$ .

Let  $(\rho, \pi)$  be a representation of  $(A, X, \chi)$  in a  $C^*$ -algebra  $B$ . We will say that  $(\rho, \pi)$  is *universal* if for any other representation  $(\rho', \pi')$  of  $(A, X, \chi)$  in a  $C^*$ -algebra  $C$ , there is a unique homomorphism  $\Phi = \Phi_{\rho', \pi'} : B \rightarrow C$  satisfying  $\Phi \circ \rho_\lambda = \rho'_\lambda$  for all  $\lambda \in \Lambda$ , and  $\Phi \circ \pi_v = \pi'_v$  for all  $v \in \Lambda^0$ . A Cuntz-Pimsner covariant representation of  $(A, X, \chi)$  is universal if it has the universal property described above with respect to Cuntz-Pimsner covariant representations  $(\rho', \pi')$ .

*Remark 3.2.2.* Since, in a regular  $\Lambda$ -system of  $C^*$ -correspondences, each  $X_\lambda$  is nondegenerate, conditions (2) and (3) of Definition 3.2.1 are then equivalent to the apparently weaker relations

- (A)  $\pi_v(A_v) \perp \pi_w(A_w)$  for distinct  $v, w \in \Lambda^0$ .
- (B)  $\rho_\alpha(x)\rho_\beta(y) = \rho_{\alpha\beta}(\chi_{\alpha,\beta}(x \otimes_{A_{s(\alpha)}} y))$  whenever  $s(\alpha) = r(\beta)$ ,  $x \in X_\alpha$  and  $y \in X_\beta$ .
- (C)  $\pi_{s(\lambda)}(\langle x, y \rangle_{A_{s(\lambda)}}) = \rho_\lambda(x)^* \rho_\lambda(y)$  for all  $\lambda \in \Lambda$  and  $x, y \in X_\lambda$ .

We discuss the complications which would arise in the absence of the regularity hypothesis at the end of the section in Remark 3.3.3

**Proposition 3.2.3.** *Let  $\Lambda$  be a row-finite  $k$ -graph with no sources, and let  $X$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences. Let  $Y = Y_X$  be the product system over  $\mathbb{N}^k$  of  $C^*$ -correspondences over  $A = \bigoplus A_v$  obtained as above.*

- (1) *If  $\psi$  is a representation of  $Y$  in a  $C^*$ -algebra  $B$ , then there is a representation  $(\rho^\psi, \pi^\psi)$  of  $X$  in  $B$  given by  $\rho_\alpha^\psi := \psi \circ \iota_\alpha$  and  $\pi_v^\psi := \psi \circ \iota_v$ .*
- (2) *Conversely if  $(\rho, \pi)$  is a representation of  $X$  in a  $C^*$ -algebra  $B$ , then there is a representation  $\psi^{(\rho, \pi)}$  of  $Y$  in  $B$  determined by  $\psi^{(\rho, \pi)}(\iota_\alpha(x)) = \rho_\alpha(x)$  for  $\alpha \in \Lambda$  and  $x \in X_\alpha$ .*

*These constructions are mutually inverse in the sense that for a representation  $\psi$  of  $Y$ , we have  $\psi^{(\rho^\psi, \pi^\psi)} = \psi$ , and for a representation  $(\rho, \pi)$  of  $X$ , we have  $(\rho^{\psi^{(\rho, \pi)}}, \pi^{\psi^{(\rho, \pi)}}) = (\rho, \pi)$ . The representation  $(\rho^{i_Y}, \pi^{i_Y})$  of  $X$  in  $\mathcal{T}_Y$  is universal in the sense described above.*

*Proof.* For (1), fix a representation  $\psi$  of  $Y$  in  $B$ , and let  $\rho^\psi$  and  $\pi^\psi$  be as in (1). We have  $\rho_v^\psi = \pi_v^\psi$  for all  $v$  by definition.

Fix  $\alpha, \beta \in \Lambda$ ,  $x \in X_\alpha$  and  $y \in X_\beta$ . We must establish Definition 3.2.1(2). When  $s(\alpha) = r(\beta)$  this follows because multiplication in  $Y$  is determined by the isomorphisms  $\chi_{\mu, \nu}$ , and  $\psi$  is multiplicative. When  $s(\alpha) \neq r(\beta)$ , the left-hand side of Definition 3.2.1(2) is equal to zero by the Hewitt-Cohen factorisation theorem because the  $\pi_v^\psi(A_v)$  are orthogonal.

To verify Definition 3.2.1(3), Fix  $\alpha, \beta \in \Lambda$ ,  $x \in X_\alpha$  and  $y \in X_\beta$ . Then

$$\rho_\alpha^\psi(x)^* \rho_\beta^\psi(y) = \psi(\iota_\alpha(x))^* \psi(\iota_\beta(y)) = \psi(\langle \iota_\alpha(x), \iota_\beta(y) \rangle_A),$$

and the desired relation holds because  $\alpha \neq \beta$  forces  $\langle \iota_\alpha(x), \iota_\beta(y) \rangle_A = 0$ , and because  $\psi$  is a representation. This establishes (1).

For (2), let  $(\rho, \pi)$  be a representation of  $X$ . By definition of the direct sums  $Y_n$ ,  $n \in \mathbb{N}^k$ , there exists a map  $\psi^{(\rho, \pi)} : Y \rightarrow B$  satisfying the given formulae, and the restriction  $\psi_0$  of  $\psi$  to  $Y_0 = A$  is a  $C^*$ -homomorphism. Fix  $x, y \in Y$ , say  $x \in Y_m$  and  $y \in Y_n$ . We must

show that  $\psi^{(\rho,\pi)}(xy) = \psi^{(\rho,\pi)}(x)\psi^{(\rho,\pi)}(y)$ . Since multiplication in  $Y$  implements bimodule isomorphisms, it suffices to consider  $x \in X_\mu$  and  $y \in Y_\nu$  for some  $\mu \in \Lambda^m$ ,  $\nu \in \Lambda^n$ , and show that

$$(3.2.1) \quad \psi^{(\rho,\pi)}(\iota_\mu(x)\iota_\nu(y)) = \psi^{(\rho,\pi)}(\iota_\mu(x))\psi^{(\rho,\pi)}(\iota_\nu(y)).$$

This follows from Definition 3.2.1(2), the Hewitt-Cohen factorisation theorem, and the definition of multiplication in  $Y$ . Now fix  $x, y \in Y$ ; we must show that  $\psi^{(\rho,\pi)}(\langle x, y \rangle_A) = \psi^{(\rho,\pi)}(x)^*\psi^{(\rho,\pi)}(y)$ . By sesqui-linearity and continuity we need only consider  $x \in X_\alpha$  and  $y \in X_\beta$  for some  $\alpha, \beta \in \Lambda$ . The desired identity then follows from routine calculations using Definition 3.2.1(3) and the definition of the inner product in  $Y$ . This completes the proof of (2).

That the two constructions are inverse to each other follows from the definitions of the maps involved. For the final statement, observe that the universal representation  $j$  of  $Y$  in  $\mathcal{T}_Y$  determines a generating representation  $(\rho^j, \pi^j)$  of  $X$  in  $\mathcal{T}_Y$ . If  $(\rho, \pi)$  is a representation of  $X$  in  $B$ , then  $(\psi^{(\rho,\pi)})$  is a representation of  $Y$  in  $B$ . The universal property of  $\mathcal{T}_Y$  gives a unique homomorphism  $\sigma : \mathcal{T}_Y \rightarrow B$  such that  $\sigma \circ j = \psi^{(\rho,\pi)}$ , and it is easy to check that this is equivalent to  $\sigma \circ \pi_v^j = \pi_v$  and  $\sigma \circ \rho_\lambda^j = \rho_\lambda$  for all  $v \in \Lambda^0$  and  $\lambda \in \Lambda$ .  $\square$

*Remark 3.2.4.* It should also be possible to consider contractive representations of a  $\Lambda$ -system of  $C^*$ -correspondences using the associated product system as in Proposition 3.2.3 and thus to formulate a corresponding dilation theory in the manner of [38, 58, 59]. We thank the referee for pointing out this connection.

Our next goal is to show that the bijection between representations of a regular  $\Lambda$ -system of  $C^*$ -correspondences  $X$  and representations of the associated product system  $Y_X$  preserves Cuntz-Pimsner covariance.

**Proposition 3.2.5.** *Let  $X$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences. Let  $Y = Y_X$  be the product system associated to  $X$  (see Proposition 3.1.7). A representation  $\psi$  of  $Y$  in a  $C^*$ -algebra  $B$  is Cuntz-Pimsner covariant if and only if the representation  $(\rho^\psi, \pi^\psi)$  of  $X$  in  $B$  is Cuntz-Pimsner covariant.*

*Proof.* Since  $\psi^{(\rho^\psi, \pi^\psi)} = \psi$  for all representations  $(\rho, \pi)$  of  $X$  and vice versa, it suffices to show that if  $\psi$  is Cuntz-Pimsner covariant, then  $(\rho^\psi, \pi^\psi)$  has the same property and that if  $(\rho, \pi)$  is Cuntz-Pimsner covariant, then  $\psi^{(\rho,\pi)}$  has the same property.

Before doing this, we establish some notation. For each  $\lambda \in \Lambda$ , there is a homomorphism  $\iota^{(\lambda)} : \mathcal{K}(X_\lambda) \rightarrow \mathcal{K}(Y_{d(\lambda)})$  determined by

$$(3.2.2) \quad \iota^{(\lambda)}(\theta_{x,y}) = \theta_{\iota_\lambda(x), \iota_\lambda(y)}$$

for  $x, y \in X_\lambda$ . For  $x, y \in X_\lambda$ , we then have

$$\begin{aligned} (\rho^\psi)^{(\lambda)}(\theta_{x,y}) &= \rho^\psi(x)\rho^\psi(y)^* \\ &= \psi(\iota_\lambda(x))\psi(\iota_\lambda(y))^* = \psi^{(d(\lambda))}(\theta_{\iota_\lambda(x), \iota_\lambda(y)}) = \psi^{(d(\lambda))}(\iota^{(\lambda)}(\theta_{x,y})); \end{aligned}$$

that is,

$$(3.2.3) \quad (\rho^\psi)^{(\lambda)} = \psi^{(d(\lambda))} \circ \iota^{(\lambda)} \text{ for all } \lambda \in \Lambda.$$

Equivalently, for a fixed representation  $(\rho, \pi)$  of  $X$ ,

$$(3.2.4) \quad \rho^{(\lambda)} = (\psi^{(\rho,\pi)})^{(d(\lambda))} \circ \iota^{(\lambda)} \text{ for all } \lambda \in \Lambda.$$



To prove the equivalence of the two notions of Cuntz-Pimsner covariance, first note that since  $A = \bigoplus_{v \in \Lambda^0} A_v$  is spanned by the  $A_v$  and since, for a representation  $\psi$  of  $Y$ , we have  $\pi_v^\psi = \psi|_{A_v}$  by definition, it will suffice to fix a representation  $\psi$  of  $Y$ , a vertex  $v \in \Lambda^0$ , an element  $n \in \mathbb{N}^k$  and an element  $a \in A_v$  and prove that

$$\sum_{\lambda \in v\Lambda^n} (\rho^\psi)^{(\lambda)}(\phi_\lambda(a)) = \psi^{(n)}(\phi_n(a)).$$

We have  $\phi_n(a) = \sum_{\lambda \in v\Lambda^n} \iota^{(\lambda)}(\phi_\lambda(a))$  by definition of the product system  $Y$ , so we may use (3.2.3) to see that

$$\begin{aligned} \sum_{\lambda \in v\Lambda^n} (\rho^\psi)^{(\lambda)}(\phi_\lambda(a)) &= \sum_{\lambda \in v\Lambda^n} \psi^{(n)}(\iota^{(\lambda)}(\phi_\lambda(a))) \\ &= \psi^{(n)}\left(\sum_{\lambda \in v\Lambda^n} \iota^{(\lambda)}(\phi_\lambda(a))\right) = \psi^{(n)}(\phi_n(a)) \end{aligned}$$

as required.  $\square$

It follows from [56, Theorem 4.1 and Proposition 5.1] that each nondegenerate product system  $Y$  in which each  $\phi_n$  is injective with range in  $\mathcal{K}(Y_n)$  has an isometric universal Cuntz-Pimsner covariant representation  $j_Y : Y \rightarrow \mathcal{O}_Y$  (it seems likely that Fowler knew this, but did not make it explicit in [19]).

**Corollary 3.2.6.** *Let  $X$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences. Let  $Y = Y_X$  be the product system associated to  $X$  (see Proposition 3.1.7). The representation  $(\rho^{j_Y}, \pi^{j_Y})$  of  $(A, X, \chi)$  in  $\mathcal{O}_Y$  is Cuntz-Pimsner covariant. This is a universal Cuntz-Pimsner covariant representation in the sense that for any other Cuntz-Pimsner covariant representation  $(\rho, \pi)$  of  $(A, X, \chi)$  in a  $C^*$ -algebra  $C$ , there is a unique homomorphism  $\Psi = \Psi_{\rho, \pi} : \mathcal{O}_Y \rightarrow C$  satisfying  $\Psi \circ \rho_\lambda^{j_Y} = \rho_\lambda$  for all  $\lambda \in \Lambda$ , and  $\Psi \circ \pi_v^{j_Y} = \pi_v$  for all  $v \in \Lambda^0$ .*

*Proof.* The results follow from the universal property of  $\mathcal{O}_Y$  and Proposition 3.2.5, as in the proof of the final statement of Proposition 3.2.3.  $\square$

**Definition 3.2.7.** Let  $X$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences, and let  $Y_X$  be the product system associated to  $X$  as in Proposition 3.1.7. We define  $C^*(A, X, \chi) := \mathcal{O}_{Y_X}$ , and let  $(\rho^X, \pi^X)$  denote the universal Cuntz-Pimsner covariant representation  $(\rho^{j_Y}, \pi^{j_Y})$  of  $(A, X, \chi)$  in  $C^*(A, X, \chi)$ . Given a Cuntz-Pimsner covariant representation  $(\rho, \pi)$  of  $(A, X, \chi)$  in a  $C^*$ -algebra  $C$ , we denote the homomorphism induced by the universal property by  $\Psi_{\rho, \pi} : C^*(A, X, \chi) \rightarrow C$ .

**3.3. The gauge-invariant uniqueness theorem.** There is a strongly continuous gauge action  $\gamma : \mathbb{T}^k \rightarrow \text{Aut}(C^*(A, X, \chi))$  (inherited from  $\mathcal{O}_Y$ ) such that for all  $z \in \mathbb{T}^k$  we have

$$\begin{aligned} \gamma_z(\pi_v^X(a)) &= \pi_v^X(a) && \text{for all } v \in \Lambda^0 \text{ and } a \in A_v, \\ \gamma_z(\rho_\lambda^X(x)) &= z^{d(\lambda)} \rho_\lambda^X(x) && \text{for all } \lambda \in \Lambda \text{ and } x \in X_\lambda, \end{aligned}$$

where  $z^{d(\lambda)}$  is multi-index notation:  $z^{d(\lambda)} = \prod_{i=1}^k z_i^{d(\lambda)_i}$ .

**Theorem 3.3.1.** *Let  $X$  be a regular  $\Lambda$ -system of  $C^*$ -correspondences. Let  $B$  be a  $C^*$ -algebra equipped with a strongly continuous action  $\beta : \mathbb{T}^k \rightarrow \text{Aut}(B)$ . Let  $(\rho, \pi)$  be a Cuntz-Pimsner covariant representation of  $(A, X, \chi)$  in  $B$ . Suppose the following conditions are satisfied:*

(1) for all  $z \in \mathbb{T}^k$  we have

$$\begin{aligned} \beta_z(\pi_v(a)) &= \pi_v(a) && \text{for } v \in \Lambda^0 \text{ and } a \in A_v, \\ \beta_z(\rho_\lambda(x)) &= z^{d(\lambda)} \rho_\lambda(x) && \text{for all } \lambda \in \Lambda \text{ and } x \in X_\lambda; \end{aligned}$$

(2) for all  $v \in \Lambda^0$ ,  $\pi_v$  is injective.

Then  $\Psi_{\rho, \pi} : C^*(A, X, \chi) \rightarrow B$  is injective.

The proof follows from Proposition 3.1.7, the definition of the gauge action and the following lemma.

**Lemma 3.3.2.** *Let  $Y$  be a nondegenerate product system over  $\mathbb{N}^k$  such that the left action on each  $Y_n$  is injective and by compacts. Let  $B$  be a  $C^*$ -algebra equipped with a strongly continuous action  $\beta : \mathbb{T}^k \rightarrow \text{Aut}(B)$ , and let  $\psi$  be a Cuntz-Pimsner covariant representation of  $Y$  in  $B$ . Suppose that the following conditions are satisfied.*

- (1) For all  $z \in \mathbb{T}^k$ ,  $n \in \mathbb{N}^k$  and  $y \in Y_n$ , we have  $\beta_z(\psi_n(x)) = z^n \psi_n(x)$ .
- (2) The homomorphism  $\psi_0 : A \rightarrow B$  is injective.

Then the induced map  $\psi_* : \mathcal{O}_Y \rightarrow B$  is injective.

*Proof.* The map  $a \mapsto \int_{\mathbb{T}^k} \gamma_z(a) dz$  is a faithful conditional expectation  $\Phi^\gamma : \mathcal{O}_Y \rightarrow \mathcal{O}_Y^\gamma$ , and condition (1) ensures that the linear map  $\Phi^\beta : \psi_*(a) \mapsto \int_{\mathbb{T}^k} \beta_z(\psi_*(a)) dz$  satisfies  $\psi_* \circ \Phi^\gamma = \Phi^\beta \circ \psi_*$ . It therefore suffices to prove that the restriction of  $\psi_*$  to  $\mathcal{O}_Y^\gamma$  is injective. We claim that  $\psi_n$  is injective for all  $n \in \mathbb{N}^k$ . Indeed, fix a nonzero  $y \in Y_n$ . Then  $\langle y, y \rangle_A^n \neq 0$ , and since  $\langle y, y \rangle_A^n \in Y_0 = A$ , we have

$$\psi_n(y)^* \psi_n(y) = \psi_0(\langle y, y \rangle_A^n) \neq 0$$

by (2). Hence  $\psi_n(y) \neq 0$ . Moreover,  $\psi^{(n)} : \mathcal{K}(Y_n) \rightarrow B$  is injective because  $\psi_0$  is (see the opening paragraph of [45, p.202]).

For all  $m, n \in \mathbb{N}^k$  with  $m \leq n$ , we have  $j_Y^{(m)}(\mathcal{K}(Y_m)) \subset j_Y^{(n)}(\mathcal{K}(Y_n))$ . These embeddings are compatible with the natural injection  $i_m^n : \mathcal{K}(Y_m) \hookrightarrow \mathcal{K}(Y_n)$  guaranteed by our assumptions on  $Y$ . Moreover,  $\bigcup_n j_Y^{(n)}(\mathcal{K}(Y_n))$  is a dense subalgebra of  $\mathcal{O}_Y^\gamma$ , and hence

$$\mathcal{O}_Y^\gamma \cong \varinjlim_{n \in \mathbb{N}^k} \mathcal{K}(Y_n).$$

It remains to show that the restriction of  $\psi_*$  to each  $j_Y^{(n)}(\mathcal{K}(Y_n))$  is injective. But this follows from the fact that each  $\psi^{(n)} : \mathcal{K}(Y_n) \rightarrow B$  is injective.  $\square$

*Remark 3.3.3.* Restricting attention to regular  $\Lambda$ -systems vastly simplifies Definition 3.2.1 as well as the proofs of the subsequent results — namely Proposition 3.2.3, Proposition 3.2.5, Corollary 3.2.6 and Theorem 3.3.1. We pause to point out how complications arise in the absence of regularity.

For a start, the results of [19] and [56] suggest that to obtain a nicely-behaved Toeplitz algebra for non-regular  $\Lambda$ -systems we would still have to restrict attention to finitely aligned  $k$ -graphs  $\Lambda$  and to  $\Lambda$ -systems which were compactly aligned in the sense that given compact operators  $S$  on  $X_\mu$  and  $T$  on  $X_\nu$  the product  $\iota_\mu^\lambda(S) \iota_\nu^\lambda(T)$  is compact for each minimal common extension of  $\mu$  and  $\nu$ . This should ensure that  $Y_X$  is compactly aligned in the sense of [19]. One would then have to add a Nica covariance relation to those already listed in Definition 3.2.1: presumably that each  $\rho^{(\mu)}(S) \rho^{(\nu)}(T)$  is equal to the sum of the  $\rho^{(\lambda)}(\iota_\mu^\lambda(S) \iota_\nu^\lambda(T))$  where  $\lambda$  ranges over all minimal common extensions of  $\mu$  and

$\nu$ . Though the technical details of the proof of Proposition 3.2.3 would be complicated by this, it seems clear that the result would generalise.

Since our interest here is in the Cuntz-Pimsner algebra rather than its Toeplitz extension, the situation is further complicated in that the issue of an appropriate notion of Cuntz-Pimsner covariance for compactly aligned  $\Lambda$ -systems over finitely aligned  $\Lambda$  arises. One would presumably have to translate the notion of Cuntz-Pimsner covariance formulated in [56] into a corresponding notion for  $\Lambda$ -systems (the introduction of [56] gives an account of the issues involved). A first guess would be to impose [56, Relation (CP)] in the product system associated to each boundary path of  $\Lambda$ . The situation here is very unclear and the technical details formidable. While versions of Proposition 3.2.5, Corollary 3.2.6 and Theorem 3.3.1 should be achievable with an appropriate definition of Cuntz-Pimsner covariance (forthcoming work of Carlsen, Larsen, Sims and Vittadello indicates how to address the gauge-invariant uniqueness theorem), one would not wish to speculate on the status of later results. For example, the definition of the  $E_x$  at the beginning of Section 4.1 is already problematic because the linking maps  $\iota_{x(0,m)}^{x(0,n)}$  may not be injective and may not map compacts to compacts; besides which, the appropriate groupoid is presumably the one described in [17] whose unit space is much more complicated than the infinite-path space in general.

#### 4. CONSTRUCTION OF A FELL BUNDLE

Fix for this section a row-finite  $k$ -graph  $\Lambda$  with no sources, and a regular  $\Lambda$ -system  $X$  of  $C^*$ -correspondences.

##### 4.1. The fibres of the Fell bundle.

**Definition 4.1.1.** Given an infinite path  $x \in \Lambda^\infty$ , let  $E_x$  be the  $C^*$ -algebraic inductive limit

$$E_x := \varinjlim_{n \in \mathbb{N}^k} \mathcal{K}(X_{x(0,n)})$$

under the connecting maps  $i_{x(0,m)}^{x(0,n)} : \mathcal{K}(X_{x(0,m)}) \rightarrow \mathcal{K}(X_{x(0,n)})$  defined by equation (3.1.1) for  $(m, n) \in \Omega_k$ . Let  $i_{x(0,n)}^x : \mathcal{K}(X_{x(0,n)}) \rightarrow E_x$  denote the canonical embedding.

**Proposition 4.1.2.** Fix  $x \in \Lambda^\infty$ . Let  $x^*X$  be the pullback  $\Omega_k$ -system given by

$$(x^*A)_m = A_{x(m)}, \quad (x^*X)_{(m,n)} = X_{x(m,n)} \quad \text{and} \quad (x^*\chi)_{(m,n),(n,p)} = \chi_{x(m,n),x(n,p)}.$$

Let  $(\rho, \pi)$  denote the universal representation of  $x^*X$  in  $C^*(x^*A, x^*X, x^*\chi)$ . Fix  $n \in \mathbb{N}^k$ . Then  $\pi_n : A_{x(n)} \rightarrow C^*(x^*A, x^*X, x^*\chi)$  extends to multiplier algebras; denote this extension  $\tilde{\pi}_n$ . Let

$$P_n := \tilde{\pi}_n(1_{(x^*A)_n}) \in \mathcal{M}(C^*(x^*A, x^*X, x^*\chi)).$$

Then  $P_n$  is full, and  $E_{\sigma^n(x)} \cong P_n C^*(x^*A, x^*X, x^*\chi) P_n$ . In particular,  $n = 0$  yields

$$E_x \cong P_0 C^*(x^*A, x^*X, x^*\chi) P_0.$$

*Proof.* To see that  $\pi_n$  extends to multiplier algebras, recall that since the  $X_{x(m,n)}$  are nondegenerate, the product system  $Y$  associated to  $x^*X$  by Proposition 3.1.7 is also nondegenerate. Hence the universal representation of  $Y$  in  $\mathcal{O}_Y$  extends to a unital homomorphism  $\tilde{j}_Y$  from  $\mathcal{M}(A) = \mathcal{M}(\bigoplus_{m \in \mathbb{N}^k} A_{x(m)})$  to  $\mathcal{M}(\mathcal{O}_Y)$ . The restriction of  $\tilde{j}_Y$  to  $\mathcal{M}(A_{x(n)})$  is then the desired extension of  $\pi_n$ .

To see that  $P_n$  is full, let  $I(P_n)$  be the ideal of  $C^*(x^*A, x^*X, x^*\chi)$  generated by  $P_n$ . Since  $(\rho, \pi)$  is Cuntz-Pimsner covariant and  $X$  is regular, each  $a \in A_{x(0)}$  satisfies  $\pi_0(a) = \rho^{(0,n)}(T)$  for some  $T \in \mathcal{K}(X_{x(0,n)})$ . Since  $\mathcal{K}(X_{x(0,n)}) = \overline{\text{span}}\{\theta_{\xi,\eta} : \xi, \eta \in X_{x(0,n)}\}$  and

$$\rho^{(0,n)}(\theta_{\xi,\eta}) = \rho_{(0,n)}(\xi)P_n\rho_{(0,n)}(\eta)^*,$$

it follows that  $A_{x(0)} \subset I(P_n)$ , and hence that  $I(P_n)$  contains  $I(P_0)$ . Now for any other  $m \in \mathbb{N}^k$ , and any  $a \in A_{x(m)}$ , that  $X$  is regular, and in particular that  $X_{x(0,m)}$  is full, ensures that  $a \in \overline{\text{span}}\{\xi, \eta\}_{A_{x(m)}} : \xi, \eta \in X_{x(0,m)}\}$ , and in particular that

$$\pi_m(a) \in \overline{\text{span}}\{\rho_{(0,m)}(\xi)^*P_0\rho_{(0,m)}(\eta) : \xi, \eta \in X_{x(0,m)}\}.$$

Thus  $\overline{\text{span}}(\bigcup_{n \in \mathbb{N}^k} \pi_n(A_{x(n)}))$  belongs to  $I(P_n)$ . Since each  $X_{x(n)}$  is nondegenerate, it follows that  $I(P_n) = C^*(x^*A, x^*X, x^*\chi)$ .

For the final statement, we will invoke the universal property of the direct limit  $E_{\sigma^n(x)} = \varinjlim (\mathcal{K}(X_{x(n,n+p)}), i_{x(n,n+p)}^{x(n,n+q)})$ . First fix  $p \in \mathbb{N}^k$ . By definition,  $\rho^{(n,n+p)}$  is a homomorphism from  $\mathcal{K}(X_{x(n,n+p)})$  to  $P_n C^*(x^*A, x^*X, x^*\chi) P_n$ . Lemma 3.10 of [45] shows that the  $\rho^{(n,n+p)}$  are compatible with the connecting maps  $i_{x(n,n+p)}^{x(n,n+q)}$  in the sense that  $\rho^{(n,n+q)} \circ i_{x(n,n+p)}^{x(n,n+q)} = \rho^{(n,n+p)}$  for all  $p$ . The universal property of the direct limit now implies that there is a unique homomorphism  $\rho^{(n,\infty)} : E_{\sigma^n(x)} \rightarrow P_n C^*(x^*A, x^*X, x^*\chi) P_n$  such that  $\rho^{(n,\infty)} \circ i_{x(n,n+p)}^{x(n,\infty)} = \rho^{(n,n+p)}$  for all  $p$ . Since each  $\pi_{n+p}$  is injective, each  $\rho^{(n,n+p)}$  is injective and hence isometric, which implies that  $\rho^{(n,\infty)}$  is isometric. Finally, since

$$\begin{aligned} & C^*(x^*A, x^*X, x^*\chi) \\ &= \overline{\text{span}}\{\rho_{(m,q)}(\xi)\rho_{(p,q)}(\eta)^* : q \in \mathbb{N}^k, m, p \leq q, \xi \in (x^*X)_{(m,q)}, \eta \in (x^*X)_{(p,q)}\}, \end{aligned}$$

we have

$$\begin{aligned} & P_n C^*(x^*A, x^*X, x^*\chi) P_n \\ &= \overline{\text{span}}\{\rho_{(n,p)}(\xi)\rho_{(n,p)}(\eta)^* : p \geq n, \xi, \eta \in X_{x(n,p)}\} \\ &= \overline{\text{span}}\{\rho^{(n,p)}(T) : p \geq n, T \in \mathcal{K}(X_{x(n,p)})\} \\ &= \rho^{(n,\infty)}(E_x), \end{aligned}$$

so that  $\rho^{(n,\infty)}$  is the desired isomorphism.  $\square$

In the Fell bundle we will construct over the  $k$ -graph groupoid  $\mathcal{G}_\Lambda$ , the  $C^*$ -algebras  $E_x$  will be the fibres over the unit space. We now turn to the construction of the other fibres. We do this using linking algebra techniques of [3]. Our first step is to associate a  $C^*$ -algebra and two complementary full projections in its multiplier algebra to each groupoid element.

*Notation 4.1.3.* Given paths  $\lambda, \mu, \nu \in \Lambda$  such that  $s(\lambda) = s(\mu) = r(\nu)$ , there is a homomorphism  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu} : \mathcal{L}(X_\lambda \oplus X_\mu) \rightarrow \mathcal{L}(X_{\lambda\nu} \oplus X_{\mu\nu})$  obtained in an analogous manner to the definition of  $i_\alpha^{\alpha\beta}$  in equation (3.1.1). Specifically, regarding  $X_\lambda \oplus X_\mu$  as a right Hilbert  $A_{r(\nu)}$  module, [36, page 42] gives a homomorphism

$$(\phi_\nu)_* : \mathcal{L}(X_\lambda \oplus X_\mu) \rightarrow \mathcal{L}((X_\lambda \oplus X_\mu) \otimes_{A_{r(\nu)}} X_\nu).$$

combining this with the isomorphism  $\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu} : (X_\lambda \oplus X_\mu) \otimes_{A_{r(\nu)}} X_\nu \rightarrow X_{\lambda\nu} \oplus X_{\mu\nu}$ , we obtain the desired homomorphism, namely

$$i_{\lambda,\mu}^{\lambda\nu,\mu\nu}(S) := (\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu}) \circ (\phi_\nu)_*(S) \circ (\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})^{-1}.$$

Proposition 4.7 of [36] and that  $X$  is regular imply that  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu}$  restricts to an injective approximately unital homomorphism from  $\mathcal{K}(X_\lambda \oplus X_\mu)$  to  $\mathcal{K}(X_{\lambda\nu} \oplus X_{\mu\nu})$ .

**Lemma 4.1.4.** *Given paths  $\lambda, \mu, \nu \in \Lambda$  such that  $s(\lambda) = s(\mu) = r(\nu)$ , let  $P_\lambda, P_\mu \in \mathcal{L}(X_\lambda \oplus X_\mu)$  be the projections onto  $X_\lambda$  and  $X_\mu$  respectively, and define  $P_{\lambda\nu}, P_{\mu\nu} \in \mathcal{L}(X_{\lambda\nu} \oplus X_{\mu\nu})$  similarly. Then*

- (1) *the homomorphism  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu}$  takes  $P_\lambda$  to  $P_{\lambda\nu}$  and  $P_\mu$  to  $P_{\mu\nu}$ ;*
- (2) *the projections  $P_\lambda$  and  $P_\mu$  are complementary full projections in  $\mathcal{M}(\mathcal{K}(X_\lambda \oplus X_\mu)) = \mathcal{L}(X_\lambda \oplus X_\mu)$ ; and*
- (3) *under the canonical identification of  $P_\lambda \mathcal{K}(X_\lambda \oplus X_\mu) P_\lambda$  with  $\mathcal{K}(X_\lambda)$ , we have*

$$P_\lambda i_{\lambda,\mu}^{\lambda\nu,\mu\nu}(S) P_\lambda = i_\lambda^{\lambda\nu}(S)$$

*for all  $S \in \mathcal{K}(X_\lambda)$ , and similarly for  $\mu$ .*

*Proof.* A typical spanning element of  $X_{\lambda\nu} \oplus X_{\mu\nu}$  is of the form  $(\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})((x_\lambda \oplus x_\mu) \otimes x_\nu)$  where  $x_\lambda \in X_\lambda$ ,  $x_\mu \in X_\mu$  and  $x_\nu \in X_\nu$ . By definition of  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu}$ , we have

$$\begin{aligned} i_{\lambda,\mu}^{\lambda\nu,\mu\nu}(P_\lambda)((\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})((x_\lambda \oplus x_\mu) \otimes x_\nu)) &= (\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})(P_\lambda(x_\lambda \oplus x_\mu) \otimes x_\nu) \\ &= (\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})(x_\lambda \oplus 0) \otimes x_\nu. \end{aligned}$$

Since  $(\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})((x_\lambda \oplus x_\mu) \otimes x_\nu) = \chi_{\lambda,\nu}(x_\lambda \otimes x_\nu) \oplus \chi_{\mu,\nu}(x_\mu \otimes x_\nu)$ , we deduce that

$$i_{\lambda,\mu}^{\lambda\nu,\mu\nu}(P_\lambda)((\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})((x_\lambda \oplus x_\mu) \otimes x_\nu)) = P_{\lambda\nu}((\chi_{\lambda,\nu} \oplus \chi_{\mu,\nu})((x_\lambda \oplus x_\mu) \otimes x_\nu)).$$

That  $P_\lambda$  and  $P_\mu$  are complementary projections is clear from their definitions. They are full because each of  $X_\lambda$  and  $X_\mu$  is a full right-Hilbert  $A_{s(\lambda)}$ -module. The last assertion is verified via a straightforward calculation using spanning elements like the one above.  $\square$

For the next result, we need the following notation. For  $g = (x, n, y) \in \mathcal{G}_\Lambda$  we define

$$D_g := \{(\lambda, \mu) : x = \lambda z, y = \mu z, d(\lambda) - d(\mu) = n\}.$$

The set  $D_g$  is ordered by the relation  $(\lambda, \mu) \leq (\lambda', \mu')$  if and only if  $d(\lambda) \leq d(\lambda')$ . Moreover, we have  $(\lambda, \mu) \leq (\lambda', \mu')$  in  $D_g$  if and only if there exists  $\nu$  such that  $\lambda' = \lambda\nu$  and  $\mu' = \mu\nu$ . We may therefore form the inductive limit

$$\varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu)$$

with respect to the connecting maps  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu}$  defined in Lemma 4.1.4. We denote the universal inclusion maps into the direct limit by

$$i_{\lambda,\mu}^g : \mathcal{K}(X_\lambda \oplus X_\mu) \rightarrow \varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu).$$

**Proposition 4.1.5.** *Fix  $g = (x, n, y) \in \mathcal{G}_\Lambda$ . The maps  $i_{\lambda,\mu}^g$  defined above extend to unital homomorphisms of multiplier algebras such that for each  $(\lambda, \mu) \in D(g)$ , the elements*

$i_{\lambda,\mu}^g(P_\lambda)$  and  $i_{\lambda,\mu}^g(P_\mu)$  are complementary full projections which do not depend on the choice of  $(\lambda, \mu) \in D(g)$ . We define  $P_x := i_{\lambda,\mu}^g(P_\lambda)$  and  $P_y := i_{\lambda,\mu}^g(P_\mu)$ . We then have

$$E_x \cong P_x \left( \varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu) \right) P_x, \text{ and}$$

$$E_y \cong P_y \left( \varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu) \right) P_y;$$

so  $\varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu)$  is a linking algebra for  $E_x$  and  $E_y$ .

*Proof.* First observe that the map  $i_{\lambda,\mu}^g$  is approximately unital for every  $(\lambda, \mu) \in D_g$ . This follows because all the linking maps  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu} : \mathcal{K}(X_\lambda \oplus X_\mu) \rightarrow \mathcal{K}(X_{\lambda\nu} \oplus X_{\mu\nu})$  are approximately unital. Hence the map  $i_{\lambda,\mu}^g$  extends to a unital map between the multiplier algebras. By the preceding lemma, the projections  $P_\lambda$  and  $P_\mu$  are complementary full projections, and since the map  $i_{\lambda,\mu}^g$  is approximately unital, their images,  $i_{\lambda,\mu}^g(P_\lambda)$  and  $i_{\lambda,\mu}^g(P_\mu)$ , are also complementary full projections. By the same lemma,  $i_{\lambda,\mu}^g(P_\lambda) = i_{\lambda',\mu'}^g(P_{\lambda'})$  and  $i_{\lambda,\mu}^g(P_\mu) = i_{\lambda',\mu'}^g(P_{\mu'})$  for all  $(\lambda, \mu), (\lambda', \mu') \in D(g)$ ; therefore  $P_x$  and  $P_y$  are well defined. Lemma 4.1.4(3) combined with the universal property of the direct limit gives the two displayed equations in the statement of the Proposition. The final statement is a consequence of the preceding ones.  $\square$

It will frequently be useful to make the identification:

$$(4.1.1) \quad \mathcal{K}(X_\mu, X_\lambda) = P_\lambda \mathcal{K}(X_\lambda \oplus X_\mu) P_\mu.$$

**Corollary 4.1.6.** *For  $g = (x, n, y) \in \mathcal{G}_\Lambda$ , the  $C^*$ -algebras  $E_x$  and  $E_y$  are Morita-Rieffel equivalent with imprimitivity bimodule*

$$E_g := P_x \left( \varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu) \right) P_y.$$

Moreover, under the identification (4.1.1) we have (as Banach spaces):

$$E_g = \varinjlim_{(\lambda,\mu) \in D_g} \mathcal{K}(X_\mu, X_\lambda).$$

*Proof.* The first statement follows immediately from Proposition 4.1.5 and [3, Theorem 1.1]. The second statement follows from the first and (4.1.1).  $\square$

*Remark 4.1.7.* Let  $g, h \in \mathcal{G}$  with  $s(g) = s(h) = x$ . Then

$$E_{gh^{-1}} \cong E_g \otimes_{E_x} E_{h^{-1}} \cong E_g \otimes_{E_x} E_h^* \cong \mathcal{K}(E_h, E_g).$$

Using the identification (4.1.1), we retain the notation  $i_{\lambda,\mu}^{\lambda\nu,\mu\nu}$  and  $i_{\lambda,\mu}^g$  for the embeddings

$$i_{\lambda,\mu}^{\lambda\nu,\mu\nu} : \mathcal{K}(X_\lambda, X_\mu) \rightarrow \mathcal{K}(X_{\lambda\nu}, X_{\mu\nu}),$$

$$i_{\lambda,\mu}^g : \mathcal{K}(X_\lambda, X_\mu) \rightarrow E_g.$$

Note that if  $\lambda = \mu$  we have  $\mathcal{K}(X_\lambda, X_\lambda) = \mathcal{K}(X_\lambda)$  and our embeddings satisfy  $i_{\lambda,\lambda}^{\lambda\nu,\lambda\nu} = i_{\lambda,\lambda}^{\lambda\nu}$ . It follows that for all  $x \in \Lambda^\infty$ , we may identify  $E_{(x,0,x)}$  with  $E_x$ , and that if  $x = \lambda z$  we have  $i_{\lambda,\lambda}^{(x,0,x)} = i_\lambda^x$ .

**4.2. The operations and topology of the Fell bundle.** The following lemma shows that the connecting maps are compatible with composition. This will be used later to define the multiplicative structure of the Fell bundle.

In this lemma and throughout the rest of the paper, given right Hilbert  $A$ -modules  $X, Y, Z$  and given  $T \in \mathcal{K}(X, Y)$  and  $S \in \mathcal{K}(Y, Z)$ , we will use the juxtaposition  $ST$  to denote the composition  $S \circ T \in \mathcal{K}(X, Z)$ .

**Lemma 4.2.1.** *Fix  $\lambda_1, \lambda_2, \lambda_3, \mu \in \Lambda$  such that  $s(\lambda_i) = r(\mu)$  for each  $i = 1, 2, 3$ . Then for  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$ ,*

$$i_{\lambda_1, \lambda_2}^{\lambda_1 \mu, \lambda_2 \mu}(T_1) i_{\lambda_2, \lambda_3}^{\lambda_2 \mu, \lambda_3 \mu}(T_2) = i_{\lambda_1, \lambda_3}^{\lambda_1 \mu, \lambda_3 \mu}(T_1 T_2).$$

*Proof.* Embed each  $\mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$  in  $\mathcal{K}(X_{\lambda_1} \oplus X_{\lambda_2} \oplus X_{\lambda_3})$  in a way analogous to (4.1.1), and then use that the linking map

$$\mathcal{K}(X_{\lambda_1} \oplus X_{\lambda_2} \oplus X_{\lambda_3}) \hookrightarrow \mathcal{K}(X_{\lambda_1 \mu} \oplus X_{\lambda_2 \mu} \oplus X_{\lambda_3 \mu})$$

is a homomorphism.  $\square$

Before setting ourselves up to define the multiplication between the various  $E_g$ , we establish a simple technical lemma that we will use a number of times.

**Lemma 4.2.2.** *Fix  $n \geq 1$ , a composable  $n$ -tuple  $(g_1, \dots, g_n) \in \mathcal{G}_\Lambda^{(n)}$ , elements  $e_i \in E_{g_i}$  for  $i \leq n$ , and  $\varepsilon > 0$ . There exist  $x \in \Lambda^\infty$ ,  $\lambda_1, \dots, \lambda_{n+1} \in \Lambda r(x)$ , and  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$  for  $i \leq n$  such that, for each  $i \leq n$ ,*

$$(4.2.1) \quad g_i = (\lambda_i x, d(\lambda_i) - d(\lambda_{i+1}), \lambda_{i+1} x) \quad \text{and} \quad \|i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) - e_i\| < \varepsilon.$$

*Proof.* We proceed by induction on  $n$ . The base case  $n = 1$  follows from the definition of  $E_{g_1}$  as  $\varinjlim_{(\lambda_1, \lambda_2) \in D_{g_1}} \mathcal{K}(X_{\lambda_2}, X_{\lambda_1})$ . Now suppose that the result holds for  $n \leq k$ , and fix  $(g_1, \dots, g_{k+1}) \in \mathcal{G}_\Lambda^{(k+1)}$  and  $e_i \in E_{g_i}$ . Apply the inductive hypothesis with  $n = k$  to  $(g_1, \dots, g_k)$ ,  $(e_1, \dots, e_k)$ ,  $\varepsilon$  and with  $n = 1$  to  $(g_{k+1})$ ,  $(e_{k+1})$ ,  $\varepsilon$  to obtain  $y, z \in \Lambda^\infty$ ,  $\mu_1, \dots, \mu_{k+1} \in \Lambda r(y)$ ,  $\nu_1, \nu_2 \in \Lambda r(z)$ ,  $R_i \in \mathcal{K}(X_{\mu_{i+1}}, X_{\mu_i})$  and  $S \in \mathcal{K}(X_{\nu_2}, X_{\nu_1})$  with the appropriate properties.

We have  $\mu_{k+1} y = s(g_k) = r(g_{k+1}) = \nu_1 z$ , so

$$\mu' := y(0, (d(\mu_{k+1}) \vee d(\nu_1)) - d(\mu_{k+1})) \quad \text{and} \quad \nu' := z(0, (d(\mu_{k+1}) \vee d(\nu_1)) - d(\nu_1))$$

satisfy  $(\mu', \nu') \in \Lambda^{\min}(\mu_{k+1}, \nu_1)$ . Let  $\lambda_i := \mu_i \mu'$  for  $i \leq k+1$ , let  $\lambda_{k+2} := \nu_2 \nu'$ , and let  $x := \sigma^{d(\mu')}(y)$ . Then

$$\lambda_{k+1} = \mu_{k+1} \mu' = \nu_1 \nu' \quad \text{and} \quad \mu' x = y \quad \text{and} \quad \nu' x = z.$$

Hence  $g_i = (\mu_i y, d(\mu_i) - d(\mu_{i+1}), \mu_{i+1} y) = (\lambda_i x, d(\lambda_i) - d(\lambda_{i+1}), \lambda_{i+1} x)$  for  $i \leq k$ , and similarly  $g_{k+1} = (\lambda_{k+1} x, d(\lambda_{k+1}) - d(\lambda_{k+2}), \lambda_{k+2} x)$ . Let  $T_i := i_{\mu_i, \mu_{i+1}}^{\mu_i \mu', \mu_{i+1} \mu'}(R_i)$  for  $i \leq k$  and let  $T_{k+1} := i_{\nu_1, \nu_2}^{\nu_1 \nu', \nu_2 \nu'}(S)$ . By the compatibility of the connecting maps in the inductive limit, we have

$$i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) = \begin{cases} i_{\mu_i, \mu_{i+1}}^{g_i}(R_i) & \text{if } i \leq k \\ i_{\nu_1, \nu_2}^{g_{k+1}}(S) & \text{if } i = k+1. \end{cases}$$

In particular, each  $\|i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) - e_i\| < \varepsilon$  by choice of the  $S_i$  and  $R_i$ .  $\square$

**Lemma 4.2.3.** *Fix composable elements  $g_1 = (x_1, n_1, x_2)$  and  $g_2 = (x_2, n_2, x_3)$  of  $\mathcal{G}_\Lambda$ . There is a bilinear map  $(e_1, e_2) \mapsto e_1 e_2$  from  $E_{g_1} \times E_{g_2}$  to  $E_{g_1 g_2}$  determined as follows: if  $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$  and  $z \in \Lambda^\infty$  satisfy  $n_i = d(\lambda_i) - d(\lambda_{i+1})$ , and  $x_i = \lambda_i z$ , (so in particular  $g_i = (\lambda_i z, n_i, \lambda_{i+1} z) \in Z(\lambda_i, \lambda_{i+1})$ ) then for  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$ ,*

$$(4.2.2) \quad i_{\lambda_1, \lambda_2}^{g_1}(T_1) i_{\lambda_2, \lambda_3}^{g_2}(T_2) = i_{\lambda_1, \lambda_3}^{g_1 g_2}(T_1 T_2).$$

Moreover,  $\|e_1 e_2\| \leq \|e_1\| \|e_2\|$ . If  $g_3 = (x_3, n_3, x_4)$ , so that  $g_2$  and  $g_3$  are composable and  $e_3 \in E_{g_3}$ , then  $(e_1 e_2) e_3 = e_1 (e_2 e_3)$ .

*Proof.* That (4.2.2) is bilinear follows from Lemma 4.2.1 and the definition of the direct limit, and we then have

$$\|i_{\lambda_1, \lambda_3}^{g_1 g_2}(T_1 T_2)\| \leq \|i_{\lambda_1, \lambda_2}^{g_1}(T_1)\| \|i_{\lambda_2, \lambda_3}^{g_2}(T_2)\|$$

because the  $i_{\lambda_i, \lambda_{i+1}}^{g_i}$  are restrictions of the injective  $C^*$ -homomorphisms

$$i_{\lambda_i, \lambda_{i+1}}^{g_i} : \mathcal{K}(X_{\lambda_i} \oplus X_{\lambda_{i+1}}) \rightarrow \varinjlim_{(\mu, \nu) \in D_{g_i}} \mathcal{K}(X_\mu \oplus X_\nu)$$

and are therefore isometric. It follows from Lemma 4.2.2 that the assignment (4.2.2) extends uniquely to the desired bilinear map  $(e_1, e_2) \mapsto e_1 e_2$ , and that this map satisfies  $\|e_1 e_2\| \leq \|e_1\| \|e_2\|$ . For the final statement, use Lemma 4.2.2 to approximate  $e_3$  by an element of the form  $i_{\lambda_3, \lambda_4}^{g_3}(T_3)$ , and then use (4.2.2) and that  $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ .  $\square$

We now define an involution on  $\prod_{g \in \mathcal{G}_\Lambda} E_g$  which is compatible with the product structure defined in Lemma 4.2.3. Recall that for right Hilbert  $A$ -modules  $X, Y$  the adjoint map  $T \mapsto T^*$  defines a conjugate linear isometry from  $\mathcal{K}(X, Y)$  to  $\mathcal{K}(Y, X)$ .

**Lemma 4.2.4.** *For each  $g = (x, n, y) \in \mathcal{G}_\Lambda$  there is a conjugate linear isometry  $e \mapsto e^*$  from  $E_g$  to  $E_{g^{-1}}$  which is determined by the following property: for  $(\lambda, \mu) \in D_g$ , and  $T \in \mathcal{K}(X_\mu, X_\lambda)$ ,*

$$(4.2.3) \quad i_{\lambda, \mu}^g(T)^* = i_{\mu, \lambda}^{g^{-1}}(T^*).$$

For  $e \in E_g$ , we have  $e^{**} = e$ , the element  $e^* e$  is positive in  $E_{s(g)}$  and  $\|e^* e\| = \|e\|^2$ . For  $g_1 = (x_1, n_1, x_2)$ ,  $g_2 = (x_2, n_2, x_3) \in \mathcal{G}_\Lambda$ ,  $e_1 \in E_{g_1}$  and  $e_2 \in E_{g_2}$  we have  $(e_1 e_2)^* = e_2^* e_1^*$ .

*Proof.* For each  $(\lambda, \mu) \in D_g$ , the involution on  $\mathcal{K}(X_\lambda \oplus X_\mu)$  restricts to the adjoint map from  $\mathcal{K}(X_\mu, X_\lambda)$  to  $\mathcal{K}(X_\lambda, X_\mu)$ . By definition of  $C^*$ -algebraic direct limits, it follows that the involution on  $\varinjlim_{(\lambda, \mu) \in D_g} \mathcal{K}(X_\lambda \oplus X_\mu)$  restricts to a conjugate linear map from  $E_g$  to  $E_{g^{-1}}$  satisfying (4.2.3). This map is an isometry because the connecting maps in the direct limit are all isometric.

To see that  $e^{**} = e$ , that  $e^* e \geq 0$  and that  $\|e^* e\| = \|e\|^2$  for all  $e \in E_x$ , note that by continuity it suffices to consider  $e = i_{\lambda, \mu}^g(T)$ . Two applications of (4.2.3) then give  $e^{**} = (i_{\mu, \lambda}^{g^{-1}}(T^*))^* = i_{\lambda, \mu}^g(T^{**}) = e$  because  $T^{**} = T$ . We have  $e^* e = i_{\mu, \lambda}^{g^{-1}}(T^*) i_{\lambda, \mu}^g(T) = i_\mu^{s(g)}(T^* T)$  as  $i_\mu^{s(g)}$  is a homomorphism. Moreover, regarding  $T$  as an element of  $\mathcal{K}(X_\lambda \oplus X_\mu)$  and  $i_{\lambda, \mu}^g$  as a homomorphism of  $\mathcal{K}(X_\lambda \oplus X_\mu)$ , we have

$$\|e^* e\| = \|i_{\lambda, \mu}^g(T^* T)\| = \|i_{\lambda, \mu}^g(T)\|^2 = \|e\|^2.$$



For the final statement, it suffices by Lemma 4.2.2 to consider  $e_i = i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i)$  for  $i = 1, 2$ , and then

$$(e_1 e_2)^* = i_{\lambda_1, \lambda_3}^{g_1 g_2} (T_1 T_2)^* = i_{\lambda_3, \lambda_1}^{-1} i_{\lambda_2, \lambda_1}^{-1} (T_2^* T_1^*) = i_{\lambda_2, \lambda_3}^{-1} (T_2^*) i_{\lambda_1, \lambda_2}^{-1} (T_1^*) = e_2^* e_1^*. \quad \square$$

The disjoint union  $\bigsqcup_{g \in \mathcal{G}_\Lambda} E_g$  will form a Fell bundle over  $\mathcal{G}_\Lambda$ , but we must first endow it with an appropriate topology. To do this, we first make it into a Banach bundle by defining a linear space of sections of continuous norm which is fibrewise dense.

**Definition 4.2.5.** Fix a pair  $\lambda, \mu \in \Lambda$  such that  $s(\lambda) = s(\mu)$ . Let  $T \in \mathcal{K}(X_\mu, X_\lambda)$ . We define an element  $f_T^{\lambda, \mu} \in \prod_{g \in \mathcal{G}_\Lambda} E_g$  by

$$f_T^{\lambda, \mu}(g) = \begin{cases} i_{\lambda, \mu}^g(T) & \text{if } g \in Z(\lambda, \mu) \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 4.2.6.** For each  $g \in \mathcal{G}_\Lambda$ , the collection

$$\{f_T^{\lambda, \mu}(g) : (\lambda, \mu) \in D_g, T \in \mathcal{K}(X_\mu, X_\lambda)\}$$

is a dense subspace of  $E_g$ . Moreover, for a fixed pair  $\lambda, \mu \in \Lambda$  with  $s(\lambda) = s(\mu)$ , the map  $T \mapsto f_T^{\lambda, \mu}$  is a linear map such that  $\|f_T^{\lambda, \mu}(g)\| = \|T\|$  for  $g \in Z(\lambda, \mu)$ ; in particular,  $g \mapsto \|f_T^{\lambda, \mu}(g)\|$  is locally constant and thus continuous.

*Proof.* The first statement follows immediately from the definitions of  $f_T^{\lambda, \mu}(g)$  and of  $E_g$ . Fix  $\lambda, \mu \in \Lambda$  with  $s(\lambda) = s(\mu)$ . The map  $T \mapsto f_T^{\lambda, \mu}$  is linear because the  $i_{\lambda, \mu}^g$  are linear, and for  $g \in Z(\lambda, \mu)$ , we have  $\|f_T^{\lambda, \mu}(g)\| = \|i_{\lambda, \mu}^g(T)\| = \|T\|$  because  $i_{\lambda, \mu}^g$  is an injective  $C^*$ -homomorphism. The final statement follows because  $Z(\lambda, \mu)$  is both open and closed in  $\mathcal{G}_\Lambda$ .  $\square$

We now define

$$\mathcal{E}_X := \text{span}\{f_T^{\lambda, \mu} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu), T \in \mathcal{K}(X_\mu, X_\lambda)\}.$$

This is the collection of sections which will determine the topology on the Fell bundle.

**Proposition 4.2.7.** Let  $E = E_X$  denote the disjoint union  $\bigsqcup_{g \in \mathcal{G}_\Lambda} E_g$ , and let  $\pi : E \rightarrow \mathcal{G}_\Lambda$  be the fibre map. There is a unique topology on  $E$  under which  $\pi : E \rightarrow \mathcal{G}_\Lambda$  becomes a Banach bundle and such that the elements of  $\mathcal{E}_X$  are continuous.

*Proof.* By [15, Proposition 1.6], it suffices to check that each  $E_g$  is a Banach space, that  $g \mapsto \|f(g)\|$  is continuous on  $\mathcal{G}_\Lambda$  for each  $f \in \mathcal{E}_X$ , and that for each  $g \in \mathcal{G}$ , the set  $\{f(g) : f \in \mathcal{E}_X\}$  is dense in  $E_g$ . Let  $f \in \mathcal{E}_X$  be given; to prove that  $g \mapsto \|f(g)\|$  is continuous on  $\mathcal{G}_\Lambda$ , it is sufficient (by Lemma 4.2.6) to observe that  $f$  agrees locally with functions of the form  $f_T^{\lambda, \mu}$ . The other statements follow from the definition of  $E_g$  and Lemma 4.2.6.  $\square$

*Remark 4.2.8.* A basis for the topology on  $E_X$  obtained from Proposition 4.2.7, is described in the second paragraph of the proof of [15, Proposition 1.6]: it consists of the sets

$$W(f, U, \varepsilon) := \{e \in E : \|f(\pi(e)) - e\| < \varepsilon \text{ and } \pi(e) \in U\}$$

where  $U$  varies over open subsets of  $\mathcal{G}_\Lambda$ ,  $f$  varies over  $\mathcal{E}_X$ , and  $\varepsilon > 0$ .

Indeed, we may restrict  $U$  to vary over any basis for the topology on  $\mathcal{G}_\Lambda$ . In particular, we may restrict  $U$  to the basis  $\mathcal{U}_\Lambda$  as defined in subsection 2.2.

The following lemma will be needed for the proof of the main theorem.

**Lemma 4.2.9.** *Fix  $f_1, f_2 \in \mathcal{E}_X$ , an element  $g \in \mathcal{G}_\Lambda$  and an open neighbourhood  $U$  of  $g$ . For any  $\varepsilon, \delta > 0$  such that  $\|f_1(g) - f_2(g)\| < \varepsilon - \delta$ , there exists an open neighbourhood  $V$  of  $g$  such that*

$$W(f_2, V, \delta) \subset W(f_1, U, \varepsilon).$$

*Proof.* By Proposition 4.2.7, each element  $f$  of  $\mathcal{E}_X$  has the property that  $g \mapsto \|f(g)\|$  is continuous. In particular, since  $\|f_1(g) - f_2(g)\| < \varepsilon - \delta$ , there exists a basic neighbourhood  $V$  of  $g$  such that for all  $h \in V$ , we have  $\|f_1(h) - f_2(h)\| < \varepsilon - \delta$ . We claim that this  $V$  suffices. Indeed, if  $e \in W(f_2, V, \delta)$ , then in particular  $\pi(e) \in V \subset U$ , and  $\|e - f_1(\pi(e))\| < \varepsilon$  by the triangle inequality.  $\square$

### 4.3. The main results.

**Theorem 4.3.1.** *Endowed with the multiplication given in Lemma 4.2.3, the involution given in Lemma 4.2.4, and the topology given in Proposition 4.2.7,  $E$  forms a saturated Fell bundle over  $\mathcal{G}_\Lambda$ .*

*Proof.* To prove that the multiplication is continuous, fix  $(g_1, g_2) \in \mathcal{G}_\Lambda^{(2)}$ , elements  $e_i \in E_{g_i}$ , and a basic neighbourhood  $W(f, U, \varepsilon)$  of  $e_1 e_2$  where  $U$  belongs to the basis  $\mathcal{U}_\Lambda$  described in Remark 4.2.8. We must find neighbourhoods  $W_i = W(f_i, U_i, \varepsilon_i)$  of the  $e_i$  such that

$$W_1 W_2 := \{v_1 v_2 : v_i \in W_i, (\pi(v_1), \pi(v_2)) \in \mathcal{G}_\Lambda^{(2)}\} \subset W(f, U, \varepsilon).$$

To do this, we first show that  $W(f, U, \varepsilon)$  contains a smaller neighbourhood of a specific form as justified by the following claim.

**Claim.** There exist  $\delta, \eta > 0$  such that, for any compatible pair of decompositions  $g_1 = (\lambda_1 z, d(\lambda_1) - d(\lambda_2), \lambda_2 z)$  and  $g_2 = (\lambda_2 z, d(\lambda_2) - d(\lambda_3), \lambda_3 z)$ , and for any pair of operators  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$  satisfying

$$\|i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) - e_i\| < \eta,$$

there exists  $n \in \mathbb{N}^k$  such that  $\nu := z(0, n)$  and  $V := Z(\lambda_1 \nu, \lambda_3 \nu)$  satisfy  $g_1 g_2 \in V \subset U$  and

$$e_1 e_2 \in W(f_{T_1 T_2}^{\lambda_1, \lambda_3}, V, \delta) \subset W(f, U, \varepsilon).$$

To prove the claim, we choose  $\delta \in (0, \varepsilon/2)$  such that  $\|e_1 e_2 - f(g_1 g_2)\| < \varepsilon - 2\delta$ . Since the multiplication map  $(e, f) \mapsto ef$  from  $E_{g_1} \times E_{g_2}$  to  $E_{g_1 g_2}$  satisfies  $\|ef\| \leq \|e\| \|f\|$ , there exists  $\eta > 0$  such that for all  $a_i \in E_{g_i}$  with  $\|a_i - e_i\| < \eta$ , we have  $\|a_1 a_2 - e_1 e_2\| < \delta$ . Fix  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$  satisfying

$$\|i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) - e_i\| < \eta.$$

Then setting  $a_i := i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i)$ , our choice of  $\eta$  forces

$$\|f_{T_1 T_2}^{\lambda_1, \lambda_3}(g_1 g_2) - f(g_1 g_2)\| = \|a_1 a_2 - f(g_1 g_2)\| < \|a_1 a_2 - e_1 e_2\| + \|e_1 e_2 - f(g_1 g_2)\| < \varepsilon - \delta.$$

By Lemma 4.2.9 applied to  $f_1 = f$  and  $f_2 = f_{T_1 T_2}^{\lambda_1, \lambda_3}$ , there exists a neighbourhood  $V_0$  of  $g_1 g_2$  such that

$$e_1 e_2 \in W(f_{T_1 T_2}^{\lambda_1, \lambda_3}, V_0, \delta) \subset W(f, U, \varepsilon).$$

There exists  $n \in \mathbb{N}^k$  such that, with  $\nu := z(0, n)$ ,

$$g_1 g_2 \in Z(\lambda_1 \nu, \lambda_3 \nu) \subset V_0 \cap Z(\lambda_1, \lambda_3)$$

since such sets form a neighborhood basis for  $g$  (see subsection 2.2). Setting  $V := Z(\lambda_1 \nu, \lambda_3 \nu)$  concludes the proof of the claim.

Let  $\delta$  and  $\eta$  be as in the claim. Let

$$M := \max\{\|e_1\|, \|e_2\|\} + 2\eta,$$

and define

$$\kappa := \min\left\{\frac{\delta}{2M}, \eta\right\}.$$

By Lemma 4.2.2, we may fix  $x \in \Lambda^\infty$  and  $\lambda_1, \lambda_2, \lambda_3 \in \Lambda r(x)$  such that  $g_i = (\lambda_i x, d(\lambda_i) - d(\lambda_{i+1}), \lambda_{i+1} x)$  and  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$ , and such that

$$(4.3.1) \quad \|i_{\lambda_i, \lambda_{i+1}}^{g_i}(T_i) - e_i\| < \kappa.$$

By the claim, there exists  $n \in \mathbb{N}^k$  such that, with  $\nu := z(0, n)$ ,  $V = Z(\lambda_1 \nu, \lambda_3 \nu)$  is a neighbourhood of  $g_1 g_2$  contained in  $U$  and satisfying

$$e_1 e_2 \in W(f_{T_1 T_2}^{\lambda_1, \lambda_3}, V, \delta) \subset W(f, U, \varepsilon).$$

Observe that  $\|T_i\| < \|e_i\| + \kappa \leq \|e_i\| + \eta$ , so  $M \geq \|e_i\| + 2\eta > (\|T_i\| - \eta) + 2\eta = \|T_i\| + \eta$  for each  $i$ .

Let  $V_i := Z(\lambda_i \nu, \lambda_{i+1} \nu)$  for  $i = 1, 2$ , and observe that  $V = V_1 V_2$ . We claim that the sets

$$W_i := W(f_{T_i}^{\lambda_i, \lambda_{i+1}}, V_i, \kappa)$$

have the desired properties. Indeed, suppose that  $b_i \in W_i$  for  $i = 1, 2$ , and let  $h_i := \pi(b_i) \in V_i$ . By Lemma 4.2.3 we have  $f_{T_1 T_2}^{\lambda_1, \lambda_3}(h_1 h_2) = f_{T_1}^{\lambda_1, \lambda_2}(h_1) f_{T_2}^{\lambda_2, \lambda_3}(h_2)$ . Using this fact and the estimate

$$\|b_i\| \leq \|T_i\| + \kappa \leq \|e_i\| + 2\kappa \leq M,$$

obtained from the definition of the  $b_i$  and (4.3.1), we may calculate:

$$\begin{aligned} \|b_1 b_2 - f_{T_1 T_2}^{\lambda_1, \lambda_3}(h_1 h_2)\| &\leq \|b_1\| \|b_2 - f_{T_2}^{\lambda_2, \lambda_3}(h_2)\| + \|b_1 - f_{T_1}^{\lambda_1, \lambda_2}(h_1)\| \|f_{T_2}^{\lambda_2, \lambda_3}(h_2)\| \\ &< \kappa \|b_1\| + \kappa \|T_2\| \\ &\leq \delta. \end{aligned}$$

We have  $e_1 e_2 \in W_1 W_2$  because  $\|T_i - e_i\| < \kappa$  forces  $e_i \in W_i$  for  $i = 1, 2$ . This completes the proof that multiplication in  $E$  is continuous.

By (4.2.3), the set  $\mathcal{E}_X$  is closed under the map  $f \mapsto f^*$  where  $f^*(g) := f(g^{-1})^*$ . Hence, the involution on  $E_X$  is continuous.

To complete the proof that  $\pi : E_X \rightarrow \mathcal{G}_\Lambda$  is a Fell bundle, we must verify that  $E_X$  is a Banach bundle, satisfies conditions (i)–(x) of Definition 2.7.1, and is saturated. It is a Banach bundle by Proposition 4.2.7. It satisfies (i)–(iv) by Lemma 4.2.3, and satisfies (v)–(x) by Lemma 4.2.4. Finally, that  $E$  is saturated follows from Corollary 4.1.6 and the remark following Definition 2.7.1.  $\square$

*Example 4.3.2.* Let  $(A, X, \chi)$  be a regular  $\Omega_k$ -system of  $C^*$ -correspondences. We have  $\Omega_k^\infty = \{x_n : n \in \mathbb{N}^k\}$  where  $x_n$  is the unique infinite path with range  $n$ . Note that for each  $n$  we have  $x_n = \sigma^n(x_0)$ . For each pair  $m, n \in \mathbb{N}^k$  there is a unique element  $g_{m,n} = (x_m, m - n, x_n)$  of  $\mathcal{G} = \mathcal{G}_{\Omega_k}$  whose range is  $x_m$  and whose source is  $x_n$ , so  $\mathcal{G}$  is isomorphic to the complete equivalence relation  $\mathbb{N}^k \times \mathbb{N}^k$ .

The pullback  $x_0^* X$  associated to the infinite path with range 0 may be identified in the obvious way with  $X$  itself. Hence Proposition 4.1.2, implies that for each  $n$ , the fibre  $E_{x_n}$  is isomorphic to the corresponding corner  $P_n C^*(A, X, \chi) P_n$ . More generally, the fibre  $E_{m,n}$  over the groupoid element  $g_{m,n}$  is the subspace  $P_m C^*(A, X, \chi) P_n$ .

*Remark 4.3.3.* Fix paths  $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$  such that  $s(\lambda_1) = s(\lambda_2) = s(\lambda_3)$ , and for  $i = 1, 2$  let  $T_i \in \mathcal{K}(X_{\lambda_{i+1}}, X_{\lambda_i})$ . It follows from Lemma 4.2.3 that

$$(4.3.2) \quad f_{T_1}^{\lambda_1, \lambda_2} f_{T_2}^{\lambda_2, \lambda_3} = f_{T_1 T_2}^{\lambda_1, \lambda_3}.$$

Recall from 2.7 that  $C_r^*(\mathcal{G}_\Lambda, E_X)$  is defined to be the closure of  $C_c(\mathcal{G}_\Lambda, E_X)$  in the operator norm.

**Lemma 4.3.4.** *Let  $E_X$  be the Fell bundle described in Theorem 4.3.1. Then  $\mathcal{E}_X$  is a dense  $*$ -subalgebra of  $C_r^*(\mathcal{G}_\Lambda, E_X)$ .*

*Proof.* Each element of  $\mathcal{E}_X$  is a linear combination of continuous sections supported on compact sets of the form  $Z(\lambda, \mu)$ . Hence  $\mathcal{E}_X \subset C_c(\mathcal{G}_\Lambda, E_X)$ . As noted above, equation (4.2.3) implies that  $\mathcal{E}_X$  is closed under involution. To show that  $\mathcal{E}_X$  is closed under convolution, fix  $\lambda, \mu, \nu, \tau \in \Lambda$  with  $s(\lambda) = s(\mu)$  and  $s(\nu) = s(\tau)$  and fix  $T_1 \in \mathcal{K}(X_\mu, X_\lambda)$  and  $T_2 \in \mathcal{K}(X_\tau, X_\nu)$ . We must show that  $f_{T_1}^{\lambda, \mu} f_{T_2}^{\nu, \tau} \in \mathcal{E}_X$ . To calculate the product, fix  $g \in \mathcal{G}_\Lambda$ . Since  $\mathcal{G}_\Lambda$  is étale, we have

$$(f_{T_1}^{\lambda, \mu} f_{T_2}^{\nu, \tau})(g) = \sum_{g_1 g_2 = g} i_{\lambda, \mu}^{g_1}(T_1) i_{\nu, \tau}^{g_2}(T_2).$$

Suppose that  $g_1, g_2 \in \mathcal{G}_\Lambda$  satisfy  $g_1 g_2 = g$  and  $i_{\lambda, \mu}^{g_1}(T_1) i_{\nu, \tau}^{g_2}(T_2) \neq 0$ . Then  $g_1 = (\lambda x, d(\lambda) - d(\mu), \mu x)$  for some  $x \in \Lambda^\infty$ , and  $g_2 = (\nu y, d(\nu) - d(\tau), \tau y)$  for some  $y \in \Lambda^\infty$ . Since  $(g_1, g_2) \in \mathcal{G}^{(2)}$ , we have

$$\mu x = s(g_1) = r(g_2) = \nu y,$$

so there is a unique  $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$  such that  $z := \sigma^{d(\mu) \vee d(\nu)}(\mu x)$  satisfies

$$\mu \alpha z = \mu x = \nu y = \nu \beta z.$$

Let  $m = d(\lambda) - d(\mu)$  and  $n = d(\nu) - d(\tau)$ . Then

$$\begin{aligned} d(\lambda \alpha) - d(\tau \beta) &= d(\lambda) + ((d(\mu) \vee d(\nu)) - d(\mu)) - (d(\tau) + ((d(\mu) \vee d(\nu)) - d(\nu))) \\ &= d(\lambda) - d(\mu) - d(\tau) + d(\nu) \\ &= m + n. \end{aligned}$$

We then have  $g = g_1 g_2 = (\lambda x, n + m, \tau y) = (\lambda \alpha z, d(\lambda \alpha) - d(\tau \beta), \tau \beta z)$ .

We conclude that if  $(f_{T_1}^{\lambda, \mu} f_{T_2}^{\nu, \tau})(g)$  is nonzero, then there is a unique  $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$  such that

$$g = (\lambda \alpha z, d(\lambda \alpha) - d(\tau \beta), \tau \beta z) \in Z(\lambda \alpha, \tau \beta),$$

and in this case,

$$\begin{aligned} (f_{T_1}^{\lambda, \mu} f_{T_2}^{\nu, \tau})(g) &= i_{\lambda, \mu}^{(\lambda x, m, \mu x)}(T_1) i_{\nu, \tau}^{(\nu y, n, \tau y)}(T_2) \\ &= i_{\lambda \alpha, \mu \alpha}^{(\lambda x, m, \mu x)}(i_{\lambda, \mu}^{\lambda \alpha, \mu \alpha}(T_1)) i_{\nu \beta, \tau \beta}^{(\nu y, n, \tau y)}(i_{\nu, \tau}^{\nu \beta, \tau \beta}(T_2)) \\ &= i_{\lambda \alpha, \tau \beta}^g(i_{\lambda, \mu}^{\lambda \alpha, \mu \alpha}(T_1) i_{\nu, \tau}^{\nu \beta, \tau \beta}(T_2)) \\ &= f_{i_{\lambda, \mu}^{\lambda \alpha, \mu \alpha}(T_1) i_{\nu, \tau}^{\nu \beta, \tau \beta}(T_2)}^{\lambda \alpha, \tau \beta}(g). \end{aligned}$$

It follows that

$$(f_{T_1}^{\lambda, \mu} f_{T_2}^{\nu, \tau})(g) = \begin{cases} f_{i_{\lambda, \mu}^{\lambda \alpha, \mu \alpha}(T_1) i_{\nu, \tau}^{\nu \beta, \tau \beta}(T_2)}^{\lambda \alpha, \tau \beta}(g) & \text{if } g \in Z(\lambda \alpha, \tau \beta) \text{ for some } (\alpha, \beta) \in \Lambda^{\min}(\mu, \nu) \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$f_{T_1}^{\lambda,\mu} f_{T_2}^{\nu,\tau} = \sum_{(\alpha,\beta) \in \Lambda^{\min(\mu,\nu)}} f_{i_{\lambda,\mu}^{\alpha,\mu}(T_1) i_{\nu,\tau}^{\beta,\tau}(T_2)}^{\lambda\alpha,\tau\beta},$$

which belongs to  $\mathcal{E}_X$  as required.

It remains to prove that  $\mathcal{E}_X$  is dense in  $C_r^*(\mathcal{G}_\Lambda, E_X)$ . Since  $C_c(\mathcal{G}_\Lambda, E_X)$  is dense in  $C_r^*(\mathcal{G}_\Lambda, E_X)$  it suffices to show that we may approximate each  $f \in C_c(\mathcal{G}_\Lambda, E_X)$  by an element of  $\mathcal{E}_X$ . Given  $f \in C_c(\mathcal{G}_\Lambda, E_X)$ , we may rewrite  $f$  as a finite sum of sections, each of which is supported on an element of  $\mathcal{U}_\Lambda$ . So it suffices to consider the case where the support of  $f$  is contained in  $U \in \mathcal{U}_\Lambda$ . On such sections, the uniform norm agrees with the  $C^*$ -norm. Fix  $\varepsilon > 0$ . For each point  $g \in U$  fix  $f_g \in \mathcal{E}_X$  such that  $\|f_g(g) - f(g)\| < \varepsilon/2$ . Then there is an element  $V_g$  of  $\mathcal{U}_\Lambda$  containing  $g$  such that  $\|f_g(h) - f(h)\| < \varepsilon$  for all  $h \in V_g$ . We pass to a finite subcover and then disjointize to obtain a cover of  $U$  by disjoint compact open sets  $V_1, \dots, V_n$  and elements  $f_1, \dots, f_n \in \mathcal{E}_X$  such that the support of each  $f_i$  is contained in  $V_i$  and each  $f_i$  is within  $\varepsilon$  of  $f$  uniformly on  $V_i$ . Now  $\sum_{i=1}^n f_i$  approximates  $f$  within  $\varepsilon$ .  $\square$

To reduce confusion in the next two results and their proofs, we adopt the following notation. For  $\lambda \in \Lambda$  and  $\xi \in X_\lambda$ , there is an element of  $\mathcal{K}(A_{s(\lambda)}, X_\lambda)$  defined by  $a \mapsto \xi \cdot a$ ; we denote this operator by  $l_\xi$ . In particular, for  $a \in A_v$ , we write  $l_a$  for the compact operator on  $X_v = A_v$  implemented by left multiplication by  $a$ . Observe that  $l_\xi^* \in \mathcal{K}(X_\lambda, A_{s(\lambda)})$  is defined by  $l_\xi^*(\eta) := \langle \xi, \eta \rangle_{A_{s(\lambda)}}$ , and in particular that  $l_\xi l_\eta^* = \theta_{\xi, \eta}$  whilst  $l_\xi^* l_\eta = l_{\langle \xi, \eta \rangle_{A_{s(\lambda)}}}$ .

**Proposition 4.3.5.** *The assignments  $\pi_v(a) := f_{l_a}^{v,v}$  and  $\rho_\lambda(x) := f_{l_x}^{\lambda, s(\lambda)}$  determine a Cuntz-Pimsner covariant representation  $(\rho, \pi)$  of  $(A, X, \chi)$  in  $C_r^*(\mathcal{G}_\Lambda, E_X)$ .*

*Proof.* Throughout this proof, we use  $1_{Z(\lambda, \mu)}$  to denote the indicator function of a cylinder set in  $\mathcal{G}_\Lambda$ .

We first show that each  $\pi_v$  is a  $C^*$ -homomorphism. They are linear by definition. We have  $1_{Z(v,v)}(g) = 1$  if and only if  $g = g^{-1} = (x, 0, x)$  for some  $x \in v\Lambda^\infty$ , and since  $a \mapsto l_a$  is a homomorphism we obtain  $\pi_v(a)^* = (f_{l_a}^{v,v})^*(g) = 1_{Z(v,v)}(g) i_v^g(l_a^*) = f_{l_a^*}^{v,v}(g) = \pi_v(a^*)$  from (4.2.3). Similarly,

$$\begin{aligned} \pi_v(a)\pi_v(b) &= (f_{l_a}^{v,v} f_{l_b}^{v,v})(g) = \sum_{g_1 g_2 = g} 1_{Z(v,v)}(g_1) 1_{Z(v,v)}(g_2) i_v^{g_1}(a) i_v^{g_2}(b) \\ &= 1_{Z(v,v)}(g) i_v^g(a) i_v^g(b) = f_{l_{ab}}^{v,v}(g) = \pi_v(ab). \end{aligned}$$

since  $i_v^g$  and  $a \mapsto l_a$  are homomorphisms.

The  $\rho_\lambda$  are clearly linear maps, and  $\rho_v = \pi_v$  for  $v \in \Lambda^0$  by definition.

We must next check the multiplicative property Definition 3.2.1(2). Fix  $\alpha, \beta \in \mathcal{G}_\Lambda$ , and fix  $\xi \in X_\alpha$  and  $\eta \in X_\beta$ . Then for  $g \in \mathcal{G}_\Lambda$ ,

$$\rho_\alpha(\xi)\rho_\beta(\eta) = \sum_{g_1 g_2 = g} 1_{Z(\alpha, s(\alpha))}(g_1) 1_{Z(\beta, s(\beta))}(g_2) i_{\alpha, s(\alpha)}^{g_1}(l_\xi) i_{\beta, s(\beta)}^{g_2}(l_\eta).$$

The conditions  $g_1 g_2 = g$ ,  $1_{Z(\alpha, s(\alpha))}(g_1) = 1$  and  $1_{Z(\beta, s(\beta))}(g_2) = 1$  combine to force  $g = (\alpha\beta x, d(\alpha\beta), x) \in Z(\alpha\beta, s(\beta))$  for some  $x \in s(\beta)\Lambda^\infty$ , and  $g_1 = (\alpha\beta x, d(\alpha), \beta x)$  and  $g_2 = (\beta x, d(\beta), x)$ ; in particular, they force  $s(\alpha) = r(\beta)$ , so  $\rho_\alpha(\xi)\rho_\beta(\eta) = 0$  if  $s(\alpha) \neq r(\beta)$ .

If  $s(\alpha) = r(\beta)$ , let  $g_\alpha := (r(g), d(\alpha), \sigma^{d(\alpha)}(r(g)))$  and  $g_\beta := (\sigma^{d(\alpha)}(r(g)), d(\beta), s(g))$ . Then we may continue our calculation:

$$\begin{aligned} \rho_\alpha(\xi)\rho_\beta(\eta) &= 1_{Z(\alpha\beta, s(\beta))}(g) i_{\alpha, s(\alpha)}^{g_\alpha}(l_\xi) i_{\beta, s(\beta)}^{g_\beta}(l_\eta) \\ &= 1_{Z(\alpha\beta, s(\beta))}(g) i_{\alpha\beta, s(\alpha\beta)}^g(i_{\alpha, s(\alpha)}^{\alpha\beta, \beta}(l_\xi) l_\eta) \\ &= 1_{Z(\alpha\beta, s(\beta))}(g) i_{\alpha\beta, s(\alpha\beta)}^g(l_{\chi_{\alpha, \beta}(\xi \otimes \eta)}) \\ &= \rho_{\alpha\beta}(\chi_{\alpha, \beta}(\xi \otimes \eta)). \end{aligned}$$

We have next to show that if  $d(\alpha) = d(\beta)$ , then  $\rho_\alpha(\xi)^* \rho_\beta(\eta) = \delta_{\alpha, \beta} \pi_{s(\alpha)}(\langle \xi, \eta \rangle_{A_{s(\alpha)}})$  for all  $\xi \in X_\alpha$  and  $\eta \in X_\beta$ . Fix such  $\alpha, \beta, \xi$  and  $\eta$ . For  $g \in \mathcal{G}_\Lambda$ ,

$$\begin{aligned} (\rho_\alpha(\xi)^* \rho_\beta(\eta))(g) &= ((f_\xi^{\alpha, s(\alpha)})^* f_\eta^{\beta, s(\beta)})(g) \\ &= \sum_{g_1 g_2 = g} f_\xi^{\alpha, s(\alpha)}(g_1^{-1})^* f_\eta^{\beta, s(\beta)}(g_2) \\ &= \sum_{g_1 g_2 = g} 1_{Z(s(\alpha), \alpha)}(g_1) 1_{Z(\beta, s(\beta))}(g_2) i_{s(\alpha), \alpha}^{g_1}(l_\xi^*) i_{\beta, s(\beta)}^{g_2}(l_\eta). \end{aligned}$$

The conditions  $g_1 g_2 = g$ ,  $1_{Z(s(\alpha), \alpha)}(g_1) = 1$  and  $1_{Z(\beta, s(\beta))}(g_2) = 1$  combine to force  $g = (x, 0, x)$  for some  $x \in s(\alpha)\Lambda^\infty$ ,  $g_1 = (x, -d(\alpha), \alpha x)$  and  $g_2 = g_1^{-1}$ . In particular, if  $\alpha \neq \beta$ , then  $(\rho_\alpha(\xi)^* \rho_\beta(\eta))(g) = 0$ . Hence, writing  $g_\alpha = (r(g), -d(\alpha), \alpha r(g))$ ,

$$\begin{aligned} (\rho_\alpha(\xi)^* \rho_\beta(\eta))(g) &= \delta_{\alpha, \beta} 1_{Z(s(\alpha), s(\alpha))}(g) i_{s(\alpha), \alpha}^{g_\alpha}(l_\xi^*) i_{\alpha, s(\alpha)}^{g_\alpha^{-1}}(l_\eta) \\ &= \delta_{\alpha, \beta} 1_{Z(s(\alpha), s(\alpha))}(g) i_{s(\alpha), s(\alpha)}^g(l_\xi^* l_\eta) = \delta_{\alpha, \beta} \pi_{s(\alpha)}(\langle \xi, \eta \rangle_{A_{s(\alpha)}}) \end{aligned}$$

as required.

It remains to show that  $(\rho, \pi)$  is Cuntz-Pimsner covariant. To do this, we first show that for  $\lambda \in \Lambda$ ,  $T \in \mathcal{K}(X_\lambda)$  and  $g \in \mathcal{G}_\Lambda$ , we have  $\rho^{(\lambda)}(T) = 1_{Z(\lambda, \lambda)}(g) i_{\lambda, \lambda}^g(T)$ . By linearity and continuity, it suffices to consider  $T = \theta_{\xi, \eta}$  for some  $\xi, \eta \in X_\lambda$ . For this we calculate

$$\begin{aligned} \rho^{(\lambda)}(\theta_{\xi, \eta}) &= \rho_\lambda(\xi)\rho_\lambda(\eta)^* \\ &= \sum_{g_1 g_2 = g} f_\xi^{\lambda, s(\lambda)}(g_1) (f_\eta^{\lambda, s(\lambda)})^*(g_2) \\ &= \sum_{g_1 g_2 = g} f_\xi^{\lambda, s(\lambda)}(g_1) (f_\eta^{\lambda, s(\lambda)})(g_2^{-1})^* \\ (4.3.3) \quad &= \sum_{g_1 g_2 = g} 1_{Z(\lambda, s(\lambda))}(g_1) 1_{Z(s(\lambda), \lambda)}(g_2) i_{\lambda, s(\lambda)}^{g_1}(l_\xi) i_{s(\lambda), \lambda}^{g_2}(l_\eta^*) \end{aligned}$$

by (4.2.3). The conditions  $g_1 g_2 = g$ ,  $1_{Z(\lambda, s(\lambda))}(g_1) = 1$  and  $1_{Z(s(\lambda), \lambda)}(g_2) = 1$  combine to force  $g = (\lambda x, 0, \lambda x)$  for some  $x \in s(\lambda)\Lambda^\infty$ ,  $g_1 = (\lambda x, d(\lambda), x)$  and  $g_2 = g_1^{-1}$ . Hence (4.3.3) together with the characterisation (4.2.2) of multiplication in  $E_X$  implies that

$$\rho^{(\lambda)}(\theta_{\xi, \eta}) = 1_{Z(\lambda, \lambda)}(g) i_{\lambda, \lambda}^g(\theta_{\xi, \eta})$$

as claimed.

By Condition 3.1.1(2), we have  $\phi_\lambda(a) = i_{r(\lambda)}^\lambda(l_a)$  for all  $a \in A_{r(\lambda)}$ . Hence

$$(4.3.4) \quad \rho^{(\lambda)}(\phi_\lambda(a)) = 1_{Z(\lambda, \lambda)}(g) i_{\lambda, \lambda}^g(i_{r(\lambda)}^\lambda(l_a)) = 1_{Z(\lambda, \lambda)}(g) i_{r(\lambda), r(\lambda)}^g(l_a).$$

We are now ready to establish that  $(\rho, \pi)$  is Cuntz-Pimsner covariant. Indeed, fix  $n \in \mathbb{N}^k$ ,  $v \in \Lambda^0$  and  $a \in A_v$ . Observe that  $Z(v, v) = \bigsqcup_{\lambda \in v\Lambda^n} Z(\lambda, \lambda)$ . Hence, for  $g \in \mathcal{G}_\lambda$ , using equation (4.3.4) to obtain the first equality, we calculate

$$\sum_{\lambda \in v\Lambda^n} \rho^{(\lambda)}(\phi_\lambda(a)) = \sum_{\lambda \in v\Lambda^n} 1_{Z(\lambda, \lambda)}(g) i_{r(\lambda), r(\lambda)}^g(l_a) = 1_{Z(v, v)}(g) i_{r(\lambda), r(\lambda)}^g(l_a) = f_{l_a}^{v, v}(g) = \pi_v(a),$$

so  $(\rho, \pi)$  is Cuntz-Pimsner covariant as required.  $\square$

Our second main theorem identifies the cross-sectional algebra of the Fell bundle with the  $C^*$ -algebra  $C^*(A, X, \chi)$  of the  $\Lambda$ -system  $X$  (see Section 3).

**Theorem 4.3.6.** *There is an isomorphism  $\Psi : C^*(A, X, \chi) \rightarrow C_r^*(\mathcal{G}_\Lambda, E_X)$  such that under the canonical identifications  $A_v = \mathcal{K}(X_v, X_v)$  and  $X_\lambda = \mathcal{K}(X_{s(\lambda)}, X_\lambda)$ , we have*

$$\begin{aligned} \Psi(\pi_v^X(a)) &= f_{l_a}^{v, v} \quad \text{for all } v \in \Lambda^0 \text{ and } a \in A_v, \text{ and} \\ \Psi(\rho_\lambda^X(x)) &= f_{l_x}^{\lambda, s(\lambda)} \quad \text{for all } \lambda \in \Lambda \text{ and } x \in X_\lambda. \end{aligned}$$

*Proof.* Let  $(\rho, \pi)$  be the Cuntz-Pimsner covariant representation of  $(A, X, \chi)$  in  $C_r^*(\mathcal{G}_\Lambda, E_X)$  obtained from Proposition 4.3.5. This gives a homomorphism

$$\Psi = \Psi_{\rho, \pi} : C^*(A, X, \chi) \rightarrow C_r^*(\mathcal{G}_\Lambda, E_X)$$

as in Definition 3.2.7 satisfying the given formulae. It remains to show that  $\Psi$  is bijective.

We use Theorem 3.3.1 to prove injectivity. Observe that for each  $v \in \Lambda^0$ , the map  $\pi_v$  is injective, so condition Theorem 3.3.1(2) holds.

Define a cocycle  $c : \mathcal{G}_\Lambda \rightarrow \mathbb{Z}^k$  by  $c(x, n, y) := n$ . Then  $c$  is locally constant and therefore continuous. Hence there is a strongly continuous action  $\beta : \mathbb{T}^k \rightarrow \text{Aut}(C_r^*(\mathcal{G}_\Lambda, E_X))$  determined by  $\beta_z(f)(g) = z^{c(g)} f(g)$ . Observe that  $\beta_z(f_T^{\lambda, \mu}) = z^{d(\lambda) - d(\mu)} f_T^{\lambda, \mu}$  for all  $T \in \mathcal{K}(X_\mu, X_\lambda)$ . In particular, for all  $v \in \Lambda^0$  and  $a \in A_v = \mathcal{K}(X_v, X_v)$  we have

$$\beta_z(\pi_v(a)) = \beta_z(f_{l_a}^{v, v}) = f_{l_a}^{v, v} = \pi_v(a),$$

and for all  $\lambda \in \Lambda$  and  $x \in X_\lambda$  we have

$$\beta_z(\rho_\lambda(x)) = \beta_z(f_{l_x}^{\lambda, s(\lambda)}) = z^{d(\lambda)} f_{l_x}^{\lambda, s(\lambda)} = z^{d(\lambda)} \rho_\lambda(x).$$

Hence Theorem 3.3.1(1) is also satisfied. Thus  $\Psi$  is injective.

For surjectivity, it suffices by Lemma 4.3.4 to show that the set  $\mathcal{E}_X$  is contained in the range of  $\Psi$ . For this it suffices by definition of  $\mathcal{E}_X$  to show that  $f_T^{\lambda, \mu}$  is in the range of  $\Psi$  for all  $\lambda, \mu \in \Lambda$  with  $s(\lambda) = s(\mu)$  and all  $T \in \mathcal{K}(X_\mu, X_\lambda)$ . Fix such  $\lambda, \mu$  and  $T$ , and let  $v := s(\lambda) = s(\mu)$ . Since the range of  $\Psi$  is closed, we may assume that  $T$  is finite rank, that is,  $T = \sum_{j=1}^n \theta_{x_j, y_j}$  where  $x_j \in X_\lambda$  and  $y_j \in X_\mu$  for  $j = 1, \dots, n$ . We have  $T = \sum_{j=1}^n l_{x_j} l_{y_j}^*$ . Hence, by the linearity of the section map  $S \mapsto f_S^{\lambda, \mu}$  and equation (4.3.2), we have:

$$\begin{aligned} f_T^{\lambda, \mu} &= \sum_{j=1}^n f_{l_{x_j} l_{y_j}^*}^{\lambda, \mu} = \sum_{j=1}^n f_{l_{x_j}}^{\lambda, v} f_{l_{y_j}^*}^{v, \mu} = \sum_{j=1}^n f_{l_{x_j}}^{\lambda, v} (f_{l_{y_j}}^{\mu, v})^* \\ &= \sum_{j=1}^n \Psi(\rho_\lambda^X(x_j)) \Psi(\rho_\lambda^X(y_j))^*. \end{aligned}$$

Thus  $f_T^{\lambda, \mu}$  is in the range of  $\Psi$ . This concludes the proof of surjectivity.  $\square$

## 5. EXAMPLES

**5.1. Higher-rank graph  $C^*$ -algebras.** Let  $\Lambda$  be a higher-rank graph. For each  $v \in \Lambda^0$ , let  $A_v$  be the 1-dimensional  $C^*$ -algebra  $\mathbb{C}$ , and for each  $\lambda \in \Lambda$ , let  $X_\lambda$  be the 1-dimensional Hilbert space  $\mathbb{C}$  regarded as an  $A_{r(\lambda)}-A_{s(\lambda)}$   $C^*$ -correspondence. The multiplication isomorphisms  $\chi_{\alpha,\beta} : w \otimes z \mapsto wz$  determine a  $\Lambda$ -system  $(A, X, \chi)$  of  $C^*$ -correspondences. Observe that the product system  $Y$  of  $C^*$ -correspondences constructed from this example as in Proposition 3.1.7 is precisely the product system associated to  $\Lambda$  by [51, Proposition 3.2]. The  $\Lambda$ -system  $(A, X, \chi)$  is regular precisely when  $\Lambda$  is row-finite and has no sources, and then  $C^*(A, X, \chi)$  is canonically isomorphic to  $C^*(\Lambda)$  by [51, Theorem 4.2].

Suppose that  $\Lambda$  is indeed row-finite with no sources. Then the Fell-bundle  $E_X$  is the trivial bundle  $\mathcal{G}_\Lambda \times \mathbb{C}$ , and its reduced cross-sectional algebra is the completion of the image of  $C_c(\mathcal{G}_\Lambda)$  under the regular representation, that is, the reduced groupoid  $C^*$ -algebra  $C_r^*(\mathcal{G}_\Lambda)$ . The isomorphism  $C^*(\Lambda) \cong C_r^*(\mathcal{G}_\Lambda)$  of Kumjian and Pask (see [31, Corollary 3.5(i)]) is a special case of Theorem 4.3.6.

**5.2. Strong shift equivalent  $C^*$ -correspondences.** Consider the situation of Example 3.1.6(i). That is, let  $A$  and  $B$  be  $C^*$ -algebras, let  $R$  be an  $A$ - $B$  correspondence, and let  $S$  be a  $B$ - $A$  correspondence. Suppose that  $R$  and  $S$  are each full and nondegenerate, and that the left action of the coefficient algebra on each is implemented by an injective homomorphism into the compact operators. Let  $(\Lambda^0, \Lambda^1)$  be the directed graph with vertices  $v, w$  and edges  $\{e, f\}$  where  $s(e) = r(f) = v$  and  $r(e) = s(f) = w$ . Define  $A_v = A$  and  $A_w = B$ ,  $X_e = R$  and  $X_f = S$ , and for  $m \geq 2$  and  $\mu \in \Lambda^m$ , define

$$X_\mu := X_{\mu_1} \otimes_{A_{s(\mu_1)}} X_{\mu_2} \otimes_{A_{s(\mu_2)}} \cdots \otimes_{s(\mu_{m-1})} X_{\mu_m}.$$

Define isomorphisms  $\chi_{\mu,\nu} : X_\mu \otimes_{A_{s(\mu)}} X_\nu$  by

$$\chi_{\mu,\nu}((x_{\mu_1} \otimes \cdots \otimes x_{\mu_m}) \otimes (x_{\nu_1} \otimes \cdots \otimes x_{\nu_n})) = x_{\mu_1} \otimes \cdots \otimes x_{\mu_m} \otimes x_{\nu_1} \otimes \cdots \otimes x_{\nu_n}.$$

The result is a regular  $\Lambda$ -system  $(A, X, \chi)$  of correspondences. Notice that our notion of regularity implies the one in [39].

Muhly, Pask and Tomforde show in [39] that  $\mathcal{O}_{R \otimes_B S}$  and  $\mathcal{O}_{S \otimes_A R}$  are Morita-Rieffel equivalent. We can reinterpret this in terms of the  $\Lambda$ -system. Indeed, if  $1_v, 1_w \in \mathcal{M}(C^*(A, X, \chi))$  are the images of the units of the multiplier algebras of  $A_v$  and  $A_w$  respectively, then each of  $1_v$  and  $1_w$  is a full projection, and two applications of the gauge-invariant uniqueness theorem show that

$$1_v C^*(A, X, \chi) 1_v \cong \mathcal{O}_{R \otimes_B S} \quad \text{and} \quad 1_w C^*(A, X, \chi) 1_w \cong \mathcal{O}_{S \otimes_A R},$$

so that  $1_v C^*(A, X, \chi) 1_w$  implements the desired Morita-Rieffel equivalence.

The graph  $\Lambda$  has precisely two infinite paths, namely  $(ef)^\infty \in v\Lambda^\infty$  and  $(fe)^\infty \in w\Lambda^\infty$ . By construction,

$$E_{(ef)^\infty} = \varinjlim_n \mathcal{K}((R \otimes_B S)^{\otimes n}) \quad \text{and} \quad E_{(fe)^\infty} = \varinjlim_n \mathcal{K}((S \otimes_A R)^{\otimes n}),$$

which are precisely the fixed-point subalgebras of the copies of  $\mathcal{O}_{R \otimes_B S}$  and  $\mathcal{O}_{S \otimes_A R}$  embedded in  $C^*(A, X, \chi)$  as above.

More generally, let  $\Lambda$  be the path-category of the directed graph consisting of a simple cycle of length  $n$ . Let  $(A, X, \chi)$  be a regular  $\Lambda$ -system. For any two vertices  $v, w \in \Lambda^0$  let  $\mu_{v,w}$  denote the unique path of minimal length from  $w$  to  $v$ , and for each vertex  $v$ , let  $\lambda_v$  denote the unique cycle of length  $n$  with range  $v$ . Proposition 4.1.2 implies that



the fibre  $E_{x_v}$  is isomorphic to the fixed point algebra  $\mathcal{O}_{X_{\lambda_v}}^\gamma$  for the gauge action on the Cuntz-Pimsner algebra of  $X_{\lambda_v}$  (see the final paragraph of Section 2.5). Then for distinct  $v, w$ , the correspondences  $X_{\mu_{v,w}}$  and  $X_{\mu_{w,v}}$  become an instance of the situation considered above, and we obtain a Morita-Rieffel equivalence

$$(5.2.1) \quad \mathcal{O}_{X_{\lambda_v}} \cong \mathcal{O}_{X_{\mu_{v,w}} \otimes_{A_w} X_{\mu_{w,v}}} \sim \mathcal{O}_{X_{\mu_{w,v}} \otimes_{A_v} X_{\mu_{v,w}}} \cong \mathcal{O}_{X_{\lambda_w}}.$$

Indeed, as argued above, each  $1_v C^*(A, X, \chi) 1_v \cong \mathcal{O}_{X_{\lambda_v}}$  and the  $1_v C^*(A, X, \chi) 1_w$  implement the Morita-Rieffel equivalences (5.2.1). For each vertex  $v \in \Lambda^0$  there is a unique infinite path  $x_v = (\lambda_v)^\infty$  such that  $v = r(x_v)$ . Recall that we may identify  $\Lambda^\infty$  with  $\mathcal{G}_\Lambda^{(0)}$ , and then for each  $v \in \Lambda^0$  the fibre  $E_{x_v}$  is isomorphic to the fixed-point algebra for the gauge action on  $\mathcal{O}_{X_{\lambda_v}}$ .

**5.3.  $\Gamma$ -systems of  $k$ -morphs.** Let  $\Gamma$  be a row-finite  $\ell$ -graph with no sources, and let  $W$  be a  $\Gamma$ -system of  $k$ -morphs satisfying the technical assumptions  $(\spadesuit)$  of [34]. Proposition 6.4 of [34] shows how to associate to each  $k$ -morph  $W_\gamma$  (where  $\gamma \in \Gamma$ ) a  $C^*(\Lambda_{r(\gamma)})$ – $C^*(\Lambda_{s(\gamma)})$  correspondence  $\mathcal{H}(W_\gamma)$ . The proof of [34, Theorem 6.6] shows how to construct isomorphisms  $\chi_{\mu,\nu} : \mathcal{H}(W_\mu) \otimes_{C^*(\Lambda_{s(\mu)})} \mathcal{H}(W_\nu) \rightarrow \mathcal{H}(W_{\mu\nu})$ , and it is routine to verify using the associativity conditions imposed on the system  $W$  of  $k$ -morphs that the  $\chi_{\mu,\nu}$  satisfy the associativity condition (3) of Definition 3.1.1. Let  $A_v := C^*(\Lambda_v)$  for all  $v \in \Gamma^0$  and let  $X_\gamma := \mathcal{H}(W_\gamma)$  for each  $\gamma \in \Gamma$ . Then  $(A, X, \chi)$  is a  $\Gamma$ -system of  $C^*$ -correspondences.

Let  $\Sigma$  be a  $\Gamma$ -bundle for the  $\Gamma$ -system of  $k$ -morphs. This is a  $(k + \ell)$ -graph  $\Sigma$  together with a functor  $f : \Sigma \rightarrow \Gamma$  called the bundle map such that the degree  $d(f(\lambda)) = (d(\lambda)_{k+1}, \dots, d(\lambda)_{k+\ell})$  for all  $\lambda \in \Sigma$ , each  $f^{-1}(v) \cong \Lambda_v$ , and each  $f^{-1}(\lambda) \cap \{\mu \in \Sigma : d(\mu)_i = 0 \text{ for } i \leq k + 1\}$  is isomorphic to  $X_\lambda$ . We claim that  $C^*(\Sigma) \cong C^*(A, X, \chi)$ . To see this, fix  $\sigma \in \Sigma$ . Factorise  $\sigma = y\lambda$  where  $d(\lambda)_i = 0$  for  $i \geq k + 1$  and  $d(y)_i = 0$  for  $i \leq k$ . Then we may identify  $\lambda$  with an element of  $\Lambda_{s(\gamma)}$  and  $y$  with an element of  $W_{f(\sigma)}$ . Let  $\gamma = f(\sigma)$ , so that  $y \in W_\gamma$ .

Proposition 6.7 of [34] shows that

$$X_\gamma = \mathcal{H}(W_\gamma) \cong \overline{C_c(W_\gamma) \otimes_{C_0(\Lambda_{s(\gamma)})} C^*(\Lambda_{s(\gamma)})}.$$

In particular if  $\delta_y \in C_c(W_\gamma)$  is the point-mass, and if  $s_\lambda$  denotes the canonical generator of  $C^*(\Lambda_{s(\gamma)})$ , then  $\delta_y \otimes s_\lambda$  is an element of  $X_\gamma$ . Let  $(\rho, \pi)$  denote the universal representation of  $(A, X, \chi)$  in  $C^*(A, X, \chi)$ , and define  $t_\sigma \in C^*(A, X, \gamma)$  by  $t_\sigma := \rho_\gamma(\delta_y \otimes s_\lambda)$ . It is routine to check that  $\{t_\sigma : \sigma \in \Sigma\}$  is a Cuntz-Krieger  $\Sigma$ -family. These elements generate  $C^*(A, X, \chi)$  because the elements  $\delta_y \otimes s_\lambda$  span a dense subspace of each  $X_\gamma$ . Hence the universal property of  $C^*(\Sigma)$  ensures that there is a surjective homomorphism  $\phi : C^*(\Sigma) \rightarrow C^*(A, X, \chi)$  such that  $\phi(s_\sigma) = t_\sigma$  for all  $\sigma$ . A straightforward application of the gauge-invariant uniqueness theorem [31, Theorem 3.4] shows that  $\phi$  is injective.

For  $g = (\alpha x, d(\alpha) - d(\beta), \beta x) \in \mathcal{G}_\Gamma$ , the fibre  $E_g$  of the Fell-bundle  $E$  can be described using the isomorphism  $\phi : C^*(\Sigma) \rightarrow C^*(A, X, \chi)$  as

$$E_g = \overline{\text{span}} \left( \bigcup_{n \in \mathbb{N}^k} \{ \phi(s_\mu s_\nu^*) : f(\mu) = \alpha x(0, n) \text{ and } f(\nu) = \beta x(0, n) \} \right).$$

**5.4. Systems of endomorphisms of a single  $C^*$ -algebra.** Consider a  $C^*$ -algebra  $A$  and a row-finite  $k$ -graph  $\Lambda$  with no sources. Suppose that  $\lambda \mapsto \varphi_\lambda$  is a contravariant functor from  $\Lambda$  to  $\text{End}_1(A)$ , the semigroup of approximately unital endomorphisms of  $A$  regarded as a category with one object  $A$ . Assuming each  $\varphi_\lambda$  is injective, we construct a regular  $\Lambda$ -system  $(A, X, \chi)$  as in Example 3.1.6(iv), where  $A_v = A$  for  $v \in \Lambda^0$  and

$X_\lambda = \varphi_\lambda A$  for  $\lambda \in \Lambda^1$ . In this case,  $E_x = \varinjlim(A_{x(n)}, \varphi_{x(m,n)})$  for  $x \in \Lambda^\infty = \mathcal{G}_\Lambda^{(0)}$ ; if all  $\varphi_\lambda$  are automorphisms,  $E_x$  is isomorphic to  $A$  for all  $x$ .

We regard the  $C^*$ -algebra  $C^*(A, X, \chi)$  as a kind of crossed-product of  $A$  by an action of  $\Lambda$ . We consider two special cases (see also section 4.3 in [46]).

*Example 5.4.1* (Crossed products by  $\mathbb{Z}^k$ ). Consider  $k$  commuting automorphisms  $\alpha_1, \dots, \alpha_k$  of a  $C^*$ -algebra  $A$ . For  $m \in \mathbb{N}^k$  let  $X_m = \alpha^m A$ , where  $\alpha^m = \alpha_1^{m_1} \cdots \alpha_k^{m_k}$ . Then  $X = (X_m)$  is a  $T_k$ -system of correspondences, where  $T_k$  is the  $k$ -graph whose path category can be identified with  $\mathbb{N}^k$ . The corresponding  $C^*$ -algebra  $C^*(A, X, \chi)$  is isomorphic to the crossed product  $A \rtimes \mathbb{Z}^k$ .

In this instance, the groupoid  $\mathcal{G}_\Lambda$  is isomorphic to  $\mathbb{Z}^k$ . Note that  $E_0 = A$ , and, more generally, for  $n \in \mathbb{Z}^k$  we have  $E_n = \alpha^n A$  with the same multiindex convention as above. Observe that the Fell bundle  $E$  is the Fell bundle over  $\mathbb{Z}^k$  obtained in the usual fashion from the dual action of  $\mathbb{T}^k$  on the crossed product.

*Example 5.4.2* (Cuntz's twisted tensor products). In [6], Cuntz constructs a twisted tensor product  $A \times_{\mathcal{U}} \mathcal{O}_n$ . Let  $A$  be a unital  $C^*$ -algebra acting nondegenerately on the Hilbert space  $\mathcal{H}$ , let  $\mathcal{U} = (U_1, \dots, U_n)$  be a family of commuting unitaries in  $\mathcal{L}(\mathcal{H})$  implementing automorphisms  $\alpha_1, \dots, \alpha_n$  of  $A$ , and let  $S_1, \dots, S_n$  be the isometries generating  $\mathcal{O}_n$ . Cuntz defines  $A \times_{\mathcal{U}} \mathcal{O}_n$  as the  $C^*$ -subalgebra of  $\mathcal{L}(\mathcal{H}) \otimes \mathcal{O}_n$  generated by  $A \otimes 1$  together with  $U_1 \otimes S_1, \dots, U_n \otimes S_n$ . Notice that we have the relations

$$(\alpha_i(a) \otimes 1)(U_i \otimes S_i) = (U_i \otimes S_i)(a \otimes 1), \text{ for } a \in A, i = 1, \dots, n.$$

Let  $\Lambda$  be the 1-graph with  $\Lambda^0 = \{v\}$  and  $\Lambda^1 = \{e_1, \dots, e_n\}$ . Let  $\alpha_1, \dots, \alpha_n$  be any collection of automorphisms of  $A$  (we need not assume that the  $\alpha_i$  commute, though this is the case in Cuntz's setting). Let  $A_v := A$ , and for each edge  $e_i$  let  $X_{e_i} := \alpha_i^{-1} A$ . Let  $(A, X, \chi)$  be the corresponding  $\Lambda$ -system. The  $C^*$ -algebra  $C^*(A, X, \chi)$  is generated by  $A$  and  $n$  isometries  $V_1, \dots, V_n$  with relations

$$V_i^* V_j = \delta_{ij}, \quad \sum_{i=1}^n V_i V_i^* = 1, \quad \alpha_i(a) V_i = V_i a, \quad i = 1, \dots, n.$$

Indeed, for  $i \leq n$ , let  $x_i = (0, \dots, 1_A, \dots, 0) \in \bigoplus_{i=1}^n X_{e_i}$ , where the  $1_A$  is in position  $i$ . Then  $\langle x_i, x_j \rangle = \delta_{ij}$  and  $x = \sum_{i=1}^n x_i \langle x_i, x \rangle$  for all  $x \in \bigoplus_{i=1}^n X_{e_i}$ . Consider the image  $V_i$  of  $x_i$  in  $C^*(A, X, \chi)$ . Then  $V_i^* V_j = \delta_{ij}$ ,  $\sum_{i=1}^n V_i V_i^* = 1$ , and since  $a \cdot x_i = x_i \langle x_i, a \cdot x_i \rangle$ , we get  $a V_i = V_i \alpha_i^{-1}(a)$  or  $\alpha_i(a) V_i = V_i a$  for  $a \in A$ . If the  $\alpha_i$  commute, the isomorphism between  $C^*(A, X, \chi)$  and  $A \times_{\mathcal{U}} \mathcal{O}_n$  is given by

$$a \mapsto a \otimes 1, \quad V_i \mapsto U_i \otimes S_i.$$

Notice that  $C^*(A, X, \chi)$  unitaly contains an isomorphic copy of  $C^*(\Lambda) = \mathcal{O}_n$ , and the groupoid  $\mathcal{G}_\Lambda$  is the Cuntz groupoid described in [53, Definition III.2.1]. As noted above, each fibre of the Fell bundle over  $\mathcal{G}_\Lambda^{(0)}$  is isomorphic to  $A$ .

**5.5. Ionescu's  $C^*$ -algebras associated to Mauldin-Williams graphs.** Our discussion is based on the class of examples considered by Ionescu in [24]. Let  $\Lambda$  be a  $k$ -graph (Ionescu considers a finite directed graph). Let  $\mathcal{T}$  be the category whose objects are compact metric spaces  $T$ , and whose morphisms are contractions. Let  $\lambda \mapsto \varphi_\lambda$  be a covariant functor from  $\Lambda$  to  $\mathcal{T}$ ; since we identify the vertices of  $\Lambda$  with its objects, we have  $\varphi_v = \text{id}_{T_v}$  for each  $v \in \Lambda^0$ .

For  $v \in \Lambda^0$ , let  $A_v := C(T_v)$ , and for  $\lambda \in \Lambda$ , let  $\varphi_\lambda^* : A_{r(\lambda)} \rightarrow A_{s(\lambda)}$  be the induced map  $\varphi_\lambda^*(f) = f \circ \varphi_\lambda$ . Now define  $X_\lambda$  to be the  $C^*$ -correspondence  $\varphi_\lambda^* A_{s(\lambda)}$  from  $A_{r(\lambda)}$  to  $A_{s(\lambda)}$ . For each composable pair  $\alpha, \beta$  in  $\Lambda$ , there is an isomorphism  $\chi_{\alpha, \beta} : X_\alpha \otimes_{A_{s(\alpha)}} X_\beta \rightarrow X_{\alpha\beta}$  given by

$$\chi_{\alpha, \beta}(f \otimes_{A_{s(\alpha)}} g)(t) = (\varphi_\beta^*(f)g)(t) = f(\varphi_\beta(t))g(t).$$

Since  $T \mapsto C(T)$  and  $\varphi \mapsto \varphi^*$  determines a contravariant functor from  $\mathcal{T}$  to the category of  $C^*$ -algebras with  $C^*$ -homomorphisms as morphisms, the triple  $(A, X, \chi)$  is a  $\Lambda$ -system of  $C^*$ -correspondences.

The  $X_\lambda$  are always full and nondegenerate with a left action by compact operators. The left actions are all injective when the  $\varphi_\lambda$  are all surjective. In particular, the system  $X$  is regular if  $\Lambda$  is row-finite with no sources, and each  $\varphi_\lambda$  is surjective.

For each element  $x \in \Lambda^\infty$ , the fibre  $E_x$  of the Fell bundle  $E$  is given by

$$E_x = \varinjlim \mathcal{K}(X_{x(0,n)}) = \varinjlim (C(T_{x(n)}), \varphi_{x(m,n)}^*) = C(\varprojlim (T_{x(n)}, \varphi_{x(m,n)})).$$

In Ionescu's setting,  $\Lambda$  is the path category of a finite directed graph with no sinks or sources and there is a constant  $c < 1$ , such that the contraction constant  $c_\lambda$  of  $\varphi_\lambda$  satisfies  $c_\lambda \leq c$  for every  $\lambda \in \Lambda^1$ . As in Remark 3.1.5, the functor is then determined by  $\{T_v : v \in \Lambda^0\}$  and  $\{\varphi_e : e \in \Lambda^1\}$ . Since  $c_\lambda \leq c < 1$  for every  $\lambda \in \Lambda^1$ , the intersection  $\bigcap_{n=1}^\infty \varphi_{x(0,n)}(T_{x(n)})$  is a singleton  $\{t_x\}$  for any infinite path  $x \in \Lambda^\infty$ . By the universal property of projective limits, it follows that  $\varprojlim (T_{x(n)}, \varphi_{x(m,n)})$  is a singleton, so  $E_x \cong \mathbb{C}$ . Unfortunately, the  $\varphi_\lambda$  will typically not be surjective in this setting. So the resulting  $\Lambda$ -system will fail to be regular.

Quigg (see [49]) considers graphs of  $C^*$ -correspondences over row-finite 1-graphs with no sources with a view towards generalizing Ionescu's result by working in the category of locally compact Hausdorff spaces with continuous proper maps.

**5.6. Pinzari, Watatani and Yonetani's systems of  $C^*$ -correspondences.** In section 5.3 of [46], Pinzari, Watatani and Yonetani study KMS states on the  $C^*$ -algebra defined using a finite family of  $C^*$ -correspondences. More precisely, let  $A_1, \dots, A_n$  be unital simple  $C^*$ -algebras. Fix a matrix  $\Sigma = (\sigma_{i,j}) \in M_n(\{0, 1\})$  with no row and no column identically zero. For each pair  $(i, j)$  such that  $\sigma_{i,j} = 1$ , let  $X_{i,j}$  be a full, finite projective  $A_i$ - $A_j$   $C^*$ -correspondence. Pinzari, Watatani and Yonetani study the KMS states on the Cuntz-Pimsner algebra of the  $C^*$ -correspondence  $X := \bigoplus_{\sigma_{i,j} \neq 0} X_{i,j}$  over  $A := \bigoplus_{i=1}^n A_i$ .

Let  $\Lambda_\Sigma$  be the 1-graph with  $\Lambda^0 = \{v_1, \dots, v_n\}$  and  $\Lambda^1 = \{e_{i,j} : \sigma_{i,j} = 1\}$  with  $s(e_{i,j}) = v_j$  and  $r(e_{i,j}) = v_i$ . As in Remark 3.1.5, setting  $A_{v_i} := A_i$  and  $X_{e_{i,j}} := X_{i,j}$  determines a regular  $\Lambda_\Sigma$ -system of  $C^*$ -correspondences. Our construction of  $C^*(A, X, \chi)$  in Definition 3.2.7 is so that  $C^*(A, X, \chi) \cong \mathcal{O}_{\bigoplus X_i}$ , the  $C^*$ -algebra studied by Pinzari-Watatani-Yonetani.

**5.7. Topological graphs fibred over directed graphs.** Let  $\Lambda$  be a 1-graph, and fix locally compact spaces  $T_v$  and  $U_e$  for each  $v \in \Lambda^0$  and  $e \in \Lambda^1$ . Suppose there are local homeomorphisms  $\sigma_e : U_e \rightarrow T_{s(e)}$  and continuous maps  $\rho_e : U_e \rightarrow T_{r(e)}$  for each  $e \in \Lambda^1$ . These data define a  $\Lambda$ -system of  $C^*$ -correspondences, by taking  $A_v = C_0(T_v)$  and  $X_e$  to be the  $A_{r(e)}$ - $A_{s(e)}$   $C^*$ -correspondence obtained from the completion of  $C_c(U_e)$ , with inner product defined by

$$\langle \xi, \eta \rangle(t) = \sum_{\sigma_e(u)=t} \overline{\xi(u)} \eta(u)$$

and with right and left multiplications defined by

$$(f \cdot \xi \cdot g)(u) = f(\rho_e(u))\xi(u)g(\sigma_e(u)),$$

where  $\xi, \eta \in C_c(U_e)$ ,  $f \in C_0(T_{r(e)})$ , and  $g \in C_0(T_{s(e)})$ . This  $\Lambda$ -system is regular if  $\Lambda$  is row-finite with no sources, the maps  $\sigma_e$  are surjective, and the  $\rho_e$  are proper with dense range. By construction,  $C^*(A, X, \chi)$  is isomorphic to the  $C^*$ -algebra of the topological graph with vertex space  $\bigsqcup_{v \in \Lambda^0} T_v$ , edge space  $\bigsqcup_{e \in \Lambda^1} U_e$ , and source and range maps defined using  $\sigma_e$  and  $\rho_e$ . An important special case of this example is given by the following.

**5.8. Katsura's realisation of nonunital Kirchberg algebras.** Building on earlier work of Deaconu (see [9]), Katsura (see [28]) constructs the topological graph  $\Lambda \times_{n,m} \mathbb{T}$  as above, with  $T_v = U_e = \mathbb{T}$  for all  $v \in \Lambda^0$  and  $e \in \Lambda^1$ . Given two maps  $n : \Lambda^1 \rightarrow \mathbb{Z}_+$  and  $m : \Lambda^1 \rightarrow \mathbb{Z}$ , define  $\sigma_e : U_e \rightarrow T_{s(e)}$  by  $\sigma_e(z) = z^{n(e)}$ , and  $\rho_e : U_e \rightarrow T_{r(e)}$  by  $\rho_e(z) = z^{m(e)}$ . He shows that every nonunital Kirchberg algebra is isomorphic to the  $C^*$ -algebra of such a topological graph where the maps  $n, m$  are chosen appropriately (see [28, Lemmas 4.2 and 4.4 and Proposition 4.5]). The associated  $\Lambda$ -system of  $C^*$ -correspondences is regular precisely when  $\Lambda$  is row-finite with no sources and  $m(e) \neq 0$  for all  $e \in \Lambda^1$ .

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