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The Immunization of Capital Projects and Other Cash Flow Streams

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by

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Abstract

Firms are continually investing resources in projects that are risky in the sense of uncertain outcomes. The need for firms to protect the net asset backing of their project portfolios and to immunise against unacceptable cash flow streams is evident in a number of contemporary practices such as factoring, sub-leasing and joint ventures. But the ad hoc farming out of projects does not provide a means of systemically deriving strategies that are optimal in terms of providing adequate protection at minimum cost. The model presented in this paper does provide such a framework. It illustrates why firms use joint-ventures and similar strategies as a form of risk sharing and shows how it is implemented in an optimal manner.

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1. Introduction

Uncertain outcomes make decision making difficult. Whilst a firm would not wish to forgo a potentially profitable project - one which adds significantly to its profitability and strengthens its balance sheet position - it would not wish to adopt projects which are marginal or where cash flows ultimately prove to be negative. This is especially so when firms have a disproportionate share of their asset portfolios invested in one project, since in this instance, an unfavourable outcome may even threaten the firm's very existence. Such risks may be reduced if the firm, after deciding on a minimum acceptable cash flow from the project, could then insure against receiving less than this minimum amount.

Attempts to insure against the undesirable outcomes attaching to risky projects are already evident in the commercial world. Obvious examples are joint ventures and the "farming out" procedures used in the oil and mining industry. Under joint ventures for example, two or more firms may agree to fund a project. The firms then participate in the outcomes in some agreed proportion, whether they be positive or negative. In this way, no one firm contributes all the required capital nor bears the full costs should the project prove to be unprofitable. Each firm gains a means of limiting its exposure in terms of funds committed to the project and its liability should the project prove unprofitable. Hence, such procedures permit participation in projects which might be beyond the resources or risk exposure of any one firm.

Similar considerations apply to the "farming out" procedures utilised in the oil exploration industry. In this instance, the holder of an exploration permit agrees to apportion a share of the proceeds from a productive well to a second party, if that party bears some or all of the costs associated with drilling the well. Benefits and costs are then shared in accordance with the drilling or "farming out" agreement. Again, the result is to limit both firms' exposure in terms of funding or liability for costs.

However, insurance arrangements such as these may be far from optimal. For example, they may provide too much or too little protection against unfavourable outcomes. Or where the protection is adequate, it may have been obtained by sacrificing a greater equity in the project than was necessary. In this paper, we address issues such as these by outlining a project immunisation model which formalises the process of providing the desired protection. Section two introduces the process of forming a hedge security under a single period binomial model. This is then expanded in section three to a multi-period scenario.

The problem may be stated as follows. A firm wishes to invest in a project where, at the time of investment, the distribution of the future cash flows is uncertain. The firm is faced with two options. Firstly, it may finance the entire project from its own resources. If it pursues this option, it will be completely exposed to the possibility of undesirably low (perhaps even negative) cash flows. Alternatively, the firm may immunise against the risk of actual cash flows falling below a minimum level. This can be achieved by constructing a hedge security which guarantees a minimum return, should the project's realised cash flow fall below a prescribed amount.

The process of forming the hedge security referred to above effectively involves the creation of a put option. As a consequence, the Black and Scholes (1973) analysis may appear to be a logical starting point[1]. However, the principal assumption of the Black-Schools model is that the underlying security (in our case the project's cash flows), follows a **geometric** Brownian motion[2]. This means that project cash flows could never be negative. To overcome this, we effectively assume that project cash flows follow an **additive** Brownian motion, which does admit the possibility of negative outcomes[3].

2. The Single Period case

We initially deal with the problem of capital project immunisation in a simple, single period binomial model world. Hence, suppose a firm is considering an investment project with an initial cash outlay of A at time t, where t is the project's remaining life. At time t - h, the net cash flow from the investment project will be either $(A + \delta)$ or $(A - \delta)$, where the latter cash flow could well be negative. The firm desires to hedge its project portfolio by constructing a security which guarantees a minimum cash inflow of E at time t - h. We thus construct the hedge security which consists of a proportionate investment Δ in the project, and the dollar amount of B in risk free government bonds. Since we can choose Δ and B in any way we want, we impose the following conditions:

$$\Delta(A + \delta) + r^h B = W_u$$
$$\Delta(A - \delta) + r^h B = W_d \tag{1}$$

where r is one plus the annual risk free rate of interest and W_u and W_d are the amounts necessary to return a cash flow of E at time t - h, should the project's cash flow be A + δ or A - δ respectively. Hence, if the project's time t - h cash flow should exceed expectations, we have:

$$W_u = \begin{cases} 0 & \text{if } (A + \delta) \geq E \\ E - (A + \delta) & \text{if } (A + \delta) < E \end{cases}$$

Alternatively, if the project's time t - h cash flow should fall below expectations, we have:

$$W_d = \begin{cases} 0 & \text{if } (A - \delta) \geq E \\ E - (A - \delta) & \text{if } (A - \delta) < E \end{cases}$$

The initial problem then is to find the values for the proportionate investment in the project (Δ) and the dollar amount to be invested in government bonds (B) which will result in the necessary values for W_u and W_d. This, given any realised cash flow and stated minimum return which the firm is just willing to accept from the project.

To do this, we solve the equations in (1) for Δ and B as follows:

$$\Delta = \frac{W_u - W_d}{2\delta} \tag{2}$$

$$B = \frac{\dfrac{(\delta - A)\, W_u}{2\delta} + \dfrac{(\delta + A)\, W_d}{2\delta}}{r^h}$$

Using these results we can compute the value W, of the hedge security at time t, as follows:

$$W = \Delta A + B$$

$$W = \frac{\frac{[(r^h-1)A+\delta]W_u}{2\delta} + \frac{[(r^h-1)]A-\delta]W_d}{2\delta}}{r^h} \qquad (3)$$

As an example, suppose A = 1 and that this is the firm's initial capital. We further suppose that $\delta = 0.20$, h = 1 and the pure annual rate of interest is r - 1 = 0.125. If we want to at least guarantee return of the firm's initial capital, we have that E = 1. To form the hedge security which guarantees this result we first note that the above information implies $W_u = 0$ and $W_d = 0.20$. Substituting these values in equation (3), it then follows that $\Delta = -0.5$ and $B = 24/45 \approx \$0.533333$. In other words, we invest approximately \$0.53 in government bonds and short the project to the extent of \$0.50. The payoff schedule associated with this scenario is as follows:

	Time t	Time t - h	
		$[A_{t-h} = A_t - \delta]$	$[A_{t-h} = A_t + \delta]$
Sell short 50% share in project	0.500000	-0.40	-0.60
Buy bonds	-0.533333	0.60	0.60
Cost (-) / Payoff (+)	-0.033333	0.20	0.00

If we create the above hedge security and go long in the underlying project, we obtain the following cost and payoff schedule:

	Time t	Time t - h	
		$[A_{t-h} = A_t - \delta]$	$[A_{t-h} = A_t + \delta]$
Purchase 50% share in project	-0.500000	0.40	0.60
Buy bonds	-0.533333	0.60	0.60
Cost (-) / Payoff (+)	-1.033333	1.00	1.20

Note that this asset portfolio has the required property - namely that the firm's end of period or time t - h capital, will be at least equal to its initial or time t capital.

A significant difficulty however, is that the required initial investment of \$1.033333 exceeds the firm's endowed capital of \$1. This problem can be overcome by a partial liquidation of the hedge security, the proceeds being invested in government bonds. We thus define the scaling factor g, such that:

$$g(W + A) + D = 1 \tag{4}$$

where g(W + A) is the firm's combined investment in the hedge security and the underlying project, and D is the required extra investment in government bonds. Note that this equation imposes the requirement that the firm cannot invest more than its initial capital of \$1. However, since we want the time t - h proceeds from this investment portfolio to be at least equal to the firm's initial capital, we impose the further condition:

$$g + r^h D = 1 \tag{5}$$

This condition applies since, in the very least, the firm's investment portfolio must return the firm's opening capital of \$1, at the end of the period. Note that the firm's combined investment in the hedge security and the underlying project g(W + A) will return, as a minimum g, at time t - h. Similarly, the investment in government bonds will have accumulated to r^hD, with the effect of interest by time t - h. Hence, we can determine the parameters necessary to insure that the initial investment will not exceed the firm's endowed capital, by solving equations (4) and (5) as follows:

$$g = \frac{1 - r^{-h}}{(W + A) - r^{-h}}$$

$$D = \frac{(W + A) - 1}{(W + A) r^h - 1} \tag{6}$$

For the example presently under consideration, W = 0.033333. A = 1. r = 1.125 and h = 1, so that from the equations (6), we have g = 10/13 whilst D = 8/39. Previously, the firm's asset portfolio was made up of a 50% share in the project and a \$24/45 ≈ \$0.533333 investment in government bonds. Hence, the new or "scaled" portfolio is composed of a $0.5^*g = 5/13 \approx 0.384615$ share in the project, and a $24/45^*g + 8/39 = \$8/13 \approx \$0.615384$ investment in government bonds. This portfolio has the following cost and payoff characteristics:

	Time t	Time t - h	
		$[A_{t-h} = A_t - \delta]$	$[A_{t-h} = A_t + \delta]$
Purchase 5/13 share in project	-0.384615	0.307692	0.461538
Buy bonds	-0.615385	0.692398	0.692308
Cost (-) / Payoff (+)	-1.000000	1.000000	1.153846

Note that the firm's investment portfolio is now self financing, but at the cost of a lower cash flow when the project's returns are higher than expected.

3. The Multi period Case[4]

To generalise the above analysis to a multi period context, where project cash flows accrue continuously in time, it is merely necessary to restate equation (3) in simpler form and use Taylor series approximations for certain key expressions. Hence. restating equation (3), we have:

$$\frac{[A(r^h-1)+\delta]W(A+\delta,t-h)}{2\delta}+\frac{[A(r^h-1)-\delta]W(A-\delta,t-h)}{2\delta}-r^h W(A,t)=0 \tag{7}$$

where $W(A + \delta,t - h) = W_u$, $W(A - \delta,t - h) = W_d$ and $W(A, t) = W$. If we expand $W(A + \delta,t - h)$ as a Taylor series about the point (A, t) we have [Apostol (1969, pp. 308-309)]:

$$W(A + \delta, t - h) \approx W(A, t) + \delta \frac{\partial W}{\partial A} + \frac{1}{2}\delta^2 \frac{\partial^2 W}{\partial A^2} - h\frac{\partial W}{\partial t} + \text{-----} \tag{8}$$

Similarly, for W(A - δ,t - h) we have:

$$W(A - \delta, t - h) \approx W(A, t) - \delta \frac{\partial W}{\partial A} + \frac{1}{2}\delta^2 \frac{\partial^2 W}{\partial A^2} - h\frac{\partial W}{\partial t} + \text{-----} \tag{9}$$

Finally, expanding r^h as a Taylor series about the origin, we have [Apostol (1967, pp. 278-279)]:

$$r^h = 1 + h \log r + \text{-----------} \tag{10}$$

Following Cox and Miller (1964, p. 206) we let $\delta = \sigma\sqrt{h}$, where σ^2 is the variance rate in the project's cash flows. Using this assumption in conjunction with equations (7) through (10), dividing by h and then letting h -----> 0, we have that the multi period version of the binomial model (3), satisfies the following partial differential equation:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial A^2} + iA\frac{\partial W}{\partial A} - \frac{\partial W}{\partial t} - i\,W(A, t) = 0 \tag{11}$$

where i = log r is the continuously compounded annual risk free rate of interest. Further, we have the <u>initial</u> condition:

$$W(A,0) = \begin{cases} 0 & \text{if } A \geq E \\ E - A & \text{if } A < E \end{cases} \tag{12}$$

To convert equation (11) to solvable form, we make the following substitutions:

$$W(A, t) = e^{-it}F(\xi,\eta)$$

where:

$$\xi = \frac{\sqrt{2}[E - A e^{it}]}{\sigma} \tag{13}$$

and:

$$\eta = \frac{e^{2it} - 1}{2i}$$

thus reducing equation (11) to the diffusion equation of mathematical physics [Crank (1975, p 11)]:

$$\frac{\partial^2 F}{\partial \xi^2} = \frac{\partial F}{\partial \eta} \tag{14}$$

Further, since:

$$\xi(A,0) = \frac{\sqrt{2}[E - A]}{\sigma}$$

the initial condition becomes:

$$F(\xi,0) = \begin{cases} \dfrac{\sigma}{\sqrt{2}}\xi & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases} \tag{15}$$

The solution to the partial differential equation (14) is [Weinberger (1965, pp. 320,328)]:

$$F(\xi,\eta) = \frac{1}{\sqrt{4\pi\eta}} \int_0^\infty F(y) \exp\left[- \frac{(\xi - y)^2}{4\eta} \right] dy$$

If we make the substitution:

$$y = \xi + z\sqrt{2\eta}$$

and use the initial condition (15), we obtain the following unique solution to the problem (11), (12):

$$W(A, t) = [Ee^{-it} - A]N(d) + \sigma\sqrt{\frac{1 - e^{-2it}}{4\pi i}}e^{-\frac{1}{2}d^2} \tag{16}$$

where:

$$d = \frac{A e^{it} - E}{\sigma\sqrt{\frac{e^{2it} - 1}{2i}}} \qquad (17)$$

and:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{d}^{\infty} e^{-\frac{1}{2}y^2} dy \qquad (18)$$

(18)

is the value of the complementary accumulated standard normal distribution.

The sum result of the above analysis is that in continuous time, the hedge security is formed by investing

$$e^{-it}[EN(d) + \sigma\sqrt{\frac{e^{2it} - 1}{4\pi i}}\, e^{-\frac{1}{2}d^2}] \qquad (19)$$

in government bonds and selling short the proportionate investment N(d), in the underlying project. We then have that the firm's asset portfolio consists of the hedge security and a long investment in the underlying project. At any time t, the value of this portfolio is:

$$W(A, t) + A = e^{-it}[EN(d) + \sigma\sqrt{\frac{e^{2it} - 1}{4\pi i}}\, e^{-\frac{1}{2}d^2}] + AN(-d) \qquad (20)$$

The first term on the right of the above expression represents the firm's investment in risk free government bonds, whilst the second term is the firm's investment in the underlying project.

As an example, suppose as in the case of the binomial model, A = 1 and that this is the firm's initial capital. We further suppose that $\sigma = 0.20$ whilst the risk free (continuously compounded annual rate of interest is i = 11.78%. If we want to at least guarantee return of the firm's initial capital, we have that E = 1. From equation (17), this information implies d = 0.59. Using equation (19) we then have at the beginning of the period, that the firm's asset portfolio consists of a \$0.31 investment in government bonds and a long proportionate investment of 72.24% in the underlying project.

As with the discrete example, the total cost of this asset portfolio, which is 0.7224 + 0.31 = \$1.0324, exceeds the firm's initial capital. However, by rebalancing the portfolio in the same manner as the discrete case, this problem is easily overcome. Thus, if we let g be the proportionate investment in the hedge security, we have:

$$g = \frac{1 - e^{-it}}{(W + A) - e^{-it}} \tag{21}$$

$$D = \frac{(W + A) - 1}{(W + A)e^{it} - 1} \tag{22}$$

Using these equations, we have g = 0.774233 and D = 0.2007. Since the previous asset portfolio is made up of a 72.24% share in the project and a \$0.31 investment in government bonds, the scaled portfolio is composed of a 0.774233*0.7224 = 55.93% share in the project and a 0.774233*0.31 + 0.2007 = \$0.4407 investment in government bonds. Note that the total cost of this asset portfolio is 0.5593 + 0.4407 = \$1, which is the firm's initial capital. In other words, the firm's investment portfolio is now self financing.

4. Discussion and Extension

The principal difficulty with this approach is that as the standard normal variate d changes, the firm's asset portfolio will need to be rebalanced. From equation (17), it will be noted d is a function of time, so that in principle rebalancing will need to occur on a continuous basis. For firms with inflexible asset portfolios this may well be a demanding requirement. However, even where only periodic or no rebalancing is achievable, the above analysis is still useful in assessing the extent to which the firm is exposed to unfavourable movements in project cash flows. In words, the project immunisation model provides a cardinal measure of how the composition of the firm's asset base varies from what it ought to be.

The technique may be particularly valuable where a firm has not one, but a number of projects making up its project portfolio. Where the firm has multiple projects, the problem is to accumulate projects which will yield an investment portfolio with the desired overall risk characteristics and cash flow requirements. To do this, the firm needs to be able to continually redefine the risk profiles of individual existing projects as compared with the desired risk profile of the total project portfolio. Over time, information on the realised risk characteristics and cash flow streams of existing projects will become known. Then, as new investment opportunities become available, the firm can select those projects which will re-balance the firm's risk and cash flow profile in the sense of moving it towards the desired base.

The project immunisation model presented here has several advantages over current ad hoc attempts to insure against unacceptably low returns from risky projects. The model shows how to optimally go about providing the desired protection against less than required cash flows,. while giving up the minimum equity in the desiring a minimum guaranteed return from only one project, or to a firm with a portfolio of projects which seeks to ensure a minimum net asset base, or a minimum cash flow stream over any period of time.

The most obvious refinement which could be made to the above model, is the incorporation of the stochastic interest rates. In this respect, the seminal procedures outlined in Merton (1973, pp. 162-167) are likely to be a useful starting point. Further, the sensitivity of the above procedures to parameter misspecification would also be an interesting empirical issue. This especially so, since it is likely that firms will have only limited information on project variances and expected cash flows, when they are first put into service.

Footnotes

1. See Bird and Tippett (1986) for a simple introduction to portfolio insurance, as it affects equity portfolios. Benninga and Blume (1985) contains a more advanced and detailed treatment.

2. See Cox, Ross and Rubenstein (1979) for a very clear exposition of the assumptions underlying the Black-Scholes model.

3. See Cox and Miller (1964, pp. 205-208) for a good introduction to this topic.

4. The <u>method</u> of proof used in this section (as distinct from the proof itself), was first suggested by Cox, Ross and Rubenstein (1979, p. 254) and amplified on in Cox and Rubenstein (1985, pp. 208-209).

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